On Observability and Controllability of Faulty Induction Machines

Mikael Stocks[†] and Alexander Medvedev[‡]

Abstract—In this paper an analysis of observability and controllability of the *electrical* part of faulty (non-symmetric) Induction Machines (IMs) is performed. A two-axis model of healthy or faulty IM is studied and the conditions under which it does not degenerate to a *descriptor* system are obtained. When degeneration does not occur and stator voltages, stator currents and the rotational angle/speed of the rotor are measured, a fault in the IM does not, in general, render the motor neither unobservable nor uncontrollable. However, it turns out that when the rotor resistance is very small, problems with observability and controllability still may arise.

NOMENCLATURE

Variables	Parameters
ψ - magnetic flux	R - electrical resistance
i - electrical current	L - electrical lumped inductance
u - feeding voltage	M - electrical mutual inductance
γ - rotor angle	

For the *variables*, subscripts S and r refer to the stator- and rotor frames in a conventional two-axis $\alpha\beta$ -description of the Induction Machine (IM), respectively. Moreover, a subscript α refers to the α -axis whereas β refers to the β -axis.

For the *parameters*, a capital letter subscript refers to the stator axis and a small letter subscript refers to the corresponding rotor axis. The units used for the variables and parameters are SI-units throughout the paper.

Bold capital letters stand for matrices, whereas small and capital letters denote respectively scalars and vectors. Identity and zero matrices of appropriate dimension (mostly 2×2) are denoted as $\mathbf{I}$ and $\mathbf{0}$. The unitary matrices

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{T}_{\bar{\gamma}} = e^{\bar{\gamma}\mathbf{J}} = \begin{bmatrix} \cos(\bar{\gamma}) & -\sin(\bar{\gamma}) \\ \sin(\bar{\gamma}) & \cos(\bar{\gamma}) \end{bmatrix}$$

where $\bar{\gamma} \in \mathcal{R}$, are frequently used in the sequel. Finally, differentiation with respect to time is denoted by $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $x^{(k)} = \frac{d^kx}{dt^k}$, or similarly, in some cases, it applies to derivation with respect to τ . It should be clear from the context which of these two alternatives is meant.

I. INTRODUCTION

Observability and controllability are fundamental properties of mathematical models of dynamic systems.

Up to date, not so much work has been done on observability and controllability of the IM, even for the case of healthy motor. Observability of the IM including the mechanical part is analyzed in [1] by means of a nonlinear approach based on Lie algebra. The case of a healthy IM with no sensors for the mechanical part is treated, highlighting the encountered observability problems. The performed observability analysis is *local* which is common to non-linear systems. For the

cases when observability is lost, it is not made clear which parts of the system become unobservable.

Another recent paper regarding healthy IMs is [2], where the mechanical part is also considered. The analysis in that paper is indeed *global* and the unobservable subspace can be derived by means of solving a differential-algebraic equation. Several interesting special cases are considered, for example the case of constant rotor speed. However, simple general conditions for observability are yet hard to derive.

No results on observability or controllability of non-symmetrical or faulty IM have been found in the literature available to the authors. This topic is important since observer-based methods of fault detection require the knowledge of plant observability and controllability both for the nominal case and under fault. It is also essential for implementing fault-tolerant control of motor drives.

The present paper aims at filling some gaps in observability and controllability analysis of IMs, both healthy and faulty ones. Only the electrical part is considered whose linearity enables *global* analysis. The parts of the system dynamics that can potentially become unobservable/uncontrollable are specified by utilizing *Kalman's Canonical Structure Theorem* (Kalman's CST) [3]. Further, the conditions under which the studied mathematical models of IMs do not lose differential order and degenerate to *descriptor* systems are given. No analysis regarding observability/controllability for descriptor models is provided yet.

The paper is composed as follows. First suitable definitions of observability and controllability of linear time-varying systems are formulated. Then mathematical models of IM with faulty stator or rotor are recapitulated. Further, controllability and observability properties of the models are proven.

II. PRELIMINARIES

Consider the descriptor system

$$\begin{aligned} \mathbf{E}\dot{x} &= \mathbf{A}x + \mathbf{B}u \\ y &= \mathbf{C}x + \mathbf{D}u \end{aligned}$$

where $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are complex-valued, not necessarily square, matrix functions of t and $x(t) \in \mathcal{C}^n$. The differential order of the system is given by the rank of $\mathbf{E}$ assuming that it is constant. If $\forall t$ $\mathbf{E}$ is (square) nonsingular, the descriptor system reduces to

$$\begin{aligned} \dot{x} &= \bar{\mathbf{A}}x + \bar{\mathbf{B}}u \\ y &= \mathbf{C}x + \mathbf{D}u \end{aligned} \quad (1)$$

where $\bar{\mathbf{A}} = \mathbf{E}^{-1}\mathbf{A}$ and $\bar{\mathbf{B}} = \mathbf{E}^{-1}\mathbf{B}$. Note that a unique solution to (1) exists for a given $x(t_0)$ if the right-hand side of the dynamical part is integrable in some well-defined

[†]Systems and Interaction, Luleå University of Technology, SE-971 87 Luleå, SWEDEN, e-mail: Stocks@ltu.se

[‡]Information Technology, Uppsala University, SE-751 05 Uppsala, SWEDEN, e-mail: Alexander.Medvedev@it.uu.se

sense. The state-transition matrix $\Phi(t, t_0)$ of (1) is the unique solution to

$$\dot{\Phi}(t, t_0) = \bar{\mathbf{A}}(t)\Phi(t, t_0); \quad \Phi(t_0^-, t_0^-) = \mathbf{I}$$

System (1) is said to be *observable* on the interval $[t_0, t_f]$ (finite or infinite) if the observability gramian [4]

$$\mathcal{O}(t_0, t_f) = \int_{t_0^-}^{t_f^+} \Phi^H(\tau, t_0) \mathbf{C}^H(\tau) \mathbf{C}(\tau) \Phi(\tau, t_0) d\tau$$

is positive definite. Above, $\cdot^H$ denotes complex conjugate transpose of a matrix. Let $|\cdot|$ denote the determinant of a matrix. Clearly, if, for some t_f^* , it holds that $|\mathcal{O}(t_0, t_f^*)| \neq 0$, then $\forall t_f \geq t_f^* : |\mathcal{O}(t_0, t_f)| \neq 0$. Observability implies that

$$\forall p \in \mathbb{C}^n, p \neq 0 : p^H \mathcal{O}(t_0, t_f) p = \int_{t_0^-}^{t_f^+} \|z_o(\tau, t_0)\|^2 d\tau > 0$$

where $z_o(\tau, t_0) = \mathbf{C}(\tau)\Phi(\tau, t_0)p$ and $\|\cdot\|$ is the vector Euclidean norm. Hence, observability is lost when there is $p \neq 0$ such that $\|z_o(\tau, t_0)\| = 0, t_0 \leq \tau \leq t_f$ for *almost* all τ , except for a set of isolated points of Lebesgue-measure zero. Taking into account the physics of the problem, distributions (generalized functions) are not considered here. Since $z_o(\tau, t_0) \equiv 0, t_0 \leq \tau \leq t_f$, it follows that $\forall k \in \mathbb{I}^+ : z_o^{(k)}(\tau, t_0) \equiv 0, t_0 \leq \tau \leq t_f$.

