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INTERPOLATION OF CESÀRO AND COPSON SPACES

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ABSTRACT. Interpolation properties of Cesàro and Copson spaces are investigated. It is shown that the Cesàro function space $Ces_p(I)$, where $I = [0,1]$ or $[0,\infty)$, is an interpolation space between $Ces_{p_0}(I)$ and $Ces_{p_1}(I)$ for $1 < p_0 < p_1 \le \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$. The same result is true for Cesàro sequence spaces. For Copson function and sequence spaces a similar result holds even in the case when $1 \le p_0 < p_1 \le \infty$. At the same time, $Ces_p[0,1]$ is not an interpolation space between $Ces_1[0,1]$ and $Ces_\infty[0,1]$ for any $1 < p < \infty$.

1. INTRODUCTION

Let us begin with some necessary definitions and notations related to the interpolation theory of operators as well as Cesàro and Copson spaces.

For two normed spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the imbedding $X \subset Y$ is continuous with the norm which is not bigger than C, i.e., $\|x\|_Y \le C\|x\|_X$ for all $x \in X$, and $X \hookrightarrow Y$ means that $X \overset{C}{\hookrightarrow} Y$ for some $C > 0$. Moreover, $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, that is, the spaces are the same and the norms are equivalent. If f and g are real functions, then the symbol $f \asymp g$ means that $c^{-1}\, g \le f \le c\, g$ for some $c \ge 1$.

For more detailed definitions of a Banach couple, intermediate and interpolation spaces with some results introduced briefly below, see [9, pp. 91-173, 289-314, 338-359] and [7, pp. 95-116].

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For a Banach couple $\bar{X} = (X_0, X_1)$ of two compatible Banach spaces X_0 and X_1 consider two Banach spaces $X_0 \cap X_1$ and $X_0 + X_1$ with its natural norms

$$\|f\|_{X_0 \cap X_1} = \max(\|f\|_{X_0}, \|f\|_{X_1}), \text{ for } f \in X_0 \cap X_1,$$

and

$$\|f\|_{X_0+X_1} = \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\},$$

for $f \in X_0 + X_1$.

A Banach space X is called an *intermediate space* between X_0 and X_1 if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$. Such a space X is called an *interpolation space* between X_0 and X_1 if, for any bounded linear operator $T : X_0 + X_1 \to X_0 + X_1$ such that the restriction $T_{|X_i} : X_i \to X_i$ is bounded for $i = 0, 1$, the restriction $T_{|X} : X \to X$ is also bounded and $\|T\|_{X \to X} \leq C \max\{\|T\|_{X_0 \to X_0}, \|T\|_{X_1 \to X_1}\}$ for some $C \geq 1$. If $C = 1$, then X is called an *exact interpolation space* between X_0 and X_1.

One of the most important interpolation methods is the *K-method* known also as the *real Lions-Peetre interpolation method.* For a Banach couple $\bar{X} = (X_0, X_1)$ the *Peetre K-functional* of an element $f \in X_0 + X_1$ is defined for $t > 0$ by

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\}.$$

Then the *spaces of the K-method of interpolation* are

$$\begin{aligned}(X_0, X_1)_{\theta,p} &= \{f \in X_0 + X_1 : \|f\|_{\theta,p} \\ &= \Big(\int_0^\infty [t^{-\theta} K(t, f; X_0, X_1)]^p \frac{dt}{t}\Big)^{1/p} < \infty\}\end{aligned}$$

if $0 < \theta < 1$ and $1 \leq p < \infty$, and

$$(X_0, X_1)_{\theta,\infty} = \{f \in X_0 + X_1 : \|f\|_{\theta,\infty} = \sup_{t>0} \frac{K(t, f; X_0, X_1)}{t^\theta} < \infty\}$$

if $0 \leq \theta \leq 1$. It is not hard to check that $(X_0, X_1)_{\theta,p}$ is an exact interpolation space between X_0 and X_1 for arbitrary $0 < \theta < 1$ and $1 \leq p \leq \infty$.

