THE K-FUNCTIONAL FOR SYMMETRIC SPACES

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The K-functional for a couple of symmetric spaces on $(0, \infty)$ is computed if there is some separation between their fundamental functions.

1. INTRODUCTION

There are several real interpolation methods introduced by Gagliardo, Krein, Lions, Oklander, Peetre and others. The most natural is the K-method constructed with the so called K-functional.

The Peetre K-functional for a Banach couple (A_0, A_1) is defined for every a in $A_0 + A_1$ and every $t > 0$ by

$$K(t, a) = K(t, a; A_0, A_1) = \inf (\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1, a_0 + a_1 = a).$$

$K(t, a)$ is a non-decreasing concave function of t , and if $A_1 \overset{C}{\hookrightarrow} A_0$ (i.e. A_1 is continuously embedded in A_0 with $\|a\|_{A_0} \leq C\|a\|_{A_1}$ for all $a \in A_1$), then we have $K(t, a) = \|a\|_{A_0}$ for all $t \geq C$.

In the K-method it is important to calculate the K-functional up to a multiplicative constant. Many authors computed the K-functional for concrete couples. For couples of some symmetric spaces on $(0, \infty)$ it was done, in chronological order, by: Peetre [25, 26, 27], Oklander [24], Krée [14], Holmstedt [13], Sagher [28], Sharpley [29, 30], Bennett [3, 4], Bergh [5], Krein, Petunin and Semenov [15], Dmitriev and Ovčinnikov [12], Torchinsky [31], Milman [21, 22], Asekritova [2], Brudnyi and Krugljak [7], Merucci [20], Nilsson [23], Arazy [1] and the author [18, 19].

PROBLEM. Compute the K-functional for a couple of symmetric spaces E_0 and E_1 on the semiaxis $(0, \infty)$.

The purpose of this paper is to present the answer to the above problem, using elementary methods, if there is some separation between

the fundamental functions of E_0 and E_1 . I would like to note that it is closely related to th.2.1 in Milman [22], th.8 in [19] and the discussion in Nilsson [23], sect.4 and sect.6.

Milman assumed that $s^{-a}\varphi_{01}(s)$ is increasing for some $a > 0$. In the paper [19] I omitted this assumption leaving only monotonicity of φ_{01} and conditions (4a). Arazy [1] assumed that the space of all multipliers from E_1 into E_0 , $M = M(E_1, E_0)$, is non-trivial, $E_0 = E_1 \cdot M$ and $\varphi_M \approx \varphi_{01}$ where φ_M is increasing with $\varphi_M(R_+) = R_+$. Our proof avoids the use of the assumption that φ_{01} is a monotone function on R_+ .

Let us point out that this result cannot be obtained, in general from results of Dmitriev and Ovčinnikov [12], Asekritova [2], Brudnyi and Krugljak [7], and Nilsson [23].

We shall use definitions and notations from the book by Krein, Petunin and Semenov [15] on symmetric spaces and interpolation theory.

2. MAIN RESULT

A non-trivial Banach subspace E of L^0 (L^0 - the F-space of Lebesgue measurable classes of real valued functions defined on the semiaxis $R_+ = (0, \infty)$) is said to be symmetric on $(0, \infty)$ if it satisfies the following conditions:

- (i) If $x \in L^0$, $y \in E$ and $|x(t)| \leq |y(t)|$ almost everywhere on $(0, \infty)$ then $x \in E$ and $\|x\|_E \leq \|y\|_E$ (i.e., E is an ideal lattice).
- (ii) If $x \in L^0$, $y \in E$ and $|x|$ equimeasurable with $|y|$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

In Western literature the rearrangement invariant space is commonly defined as a symmetric space with an additional Fatou condition imposed on the norm.

The fundamental function $\omega = \omega_E$ of a symmetric space E on $(0, \infty)$ is defined for $t > 0$ as $\omega_E(t) = \|1_{(0,t)}\|_E$, where $1_{(0,t)}$ is the characteristic function of the interval $(0, t)$.

The smallest (largest) symmetric space contained in (containing) E with the same fundamental function is defined by

$$\Lambda(E) = \{x \in L^0: \|x\|_{\Lambda(E)} = \int_0^\infty x^*(s) d\omega_E(s) < \infty\}$$

$$(M(E) = \{x \in L^0: \|x\|_{M(E)} = \sup_{t>0} (\varphi_E(t)/t) \int_0^t x^*(s) ds < \infty\}).$$

Here x^* means the non-increasing rearrangement of x .