Analogously, (1) is said to be *controllable* on $[t_0, t_f]$ if the controllability gramian

$$\mathcal{C}(t_0, t_f) = \int_{t_0^-}^{t_f^+} \Phi(t_0, \tau) \bar{\mathbf{B}}(\tau) \bar{\mathbf{B}}^H(\tau) \Phi^H(t_0, \tau) d\tau$$

is positive definite. Controllability implies that

$$\forall p \in \mathbb{C}^n, p \neq 0 : p^H \mathcal{C}(t_0, t_f) p = \int_{t_0^-}^{t_f^+} \|z_c(t_0, \tau)\|^2 d\tau > 0$$

where $z_c(t_0, \tau) = \bar{\mathbf{B}}^H(\tau)\Phi^H(t_0, \tau)p$. Similar to the observability case, for (1) to be uncontrollable, the condition $z_c(t_0, \tau) \equiv 0, t_0 \leq \tau \leq t_f$ has to be satisfied for some $p \neq 0$.

In the sequel, the notion of unobservable (uncontrollable) states is also utilized. It is agreed that elements of the state vector x of (1) are unobservable (uncontrollable) on $[t_0, t_f]$ if $|\mathcal{O}(t_0, t_f)| = 0$ ($|\mathcal{C}(t_0, t_f)| = 0$) for the gramian evaluated for the whole system but $|\mathcal{O}(t_0, t_f)| \neq 0$ ($|\mathcal{C}(t_0, t_f)| \neq 0$) when the gramian is evaluated for the system model with all unobservable (uncontrollable) state vector elements excluded.

III. MODELLING OF INDUCTION MACHINES

In this section, a mathematical model describing how motor parameters are influenced by electrical faults is summarized. A general model is derived in detail in [5] and extensively validated in experiments with industrial IMs by intentionally inflicting electrical and magnetic asymmetry on them. Condensed model derivations for faulty IMs can be found in [6], [7] and validation results in [8].

Under idealizing assumptions, a model of the IM is derived by defining three coordinate axes for the stator (A, B, C) and three for the rotor (a, b, c), with both coordinate systems lying in the same plane, see Fig. 1. The angle between two consecutive stator- or rotor axes is $\frac{2\pi}{3}$. The angle between

the stator- and rotor frameworks (the electrical rotor angle) is then defined as $\bar{\gamma} = p\gamma$, where p is the number of pole-pairs in the stator.

Fig. 1. Three-axis description of the IM. The stator and rotor frames are separated by the electrical angle $\bar{\gamma}$.

The electrical parameters of each axis are thereafter defined in a straightforward way. Fig.2 illustrates the model parameters definitions with respect to stator phase A and rotor phase a . Note the interaction between stator phase A and rotor phase a through M_{Aa} . The situation for the other phases is similar.

Fig. 2. Stator phase A , rotor phase a and the mutual inductance M_{Aa} between phases.

After defining all electrical parameters of the stator and rotor, fundamental laws of physics can then be used to derive a dynamical model of the *electrical* part of an IM in three axes. In order to obtain a mathematically simpler model, a transformation from three- to two axes is often applied.

A. General induction machine

A conventional description of the *electrical* part of an IM in two axes is given by

$$\begin{aligned} \dot{\psi}_S &= -\mathbf{R}_S \mathbf{i}_S + u_S \\ \dot{\psi}_r &= -\mathbf{R}_r \mathbf{i}_r \\ \begin{bmatrix} \psi_S \\ \psi_r \end{bmatrix} &= \mathbf{L} \begin{bmatrix} \mathbf{i}_S \\ \mathbf{i}_r \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} \mathbf{L}_S & \mathbf{M} \\ \mathbf{M}^T & \mathbf{L}_r \end{bmatrix} \end{aligned} \quad (2)$$

where

$$\begin{aligned} \mathbf{i}_S &= [\mathbf{i}_{\alpha S} \quad \mathbf{i}_{\beta S}]^T; & \mathbf{i}_r &= [\mathbf{i}_{\alpha r} \quad \mathbf{i}_{\beta r}]^T \\ \psi_S &= [\psi_{\alpha S} \quad \psi_{\beta S}]^T; & \psi_r &= [\psi_{\alpha r} \quad \psi_{\beta r}]^T \\ u_S &= [u_{\alpha S} \quad u_{\beta S}]^T \end{aligned}$$

To get rid of the algebraic equations and at the same time avoid problems with possible singularities, the algebraic

equations are differentiated. After collecting terms, one gets

$$\mathbf{L} \begin{bmatrix} \dot{i}_S \\ \dot{i}_r \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_S & -\dot{\mathbf{M}} \\ -\dot{\mathbf{M}}^T & -\mathbf{R}_r \end{bmatrix} \begin{bmatrix} i_S \\ i_r \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} u_S \quad (3)$$

If the inductance matrix $\mathbf{L}$ is singular, then (3) is a descriptor system, *i. e.* a system of differential and algebraic equations, see [9] and [10]. If $\mathbf{L}$ is non-singular, (3) is simply a system of ordinary differential equations. The non-singularity conditions for $\mathbf{L}$ are discussed further.

B. Faulty Stator

Assume without loss of generality that a fault occurs in stator phase *A*. Examples of motor faults that are covered by the mathematical model are: stator winding shortcut, increased stator winding resistance and decreased air gap in front of stator phase *A*. The motor parameters can then be divided into two groups, one group for the parameters related to the healthy IM and another one for the parameters whose value is altered by the fault, see Table I. When a stator fault

Fault-invariant parameters		Fault dependent parameters	
Value	Model parameter	Value	Model parameter
R_S	R_B, R_C	R_A	R_A
R_r	R_a, R_b, R_c	L_A	L_A
L_S	L_B, L_C	M_{AS}	M_{AB}, M_{AC}
L_r	L_a, L_b, L_c	M_{Ar}	M_{Aa}, M_{Ab}, M_{Ac}
M_S	M_{BC}		
M_r	M_{ab}, M_{ac}, M_{bc}		
M	$M_{Ba}, M_{Bb}, M_{Bc},$ M_{Ca}, M_{Cb}, M_{Cc}		