Very useful in calculations is the so-called *reiteration formula* showing the stability of the K-method of interpolation. If $1 \leq p_0, p_1, p \leq \infty, 0 < \theta_0, \theta_1, \theta < 1$ and $\theta_0 \neq \theta_1$, then

$$\big((X_0, X_1)_{\theta_0,p_0}, (X_0, X_1)_{\theta_1,p_1}\big)_{\theta,p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1,p}, \tag{1}$$

with equivalent norms (see [7, Theorem 2.4, p. 311] or [8, Theorems 3.5.3] or [19, Theorem 1.10.2]) and in the extreme cases

$$\left(X_0, (X_0, X_1)_{\theta_1, p_1}\right)_{\theta, p} = (X_0, X_1)_{\theta\theta_1, p}, \tag{2}$$

and

$$\left((X_0, X_1)_{\theta_0, p_0}, X_1\right)_{\theta, p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta, p}, \tag{3}$$

with equivalent norms (see [12], formulas 3.16 and 3.17).

Now, to treat interpolation results for Cesàro and Copson spaces we need to define these spaces. The *Cesàro sequence spaces* ces_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{ces(p)} = \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right]^{1/p} < \infty, \text{ for } 1 \le p < \infty,$$

and

$$\|x\|_{ces(\infty)} = \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n}|x_k| < \infty, \text{ for } p = \infty.$$

The *Cesàro function spaces* $Ces_p = Ces_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0,1]$ or $I = [0,\infty)$ such that

$$\|f\|_{Ces(p)} = \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty, \text{ for } 1 \le p < \infty,$$

and

$$\|f\|_{Ces(\infty)} = \sup_{0<x\in I} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty, \text{ for } p = \infty.$$

The Cesàro spaces are Banach lattices which are not symmetric except as they are trivial, namely, $ces_1 = \{0\}, Ces_1[0,\infty) = \{0\}$. By a *symmetric space* we mean a Banach lattice X on I satisfying the additional property: if $g^*(t) = f^*(t)$ for all $t > 0$, $f \in X$ and $g \in L^0(I)$ (the set of all classes of Lebesgue measurable real functions on I) then $g \in X$ and $\|g\|_X = \|f\|_X$ (cf. [7], [13]). Here and next f^* denotes the non-increasing rearrangement of $|f|$ defined by $f^*(s) = \inf\{\lambda > 0 : m(\{x \in \Omega : |f(x)| > \lambda\}) \le s\}$, where m is the usual Lebesgue measure (see [13, pp. 78-79] or [7, Theorem 6.2, pp. 74-75]). Moreover, by the classical Hardy inequalities (cf. [11, Theorems 326 and 327] and [14, Chapter 3]),

$$l_p \overset{p'}{\hookrightarrow} ces_p,\ L_p(I) \overset{p'}{\hookrightarrow} Ces_p(I),\ 1 < p \le \infty$$

(in what follows $\frac{1}{p} + \frac{1}{p'} = 1$), and if $1 < p < q < \infty$, then $ces_p \overset{1}{\hookrightarrow} ces_q \overset{1}{\hookrightarrow} ces_\infty$. Also for $I = [0,1]$ and $1 < p < q < \infty$ we have

$$L_\infty \overset{1}{\hookrightarrow} Ces_\infty \overset{1}{\hookrightarrow} Ces_q \overset{1}{\hookrightarrow} Ces_p \overset{1}{\hookrightarrow} Ces_1 = L_1(\ln 1/t) \text{ and } Ces_\infty \overset{1}{\hookrightarrow} L_1.$$

For $1 \leq p < \infty$ the *Copson sequence spaces* cop_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{cop(p)} = \left[\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{|x_k|}{k}\right)^p\right]^{1/p} < \infty,$$

and the *Copson function spaces* $Cop_p = Cop_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0,\infty)$ or $I = [0,1]$ such that

$$\|f\|_{Cop(p)} = \left[\int_0^\infty \left(\int_x^\infty \frac{|f(t)|}{t}\,dt\right)^p dx\right]^{1/p} < \infty, \text{ for } I = [0,\infty),$$

and

$$\|f\|_{Cop(p)} = \left[\int_0^1 \left(\int_x^1 \frac{|f(t)|}{t}\,dt\right)^p dx\right]^{1/p} < \infty, \text{ for } I = [0,1].$$

We have $cop_1 = l_1, Cop_1(I) = L_1(I)$ and by the classical Copson inequalities (cf. [11, Theorems 328 and 331], [6, p. 25] and [14, p. 159]), which are valid for $1 < p < \infty$, we obtain $l_p \overset{p}{\hookrightarrow} cop_p, L_p(I) \overset{p}{\hookrightarrow} Cop_p(I)$.