In the sequel, let E_0 and E_1 be two symmetric spaces defined on $(0, \infty)$ with the fundamental functions φ_0, φ_1 , and let us denote $\varphi_{01}(t) = \varphi_0(t)/\varphi_1(t)$ for a couple (E_0, E_1) .

In the proof of the main result we use the following facts ($i=0,1$):

- (1) $(x_0 + x_1)^*(s) \leq x_0^*(s/2) + x_1^*(s/2)$ [15, p.67],
- (2) $\|x^*(\cdot/2)\|_{E_i} \leq a_{E_i} \|x\|_{E_i}$, where $1 \leq a_{E_i} \leq 2$ [15, pp.96-98],
- (3) $x^*(s/2)\varphi_i(s) \leq \|x^*(\cdot/2)1_{(0,s)}\|_{E_i} \leq a_{E_i} \|x\|_{E_i} \quad \forall x \in E_i$.

Theorem 1. We assume that for each $t \in R_+$ there exists a partition $R_+ = A_t \cup B_t$ and a function $u: R_+ \rightarrow R_+$ such that the conditions

$$(4) \quad \left\| \frac{1_{A_t}}{\varphi_1} \right\|_{E_0} \leq c_0 u(t), \quad \left\| \frac{1_{B_t}}{\varphi_0} \right\|_{E_1} \leq c_1 / u(t)$$

hold uniformly in $t > 0$ for some $c_0, c_1 \geq 1$. Then for all $x \in E_0 + E_1$ and all $t > 0$ we have

$$(5) \quad K(u(t), x; E_0, E_1) \leq \|x^*1_{A_t}\|_{E_0} + u(t) \|x^*1_{B_t}\|_{E_1} \leq C K(u(t), x; E_0, E_1),$$

where $C = \max(a_{E_0} + a_{E_0}c_1, a_{E_1} + a_{E_1}c_0)$.

P r o o f. Let $x \in E_0 + E_1$ and let $x = x_0 + x_1$ be a decomposition of x with $x_0 \in E_0$ and $x_1 \in E_1$. Then

$$\begin{aligned} & \|x^*1_{A_t}\|_{E_0} + u(t) \|x^*1_{B_t}\|_{E_1} \quad [\text{from property (1) of } x^*] \\ & \leq \|x_0^*(\cdot/2)1_{A_t} + x_1^*(\cdot/2)1_{A_t}\|_{E_0} + u(t) \|x_0^*(\cdot/2)1_{B_t} + x_1^*(\cdot/2)1_{B_t}\|_{E_1} \end{aligned}$$

[by triangle inequality and by the monotonicity of $\|\cdot\|_{E_i}$]

$$\leq \|x_0^*(\cdot/2)\|_{E_0} + u(t) \|x_1^*(\cdot/2)\|_{E_1} + u(t) \|x_0^*(\cdot/2)1_{B_t}\|_{E_1} + \|x_1^*(\cdot/2)1_{A_t}\|_{E_0}$$

[by property (2) and (3)]

$$\leq a_{E_0} \|x_0\|_{E_0} + a_{E_1} u(t) \|x_1\|_{E_1} + a_{E_0} u(t) \left\| \frac{1_{B_t}}{\varphi_0} \|x_0\|_{E_0} \right\|_{E_1} + a_{E_1} \left\| \frac{1_{A_t}}{\varphi_1} \|x_1\|_{E_1} \right\|_{E_0}$$

[by assumptions (4)]

$$\leq a_{E_0} \|x_0\|_{E_0} + a_{E_1} u(t) \|x_1\|_{E_1} + a_{E_0} c_1 \|x_0\|_{E_0} + a_{E_1} c_0 u(t) \|x_1\|_{E_1}$$

$$\leq \max(a_{E_0} + a_{E_0} c_1, a_{E_1} + a_{E_1} c_0) (\|x_0\|_{E_0} + u(t) \|x_1\|_{E_1}).$$

Hence

$$\|x^* 1_{A_t}\|_{E_0} + u(t) \|x^* 1_{B_t}\|_{E_1} \leq C K(u(t), x).$$

On the other hand, if $x \in E_0 + E_1$ and $t > 0$ is fixed, let $x_{0t} = x^* 1_{A_t}$ and $x_{1t} = x^* 1_{B_t}$, then by the above inequality $x_{0t} \in E_0$, $x_{1t} \in E_1$. Since $x^* = x_{0t} + x_{1t}$, we have

$$K(u(t), x) = K(u(t), x^*) \leq \|x^* 1_{A_t}\|_{E_0} + u(t) \|x^* 1_{B_t}\|_{E_1}$$

This concludes the proof.