TABLE I
MODEL PARAMETERS FOR IM WITH FAULTY STATOR

occurs, one or several of the following inequalities apply

$$R_A \neq R_S; L_A \neq L_S; M_{AS} \neq M_S; M_{Ar} \neq M \quad (4)$$

From theory of linear electrical circuits

$$M_{AS} = k_{AS} \sqrt{L_S L_A}; M_S = k_S L_S; M = k \sqrt{L_S L_r};$$

$$M_r = k_r L_r; M_{Ar} = k_{Ar} \sqrt{L_A L_r} \quad (5)$$

where the coupling coefficients are bounded by $0 \leq k_{AS}, k_S, k_r, k_{Ar}, k \leq 1$. Recall that a zero value of a coupling coefficient corresponds to no coupling and unit value arises under perfect coupling between the corresponding coils. Defining parameters as in Table I, using (5) and transforming to a two-axis framework [5] (and [6]), one arrives at

$$\mathbf{R}_S = \begin{bmatrix} \frac{3}{2} R'_A & 0 \\ 0 & \frac{3}{2} R_S \end{bmatrix}; \quad \mathbf{R}_r = \frac{3}{2} R_r \mathbf{I};$$

$$\mathbf{L}_S = \begin{bmatrix} \frac{9}{4} L'_A & 0 \\ 0 & \frac{9}{4} L'_S \end{bmatrix}; \quad \mathbf{L}_r = \frac{9}{4} L'_r \mathbf{I}$$

$$\mathbf{M} = \bar{\mathbf{M}} \mathbf{T}_{\bar{\gamma}}; \quad \bar{\mathbf{M}} = \begin{bmatrix} \frac{9}{4} M'_{Ar} & 0 \\ 0 & \frac{9}{4} M \end{bmatrix}$$

where

$$\begin{aligned} \frac{3}{2} R'_A &= R_A + \frac{1}{2} R_S \\ \frac{9}{4} L'_A &= \frac{1}{2} \left(1 - \frac{1}{2} k_S \right) L_S + L_A + k_{AS} \sqrt{L_S L_A} \\ \frac{9}{4} L'_S &= \frac{3}{2} \left(1 + \frac{1}{2} k_S \right) L_S; \quad \frac{9}{4} L'_r = \frac{3}{2} \left(1 + \frac{1}{2} k_r \right) L_r \\ \frac{9}{4} M'_{Ar} &= \frac{3}{2} \left(k_{Ar} \sqrt{L_A L_r} + \frac{1}{2} k \sqrt{L_S L_r} \right) \end{aligned} \quad (6)$$

The inductance matrix $\mathbf{L}$ becomes singular iff $|\mathbf{L}| = \sigma \sigma' = 0$ where

$$\sigma' = L'_A L'_r - M'^2_{Ar}; \quad \sigma = L'_S L'_r - M^2$$

The case of $\sigma \sigma' = 0$ occurs under perfect coupling between the stator and the rotor in the two-axis framework. Note that σ' and σ can be written as

$$\begin{aligned} \sigma' &= - \left[\frac{4}{9} \left(k_{Ar} \sqrt{L_A} + \frac{1}{2} k \sqrt{L_S} \right)^2 - \frac{2}{3} L'_A \left(1 + \frac{1}{2} k_r \right) \right] L_r \\ \sigma &= - \left[k^2 - \frac{4}{9} \left(1 + \frac{1}{2} k_S \right) \left(1 + \frac{1}{2} k_r \right) \right] L_S L_r \end{aligned}$$

Now the following implications can be established

$$\begin{aligned} R'_A = 0 &\Rightarrow R_S = 0; \quad L'_S = 0 \Rightarrow \sigma = 0 \\ L'_A = 0 &\Rightarrow L'_S = M'_{Ar} = \sigma' = \sigma = 0 \\ L'_r = 0 &\Rightarrow M'_{Ar} = \sigma' = \sigma = 0; \quad M'_{Ar} = 0 \Rightarrow M = 0 \end{aligned} \quad (7)$$

To simplify notation later on, introduce the following matrix

$$\boldsymbol{\Sigma} = \frac{81}{16} \begin{bmatrix} \sigma' & 0 \\ 0 & \sigma \end{bmatrix}$$

In order to get a neater description of the IM, without trigonometric terms, the rotor currents are projected onto the stator axes, $i_p = \mathbf{T}_{\bar{\gamma}} i_r$, yielding

$$\bar{\mathbf{L}} \begin{bmatrix} \dot{i}_S \\ \dot{i}_p \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_S & \mathbf{0} \\ \bar{\omega} \mathbf{J} \bar{\mathbf{M}} & -\mathbf{R}_r + \bar{\omega} \mathbf{J} \mathbf{L}_r \end{bmatrix} \begin{bmatrix} i_S \\ i_p \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} u_S \quad (8)$$

where $\bar{\omega} = \dot{\bar{\gamma}}$ is the electrical speed and

$$\bar{\mathbf{L}} = \begin{bmatrix} \mathbf{L}_S & \bar{\mathbf{M}} \\ \bar{\mathbf{M}} & \mathbf{L}_r \end{bmatrix}$$

The matrix $\bar{\mathbf{L}}$ may be viewed as a modified inductance matrix. Model (8) is a valid description of the dynamics of the IM for all admissible values of motor parameters.

Consider the singularity conditions for $\bar{\mathbf{L}}$. Note that $|\bar{\mathbf{L}}| = \sigma \sigma'$ and $\bar{\mathbf{L}}$ becomes singular whenever $\mathbf{L}$ does. Assume that $L_S, L_r \neq 0$. Then $\sigma = 0$ when k, k_S, k_r fulfil

$$k^2 = \frac{4}{9} \left(1 + \frac{1}{2} k_S \right) \left(1 + \frac{1}{2} k_r \right)$$

Considering the extreme values of $k_S, k_r \in [0, 1]$, it can be concluded that

$$\frac{4}{9} \leq \frac{4}{9} \left(1 + \frac{1}{2} k_S \right) \left(1 + \frac{1}{2} k_r \right) \leq 1$$

which is within the range of k^2 . However, from the physics of the IM, the parameters L'_S, L'_r and M are mutually dependent and $\sigma = 0$ is only possible under extreme idealizing

assumptions, such as zero leakage flux, [11]. Therefore, it can be safely concluded that perfect coupling is unrealistic in the *two-axis* framework as well as in the *three-axis* representation. At this point it is not clear whether $\sigma' = 0$ is possible, under the idealizing assumptions imposed. More physical analysis is needed to draw a definite conclusion.