We can define similarly the spaces cop_∞ and Cop_∞ but then it is easy to see that $cop_\infty = l_1(1/k)$ and $Cop_\infty(I) = L_1(1/t)(I)$. Moreover, for $I = [0,1]$ we have $L_p \overset{p}{\hookrightarrow} Cop_p \overset{1}{\hookrightarrow} Cop_1 = L_1$.

It is important to mention that if $1 < p < \infty$, then

$$ces_p = cop_p \text{ and } Ces_p[0,\infty) = Cop_p[0,\infty). \tag{4}$$

The first equality was proved by Bennett (cf. [6], Theorems 4.5 and 6.6) and the second one in the paper [4], Theorem 1(ii). Moreover, if $1 < p \leq \infty$, then

$$Cop_p[0,1] \overset{p'}{\hookrightarrow} Ces_p[0,1] \text{ and } Cop_p[0,1] \neq Ces_p[0,1], \tag{5}$$

which was proved in [4], Theorem 1(iii).

Structure of the Cesàro sequence and function spaces was investigated by several authors (see, for example, [6], [15] and [1], [2], [3] and the references given there). Here we are interested in studying interpolation properties of Cesàro and Copson spaces.

The main aim of this paper is to survey and supplement the results of our recent paper [4].

2. Interpolation of Copson spaces

Interpolation properties of Copson spaces are rather completely described by the following theorem.

Theorem 1. *Let* $I = [0,1]$ *or* $[0,\infty)$*. If* $1 \le p_0 < p_1 \le \infty$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *with* $0<\theta<1$*, then*

$$(cop_{p_0}, cop_{p_1})_{\theta,p} = cop_p \quad and \quad (Cop_{p_0}(I), Cop_{p_1}(I))_{\theta,p} = Cop_p(I). \tag{6}$$

Proof. In the case of sequence spaces we shall use the following identification of Copson spaces as interpolation spaces with respect to weighted l_1-spaces which was recently obtained in [4, Theorem 1 (i)]:

$$(l_1, l_1(1/k))_{1-1/p,p} = cop_p, \tag{7}$$

for every $1<p<\infty$. Therefore, assuming firstly that $1<p_0<p_1<\infty$, by reiteration equality (1), we have

$$\begin{aligned}(cop_{p_0}, cop_{p_1})_{\theta,p} &= \big((l_1, l_1(1/k))_{1-1/p_0,p_0}, (l_1, l_1(1/k))_{1-1/p_1,p_1}\big)_{\theta,p} \\ &= (l_1, l_1(1/k))_{(1-\theta)(1-1/p_0)+\theta(1-1/p_1),p}\end{aligned}$$

and since $(1-\theta)(1-1/p_0)+\theta(1-1/p_1) = 1 - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} = 1 - \frac{1}{p}$ it follows that the last space is $(l_1, l_1(1/k))_{1-1/p,p} = cop_p$.

In the case when $1<p_0<p_1=\infty$, using reiteration formula (3), the equality $cop_\infty = l_1(1/k)$ and (7) twice, we obtain

$$\begin{aligned}(cop_{p_0}, cop_\infty)_{\theta,p} &= (cop_{p_0}, l_1(1/k))_{\theta,p} = \big((l_1, l_1(1/k))_{1-1/p_0,p_0}, l_1(1/k)\big)_{\theta,p} \\ &= (l_1, l_1(1/k))_{(1-\theta)(1-1/p_0)+\theta,p} = (l_1, l_1(1/k))_{1-\frac{1-\theta}{p_0},p} \\ &= (l_1, l_1(1/k))_{1-1/p,p} = cop_p.\end{aligned}$$

Analogously, if $1=p_0<p_1<\infty$, by the equality $cop_1 = l_1$ and formulas (7) and (2), we obtain

$$\begin{aligned}(cop_1, cop_{p_1})_{\theta,p} &= \big(l_1, (l_1, l_1(1/k))_{1-1/p_1,p_1}\big)_{\theta,p} \\ &= (l_1, l_1(1/k))_{\theta(1-1/p_1),p} = (l_1, l_1(1/k))_{1-1/p,p} = cop_p.\end{aligned}$$

Finally, if $p_0 = 1$ and $p_1 = \infty$, the result follows from (7) and equalities $cop_\infty = l_1(1/k)$ and $cop_1 = l_1$.