Remark 1. If E_0 and E_1 are interpolation spaces between L^1 and L^∞ , then $C = \max(1 + c_0, 1 + c_1)$ since in this case

$$K(t, x^*) = \inf \{ \|x_0\|_{E_0} + t \|x_1\|_{E_1} : 0 \leq x_0 \in E_0, 0 \leq x_1 \in E_1, x_0, x_1 \text{ non-increasing and } x_0 + x_1 = x^* \}.$$

2. If $\varphi_0, \varphi_1, E_1 \xrightarrow{C} E_0$ and $1/\varphi_0 \notin E_0$, then for any partition $R_+ = A_t \cup B_t$ we have either $1_{A_t}/\varphi_1 \notin E_0$ or $1_{B_t}/\varphi_0 \notin E_1$.

From the above theorem we get the following corollary.

Theorem 1a ([19], Th.8). If φ_{01} is a non-decreasing function on R_+ and

$$(4a) \quad \left\| \frac{1_{(0,t)}}{\varphi_1} \right\|_{E_0} \leq c_0 \varphi_{01}(t), \quad \left\| \frac{1_{[t,\infty)}}{\varphi_0} \right\|_{E_1} \leq c_1 / \varphi_{01}(t)$$

for some $c_0, c_1 \geq 1$ and all $t > 0$, then for all $x \in E_0 + E_1$ and all $t > 0$

$$(5a) \quad K(\varphi_{01}(t), x; E_0, E_1) \leq \|x^{*1}_{(0,t)}\|_{E_0} + \varphi_{01}(t) \|x^{*1}_{[t,\infty)}\|_{E_1} \\ \leq C K(\varphi_{01}(t), x; E_0, E_1).$$

The first inequality can be proved in a different way with constant 2 using the decomposition introduced by Calderón in 1966 (cf. Milman[22]).

Remark 3. From the assumptions that $x \in E_0 + E_1$ and φ_{01} is a non-decreasing function it does not always follow that $x^{*1}_{(0,t)} \in E_0$ and $x^{*1}_{[t,\infty)} \in E_1$ (cf. remark 2).

4. Since $K(\varphi_{01}(t), x; E_0, E_1) = \varphi_{01}(t) K(1/\varphi_{01}(t), x; E_1, E_0)$, if φ_{01} is a non-increasing function on R_+ , then in (4a) and (5a) one must interchange the sets $(0, t)$ and $[t, \infty)$.

3. SOME EXAMPLES AND PROBLEMS

Now we give examples of symmetric spaces which do or do not satisfy conditions of type (4).

Example 1. If $t^{-a}\varphi_{01}(t)$ is a non-decreasing function on R_+ for some $0 < a < 1$, then the conditions (4a) hold and $\varphi_{01}(R_+) = R_+$. Hence from Theorem 1a,

$$(5a_1) \quad K(t, x; E_0, E_1) \leq \|x^{*1}_{(0, \varphi_{01}^{-1}(t))}\|_{E_0} + t \|x^{*1}_{[\varphi_{01}^{-1}(t), \infty)}\|_{E_1} \\ \leq C K(t, x; E_0, E_1)$$

for all $x \in E_0 + E_1$ and all $t > 0$.

In particular, we have Milman's results (see [22], Th.2.1); Milman assumed additionally that $\int_0^t \varphi_i(s) ds/s \approx \varphi_i(t)$, $i = 0, 1$.

P r o o f. We have, for all $t > 0$,

$$2^{-1} \left\| \frac{1_{(0,t)}}{\varphi_1} \right\|_{E_0} \leq \left\| \frac{1_{(0,t)}}{\varphi_1} \right\|_{\Lambda(E_0)} = \int_0^t \frac{d\varphi_0(s)}{\varphi_1(s)}$$

$$\leq \int_0^t \varphi_{01}(s) ds/s \leq t^{-a} \varphi_{01}(t) \int_0^t s^{a-1} ds = \varphi_{01}(t)/a$$

and

$$\begin{aligned} 2^{-1} \left\| \frac{1_{[t, \infty)}}{\varphi_0} \right\|_{E_1} &\leq \left\| \frac{1_{[t, \infty)}}{\varphi_0} \right\|_{\Lambda(E_1)} = \int_0^\infty \left(\frac{1_{[t, \infty)}}{\varphi_0} \right)^*(s) d\varphi_1(s) \\ &= \int_0^\infty \frac{d\varphi_1(s)}{\varphi_0(s+t)} \leq \frac{1}{\varphi_0(t)} \int_0^t d\varphi_1(s) + \int_t^\infty \frac{1}{\varphi_{01}(s)} \frac{ds}{s} \\ &\leq \frac{1}{\varphi_{01}(t)} + \frac{t^a}{\varphi_{01}(t)} \int_t^\infty s^{-a-1} ds = \left(1 + \frac{1}{a}\right) \frac{1}{\varphi_{01}(t)}. \end{aligned}$$