For non-singular $\bar{\mathbf{L}}$, (8) can be expressed in several useful forms. First note that

$$\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \bar{\mathbf{L}}^{-1} = \begin{bmatrix} \mathbf{L}_r & -\bar{\mathbf{M}} \\ -\bar{\mathbf{M}} & \mathbf{L}_s \end{bmatrix}$$

hence (8) can be rewritten as

$$\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \begin{bmatrix} i_S \\ i_p \end{bmatrix} = \mathbf{A}_{so} \begin{bmatrix} i_S \\ i_p \end{bmatrix} + \begin{bmatrix} \mathbf{L}_r \\ -\bar{\mathbf{M}} \end{bmatrix} u_S \quad (9)$$

$$y = \mathbf{C}_{so} \begin{bmatrix} i_S \\ i_p \end{bmatrix}$$

where

$$\mathbf{A}_{so} = \begin{bmatrix} -\mathbf{L}_r \mathbf{R}_s - \bar{\omega} \bar{\mathbf{M}} \mathbf{J} \bar{\mathbf{M}} & \bar{\mathbf{M}} (\mathbf{R}_r - \bar{\omega} \mathbf{J} \mathbf{L}_r) \\ (\mathbf{R}_s + \bar{\omega} \mathbf{L}_s \mathbf{J}) \bar{\mathbf{M}} & -\mathbf{L}_s (\mathbf{R}_r - \bar{\omega} \mathbf{J} \mathbf{L}_r) \end{bmatrix}$$

$$\mathbf{C}_{so} = [\mathbf{I} \quad \mathbf{0}]$$

This form turns out to be convenient for *observability* analysis. After transformation to the form of ordinary differential equations (1), the system matrix is

$$\bar{\mathbf{A}}_{so} = \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-1} \end{bmatrix} \mathbf{A}_{so}$$

Another useful form is obtained by using the transformation to the fluxes

$$\begin{bmatrix} \psi_S \\ \psi_p \end{bmatrix} = \bar{\mathbf{L}} \begin{bmatrix} i_S \\ i_p \end{bmatrix}$$

which yields

$$\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \begin{bmatrix} \dot{\psi}_S \\ \dot{\psi}_p \end{bmatrix} = \mathbf{A}_{sc} \begin{bmatrix} \psi_S \\ \psi_p \end{bmatrix} + \begin{bmatrix} \Sigma \\ \mathbf{0} \end{bmatrix} u_S \quad (10)$$

$$\Sigma i_S = [\mathbf{L}_r \quad -\bar{\mathbf{M}}] \begin{bmatrix} \psi_S \\ \psi_p \end{bmatrix}$$

where

$$\mathbf{A}_{sc} = \begin{bmatrix} -\mathbf{R}_s \mathbf{L}_r & \mathbf{R}_s \bar{\mathbf{M}} \\ \mathbf{R}_r \bar{\mathbf{M}} & \mathbf{R}_r \mathbf{L}_s + \bar{\omega} \Sigma \mathbf{J} \end{bmatrix}$$

This form is suitable for *controllability* analysis. Note that the states of both (9) and (10) still have physical meaning. After transformation of (10) to the form of (1), the following matrices are obtained

$$\bar{\mathbf{A}}_{sc} = \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-1} \end{bmatrix} \mathbf{A}_{sc}; \quad \bar{\mathbf{B}}_{sc} = [\mathbf{I} \quad \mathbf{0}]^T$$

It should be observed that (9), (10) are *not* equivalent to (8) when $\sigma\sigma' = 0$. This can be easily demonstrated by a counterexample. For the case $\sigma = \sigma' = 0$ all the dynamics in (9) and (10) are cancelled, while $\text{rank}(\bar{\mathbf{L}}) = 2$ in (8). This highlights the modeling problems due to the singularities. Not even a limiting approach is safe, because the matrix inverse is not a continuous operator in vicinity of singularities.

C. Faulty Rotor

Assume now that a fault occurs in rotor phase *a*. Relevant examples of faults are in this case: broken rotor bar, broken end ring and diminished air gap in front of rotor phase *a*. Following the lines of the model derivation for faulty stator, introduce two groups of parameters (Table II). Thus, a rotor

Fault-invariant parameters		Fault dependent parameters	
Value	Model parameter	Value	Model parameter
R_S	R_A, R_B, R_C	R_a	R_a
R_r	R_b, R_c	L_a	L_a
L_S	L_A, L_B, L_C	M_{ar}	M_{ab}, M_{ac}
L_r	L_b, L_c	M_{Sa}	M_{Aa}, M_{Ba}, M_{Ca}
M_S	M_{AB}, M_{AC}, M_{BC}		
M_r	M_{bc}		
M	$M_{Ab}, M_{Ac}, M_{Bb}, M_{Bc}, M_{Cb}, M_{Cc}$		

TABLE II
MODEL PARAMETERS FOR IM WITH FAULTY ROTOR

fault, depending on its nature, results in one or several of the following model parameter alternations

$$R_a \neq R_r; L_a \neq L_r; M_{ar} \neq M_r; M_{Sa} \neq M \quad (11)$$

Similar to the case of faulty stator,

$$M_S = k_S L_S; M_{ar} = k_{ar} \sqrt{L_r L_a}; M_r = k_r L_r$$

$$M_{Sa} = k_{Sa} \sqrt{L_S L_a}; M = k \sqrt{L_S L_r}$$

where $0 \leq k_S, k_{ar}, k_r, k_{Sa}, k \leq 1$. Hence, for the case of faulty rotor ([5], [7])

$$\mathbf{R}_r = \begin{bmatrix} \frac{3}{2} R'_a & 0 \\ 0 & \frac{3}{2} R_r \end{bmatrix}; \quad \mathbf{R}_s = \frac{3}{2} R_S \mathbf{I}$$

$$\mathbf{L}_r = \begin{bmatrix} \frac{9}{4} L'_a & 0 \\ 0 & \frac{9}{4} L_r \end{bmatrix}; \quad \mathbf{L}_s = \frac{9}{4} L'_S \mathbf{I}$$

$$\mathbf{M} = \mathbf{T}_{\bar{\gamma}} \bar{\mathbf{M}}; \quad \bar{\mathbf{M}} = \begin{bmatrix} \frac{9}{4} M'_{Sa} & 0 \\ 0 & \frac{9}{4} M \end{bmatrix}$$

where

$$\frac{3}{2} R'_a = R_a + \frac{1}{2} R_r; \quad \frac{9}{4} L'_S = \frac{3}{2} \left(1 + \frac{1}{2} k_S \right) L_S$$

$$\frac{9}{4} L'_a = \frac{1}{2} \left(1 - \frac{1}{2} k_r \right) L_r + L_a + k_{ar} \sqrt{L_r L_a}$$

$$\frac{9}{4} L'_r = \frac{3}{2} \left(1 + \frac{1}{2} k_r \right) L_r \quad (12)$$

$$\frac{9}{4} M'_{Sa} = \frac{3}{2} \left(k_{Sa} \sqrt{L_S L_a} + \frac{1}{2} k \sqrt{L_S L_r} \right)$$