The proof is completely similar for Copson function spaces only instead of (7) we use the corresponding identification of Copson function spaces as interpolation spaces with respect to weighted L_1-spaces [4, Theorem 1 (ii) and (iii)]:

$$(L_1(I), L_1(1/t)(I))_{1-1/p,p} = Cop_p(I)$$

and equalities $Cop_\infty(I) = L_1(1/t)(I)$ and $Cop_1(I) = L_1(I)$. $\square$

3. Interpolation of Cesàro spaces

Interpolation properties of Cesàro spaces are more non-trivial and interesting.

Theorem 2. *Let* $I = [0,1]$ *or* $[0,\infty)$*. If* $1 < p_0 < p_1 \leq \infty$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *with* $0 < \theta < 1$*, then*

$$(8) \qquad (ces_{p_0}, ces_{p_1})_{\theta,p} = ces_p \ \ and \ \ (Ces_{p_0}(I), Ces_{p_1}(I))_{\theta,p} = Ces_p(I).$$

Proof. If $p_1 < \infty$, equalities (8) are immediate consequence of equalities (4) and (6). Moreover, if $p_1 = \infty$, the second formula in (8) in the case $I = [0,\infty)$ is proved in [4, Corollary 2]. Let us prove the first one assuming that $p_1 = \infty$.

We claim that the operator

$$Pf(t) := \sum_{k=1}^{\infty} \int_k^{k+1} f(s)\, ds \cdot \chi_{[k,k+1)}(t), \quad t > 0,$$

where by χ_A is denoted the characteristic function of a set A, is bounded in $Ces_p[0,\infty)$ for all $1 < p \leq \infty$. In fact, if $i \leq x < i+1$, $i = 1, 2, \ldots$, we have

$$\int_1^x |Pf(s)|\, ds = \sum_{k=1}^{i-1} \int_k^{k+1} |f(s)|\, ds + \int_i^{i+1} |f(s)|\, ds \cdot (x-i) \leq \int_1^{x+1} |f(s)|\, ds,$$

and in the case when $1 < p < \infty$ we obtain

$$\begin{aligned} \|Pf\|_{Ces(p)}^p &\le \int_1^\infty \left(\frac{1}{x} \int_1^{x+1} |f(s)|\, ds \right)^p \, dx \\ &\le 2^p \int_1^\infty \left(\frac{1}{x+1} \int_1^{x+1} |f(s)|\, ds \right)^p \, dx \\ &= 2^p \int_2^\infty \left(\frac{1}{u} \int_1^{u} |f(s)|\, ds \right)^p \, du \le 2^p \|f\|_{Ces(p)}^p . \end{aligned}$$

Similarly, $\|Pf\|_{Ces(\infty)} \le 2\|f\|_{Ces(\infty)}$. Thus, our claim is proved.

Next, it is easy to see that $Pf = f$ for arbitrary function f from the subspace U_p of the space $Ces_p[0,\infty)$ generated by the sequence $\{\chi_{[k,k+1)}(t)\}_{k=1}^\infty$. Therefore, U_p is a complemented subspace of the space $Ces_p[0,\infty)$ for arbitrary $1 < p \le \infty$, and applying the well-known result of Baouendi and Goulaouic [5, Theorem 1] (see also [19, Theorem 1.17.1]) and the second equality from (8) in the case $I = [0,\infty)$ and $p_1 = \infty$, we have

$$(U_{p_0}, U_\infty)_{\theta,p} = U_p$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$. On the other hand, it is not hard to show (see also [18]) that, for every $1 < p \le \infty$,

$$\Big\| \sum_{k=1}^\infty c_k \chi_{[k,k+1)} \Big\|_{Ces(p)} \asymp \|(c_k)\|_{ces(p)},$$

whence the mapping

$$(c_k)_{k=1}^\infty \longmapsto \sum_{k=1}^\infty c_k \chi_{[k,k+1)}$$

is an isomorphism from ces_p onto U_p, $1 < p \le \infty$. Combining this with the previous equality, we obtain the result.

