Blind Adapted, Pre-Whitened Constant Modulus Algorithm

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Abstract—We examine the use of a blind adaptive “pre-whitening” filter to precede an equalizer adapted by the constant modulus algorithm (CMA). The idea is based on results presented in which the use of a (fixed, or non-adaptive) pre-whitening filter provides an isometry (i.e. geometry preserving transformation) between the combined channel-equalizer (or global space) and the equalizer tap space. As much analysis found in CMA literature is done in “global space” now applies to the pre-whitened “equalizer tap space”, considerable exploitation of a known geometric structure is now possible. This paper’s main result demonstrates a method for improving the convergence rates of CMA adapted equalizers using the results of pre-whitening analysis. The slow convergence of CMA equalizers is a common negative point associated with this well-known, robust blind equalization algorithm.

Keywords—Constant Modulus Algorithm (CMA), Blind Equalization, Whitening, Convergence Analysis

I. INTRODUCTION

THE constant modulus algorithm (CMA) [1], [2], is a well-known algorithm for blind equalization which has appeared in the recent literature ([3], [4] contain generous lists of references). Its popularity stems from its implementational simplicity and robustness properties. However, it suffers from slow convergence.

The associated CMA error surface has been analyzed extensively (for example, [5], [6], [7], [8]). However, much of its analysis is done in the combined “channel-equalizer” (or “global”) space. This is often needed to provide understanding of the algorithm that is independent of the actual channel of interest and/or to allow for tractable analysis. However, there are drawbacks to restricting such analysis to the combined channel-equalizer space. Specifically, one loses understanding of many convergence issues and transient characteristics of the *actual equalizer* parameterizations, since it is indeed only the equalizer that is adapted (i.e. not the combined channel-equalizer).

Analysis of the use of a pre-whitening filter on CMA shows that whitening of the received signal that a CMA equalizer

with pre-whitened input will have the exact same transient and asymptotic behavior as a system adapted in the combined channel-equalizer space. Such results provide a basis for the use of the adaptive pre-whitener explored herein.

II. MATHEMATICAL FRAMEWORK

Consider Figure 1 in which a zero-mean, unit-variance, independent, identically distributed (IID) source sequence $\{s(n)\}$ drawn from a finite alphabet passes through a channel with memory $C(z)$ introducing intersymbol interference (ISI) in the received sequence $\{x(n)\}$. An equalizer $D(z)$ is to be used to remove (or reduce) the ISI. The cascaded channel-equalizer is known as the *global* response and denote $G(z)$. The transmitted source may then be estimated from the equalizer output sequence $\{y(n)\}$ using a symbol-by-symbol decision device.

Fig. 1. Simple Channel-Equalizer Block Diagram

Proper equalizer design usually requires knowledge (or estimation) of the channel. Some adaptive methods such as LMS require a bandwidth consuming training sequence (see [9]). Blind equalization [10] allows adaptation to ISI reducing equalizer settings without the need for such training sequences or channel estimates. CMA [2] is a popular blind method which penalizes the deviation of the equalizer output’s modulus from some constant γ . The CMA cost function may be written,

$$J = \frac{1}{4} E \left\{ \left(|y(n)|^2 - \gamma \right)^2 \right\}. \quad (1)$$

The length M equalizer’s tap weights $D(n)$ are adapted using an instantaneous estimate of the gradient of J with respect to

tap weight vector D via

$$y(n) = D^T(n)X(n) \quad (2)$$

$$D(n+1) = D(n) - \alpha (|y(n)|^2 - \gamma) y(n) X^*(n) \quad (3)$$

where $X(n) = [x(n) \ x(n-1) \ \dots \ x(n-(M-1))]^T$ is the M -tap equalizer state vector, $*$ denotes complex conjugation and α is the adaptation stepsize.

III. KNOWN RESULTS IN GLOBAL SPACE

The CMA error surface is known to be multimodal and contains many saddle points [12], [7], [13]. However, it is also known to exhibit robust performance in a variety of conditions. Due to the nonlinearity of the cost function, the analysis of CMA behavior is difficult and often only possible with some simplifying assumptions.

A common approach is to use an infinite-tap equalizer which means all points in combined channel-equalizer space (G -space) are reachable (assuming an invertible channel $C(z)$). In this way, we have an *isomorphism* (one-to-one and onto relation) between the point in the equalizer space D and points in the combined channel-equalizer space G . Using this it has been shown for the noiseless cases [14] that

- CM minima correspond to global transfer function $G(z) = z^{-\nu}$.
- CM saddles are strictly inside the unit sphere.
- CMA (gradient descent adaptation) converges to the minima asymptotically

Additionally, robustness of CMA in additive noise is shown in [17].