The inductance matrix $\mathbf{L}$ becomes singular iff $\sigma\sigma' = 0$ and

$$\sigma' = L'_S L'_a - M'^2_{Sa}; \quad \sigma = L'_S L'_r - M^2$$

Again, $\sigma\sigma' = 0$ implies a perfect coupling between the stator and the rotor in the two-axis framework. To proceed, note that σ' and σ can be written as

$$\sigma' = - \left[\frac{4}{9} \left(k_{Sa} \sqrt{L_a} + \frac{1}{2} k \sqrt{L_r} \right)^2 - \frac{2}{3} L'_a \left(1 + \frac{1}{2} k_S \right) \right] L_S$$

$$\sigma = - \left[k^2 - \frac{4}{9} \left(1 + \frac{1}{2} k_S \right) \left(1 + \frac{1}{2} k_r \right) \right] L_S L_r$$

Thus the following implications can be established

$$\begin{aligned} R'_a = 0 &\Rightarrow R_r = 0; L'_S = 0 \Rightarrow M'_{Sa} = \sigma' = \sigma = 0 \\ L'_a = 0 &\Rightarrow L'_r = M'_{Sa} = \sigma' = \sigma = 0 \\ L'_r = 0 &\Rightarrow \sigma = 0; M'_{Sa} = 0 \Rightarrow M = 0 \end{aligned} \quad (13)$$

Now, contrary to the case of stator fault, the stator currents are projected onto the rotor axes *i.e.* $i_S = \mathbf{T}_{\bar{\gamma}} i_P$, yielding

$$\bar{\mathbf{L}} \begin{bmatrix} i_P \\ i_r \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_S - \bar{\omega} \mathbf{J} \mathbf{L}_S & -\bar{\omega} \mathbf{J} \bar{\mathbf{M}} \\ \mathbf{0} & -\mathbf{R}_r \end{bmatrix} \begin{bmatrix} i_P \\ i_r \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{\bar{\gamma}}^T \\ \mathbf{0} \end{bmatrix} u_S \quad (14)$$

Notably, (14) is always a valid description of the IM dynamics.

Assuming non-singularity of $\bar{\mathbf{L}}$, (14) can be expressed in alternative forms useful for observability and controllability analysis. Noting that

$$\begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \bar{\mathbf{L}}^{-1} = \begin{bmatrix} \mathbf{L}_r & -\bar{\mathbf{M}} \\ -\bar{\mathbf{M}} & \mathbf{L}_S \end{bmatrix}$$

(14) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \begin{bmatrix} i_P \\ i_r \end{bmatrix} &= \mathbf{A}_{ro} \begin{bmatrix} i_P \\ i_r \end{bmatrix} + \begin{bmatrix} \mathbf{L}_r \\ -\bar{\mathbf{M}} \end{bmatrix} u_P \quad (15) \\ y &= \mathbf{C}_{ro} \begin{bmatrix} i_P \\ i_r \end{bmatrix} \end{aligned}$$

where $y = i_S$ and $u_P = \mathbf{T}_{\bar{\gamma}}^T u_S$

$$\begin{aligned} \mathbf{A}_{ro} &= \begin{bmatrix} -\mathbf{L}_r(\mathbf{R}_S + \bar{\omega} \mathbf{J} \mathbf{L}_S) & (\mathbf{R}_r - \bar{\omega} \mathbf{L}_r \mathbf{J}) \bar{\mathbf{M}} \\ \bar{\mathbf{M}}(\mathbf{R}_S + \bar{\omega} \mathbf{J} \mathbf{L}_S) & -\mathbf{L}_S \mathbf{R}_r + \bar{\omega} \bar{\mathbf{M}} \mathbf{J} \bar{\mathbf{M}} \end{bmatrix} \\ \mathbf{C}_{ro} &= [\mathbf{T}_{\bar{\gamma}} \quad \mathbf{0}] \end{aligned}$$

This form is convenient for *observability* analysis. After transformation to the form of ordinary differential equations (1), the system matrix is

$$\bar{\mathbf{A}}_{ro} = \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-1} \end{bmatrix} \mathbf{A}_{ro}$$

To investigate controllability, the model below is more suitable. The transformation

$$\begin{bmatrix} \psi_P \\ \psi_r \end{bmatrix} = \bar{\mathbf{L}} \begin{bmatrix} i_P \\ i_r \end{bmatrix}$$

yields

$$\begin{aligned} \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix} \begin{bmatrix} \dot{\psi}_P \\ \dot{\psi}_r \end{bmatrix} &= \mathbf{A}_{rc} \begin{bmatrix} \psi_P \\ \psi_r \end{bmatrix} + \begin{bmatrix} \Sigma \mathbf{T}_{\bar{\gamma}}^T \\ \mathbf{0} \end{bmatrix} u_S \quad (16) \\ \Sigma i_P &= [\mathbf{L}_r \quad -\bar{\mathbf{M}}] \begin{bmatrix} \psi_P \\ \psi_r \end{bmatrix} \end{aligned}$$

where

$$\mathbf{A}_{rc} = \begin{bmatrix} -\mathbf{R}_S \mathbf{L}_r - \bar{\omega} \Sigma \mathbf{J} & \mathbf{R}_S \bar{\mathbf{M}} \\ \mathbf{R}_r \bar{\mathbf{M}} & -\mathbf{R}_r \mathbf{L}_S \end{bmatrix}$$

After transformation of (16) to the form of (1), the following matrices are obtained

$$\bar{\mathbf{A}}_{rc} = \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-1} \end{bmatrix} \mathbf{A}_{rc}; \quad \bar{\mathbf{B}}_{rc} = [\mathbf{T}_{\bar{\gamma}} \quad \mathbf{0}]^T$$

Again, (15) and (16) are *not* equivalent to (14) when $\bar{\mathbf{L}}$ is singular.

IV. MAIN RESULTS

Since all existing motor drive schemes rely on observability and controllability of the IM under a broad range of working conditions, it is important to know whether observability and/or controllability of the IM can be lost as a consequence of a fault in one of the phases of either the stator or the rotor. The two Theorems below prove that this is *not* the case for any of the IM models considered above.

Theorem 1: (Stator Fault) Let the IM be subject to a fault in stator phase *A* resulting in that one or several inequalities in (4) apply. Further, assume that $\sigma\sigma' \neq 0$ and $R_r, M > 0$. Then model (8) is observable and controllable on any non-trivial $[t_0, t_f]$.

Proof: Observability: The assumption $\sigma\sigma' \neq 0$ enables the use of (9) to model the dynamics of the IM. Let Φ be the transition matrix corresponding to the system matrix $\bar{\mathbf{A}}_{so}$.

