Relations between functional norms of a non-negative function and its square root on the positive cone of Besov and Triebel-Lizorkin spaces¹

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Abstract. In this communication we study in detail the relations between the smoothness of f and $\sqrt{f}$ in the case when the smoothness of the univariate non-negative functions f is measured via Besov and Triebel-Lizorkin space scales. The results obtained can be considered also as embedding theorems for usual Besov and Triebel-Lizorkin spaces and their analogues in Hellinger metric. These results can be used in constrained approximation using wavelets, with applications to probability density estimation in speech recognition, non-negative non-parametric regression-function estimation in positron-emission tomography (PET) imaging, shape/order-preserving and/or one-sided approximation and many others.

Keywords: positive cone, isoperimetric constraint, shape-preserving, one-sided approximation, Besov space, Triebel-Lizorkin space, Hellinger metric, wavelet, embedding theorem, interpolation functor, pointwise multiplier

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INTRODUCTION

Wavelets have remarkable approximation properties, but the very essence of the construction of orthogonal or biorthogonal scaling functions and mother wavelets prevents the wavelet approximants from retaining certain shape-preserving properties such as positivity, monotonicity, k -monotonicity, one-sidedness of the approximation, etc. Wavelet approximation does not obey 'per se' also isoperimetric constraints such as preserving the value of their integral and higher-order moments. On the other hand, many engineering problems where wavelets find applications require preservation of some of the afore-mentioned types of constraints. One example is probability density estimation for speech recognition where the approximation is desirable to be a density itself, i.e.,

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to be non-negative and to have integral equal to one. Another example is non-negative non-parametric regression-function estimation in positron-emission tomography (PET) imaging. In [1, 2] shape-preserving statistical density estimators were proposed by considering $\sqrt{f} \in L_2$, where $f \in L_1$ is an unknown density. For this approach, optimal estimation rates of the risk were obtained under the assumptions that $\sqrt{f}$ belongs, more specifically, to certain function spaces with additional smoothness, continuously embedded in L_2 . Since usually information is available about the smoothness of f itself, this required the study of the relations between the smoothness of f and $\sqrt{f}$. In this communication we study in detail these relations in the case when the smoothness of f is measured via Besov and Triebel-Lizorkin space scales. The results obtained can be considered also as embedding theorems for usual Besov and Triebel-Lizorkin spaces and their analogues in Hellinger metric. The importance of this research topic was emphasized in [3, Subsection 8.1].

In this paper we consider only the univariate case. All functions considered are real-valued.

The organization of the paper is as follows.

In the next (second) section, we give an exposition of the main results, which we classify into *direct estimates* of functionals depending on f by respective functionals depending on $\sqrt{f}$ and *inverse estimates*, where the role of f and $\sqrt{f}$ is reversed. In the first subsection these functionals are wavelet coefficients. In the second subsection they are Sobolev norms with variable metric indices and fixed constant integer smoothness index. In the third subsection, the variability of the metric and smoothness indices is reversed: the considered functionals are (quasi-)norms of Besov and Triebel-Lizorkin spaces with fixed metric index and variable, possibly fractional, smoothness indices. In the concluding fourth subsection we consider interpolation of the estimates from the previous subsections by the real and complex interpolation method, to obtain new, more general, direct and inverse results.

The next (third) section contains all proofs.

Some concluding remarks are given in the last (fourth) section.

MAIN RESULTS

Relations between the inner-product wavelet coefficients of f and $\sqrt{f}$

In [4] asymptotic minimax rates of statistical estimation of distribution density f were studied by using a wavelet approximation which does not preserve two important properties of the density f : the property that $f \geq 0$ on its definition domain, and the fact that the integral of f over the whole definition domain is 1. The definition domain considered was $\mathbb{R}^n$, $n = 1, 2, \dots$ and the minimax rates were in terms of functional norms of f , obtained by measuring the size of the wavelet coefficients of f in appropriate weighted sequence spaces. In [1, 2], for $n = 1$, were considered different wavelet approximations of the density of f that preserve its non-negativity and equality of the integral to 1. Moreover, in [1] it was also proved that the statistical estimator based on the new positivity- and integral-preserving wavelet approximation, also achieves certain asymptotically minimax rates. However, these rates were in terms of functional norms

obtained by measuring the size of the wavelet coefficients of $\sqrt{f}$, rather than the ones of f . Thus the problem of comparing the wavelet coefficients of f and $\sqrt{f}$ arose, and in this subsection we derive some simple but useful results about this comparison.