In the above example it is sufficient to assume that $\int_0^t \varphi_{01}(s) ds/s \approx \varphi_{01}(t)$. Indeed, defining $\overline{\varphi}_{01}(t) = \int_0^t \varphi_{01}(s) ds/s$ we have $\overline{\varphi}_{01} \approx \varphi_{01}$ and $t^{-a} \overline{\varphi}_{01}(t)$ is an increasing function on R_+ if $a > 0$ is sufficiently small. Thus we have reduced this case to that of Example 1.

Example 2. If $t^{-a} \varphi_1(t)$ is a non-decreasing function on R_+ for some $0 < a < 1$ and the upper Boyd index satisfies the inequality $\beta(E_1) < 1$, then for all $x \in L^1 + E_1$ and all $t > 0$

$$(5a_2) \quad K(t, x; L^1, E_1) \approx \int_0^{\varphi^{-1}(t)} x^*(s) ds + t \left\| x^{*1}_{[\varphi^{-1}(t), \infty)} \right\|_{E_1}$$

$$\approx t \left\| x^{**1}_{[\varphi^{-1}(t), \infty)} \right\|_{E_1}$$

where $\varphi(t) = t/\varphi_1(t)$ and $x^{**}(t) = t^{-1} \int_0^t x^*(s) ds$.

P r o o f. From Theorem 1a we get our first equivalence.

On the other hand, we have

$$\int_0^t x^*(s) ds \approx \int_0^t x^*(s) ds \varphi(t) \left\| 1_{[t, \infty)}(s)/s \right\|_{E_1} \leq \varphi(t) \left\| x^{*1}_{[t, \infty)} \right\|_{E_1}$$

and

$$\begin{aligned} \left\| x^{*1}_{[t, \infty)} \right\|_{E_1} &= \left\| (x^{*1}_{[t, \infty)})^* \right\|_{E_1} = \left\| x^{**}(\cdot + t) \right\|_{E_1} \\ &= \left\| x^{**}(\cdot + t) 1_{(0, t)} + x^{**}(\cdot + t) 1_{[t, \infty)} \right\|_{E_1} \leq c_2 x^{**}(t) \varphi_1(t) + \\ &+ \left\| (x^{*1}_{[t, \infty)})^{**} \right\|_{E_1} \leq c_2 x^{**}(t) \varphi_1(t) + c_3 \left\| x^{*1}_{[t, \infty)} \right\|_{E_1}. \end{aligned}$$

Consequently, we obtain the second equivalence, and the proof is finished.

Example 3. If $E_1 = L^\infty$, then the conditions (4a) hold with $c_0 = c_1 = 1$ and for any $x \in E_0 + L^\infty$

$$(5a_3) \quad K(\varphi_0(t), x; E_0, L^\infty) \approx \|x^{*1}_{(0,t)}\|_{E_0}$$

(cf. [19], Th.3 where, moreover, the best constants are computed). In particular,

$$(5a'_3) \quad K(t, x; L^p + L^\infty, L^\infty) \approx \|x^{*1}_{(0, \min(1, t^p))}\|_{L^p}$$

$$(5a''_3) \quad K(t, x; L^p \cap L^\infty, L^\infty) \approx \max(\|x^{*1}_{(0, t^p)}\|_{L^p}, \min(1, t) \|x\|_{L^\infty}).$$

The last two equivalences cannot be obtained from Milman's result.