For a constant vector p , define $P(\tau, t_0) = \Phi(\tau, t_0)p$ so that $z_o(\tau, t_0) = \mathbf{C}_{so}(\tau)P(\tau, t_0)$ and therefore

$$z_o(\tau, t_0) = \begin{bmatrix} P_1(\tau, t_0) \\ P_2(\tau, t_0) \end{bmatrix}$$

where $P_i(\cdot, \cdot), i = 1, \dots, 4$ are the elements of the vector $P(\cdot, \cdot)$. Recall from the discussion in Section II that singularity of $\mathcal{O}(t_0, t_f)$ is equivalent to $z_o(\tau, t_0) = 0, \tau \in [t_0, t_f], p \neq 0$ and thus

$$z_o(\tau, t_0) = \begin{bmatrix} P_1(\tau, t_0) \\ P_2(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (17)$$

Noting that $\forall \tau, t_0 : |\Phi(\tau, t_0)| \neq 0$, it follows that $P(\tau, t_0) \equiv 0 \Leftrightarrow p = 0$. Hence, the observability part of the theorem is proven if it can be shown that $z_o(\tau, t_0) = 0 \Rightarrow P(\tau, t_0) = 0, \tau \in [t_0, t_f], \forall t_0, t_f$.

Note that only $P_1(\tau, t_0) \equiv P_2(\tau, t_0) \equiv 0$ has been shown, hence further analysis is needed. By differentiating both sides of (17), it follows

$$\Sigma \dot{z}_o(\tau, t_0) = \bar{\mathbf{M}}(\mathbf{R}_r - \bar{\omega} \mathbf{J} \mathbf{L}_r) \begin{bmatrix} P_3(\tau, t_0) \\ P_4(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where a solution exists for $[P_3(\tau, t_0) \ P_4(\tau, t_0)]^T \neq 0$ iff $|\bar{\mathbf{M}}(\mathbf{R}_r - \bar{\omega} \mathbf{J} \mathbf{L}_r)| = 0 \Leftrightarrow M M'_{Ar} (\frac{4}{9} R_r^2 + \bar{\omega}^2 L_r'^2) = 0$. Since $R_r, M > 0$ by assumption, only $M'_{Ar} = 0$ is feasible. From previous analysis in (7), it is known that $M'_{Ar} = 0 \Rightarrow M = 0$. However, the latter equality is not possible according to the assumptions made, hence $P_3(\tau, t_0) \equiv P_4(\tau, t_0) \equiv 0$. Since the argument is independent of the chosen time interval, the observability gramian is nonsingular on any interval $[t_0, t_f]$.

Controllability: The controllability part of proof is similar to the proof of observability and provided in less detail. The assumption $\sigma\sigma' \neq 0$ enables the use of (10). Let Φ be the transition matrix corresponding to the system matrix $\bar{\mathbf{A}}_{sc}$. Define

$$P(t_0, \tau) = \Phi^H(t_0, \tau)p, \quad P(t_0, \tau) = [P_1(t_0, \tau) \ \dots \ P_4(t_0, \tau)]^T$$

With the adopted notation, it follows

$$z_c(t_0, \tau) = \begin{bmatrix} P_1(t_0, \tau) \\ P_2(t_0, \tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (18)$$

implying that $P_1(t_0, \tau) \equiv P_2(t_0, \tau) \equiv 0$. Differentiating (18) gives

$$\Sigma \dot{z}_c(t_0, \tau) = -\mathbf{R}_r \bar{\mathbf{M}} \begin{bmatrix} P_3(t_0, \tau) \\ P_4(t_0, \tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which lacks a solution for $P_3(t_0, \tau) \neq P_4(t_0, \tau) \neq 0$ unless $R_r^2 M M'_{Ar} = 0$. The latter contradicts the assumptions of the Theorem. $\square$

Theorem 2: (Rotor Fault) Let the IM be subject to a fault in rotor phase a resulting in that one or several inequalities in (11) apply. Further, assume that $\sigma\sigma' \neq 0$ and $R_r, M > 0$. Then model (14) is observable and controllable on any $[t_0, t_f]$.

Proof: *Observability:* Assuming $\sigma\sigma' \neq 0$, i.e. non-singularity of $\bar{\mathbf{L}}$, enables the use of (15) instead of (14). Let Φ be the transition matrix corresponding to the system matrix $\bar{\mathbf{A}}_{ro}$. Consider the observability gramian of (15)

$$\begin{aligned} \mathcal{O}(t_0, t_f) &= \int_{t_0}^{t_f} \Phi^H(\tau, t_0) \bar{\mathbf{C}}^H(\tau) \mathbf{T}_{\bar{\gamma}}^H \mathbf{T}_{\bar{\gamma}} \bar{\mathbf{C}}(\tau) \Phi(\tau, t_0) d\tau \\ &= \int_{t_0}^{t_f} \Phi^H(\tau, t_0) \bar{\mathbf{C}}^H(\tau) \bar{\mathbf{C}}(\tau) \Phi(\tau, t_0) d\tau \end{aligned}$$

where $\bar{\mathbf{C}} = [\mathbf{I} \ 0]$. Clearly the unitary transformation $\mathbf{T}_{\bar{\gamma}}$ has no effect on observability. Hence, in the sequel, (15) is used for observability analysis, but with v_P defined as output. This change of output considerably simplifies the analysis since $\bar{\mathbf{C}}^{(k)} = 0$. Following the scheme of the proof of Theorem 1, singularity of $\mathcal{O}$ on the interval $[t_0, t_f]$ implies for $p \neq 0$

$$z_o(\tau, t_0) = \begin{bmatrix} P_1(\tau, t_0) \\ P_2(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (19)$$

and the upper two elements of P have to be zero. Differentiating (19) yields

$$\Sigma \dot{z}_o(\tau, t_0) = (\mathbf{R}_r - \bar{\omega} \mathbf{L}_r \mathbf{J}) \bar{\mathbf{M}} \begin{bmatrix} P_3(\tau, t_0) \\ P_4(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. there is only a trivial solution with respect to P_3, P_4 unless $MM'_{Sa}(\frac{4}{9}R_r R'_a + \bar{\omega}^2 L'_r L'_a) = 0$. Since $R_r, M > 0$ by assumption, the cases $M'_{Sa} = 0$ and/or $R'_a = 0$ are left to sort out. From analysis in (13) it is known that $M'_{Sa} = 0 \Rightarrow M = 0$ and $R'_a = 0 \Rightarrow R_r = 0$, both contradicting with the assumptions of the Theorem.