As customary in wavelet analysis, for a function ϕ defined on $\mathbb{R}$ we consider the functions $\phi_{j,k}$ as follows.

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k), \quad x \in \mathbb{R}, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}.$$

The normalization $2^{\frac{j}{2}}$ is chosen so that if $\|\phi\|_2 = 1$, then also $\|\phi_{j,k}\|_2 = 1$ ($\|\cdot\|_2$ denoting L_2 -norm). In wavelet multiresolution analysis two different types of functions $\phi_{j,k}$ are considered: scaling functions (father wavelets), with integral equal to 1, and (mother) wavelets, with integral equal to 0. Moreover, in most wavelet-related applications the functions $\phi_{j,k}$ form an orthonormal or biorthonormal basis (paired with another similar basis) of $L_2(\mathbb{R})$. Here we shall consider the problem of comparing the wavelet coefficients of f and $\sqrt{f}$ in bigger generality, thus including other possible non-biorthogonal wavelet constructions as considered, e.g., in [5, 6]. We shall not be making assumptions about orthonormality or biorthogonality, but we shall assume that ϕ is compactly supported and bounded on its support.

Lemma 1. Fix j, k and let $h = \phi_{j,k}$; consider $h_+ = \max(h, 0)$, $h_- = -\min(0, h)$ (so that $h_+, h_- \geq 0$ and $h = h_+ - h_-$).

Assume that f is an (almost everywhere) non-negative and (essentially) bounded integrable function on $\mathbb{R}$, i.e., $f \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. Assume also that the wavelet coefficients $\langle f, \phi_{j,k} \rangle = \int_{\mathbb{R}} f(x) \phi_{j,k}(x) dx$ are hard-thresholded with the value $d > 0$ ($d = d_{j,k}$), so that either $|\langle f, \phi_{j,k} \rangle| \geq d$ or $\langle f, \phi_{j,k} \rangle = 0$.

Then there exist constants of equivalence $0 < c_1 < c_2 < \infty$, $0 < c_3 < c_4 < \infty$, where c_2 and c_4 are absolute constants, and c_1 and c_3 depend on d and/or $\|f\|_\infty$, so that

$$\begin{aligned} c_1(\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) &\leq \langle f, h \rangle^2 \leq c_2(\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) \quad \text{and} \\ c_3(\langle \sqrt{f}, h_+ \rangle^2 + \langle \sqrt{f}, h_- \rangle^2) &\leq \langle \sqrt{f}, h \rangle^2 \leq c_4(\langle \sqrt{f}, h_+ \rangle^2 + \langle \sqrt{f}, h_- \rangle^2). \end{aligned}$$

This lemma can be used to obtain direct and inverse results for comparison between $|\langle f, h \rangle|$ and $|\langle \sqrt{f}, h \rangle|$, as follows.

Proposition 1 (Direct results). Under the assumptions of Lemma 1,

$$|\langle f, h \rangle| \leq c_5 \left| \langle \sqrt{f}, h \rangle \right|,$$

where $0 < c_5 < \infty$, c_5 depends on $\|f\|_\infty$ and d , as defined in Lemma 1.

Proposition 2 (Inverse results). Under the assumptions of Lemma 1, and recalling that $h = \phi_{j,k}$,

$$\left| \langle \sqrt{f}, h \rangle \right| \leq c_6 2^{-\frac{j}{4}} |\langle f, h \rangle|^{\frac{1}{2}},$$

where $0 < c_6 < \infty$ and c_6 depends on $\|f\|_\infty$, d , $\|\phi\|_\infty$ and on the length of $\text{supp } \phi$.