Similar methods rely on temporal (or spatial) diversity of fractionally-spaced equalizers (FSE's) (or antenna arrays) to establish this isomorphic relation between the channel-equalizer space and equalizer space. In this way, finite length equalizers may be used also to gain similar results to the infinite-tap equalizers (see [15], [18], [12], [16]).

IV. USE OF WHITENING FILTER

Next we consider the use of a whitening filter $W(z)$ prior to the equalizer $D_w(z)$ as in Figure 2. It is well-known that for broad class of processes that a causal, stable whitening filter exists [19]. (Note that pre-whitening does *not* in general correspond to ISI removal.) Such prewhitening may be done as in the algorithm proposed in [11] or in a standard linear prediction method (in an optimal MSE/Weiner Sense). We denote the cascade of the channel and whitening filter as $Q(z) = W(z)C(z)$.

Fig. 2. Pre-whitened Block Diagram

Notice (from Figure 2) that with

$$Q = \begin{bmatrix} q_0 & q_1 & q_2 & \dots \\ 0 & q_0 & q_1 & q_2 & \dots \\ 0 & 0 & q_0 & q_1 & q_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (4)$$

being the convolution matrix of the cascade channel-whitening filter $Q(z)$ we have several useful relations. The relation between the vectors of source sequence S and vectors of the whitener output U is given by

$$U = Q^H S \quad (5)$$

Relating the source autocorrelation matrix to the whitening filter output, we have

$$\begin{aligned} \mathcal{R}_{UU} &= E\{UU^H\} \\ &= E\{Q^H S S^H Q\} \\ &= Q^H E\{S S^H\} Q \\ &= Q^H \mathcal{R}_{SS} Q \end{aligned}$$

Using the whiteness assumption on the source, and definition of whiteness on U , we have $\mathcal{R}_{SS} = \mathcal{R}_{UU} = \mathcal{I}$, and

$$\mathcal{I} = Q^H Q.$$

Hence the combined channel-whitening transformation is *unitary*, that is $Q^{-1} = Q^H$. A unitary transformation from one space to another has all eigenvalues equal to one and is well-known to preserve geometries between the two spaces. In other words, it provides an *isometry* between the two spaces in which distances between points are preserved under the transformation. To see this, use the property of the adjoint being the unique operator which commutes across the inner product operation and consider the distance between two whitening filter output sequences u_1 and u_2 when $U = Q^H S$

$$\begin{aligned} \|u_1 - u_2\|^2 &= \langle (u_1 - u_2), (u_1 - u_2) \rangle \\ &= \langle Q^H (s_1 - s_2), Q^H (s_1 - s_2) \rangle \\ &= \langle (s_1 - s_2), Q Q^H (s_1 - s_2) \rangle \\ &= \|s_1 - s_2\|^2. \end{aligned} \quad (6)$$

In the non-unitary cases, the distance distortion appears in the $\mathcal{R}_{UU}$ term.

We now explore the relationship between the CMA cost function parameterized by the global-space, $J(G)$, and the CMA cost function parameterized by the equalizer in the pre-whitened case $J(D_w)$. For the global space, with $|y|^2 = (G^H S)(G^H S)^H = G^H S S^H G$ where S is the state vector of the global space and G the global impulse response vector, we get,

$$\begin{aligned} J(G) &= \frac{1}{4} \mathbb{E} \left\{ (|y(n)|^2 - \gamma)^2 \right\} \\ &= \frac{1}{4} \mathbb{E} \left\{ (G^H S S^H G - \gamma)^2 \right\} \\ &= \frac{1}{4} \mathbb{E} \left\{ G^H S S^H G G^H S S^H G - 2\gamma G^H \mathbb{E} \{ S S^H \} G + \gamma^2 \right\} \\ &= \frac{1}{4} (G^H \mathbb{E} \{ S S^H G G^H S S^H \} G - 2\gamma G^H G + \gamma^2). \end{aligned} \quad (7)$$