Controllability: With the assumption $\sigma\sigma' \neq 0$, model (16) can be used for controllability analysis instead of more general (14). Let Φ be the transition matrix corresponding to the system matrix $\bar{\mathbf{A}}_{rc}$. By evaluating the controllability gramian for (16), similar to the case of observability, it follows that u_P (defined as in (15)) can be considered as input instead of u_S with no effect on controllability conditions. This yields a time-invariant mapping of the control signal to the derivative of the state vector and hence simplifies the analysis. Once again, singularity of the controllability gramian $\mathcal{C}$ of model (16) on an interval is equivalent to

$$z_c(t_0, \tau) = \begin{bmatrix} P_1(t_0, \tau) \\ P_2(t_0, \tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (20)$$

Differentiating (20) gives

$$\Sigma \dot{z}_c(t_0, \tau) = -\mathbf{R}_r \bar{\mathbf{M}} \begin{bmatrix} P_3(t_0, \tau) \\ P_4(t_0, \tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where the second equality has only trivial solution with respect to elements of P unless $R_r R'_a M M'_{Sa} = 0$. However,

as proved before, none of the involved factors can be zero without violating the assumptions of the Theorem. $\square$

Naturally one wonders what happens if one of the three assumptions in Theorem 1 and Theorem 2 is violated. Although $\sigma\sigma' = 0$ is theoretically interesting since it would turn the IM models into *descriptor* systems, this case is outside of the scope of the paper. Next, from (5), $M = 0 \Leftrightarrow (k = 0 \text{ and/or } L_S L_r = 0)$, where only $k = 0$ is feasible since $L_S = 0 \Leftrightarrow L'_S = 0 \Rightarrow \sigma = 0$ and $L_r = 0 \Leftrightarrow L'_r = 0 \Rightarrow \sigma = \sigma' = 0$, both leading to contradictions with $\sigma\sigma' \neq 0$. However, $k = 0$ implies an IM with no coupling between the (healthy) stator and rotor phases at all, which is impractical since no torque can be generated in the rotor. Thus, the case of $M = 0$ is omitted. Of course, the assumptions $R_r, M > 0$ are consistent with the physics and valid for any real-life IM. However, the case of $R_r = 0$ is interesting, since it reduces the total losses in the IM. The following four Theorems look deeper into what happens when the rotor windings resistance is small.

Theorem 3: (Stator Fault with Small losses at Standstill) Let the IM be subject to a fault in stator phase A resulting in that one or several inequalities in (4) apply. Further, assume that $\sigma\sigma' \neq 0$, $R_r = 0$ and $M > 0$. Then, iff $\bar{\omega} = 0, t \in [t_0, t_f]$, the rotor current i_r is not observable from the output i_S on $[t_0, t_f]$.

Proof: Let's show that (9) being unobservable implies $\bar{\omega}(t) = 0, t \in [t_0, t_f]$. The notation of Theorem 1 apply. By inspecting M in (5) and the last equation in (6), it can be concluded that $M > 0 \Rightarrow M'_{Ar} > 0$. Recall from the proof of Theorem 1 that if $MM'_{Ar}(\frac{4}{9}R_r^2 + \bar{\omega}^2 L_r'^2) \neq 0$ then the IM is observable. Assume that this condition fails, i.e. $(\frac{4}{9}R_r^2 + \bar{\omega}^2 L_r'^2) = 0$. By assumption $R_r = 0$ and from (7), $L_r' = 0 \Rightarrow \sigma = \sigma' = 0$, which is in contradiction with the assumptions of the Theorem. Hence, $\bar{\omega} = 0$.

Now it can be shown that also all the derivatives of $\bar{\omega}$ have to be zero under the conditions of the Theorem. Note that $\bar{\omega} = 0 \Rightarrow \dot{z}_o(\tau, t_0) = 0$. Further differentiation yields

$$\Sigma \ddot{z}_o(\tau, t_0) = -\dot{\bar{\omega}} \bar{\mathbf{M}} \mathbf{J} \mathbf{L}_r \begin{bmatrix} P_3(\tau, t_0) \\ P_4(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose solution with respect to P_3, P_4 is non-trivial iff $\dot{\bar{\omega}} = 0$. Carrying on with differentiation gives

$$\begin{aligned} z_o^{(3)}(\tau, t_0) &= \mathbf{C}_{so} \left(\ddot{\bar{\omega}}_{so} + \bar{\mathbf{A}}_{so}^3 \right) P(\tau, t_0) \\ &= -\ddot{\bar{\omega}} \Sigma^{-1} \bar{\mathbf{M}} \mathbf{J} \mathbf{L}_r \begin{bmatrix} P_3(\tau, t_0) \\ P_4(\tau, t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

if $\dot{\bar{\omega}} = 0$, lacking a non-trivial solution iff $\ddot{\bar{\omega}} = 0$. Generally, it can be shown that

$$z_o^{(k)}(\tau, t_0) = \mathbf{C}_{so} \left(\bar{\mathbf{A}}_{so}^{(k-1)} + \bar{\mathbf{A}}_{so}^k \right) P(\tau, t_0), \quad k \geq 2$$

if $\dot{\bar{\omega}} = \ddot{\bar{\omega}} = \dots = \bar{\omega}^{(k-2)} = 0$. By induction one gets

$$\begin{aligned} \bar{\mathbf{A}}_{so}^{(k)} &= \bar{\omega}^{(k)} \begin{bmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-1} \end{bmatrix} \begin{bmatrix} -\bar{\mathbf{M}} \mathbf{J} \bar{\mathbf{M}} & -\bar{\mathbf{M}} \mathbf{J} \mathbf{L}_r \\ \mathbf{L}_s \mathbf{J} \bar{\mathbf{M}} & \mathbf{L}_s \mathbf{J} \mathbf{L}_r \end{bmatrix} \\ \bar{\mathbf{A}}_{so}^k &= (-1)^k \begin{bmatrix} \Sigma^{-k} & \mathbf{0} \\ \mathbf{0} & \Sigma^{-k} \end{bmatrix} \begin{bmatrix} \mathbf{I}_r^k \mathbf{R}_s^k & \mathbf{0} \\ -\mathbf{I}_r^{k-1} \mathbf{R}_s^k \bar{\mathbf{M}} & \mathbf{0} \end{bmatrix} \quad (21) \end{aligned}$$

Using (21) one arrives to

$$\Sigma z_o^{(k)}(\tau, t_0) = -\bar{\omega}^{(k-1)} \bar{\mathbf{M}} \mathbf{J} \mathbf{L}_r \begin{bmatrix} P_3(\tau, t_0) \\ P_4(\tau, t_0) \end{bmatrix}$$

if $\dot{\omega} = \ddot{\omega} = \dots = \bar{\omega}^{(k-2)} = 0, k \geq 2$. Hence, under the assumptions imposed, observability is lost if $\bar{\omega} = \bar{\omega}^{(k)} = 0, 1 \leq k < \infty \Leftrightarrow \bar{\omega}(t) = 0, t \in [t_0, t_f]$.

In order to show which part of the IM becomes unobservable, Kalman's CST is used. Note that from (9) for $R_r = \bar{\omega} = 0$

$$\mathbf{A}_{so} = \begin{bmatrix} -\mathbf{L}_r \mathbf{R}_s & \mathbf{0} \\ \mathbf{R}_s \bar{\mathbf{M}} & \mathbf{0} \end{bmatrix} \quad (22)$$

hence the projected rotor currents i_p are clearly not related to the stator currents during the time-interval under consideration if $\bar{\omega}(t) = 0, [t_0, t_f]$. To check if these are the only non-observable states, the superfluous states are removed. Now consider the reduced system

$$\Sigma \dot{i}_S = -\mathbf{L}_r \mathbf{R}_s i_S + \mathbf{L}_r u_S$$

It is LTI, has i_S as the output and therefore is observable. Finally, recall that $i_p = \mathbf{T}_{\bar{\gamma}} i_r$, i.e. the fact that i_p is unobservable is equivalent to the unobservability of i_r .