If, additionally, it is also known that $f|_{\text{supp } \phi_{j,k}} \geq c_f > 0$ ($c_f = c_f(j, k)$), then $|\langle \sqrt{f}, h \rangle| \leq c_7 \leq |\langle f, h \rangle|$ holds, where $0 < c_7 < \infty$ and c_7 depends on c_f .

Variable metric indices, constrained integer smoothness index

Direct results

As customary, we define the (inhomogeneous) Sobolev spaces with metric index $0 < p \leq \infty$ and smoothness index $m = 0, 1, 2, \dots$ via their (quasi)norms

$$\|f\|_{W_p^m} \stackrel{\text{def}}{=} \|f\|_{L_p} \stackrel{\text{def}}{=} \|f\|_p, \quad 0 < p \leq \infty, \quad m = 0$$

and

$$\|f\|_{W_p^m} \stackrel{\text{def}}{=} \|f\|_{L_p} + \left\| f^{(m)} \right\|_{L_p}, \quad 0 < p \leq \infty, \quad m = 1, 2, \dots$$

Here we shall compute direct estimates (for f -density) of $\|f\|_{L_p}$ and $\|f\|_{W_p^m}$ via $\|\sqrt{f}\|_{L_q}$ and $\|\sqrt{f}\|_{W_q^m}$, respectively.

Theorem 1. For $0 < p \leq \infty$ and $m = 0, 1, 2, \dots$

$$\|f\|_{W_p^m} \leq c_2(p, m) \left\| \sqrt{f} \right\|_{W_{2p}^m}^2.$$

Inverse results

Theorem 2. Let $0 < p \leq \infty$. If $m = 0$, then the direct result in Theorem 1 is also inverse:

$$\left\| \sqrt{f} \right\|_p = \|f\|_{p/2}^{1/2}.$$

If $m = 1, 2, \dots$, assume additionally that $\frac{1}{\sqrt{f}} \in W_q^{m-1}$ for $p \leq q \leq \infty$. Then

$$\left\| \sqrt{f} \right\|_{W_p^m} \leq c(p, m) \left\| \frac{1}{\sqrt{f}} \right\|_{W_q^{m-1}} \cdot \|f\|_{W_r^m} \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Constant metric index, fractional (possibly variable) smoothness index

To obtain direct and inverse results with preservation of the metric, we shall use pointwise multipliers in the respective function spaces. The concept of pointwise multipliers in Besov and Triebel-Lizorkin function spaces is introduced, e.g., in [7]. For our purposes here it is sufficient to note (see, e.g., [7, p.143/(24)]) that if $g \in B_{\infty q}^s$ holds, then

$$\|gh\|_{B_{p,q}^s} \leq c \|g\|_{B_{\infty,q}^s} \cdot \|h\|_{B_{p,q}^s}, \quad s > 0, \quad 1 \leq p \leq \infty, \quad 0 < q \leq \infty \quad (1)$$

also holds, and thus, g is a pointwise multiplier (p.m.) in $B_{p,q}^s$.

Direct results

For $g = h = \sqrt{f} \in B_{\infty,q}^s$, the formula (1) clearly implies that

$$\|f\|_{B_{\infty,q}^s} \leq c \left\| \sqrt{f} \right\|_{B_{\infty,q}^s}^2$$

which coincides with the partial case of Theorem 12 for $p = \infty$ (see below). However, the use of p.m. makes it possible to extend the direct results to $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$.