The mean gradient of this cost function with respect to G determines the trajectory of a CMA equalizer for the special case when the global parameterization (combined channel-equalizer space) is adjustable is given by

$$\begin{aligned} \frac{\partial J(G)}{\partial G} &= \frac{\frac{1}{4} \frac{\partial \mathbb{E} \left\{ (G^H S S^H G - \gamma)^2 \right\}}{\partial G}}{\partial G} \\ &= \mathbb{E} \{ G^H S S^H G S S^H G \} - \gamma \mathbb{E} \{ S S^H \} G \\ &= \mathbb{E} \{ G^H S S^H G S S^H G \} - \gamma G. \end{aligned} \quad (8)$$

Now consider the prewhitened case with $y_w = D_w^H U = U^H D_w$ where U is the state vector of the equalizer. Similar steps obtain

$$\begin{aligned} J(D_w) &= \frac{1}{4} (D_w^H \mathbb{E} \{ U U^H D_w D_w^H U U^H \} D_w \\ &\quad - 2\gamma D_w^H D_w + \gamma^2) \end{aligned} \quad (9)$$

Similarly, the mean gradient of this cost function with respect to D_w (the equalizer taps) determines the actual trajectory of a pre-whitened CMA equalizer. Since $U = \mathcal{Q}^H S$, the CMA cost function mean gradient for pre-whitened CMA ($\frac{\partial J(D_w)}{\partial D_w}$) can be written in terms of the global system gradient $\frac{\partial J(G)}{\partial G}$ as follows,

$$\frac{\partial J(D_w)}{\partial D_w} = \mathcal{Q}^H \frac{\partial J(G)}{\partial G}. \quad (10)$$

Accordingly, the Hessian with respect to G and D_w are related through:

$$\mathcal{H}_{D_w}(D_w) = \mathcal{Q}^H \mathcal{H}_G(G) \mathcal{Q}. \quad (11)$$

where the i, j entry of $\mathcal{H}_G(G)$ is given by $\frac{\partial^2 J(G)}{\partial G_i \partial G_j}$.

So, in fact, the gradients and Hessians of the CMA cost function between the globally adaptable system and pre-whitened CMA agree up to a unitary transformation. In other words, the

behavior of prewhitened CMA is “equivalent to” CMA adapted in the global space (modulo a unitary transformation). The effect of this unitary transformation doesn’t affect asymptotic convergence but may affect the convergence speed (compared to the non-prewhitened case). This is similar to the effect of reducing eigenvalue disparity in the filter input autocorrelation matrix in standard LMS. High eigenvalue disparity introduces a “slow” mode which adversely affects convergence speed.

Moreover, we are looking for D_w solving for $\mathcal{Q} D_w = G$ with G having one non-zero entry. Therefore, $D_w = \mathcal{Q}^H G$ should belong to the unit sphere. This may lead us to a better initialization of D_w for the CMA adaptation and to eventual modifications of the CMA.

V. ADAPTIVE PRE-WHITENING

The above analysis considered a fixed, optimal prewhitened (in the Weiner sense) prewhitening filter. Next we introduce a blind adaptive pre-whitening filter using the robust least mean squares (LMS) algorithm in a linear prediction configuration prior to the CMA adapted equalizer as shown in Figure 3

Fig. 3. Adaptive Pre-whitening with Linear Prediction

In this configuration the whitening filter is an LMS linear predictor. It is the error signal (or innovations) that is fed on to the CMA equalizer. The linear predictor’s N -tap weights are adjusted using LMS as

$$u(n) = x(n) - W^T(n) \tilde{X}(n) \quad (12)$$

$$W(n+1) = W(n) - \beta_{lp} u(n) \tilde{X}(n) \quad (13)$$

where $\tilde{X}(n) = [x(n-1) \ x(n-2) \ \dots \ x(n-N)]^T$. The adaptation stepsize parameter for this linear predictor is denoted β_{lp} .

The equalizer’s M tap weights are adjusted using CMA in the usual fashion.

$$y(n) = D_W^T(n) U(n) \quad (14)$$

$$D_W(n+1) = D_W(n) - \beta_{cma} (|y(n)|^2 - \gamma) y(n) U^*(n) \quad (15)$$

where $U(n) = [u(n) \ u(n-1) \ \dots \ u(n-(M-1))]^T$ is the M -tap equalizer state vector. The stepsize parameter for this CMA equalizer portion of the prewhitened CMA system is denoted β_{cma} .

