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Using reiterated homogenization for stiffness computation of woven composites

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The homogenization theory is a branch of mathematics that has been developed in the studies of partial differential equations with rapidly oscillating coefficients, and is based on some type of averaging. This method can for instance be used in the computation of mechanical properties of composite materials, since a common feature for these materials is that they behave like a homogeneous medium when the characteristic size of the microstructure is much smaller than the whole sample. In this paper, we will investigate a woven composite (which can be considered to have two different local scales) by so called reiterated homogenization.

Let us consider linear elastic body that occupies a region Ω in R^3. We introduce a Cartesian coordinate system (x_i) and σ_{ij}, e_{ij}, f_i, t_i, u_i and n_i as the stress tensor, the strain tensor, the internal force field, the surface force field, the displacement field and the outer unit normal to the boundary $\partial\Omega$ of Ω, respectively. We assume that the body consists of a resin matrix and woven fiber tows. Each fiber tow consist in itself of isotropic fibers distributed in a continuous phase of resin matrix. We assume that the fibers are periodically distributed in the fiber tows in the sense that we can define a unit cell Z which is periodically repeated. Moreover, we assume that the fiber tows in them selves are periodically distributed in the resin matrix so that we can define a unit cell Y. We introduce the local variables $\mathbf{y} = \mathbf{x}/\varepsilon$ and $\mathbf{z} = \mathbf{x}/\varepsilon^2$ and assume that $C^{\varepsilon}_{ijkl} = C_{ijkl}(\mathbf{x}/\varepsilon, \mathbf{x}/\varepsilon^2) = C_{ijkl}(\mathbf{y}, \mathbf{z})$ is $Y - Z$ periodic. By considering ε as a parameter for varying the fineness of the structure we now study the following class of problems

$$\begin{cases} -\frac{\partial}{\partial x_j}(C^{\varepsilon}_{ijkl} e_{kl}(\mathbf{u}^{\varepsilon})) = f^{\varepsilon}_i \text{ in } \Omega, \\ u^{\varepsilon}_i = 0 \text{ on } \Gamma_1 \text{ and } C^{\varepsilon}_{ijkl} e_{kl}(\mathbf{u}^{\varepsilon}) n_j = t_i \text{ on } \Gamma_2, \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The main idea in the homogenization method is to approximate the solutions $\mathbf{u}^{\varepsilon}$ of the above class of problems with a function $\mathbf{u}$ which solves the problem corresponding to a homogenized material

$$\begin{cases} -\overline{C}_{ijkl} \frac{\partial}{\partial x_j} e_{kl}(\mathbf{u}) = f_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \Gamma_1 \text{ and } C_{ijkl} e_{kl}(\mathbf{u}) n_j = t_i \text{ on } \Gamma_2, \end{cases}$$

where $\overline{C}_{ijkl}$ is a constant tensor. It can be shown that the tensor $\overline{C}_{ijkl}$ is given by

$$\overline{C}_{ijkl} = \frac{1}{|Y|\,|Z|} \iint_{Y \times Z} C_{ijkl} + C_{ijmn} e_{mnz}(\chi^{kl}_y) +$$

$$+ C_{ijmn} e_{mnz}(\chi^{rs}_y) e_{rsy}(\chi^{kl}) + C_{ijmn} e_{mny}(\chi^{kl}) d\mathbf{y} d\mathbf{z},$$

where $\chi_y^{rs} = \chi_y^{rs}(\mathbf{z})$ is determined by the cell problem

$$\begin{cases} (A_1(\chi_y^{rs} + V^{rs}))_i = 0 \text{ on } Z, \\ \chi_y^{rs} \text{ is } Z-periodic, \; V_i^{rs} = \delta_{ir} z_s, \end{cases}$$

and $\chi^{rs} = \chi^{rs}(\mathbf{y})$ is determined by the cell problem

$$\begin{cases} (a_1(\chi^{rs} + W^{rs}))_i = 0 \text{ on } Y, \\ \chi^{rs} \text{ is } Y-periodic, \; W_i^{rs} = \delta_{ir} y_s, \end{cases}$$

in which the elliptic operators A_1 and a_1 are defined by

$$(A_1\mathbf{\Phi})_i = -\frac{\partial}{\partial z_j}(C_{ijkl}e_{klz}(\mathbf{\Phi})),$$

$$(a_1\mathbf{\Phi})_i = -\frac{\partial}{\partial y_j}(\widehat{C}_{ijkl}e_{kly}(\mathbf{\Phi})),$$

$$\widehat{C}_{ijkl} = \frac{1}{|Z|}\int_Z C_{ijkl} + C_{ijmn}e_{mnz}(\chi_y^{kl})d\mathbf{z}.$$

It can now be shown that the expression for $\overline{C}_{ijkl}$ can be computed by first solving the Z cell problem and compute a homogenized tensor $\widehat{C}_{ijkl}$ for this cell, and then use this to tensor and then solve the Y cell problem and compute the overall homogenized tensor by

$$\overline{C}_{ijkl} = \frac{1}{|Y|}\int_Y \widehat{C}_{ijkl} + \widehat{C}_{ijmn}e_{mny}(\chi^{kl})d\mathbf{y},$$

We used this reiterated homogenization procedure to compute the stiffness of a woven composite. The composite consisted of isotropic resin matrix with properties

$$E_m = 3.0 \; GPa, \; \nu_m = 0.38,$$

and fiber tows consisting of 34% resin matrix and 66% glass fibers with properties

$$E_f = 72 \; GPa, \; \nu_f = 0.22.$$

We thus first solved the Z cell problem and computed the stiffness tensor for the tows, and then used this to solve the Y cell problem and compute the overall stiffness tensor. We also compared the result from the calculations with experimental results obtained at the division of polymer engineering. The results are

	$ReiteratedHomog.$	$Exp.$
E_L	27.0	26.4
ν_{LT}	0.146	0.150

This reiterated homogenization could of course be generalized to n scales, and we conclude that it could be a useful tool for determination of the stiffness tensors of periodic media with different local scales. The subject of reiterated homogenization could be found in the book by Bensoussan et al. [1] and the thesis by Lukkassen [2], whereas an introduction to the subject of homogenization could be found in Persson et al. [3] and the thesis by Wall [4].

References

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