Example 4. Let E be a symmetric space on $(0, \infty)$ with the fundamental function φ such that $\varphi(R_+) = R_+$. For every $1 \leq p < \infty$ let $E^{(p)} = \{x \in L^0 : |x|^p \in E\}$ be normed by $\|x\|_{E^{(p)}} = \| |x|^p \|_E^{1/p}$. It is well known that $E^{(p)}$ is a symmetric space. If $1 \leq p < q < \infty$, then for all $x \in E^{(p)} + E^{(q)}$ and all $t > 0$ we have

$$(5a_4) \quad \begin{aligned} K(\varphi(t)^{1/p-1/q}, x; E^{(p)}, E^{(q)}) &\approx \|x^{*1}_{(0,t)}\|_{E^{(p)}} + \\ &+ \varphi(t)^{1/p-1/q} \|x^{*1}_{[t,\infty)}\|_{E^{(q)}}. \end{aligned}$$

P r o o f. We have

$$\begin{aligned} 2^{-1} \left\| \frac{1_{(0,t)}}{\varphi^{1/q}} \right\|_{E^{(p)}}^p &\leq \left\| \frac{1_{(0,t)}}{\varphi^{1/q}} \right\|_{\Lambda(E)^{(p)}}^p = \int_0^t \frac{d\varphi(s)}{\varphi(s)^{p/q}} \\ &= \frac{q}{q-p} \int_0^t d[\varphi(s)^{1-p/q}] = \frac{q}{q-p} \varphi(t)^{1-p/q} \end{aligned}$$

and

$$\begin{aligned} 2^{-1} \left\| \frac{1_{[t,\infty)}}{\varphi^{1/p}} \right\|_{E^{(q)}}^q &\leq \left\| \frac{1_{[t,\infty)}}{\varphi^{1/p}} \right\|_{\Lambda(E)^{(q)}}^q = \int_0^\infty \left(\frac{1_{[t,\infty)}}{\varphi^{1/p}} \right)^*(s)^q d\varphi(s) \\ &= \int_0^\infty \frac{d\varphi(s)}{\varphi(s+t)^{q/p}} \leq \frac{1}{\varphi(t)^{q/p}} \int_0^t d\varphi(s) + \frac{p}{p-q} \int_t^\infty d[\varphi(s)^{1-q/p}] \end{aligned}$$

$$= \varphi(t)^{1-q/p} - \frac{p}{p-q} \varphi(t)^{1-q/p} = \frac{q}{q-p} \varphi(t)^{1-q/p}.$$

Hence the conditions (4a) are fulfilled and by Theorem 1a we have the equivalence (5a₄).

Arazy in [1] has given another proof of the above equivalence.

Remark 5. Since $K(t, |x|; E_0, E_1) = \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1} : 0 \leq x_0 \in E_0, 0 \leq x_1 \in E_1, |x| \leq x_0 + x_1\}$, it follows that for all $x \in E_0^{(p)} + E_1^{(p)}$ and all $t > 0$ we have

$$(6) \quad K(t^p, |x|^p; E_0, E_1)^{1/p} \leq K(t, x; E_0^{(p)}, E_1^{(p)}) \leq 2^{1-1/p} K(t^p, |x|^p; E_0, E_1)^{1/p}.$$

The constant $2^{1-1/p}$ cannot be improved (cf. [5]).

Example 5. Let $1 \leq p_0 < p_1 < \infty$. For the couples $(L^{p_0} \cap L^\infty, L^{p_1} \cap L^\infty)$ and $(L^{p_0} + L^\infty, L^{p_1} + L^\infty)$ the conditions (4a) are satisfied.

Example 6. Let $1 \leq p_0 < p_1 \leq \infty$. For the couple $(L^{p_0} + L^{p_1}, L^{p_0} \cap L^{p_1})$ we have $\varphi_{01}(t) = \min(t^{1/p_0 - 1/p_1}, t^{1/p_1 - 1/p_0})$. If we choose $A_t = (0, \min(t, t^{-1})) \cup (\max(t, t^{-1}), \infty)$, then the conditions (4) are fulfilled with $u(t) = \varphi_{01}(t)$. By Theorem 1, for $0 < t \leq 1$, we have

$$(5') \quad K(t, x; L^{p_0} + L^{p_1}, L^{p_0} \cap L^{p_1}) \approx K(t, x; L^{p_0}, L^{p_1}) + t K(t^{-1}, x; L^{p_0}, L^{p_1})$$

For some couples of symmetric spaces the conditions (4) are not satisfied.

(Counter)Example 7. Let $1 \leq p_0 < p_1 \leq \infty$. For the couples $(L^{p_0}, L^{p_0} + L^{p_1}), (L^{p_0}, L^{p_0} \cap L^{p_1})$ and, if $p_1 < \infty$, for the couples $(L^{p_0} + L^{p_1}, L^{p_1}), (L^{p_0} \cap L^{p_1}, L^{p_1})$ as well, the function φ_{01} is non-decreasing on R_+ and the conditions (4a) do not hold. More generally, if $R_+ = A_t \cup B_t$, then either $1_{A_t}/\varphi_1 \notin E_0$ or $1_{B_t}/\varphi_0 \notin E_1$, i.e., the conditions (4) do not hold.