In the case $I = [0,1]$ the space $Ces_p[0,1]$, for every $1 \le p < \infty$, is not an intermediate space between $L_1[0,1]$ and $Ces_\infty[0,1]$. On the other hand, we have

$$Ces_\infty[0,1] \overset{1}{\hookrightarrow} Ces_p[0,1] \overset{1}{\hookrightarrow} Ces_1[0,1] = L_1(\ln 1/t)[0,1] \overset{1}{\hookrightarrow} L_1(1-t)[0,1].$$

Moreover, as it was shown in [4, Theorem 2], if $1 < p < \infty$, then

$$(L_1(1-t)[0,1], Ces_\infty[0,1])_{1-1/p,p} = Ces_p[0,1]. \tag{9}$$

Therefore, if $I = [0, 1]$, equality (8) can be proved in the same way as in Theorem 1 by using reiteration formulas (1) and (3). $\square$

Remark 1. The space ces_p for $1 < p < \infty$ can be obtained as an interpolation space with respect to the couple $(l_1, l_1(2^{-n}))$ by the so-called K^+-method being a version of the standard K-method, precisely, $ces_p = (l_1, l_1(2^{-n}))^{K^+}_{l_p(1/n)}$ (cf. [10, the proof of Theorem 6.4]) but, by now, for this interpolation method there isn't suitable reiteration theorem.

Remark 2. Another proof of the second equality in (8) for the spaces on $I = [0, \infty)$ was also given by Sinnamon [17, Corollary 2]. Moreover, it is contained implicitly in the paper [16] (cf. explanation in [4], Section 3).

Remark 3. If $1 < p < \infty$, then the restriction of the space $Ces_p[0, \infty)$ to the interval $[0, 1]$ coincides with the intersection $Ces_p[0, 1] \cap L_1[0, 1]$ (cf. [4], Remark 5). Therefore, if we "restrict" second formula in (8) for $[0, \infty)$ to $[0, 1]$ we obtain only

$$(Ces_{p_0}[0,1] \cap L_1[0,1], Ces_{p_1}[0,1] \cap L_1[0,1])_{\theta,p} = Ces_p[0,1] \cap L_1[0,1],$$

where $1 < p_0 < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This also shows that the real method $(\cdot, \cdot)_{\theta,p}$ "well" interpolates the intersection of Cesàro spaces on the segment $[0, 1]$ with the space $L_1[0, 1]$ or, more precisely, we have

$$\begin{aligned}(Ces_{p_0}[0,1] \cap L_1[0,1], Ces_{p_1}[0,1] \cap L_1[0,1])_{\theta,p} \\ = (Ces_{p_0}[0,1], Ces_{p_1}[0,1])_{\theta,p} \cap L_1[0,1],\end{aligned}$$

for all $1 < p_0 < p_1 \leq \infty, 0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Recalling that $Ces_1[0, 1] = L_1(\ln 1/t)$, let us consider the problem whether $Ces_p[0, 1]$, $1 < p < \infty$, is an interpolation space between $Ces_1[0, 1]$ and $Ces_\infty[0, 1]$.

Note that for arbitrary $1 < p < \infty$ the following embedding holds:

$$(Ces_1[0,1], Ces_\infty[0,1])_{1-1/p,p} \overset{1}{\hookrightarrow} Ces_p[0,1]. \tag{10}$$