Theorem 3. Let $-\infty < s < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $\sigma > \max \left\{ s, \frac{1}{\min\{p,1\}} - 1 - s \right\}$. If

$$\sqrt{f} \in B_{\infty,\infty}^\sigma \cap B_{p,q}^s \quad (2)$$

(which is possible only for a density with bounded support), then

$$f \in B_{p,q}^s \quad \text{and} \quad \|f\|_{B_{p,q}^s} \leq c \left\| \sqrt{f} \right\|_{B_{\infty,\infty}^\sigma} \left\| \sqrt{f} \right\|_{B_{p,q}^s}.$$

Theorem 4. Let $-\infty < s < \infty$, $0 < p < \infty$, $0 < q < \infty$ and $\sigma > \max \left\{ s, \frac{1}{\min\{p,q,1\}} - 1 - s \right\}$. If $\sqrt{f} \in B_{\infty,\infty}^\sigma \cap F_{p,q}^s$ (compare also with (2)), then

$$f \in F_{p,q}^s \quad \text{and} \quad \|f\|_{F_{p,q}^s} \leq c \left\| \sqrt{f} \right\|_{B_{\infty,\infty}^\sigma} \left\| \sqrt{f} \right\|_{F_{p,q}^s}.$$

The next direct result is related to the continuity of f and $\sqrt{f}$ (see [7, Section 2.8.3]), in view of the range of s and p for which Sobolev-type embeddings hold true.

Theorem 5. Let $0 < p \leq \infty$ and either $1 < q \leq \infty$, $s > 1/p$ or $0 < q \leq 1$, $s \geq 1/p$. If $\sqrt{f} \in B_{p,q}^s$, then

$$f \in B_{p,q}^s \quad \text{and} \quad \|f\|_{B_{p,q}^s} \leq c \left\| \sqrt{f} \right\|_{B_{p,q}^s}^2.$$

Theorem 6. Let either $0 < p < q \leq \infty$, $s > \frac{1}{p}$ or $0 < p \leq q < \infty$, $s > \frac{1}{2} \left(\frac{1}{p} + \frac{1}{q} \right)$. If $\sqrt{f} \in F_{p,q}^s$, then

$$f \in F_{p,q}^s \quad \text{and} \quad \|f\|_{F_{p,q}^s} \leq c \left\| \sqrt{f} \right\|_{F_{p,q}^s}^2$$

Inverse results

All the remaining results in this subsection (Theorem 7–11) are based on modifications of the ideas of proof of the respective direct results (see also [8, 9, 10]). Note also that in this inverse case it is convenient to estimate explicit derivatives of $\sqrt{f}$ that leads to some variability of the smoothness index in the lefthand and righthand side of the resulting inequalities.

The inverse estimates depend essentially on the behaviour of $\frac{1}{\sqrt{f}}$: namely, in which Besov and Triebel-Lizorkin spaces it is a pointwise multiplier. More precisely, we are considering the function $\frac{1}{\sqrt{f}}$ defined in the following way:

$$\begin{cases} \frac{1}{\sqrt{f(x)}} & \text{for } x \text{ such that } f(x) > 0, \\ 0 & \text{for } x \text{ such that } f \equiv 0 \end{cases}$$

with a $0/\infty$ -discontinuity point for x such that $f(x) = 0$ but every neighbourhood of x contains some point y such that $f(y) > 0$ (i.e., x is on the boundary of $\text{supp } f$). The following is an inverse analogue of Theorem 3.

Theorem 7. *Let $1 \leq p \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$, $s > m$. If $\left(\frac{1}{\sqrt{f}}\right)^{(\mu)} \in B_{\infty,q}^{s-m}$, $\mu = 0, 1, \dots, m$ and if $f \in B_{p,q}^s$, then $(\sqrt{f})^{(m)} \in B_{p,q}^{s-m}$ and $\left\|(\sqrt{f})^{(m)}\right\|_{B_{p,q}^{s-m}} \leq c_m \left(\sum_{\mu=0}^m \left\|\left(\frac{1}{\sqrt{f}}\right)^{(\mu)}\right\|_{B_{\infty,\infty}^{s-m}}\right) \cdot \|f\|_{B_{p,q}^s}$*

This result can be extended for p or $q < \infty$ and for Triebel-Lizorkin spaces. The following is an “inverse” analogue of Theorem 4, taking also in consideration [7, p. 113, Remark 2].