Problem 1. Compute the K-functional for the couple of arbitrary Orlicz spaces (L^{F_0}, L^{F_1}) .

Let us remark that the above considerations do not include the cases when the fundamental functions are equivalent. However, applying, if needed, Theorem 1 of Dmitriev-Ovčinnikov [12] or Theorem 1 of Asekritova [2], or Theorem 8.1 of Brudnyi-Krugljak [7], we can obtain, among other things, the K-functional for the couples (L^{pq_0}, L^{pq_1}) , $(\Lambda(E), M(E))$ and $(\Lambda^{q_0}(E), \Lambda^{q_1}(E))$. The method for doing this was shown to the author by Per Nilsson in a private communication.

Let us consider the couple (L^{pq_0}, L^{pq_1}) , $1 < p < \infty$ and $1 \leq q_0 < q_1 \leq \infty$. Pick out $1 \leq p_0 < p < \infty$ and θ such that $L^{pq_0} = (L^{p_0}, L^\infty)_{\theta, q_0; K}$.

Then Theorem 1 of Dmitriev-Ovčinnikov [12] implies that

$$\begin{aligned} K(t, x; L^{pq_0}, L^{pq_1}) &\approx K(t, x; (L^{p_0}, L^\infty)_{\theta, q_0; K}, (L^{p_0}, L^\infty)_{\theta, q_1; K}) \\ &\approx K(t, K(s, x; L^{p_0}, L^\infty); L_*^{q_0}(t^{-\theta}), L_*^{q_1}(t^{-\theta})) \\ &\approx K(t, s^{-\theta} \int_0^s x^*(s) p_0 ds)^{1/p_0}; L_*^{q_0}, L_*^{q_1}) \end{aligned}$$

this can be obtained explicitly using the results of Holmstedt [13].

Problem 2. Compute the K-functional for couples $(E, \Lambda(E))$ and $(M(E), E)$.

4. THE K-FUNCTIONAL FOR SOME BANACH COUPLES

Now we consider any Banach couple $\bar{A} = (A_0, A_1)$. We let

$$\Delta(\bar{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\bar{A}) = A_0 + A_1$$

Examples 6 and 7 evoke the need of computing the K-functional for the following couples of Banach spaces: $(\Sigma(\bar{A}), \Delta(\bar{A}))$, $(\Sigma(\bar{A}), A_0)$, $(A_0, \Delta(\bar{A}))$, $(\Sigma(\bar{A}), A_1)$ and $(A_1, \Delta(\bar{A}))$ (of course for $0 < t \leq 1$). It can be done either by Cwikel [10] and Brudnyi-Krugljak [7] or Asekritova [2] results, or by direct computation.

If $0 < t \leq 1$, then Lemma 2 in Cwikel [10], Theorem 8.1 in Brudnyi-Krugljak [7] and Corollary 1 in [19] imply that

$$K(t, a; \Sigma(\bar{A}), \Delta(\bar{A})) = K(t, a; \bar{A}_{\Sigma(\bar{L}^\infty)}; K, \bar{A}_{\Delta(\bar{L}^\infty)}; K)$$

$$\approx K(t, K(\cdot, a; \bar{A}); \Sigma(\bar{L}^\infty), \Delta(\bar{L}^\infty)) \approx K(t, a; \bar{A}) + tK(t^{-1}, a; \bar{A}),$$

$$K(t, a; \Sigma(\bar{A}), A_0) = K(t, a; \bar{A}_{\Sigma(\bar{L}^\infty)}; K, \bar{A}_{L^\infty}; K)$$

$$\approx K(t, K(\cdot, a; \bar{A}); \Sigma(\bar{L}^\infty), L^\infty) = tK(t^{-1}, a; \bar{A})$$

etc.

In the above equivalences we applied the Brudnyĭ-Krugljak formula $K(t, a; \bar{A}_{\phi_0}; K, \bar{A}_{\phi_1}; K) \approx K(t, K(\cdot, a; \bar{A}); \tilde{\phi}_0, \tilde{\phi}_1)$, whose proof depends in an essential way on K-divisibility of the K-functional. Therefore we give simple computations of the K-functional for the above couples.