To prove (10), let us show, firstly, that for any $f \in Ces_1 = Ces_1[0, 1]$ and all $0 < t \leq 1$ we have

$$K(t, f) := K(t, f; Ces_1, Ces_\infty) \geq \int_0^t (Cf)^*(s)\, ds, \tag{11}$$

where $Cf(x) = \frac{1}{x}\int_0^x |f(s)|\,ds, x \in (0,1]$. In fact, we can assume that $f \geq 0$. If $f = g+h, g \geq 0, h \geq 0, g \in Ces_1, h \in Ces_\infty$, then $Cf = Cg+Ch$ and, therefore, by the well-known formula for K-functional with respect to the couple (L_1, L_∞) (see, for example, [13, Chapter II, § 3]),

$$\begin{aligned} & \|g\|_{Ces(1)} + t\,\|h\|_{Ces(\infty)} \\ = \;& \|Cg\|_{L_1} + t\,\|Ch\|_{L_\infty} \\ \geq \;& \inf\{\|y\|_{L_1} + t\,\|z\|_{L_\infty} : Cf = y+z, y \in L_1, z \in L_\infty\} \\ = \;& K(t, Cf; L_1, L_\infty) = \int_0^t (Cf)^*(s)\,ds. \end{aligned}$$

Taking the infimum over all suitable g and h we get (11). Next, by the definition of the real interpolation spaces, we obtain

$$\begin{aligned} \|f\|_{1-1/p,p}^p \;&\geq\; \int_0^1 \left[t^{1/p-1}K(t,f)\right]^p \frac{dt}{t} = \int_0^1 t^{-p}K(t,f)^p\,dt \\ &\geq\; \int_0^1 t^{-p}\left[\int_0^t (Cf)^*(s)\,ds\right]^p dt \geq \|Cf\|_{L_p[0,1]}^p = \|f\|_{Ces(p)}^p, \end{aligned}$$

and the proof of imbedding (10) is complete.

However, the opposite imbedding does not hold. Moreover, in [4, Theorem 6] the following result is proved.

Theorem 3. *For any $1 < p < \infty$ the space $Ces_p[0,1]$ is not an interpolation space between the spaces $Ces_1[0,1]$ and $Ces_\infty[0,1]$.*

Remark 4. Equality (9) and the last theorem show that the weighted space $L_1(1-t)[0,1]$ is in a sense the "proper" end of the scale of Cesàro spaces $Ces_p[0,1]$, $1 < p \leq \infty$.

Remark 5. It would be worth to find an example of the operator which is bounded in $Ces_1[0,1]$ and $Ces_\infty[0,1]$ but unbounded in $Ces_p[0,1]$ for any $1 < p < \infty$.

After the negative answer given in Theorem 3 it is interesting to identify a space which we get by the K-method applied to the couple $(Ces_1[0,1], Ces_\infty[0,1])$. A long calculation in [4, Theorems 3 and 5] shows the following

Theorem 4. *For every $1 < p < \infty$ we have*

$$(Ces_1[0,1], Ces_\infty[0,1])_{1-1/p,p} = Ces_p(\ln\frac{e}{t})[0,1], \tag{12}$$

where the weighted Cesàro function space $Ces_p(\ln \frac{e}{t})[0,1]$ *is a Banach space generated by the norm*

$$\|f\|_{Ces(p,\ln)} := \Big(\int_0^1 \Big(\frac{1}{x} \int_0^x |f(t)|\, dt \Big)^p \ln \frac{e}{x}\, dx \Big)^{1/p}.$$

The crucial point in proving Theorem 4 is the following description of the K-functional for the couple $(Ces_1[0,1], Ces_\infty[0,1])$: for every $f \in Ces_1[0,1]$ and for all $0 < t \leq 1$ we have

$$K(t, f; Ces_1[0,1], Ces_\infty[0,1]) \asymp \|f \chi_{[0,\tau_1(t)] \cup [\tau_2(t),1]}\|_{Ces(1)} + t\, \|f \chi_{[\tau_1(t),\tau_2(t)]}\|_{Ces(\infty)},$$

where $\tau_1(t) = t/\ln(e/t)$ and $\tau_2(t) = e^{-t}$ (cf. [4, Theorem 3]). Clearly, if $t \geq 1$, we have $K(t, f; Ces_1[0,1], Ces_\infty[0,1]) = \|f\|_{Ces(1)}$.

Note that $Ces_p(\ln \frac{e}{t})[0,1] \overset{1}{\hookrightarrow} Ces_p[0,1]$ for every $1 < p < \infty$, and this imbedding is strict.

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