Proving that $R_r = 0$ and $\bar{\omega}(t) = 0$ on $t \in [t_0, t_f]$ implies unobservability of i_r is straightforward by evaluating the observability gramian and taking into account the structure of $\mathbf{A}_{so}$ given by (22). Thus the necessity is also shown. $\square$

To save space, only the key steps of the proofs, mainly the ones related Kalman's CST, are provided in the sequel. The rest can be easily filled in following the method of Theorem 3.

Theorem 4: (Stator Fault with Small Losses) Let the IM be subject to a fault in stator phase A resulting in that one or several inequalities in (4) apply. Further, assume that $\sigma\sigma' \neq 0, R_r = 0$ and $M > 0$. Then, the rotor flux ψ_r is not controllable from the input u_S on any $[t_0, t_f]$.

Proof: Model (10) is considered and Kalman's CST is utilized. If $R_r = 0$, note that the matrix $\mathbf{A}_{sc}$ in (10) becomes

$$\mathbf{A}_{sc} = \begin{bmatrix} -\mathbf{R}_s \mathbf{L}_r & \mathbf{R}_s \bar{\mathbf{M}} \\ \mathbf{0} & \bar{\omega} \Sigma \mathbf{J} \end{bmatrix}$$

The input u_S can only influence ψ_S due to the block structure of the model matrices. Hence, it can be concluded that the projected rotor flux ψ_p is not controllable. Further examination reveals that the rest of the state variables are controllable. Finally, $\psi_p = \mathbf{T}_{\bar{\gamma}} \psi_r$ where $\mathbf{T}_{\bar{\gamma}}$ is unitary and thus ψ_r is uncontrollable as well. $\square$

Theorem 5: (Rotor Fault with Small Losses at Standstill) Let the IM be subject to a fault in rotor phase a resulting in that one or several inequalities in (11) apply. Further, assume that $\sigma\sigma' \neq 0, R_r = 0$ and $M > 0$. Then, iff $\bar{\omega} = 0, [t_0, t_f]$, the rotor current $i_{\beta r}$ is not observable from the output i_S on $[t_0, t_f]$.

Proof: Model (15) is considered and Kalman's CST is applied. Then, for $R_r = \bar{\omega} = 0$,

$$\mathbf{A}_{ro} = \begin{bmatrix} -\mathbf{L}_r \mathbf{R}_s & \frac{27}{8} R'_a M'_S & 0 \\ 0 & 0 & 0 \\ \bar{\mathbf{M}} \mathbf{R}_s & -\frac{27}{8} R'_a L'_S & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (23)$$

and it can be concluded that the last element of the state vector $i_{\beta r}$ is not observable since the corresponding observability gramian is singular iff $\bar{\omega} = 0, t_0 \leq t \leq t_f$. Further examination reveals that this is the only unobservable state. $\square$

Corollary: If in Theorem 5 it is additionally assumed that $R_a = 0$, then the vector of rotor currents i_r is unobservable from the output i_S .

Proof: From (12), $R_r = R_a = 0 \Rightarrow R'_a = 0$ and the result follows by inspection of (23). $\square$

Theorem 6: (Rotor Fault with Small Losses) Let the IM be subject to a fault in rotor phase a resulting in that one or several inequalities in (11) apply. Further, assume that $\sigma\sigma' \neq 0, R_r = 0$ and $M > 0$. Then, the rotor flux $\psi_{\beta r}$ is not controllable from u_S on any $[t_0, t_f]$.

Proof: For this case, Kalman's CST is used once again. If $R_r = 0$, note that the matrix $\mathbf{A}_{rc}$ in (16) turns to

$$\mathbf{A}_{rc} = \begin{bmatrix} -\mathbf{R}_s \mathbf{L}_r - \bar{\omega} \Sigma \mathbf{J} & \mathbf{R}_s \bar{\mathbf{M}} \\ \frac{27}{8} R'_a M'_S & 0 & -\frac{27}{8} R'_a L'_S & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (24)$$

By inspection of (16) it is immediately seen that u_S does not influence the last element of the state vector neither directly nor via other states. Hence the state $\psi_{\beta r}$ is uncontrollable. Moreover, the rest of the state vector is controllable. $\square$

Corollary: If in Theorem 6 it is additionally assumed that $R_a = 0$, then the vector of rotor fluxes ψ_r is uncontrollable.

Proof: From (12), $R_r = R_a = 0 \Rightarrow R'_a = 0$ and the result follows by inspection of (24). $\square$

V. CONCLUSION

A systematic analysis of the observability and controllability of a model of the *electrical* part of a possibly asymmetrical IM has been performed.

It is demonstrated that for most considered faults and asymmetries, the model does not normally degenerate to a *descriptor* system and neither observability nor controllability is affected. It is also shown that for the ideal case of zero rotor resistance, the IM becomes uncontrollable. If, in addition, the motor is at standstill, it is unobservable.

REFERENCES

- [1] C. C. De Wit, A. Youssef, J. P. Barbot, P. Martin, and F. Malrait, "Observability conditions of induction motors at low frequencies," in *39th IEEE Conference On Decision And Control CDC*, 2000.
- [2] S. Ibarra-Rojas, J. Moreno, and G. Espinosa-Pérez, "Global observability analysis of sensorless induction motors," *Automatica*, vol. 40, 2004.
- [3] R. E. Kalman, "Mathematical description of linear dynamical systems," *SIAM Journal On Control*, vol. 1, 1963.
- [4] T. Kailath, *Linear Systems*. Prentice-Hall, 1980.
- [5] M. Stocks, F. Rodyukov, and A. Medvedev, "Idealized two-axis models of faulty induction machines," CSEE / Control Engineering Group, Tech. Rep. ISSN 1402-1528 / ISRN LTU-FR-05/05-SE / NR 2005:05, 2005.
- [6] —, "Idealized two-axis model of induction machines under stator fault," in *1st IEEE Conference on Industrial Electronics and Applications ICIEA*, Singapore, 2006.
- [7] —, "Idealized two-axis model of induction machines under rotor fault," in *1st IEEE Conference on Industrial Electronics and Applications ICIEA*, Singapore, 2006.
- [8] —, "Experimental validation of idealized two-axis models of faulty induction machines," in *25th American Control Conference ACC*, Minneapolis, USA, 2006.
- [9] D. Cobb, "On the solutions of linear differential equations with singular coefficients," *Journal Of Differential Equations*, vol. 46, 1982.
- [10] T. Geerts, "Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant singular systems: The general case," *Linear Algebra And Its Applications*, vol. 181, 1993.
- [11] D. C. White and H. H. Woodson, *Electromechanical Energy Conversion*. New York: John Wiley & Sons, 1959.