VI. EXAMPLE

The following computer simulation demonstrates the performance advantages that the adaptive pre-whitening input to a standard CMA equalizer provides. In this case a comparison is made with a standard CMA equalizer (denoted CMA) and an adaptive (linear prediction) prewhitened CMA equalizer (denoted LP-CMA) as well as an optimal pre-whitened CMA (denoted WCMA). The channel used is somewhat arbitrary, but deliberately chosen to have a large range of frequency response (i.e. a case wherein CMA is known to have long convergence times). It is interesting to note, that LMS adapted filters also would exhibit a “slow mode” convergence in such cases. However, the quadratic nature of the LMS algorithm appears to be less sensitive to the eigenvalue spread than the CMA’s quartic error surface.

The channel taps C are

$$C = \begin{bmatrix} .0003 & .0029 & .0090 & -.0440 & .0468 & -.0485 \\ & .3603 & -.7401 & .5451 & -.1326 & .0350 \end{bmatrix}^T$$

which has frequency response as in Figure 4.

Fig. 4. Channel Frequency Response

The source sequence is iid BPSK with $s(n) \in \{-1, +1\}$, and the CMA dispersion constant is thus $\gamma = 1$. The $M(= 25)$ initial linear predictor weights and the initial $N(= 25)$ CMA equalizer weights for the all examples (CMA and LP-CMA) are all zero, except for the center tap which equals 1 (using the “standard” center tap initialization).

The adaptation stepsize for the three experiments are shown in Table I. Note that the standard CMA is given a stepsize *twice* as large as the LP-CMA combination and the WCMA case. This is to allow fairer comparison since the LP-CMA combination enjoys two adaptations (one for the linear predictor, and one for the CMA equalizer) per sample and the “omniscient” WCMA, while standard CMA has only one. Doubling standard CMA’s adaptation stepsize is meant to level the playing field. (Although note that the LP-CMA system has more parameters to adjust. Typically adaptive systems with more parameters will take *longer* time to converge than systems with less parameters.)

TABLE I

STEPWISE PARAMETERS FOR EXPERIMENTS

Experiment Name	LP Stepsize	CMA Stepsize
CMA		$\alpha = 0.02$
LP-CMA I	$\beta_{lp} = 0.005$	$\beta_{cma} = 0.01$
LP-CMA II	$\beta_{lp} = 0.01$	$\beta_{cma} = 0.01$
LP-CMA III	$\beta_{lp} = 0.05$	$\beta_{cma} = 0.01$
WCMA		$\alpha = 0.01$

Also, in order to allow for proper comparison of the systems with these simulation the adaptation equations used a *normalized* form (à la normalized LMS – NLMS) in which the adaptation stepsizes are normalized by the filter’s regressor’s power. Specifically,

- in (3) replace α with $\alpha/X^H X$
- in (13) replace β_{lp} with $\beta_{lp}/\tilde{X}^H \tilde{X}$
- in (15) replace β_{cma} with $\beta_{cma}/U^H U$.

Although certainly not necessary for actual implementation, it was used here since the CMA filter’s (D_W input power) in the LP-CMA system input power may be fluctuating due to the adapting linear predictor. The normalized adaptation algorithm versions were used in the simulations here for fairer comparison. But it is suspected that prewhitening the CMA input would actually allow a *larger* adaptation stepsize to be used. This would allow for even faster convergence for the prewhitened CMA case.

Figure 5 shows the equalizer outputs of the two systems. Notice that the prewhitened CMA equalizer achieves faster eye-opening. For a better view of the convergence behavior of these two systems, define a measure of the intersymbol interference (ISI) as

$$ISI = \frac{\sum_i h_i^2 - \max_i(h_i^2)}{\max_i(h_i^2)}. \quad (16)$$

Figure 6 shows the ISI trajectories (in dB) during adaptation for the two systems. Notice that large differences in the slopes on the log scale between the two systems. Also note that the difference between the slopes is increasing: the standard CMA system is “decelerating” while the LP-CMA system maintains the same exponential-constant of convergence.

VII. CONCLUSION

This work introduces an exploit of the analysis of the prewhitened CMA error surface. Such prewhitening allows the adapted equalizer to traverse a trajectory similar to the trajectory

Fig. 5. Equalizer Outputs

which a non-realizable globally (channel-equalizer) adapted system. The geometries of such global channel-equalizer spaces are well understood from the wealth of existing research on CMA. The immediate impact of this allows reduction of “slow” modes of convergence of the CMA equalizer through the use of prewhitening. Specifically, the use of a LMS adaptive filter in a linear predictor configuration is introduced to prewhiten the input to a conventional CMA equalizer. The convergence speed is seen to be markedly improved for channels which exhibits a large frequency response deviation.

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Fig. 6. ISI Trajectory Comparison

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