Theorem 2. For all $a \in \Sigma(\bar{A})$ and all $t > 0$ we have

$$(7) \quad K(t, a; \Sigma(\bar{A}), A_i) = K(\min(1, t), a; A_{1-i}, A_i), \quad i = 0, 1.$$

P r o o f. It is enough to prove this for $i = 0$. Fix $a \in \Sigma(\bar{A})$ and $0 < t \leq 1$. For each $\epsilon > 0$ there exists $a_t \in A_0$ such that

$$\|a - a_t\|_{\Sigma(\bar{A})} + t\|a_t\|_{A_0} \leq (1 + \epsilon)K(t, a; \Sigma(\bar{A}), A_0).$$

We can find a decomposition $a - a_t = a_{0t} + a_{1t}$ such that $a_{0t} \in A_0$, $a_{1t} \in A_1$ and

$$\|a_{0t}\|_{A_0} + \|a_{1t}\|_{A_1} \leq (1 + \epsilon)\|a - a_t\|_{\Sigma(\bar{A})}.$$

It follows that $a = a_{0t} + a_t + a_{1t}$ and

$$\begin{aligned} K(t, a; A_1, A_0) &\leq \|a_{1t}\|_{A_1} + t\|a_{0t} + a_t\|_{A_0} \leq \|a_{0t}\|_{A_0} + \\ &+ \|a_{1t}\|_{A_1} + t\|a_t\|_{A_0} \leq (1 + \epsilon)\|a - a_t\|_{\Sigma(\bar{A})} + t\|a_t\|_{A_0} \\ &\leq (1 + \epsilon)^2 K(t, a; \Sigma(\bar{A}), A_0). \end{aligned}$$

This proves (the non-trivial part of) the theorem.

In particular, for all $x \in L^1 + L^\infty$ and all $t > 0$ we have the equalities:

$$K(t, x; L^1 + L^\infty, L^\infty) = \int_0^{\min(1, t)} x^*(s) ds,$$

$$K(t, x; L^1, L^1 + L^\infty) = \min(1, t) \int_0^{\max(1, t)} x^*(s) ds.$$

Theorem 3. Let $a \in \Sigma(\bar{A})$ and $0 < t < 1$. Then

$$(8) \quad K(t, a; \Sigma(\bar{A}), \Delta(\bar{A})) \approx K(t, a; \bar{A}) + tK(t^{-1}, a; \bar{A}).$$

P r o o f. From Theorem 2 we have

$$\begin{aligned} \max(K(t, a; \bar{A}), tK(t^{-1}, a; \bar{A})) &= \max(K(t, a; \Sigma(\bar{A}), A_1), K(t, a; \Sigma(\bar{A}), A_0)) \\ &\leq K(t, a; \Sigma(\bar{A}), \Delta(\bar{A})). \end{aligned}$$

On the other hand, for each $\epsilon > 0$ there exist decompositions $a = a_{0t} + a_{1t} = a'_{0t} + a'_{1t}$ with $a_{0t}, a'_{0t} \in A_0$ and $a_{1t}, a'_{1t} \in A_1$ such that

$$\|a_{0t}\|_{A_0} + t\|a_{1t}\|_{A_1} \leq (1+\epsilon)K(t, a; \bar{A})$$

$$\|a_{0t}\|_{A_0} + t^{-1}\|a'_{1t}\|_{A_1} \leq (1+\epsilon)K(t^{-1}, a; \bar{A}).$$

Put

$$a_t = a'_{0t} - a_{0t} = a_{1t} - a'_{1t}.$$

Then $a_t \in \Delta(\bar{A})$. Furthermore,

$$\|a - a_t\|_{\Sigma(\bar{A})} = \|a_{0t} + a'_{1t}\|_{\Sigma(\bar{A})} \leq \|a_{0t}\|_{A_0} + \|a'_{1t}\|_{A_1}$$

$$\leq (1+\epsilon)K(t, a; \bar{A}) + (1+\epsilon)tK(t^{-1}, a; \bar{A})$$

and

$$t\|a_t\|_{\Delta(\bar{A})} = t\max(\|a'_{0t} - a_{0t}\|_{A_0}, \|a_{1t} - a'_{1t}\|_{A_1})$$

$$\begin{aligned}
&\leq t \max(\|a'_{0t}\|_{A_0} + \|a_{0t}\|_{A_0}, \|a_{1t}\|_{A_1} + \|a'_{1t}\|_{A_1}) \\
&\leq (1+\varepsilon) \max(K(t^{-1}, a; \bar{A}) + K(t, a; \bar{A}), t^{-1}K(t, a; \bar{A}) + tK(t^{-1}, a; \bar{A}))t \\
&\leq (1+\varepsilon) (K(t, a; \bar{A}) + tK(t^{-1}, a; \bar{A})).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
K(t, a; \Sigma(\bar{A}), \Delta(\bar{A})) &\leq \|a - a_t\|_{\Sigma(\bar{A})} + t \|a_t\|_{\Delta(\bar{A})} \\
&\leq 2(1+\varepsilon) (K(t, a; \bar{A}) + tK(t^{-1}, a; \bar{A})),
\end{aligned}$$

and the proof is finished.

The simple proof in the remaining cases is left to the reader. From Theorem 3 it follows immediately that for $0 < \theta < 1$ and $1 \leq p < \infty$ or $0 \leq \theta \leq 1$ and $p = \infty$ we have

$$(\Sigma(\bar{A}), \Delta(\bar{A}))_{\theta, p; K} = \begin{cases} \bar{A}_{\theta, p; K} + \bar{A}_{1-\theta, p; K}, & \theta \leq 1/2 \\ \bar{A}_{\theta, p; K} \cap \bar{A}_{1-\theta, p; K}, & \theta \geq 1/2. \end{cases}$$

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1070

Interpolation Spaces and Allied Topics in Analysis

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Maligranda, Lech (PL-PAN)

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The K -functional for symmetric spaces.

Interpolation spaces and allied topics in analysis (Lund, 1983), 169–182, *Lecture Notes in Math.*, 1070, Springer, Berlin-New York, 1984.

The author proves the following interesting result: Assume that for each $t \in \mathbb{R}_+$ there exists a partition $\mathbb{R}_+ = A_t \cup B_t$ and a function $u: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the conditions $\|1_{A_t}/\varphi_1\|_{E_0} \leq c_0 u(t)$, $\|1_{B_t}/\varphi_0\|_{E_1} \leq c_1 u(t)$ hold uniformly in $t > 0$ for some $c_0, c_1 \geq 1$, where the E_i are symmetric spaces on $(0, \infty)$ and $\varphi_i(t) = \|1_{(0,t)}\|_{E_i}$, $i = 0, 1$. Then

$$K(u(t), x; E_0, E_1) \approx \|x^* 1_{A_t}\|_{E_0} + u(t) \|x^* 1_{B_t}\|_{E_1}$$

for all $x \in E_0 + E_1$.

{For the entire collection see MR 85g:46003.}

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Zbl 573

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Interpolation spaces and allied topics in analysis, Proc. Conf., Lund/Swed. 1983, Lect. Notes Math. 1070, 169 – 182 (1984).

[For the entire collection see Zbl. 534.00013.]

Si A_0 et A_1 désignent un couple d'espace de Banach la fonction K :

$$a = a_0 + a_1 \in A_0 + A_1 \rightarrow K(t, a) = \inf(\|a_0\|_{A_0} + t\|a_1\|_{A_1}) \in \mathbb{R}^+$$

joue un rôle important dans la théorie de l'interpolation au sens de Lions-Magenese et Peetre. L'auteur s'intéresse au cas d'espaces de fonctions invariants par réarrangement, ainsi dans son théorème I énonce-t-il des hypothèses sous les-elles on obtient une forme équivalente de cette fonction, plus facile à utiliser dans les applications. Puis il donne des exemples de ce théorème.

M. - Th. Lacroix.

3 Б898. K -функционал для симметричных пространств. The K -functional for symmetric spaces. Maligranda Lech. «Lect. Notes Math.», 1984, 1070, 169–182 (англ.)

Пусть E_0, E_1 — симметричные пространства на $[0, \infty)$ с фундаментальными функциями φ_0, φ_1 . Предполагается, что существует неотрицательная функция $u(t)$ на полуоси и для каждого $t > 0$ существует такое разложение $(0, \infty) = A_t \cup B_t$, что

$$\left\| \frac{1_{A_t}}{\varphi_1} \right\|_{E_0} \leq c_0 u(t), \quad \left\| \frac{1_{B_t}}{\varphi_0} \right\|_{E_1} \leq c_1 / u(t).$$

Тогда

$$K(u(t), x, E_0, E_1) \approx \|x^* 1_{A_t}\|_{E_0} + u(t) \|x^* 1_{B_t}\|_{E_1}.$$

Приводятся несколько конкретных реализаций этой теоремы. Библи. 32. Е. Семёнов