Theorem 8. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{N}_0$, $a < s \max\{1, 1/p\}$ and $\sigma > s > m - a + \frac{1}{\min\{p,1\}}$. If $\left(\frac{1}{\sqrt{f}}\right)^{(\mu)} \in B_{\infty,\infty}^{s-m}$, $\mu = 0, 1, \dots, m$ and if $f \in B_{p,q}^s$, then $(\sqrt{f})^{(m)} \in B_{p,q}^{s-m}$ and $\left\|(\sqrt{f})^{(m)}\right\|_{B_{p,q}^{s-m}} \leq c_{m,p,a,\sigma} \left(\sum_{\mu=0}^m \left\|\left(\frac{1}{\sqrt{f}}\right)^{(\mu)}\right\|_{B_{\infty,\infty}^{s-m}}\right) \cdot \|f\|_{B_{p,q}^s}$*

The following is an “inverse” analogue of Theorem 5, taking in consideration [7, p. 104, Remark 1].

Theorem 9. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{N}_0$ and $\sigma > s > m + \max\left\{\frac{1}{\min\{p,q\}}, \frac{1}{\min\{p,q,1\}} - 1\right\}$. If $\left(\frac{1}{\sqrt{f}}\right)^{(\mu)} \in B_{\infty,\infty}^{s-m}$, $\mu = 0, 1, \dots, m$ and if $f \in F_{p,q}^s$, then $(\sqrt{f})^{(m)} \in F_{p,q}^{s-m}$ and $\left\|(\sqrt{f})^{(m)}\right\|_{F_{p,q}^{s-m}} \leq c_{m,p,q,\sigma} \left(\sum_{\mu=0}^m \left\|\left(\frac{1}{\sqrt{f}}\right)^{(\mu)}\right\|_{B_{\infty,\infty}^{s-m}}\right) \cdot \|f\|_{F_{p,q}^s}$*

The following is an “inverse” analogue of Theorem 6, considering [7, p. 113, Remark 2]

Theorem 10. *Let $0 < p \leq \infty$, $m \in \mathbb{N}_0$ and either $1 < q \leq \infty$, $s > m + 1/p$ or $0 < q \leq 1$, $s \geq m + 1/p$. If $\left(\frac{1}{\sqrt{f}}\right)^{(\mu)} \in B_{p,q}^{s-m}$, $\mu = 0, 1, \dots, m$ and if $f \in B_{p,q}^s$, then $(\sqrt{f})^{(m)} \in B_{p,q}^{s-m}$ and $\left\|(\sqrt{f})^{(m)}\right\|_{B_{p,q}^{s-m}} \leq c_{m,p} \left(\sum_{\mu=0}^m \left\|\left(\frac{1}{\sqrt{f}}\right)^{(\mu)}\right\|_{B_{p,q}^{s-m}}\right) \cdot \|f\|_{B_{p,q}^s}$*

The following is an “inverse” analogue of Theorem 6, considering [7, p. 104, Remark 1].

Theorem 11. *Let $m \in \mathbb{N}_0$, $0 < p \leq \infty$, $0 < q \leq \infty$, $s > \frac{1}{\min\{p,q\}}$. If $\left(\frac{1}{\sqrt{f}}\right)^{(\mu)} \in F_{p,q}^{s-m}$, $\mu = 0, 1, \dots, m$ and if $f \in F_{p,q}^s$, then $(\sqrt{f})^{(m)} \in F_{p,q}^{s-m}$ and $\left\|(\sqrt{f})^{(m)}\right\|_{F_{p,q}^{s-m}} \leq c_{m,p} \left(\sum_{\mu=0}^m \left\|\left(\frac{1}{\sqrt{f}}\right)^{(\mu)}\right\|_{F_{p,q}^{s-m}}\right) \cdot \|f\|_{F_{p,q}^s}$.*

Interpolation estimates

In this subsection we shall invoke the real and complex interpolation methods for interpolation of operators to obtain further upgrades of the direct results in Theorem 1 and the inverse results in Theorem 2.

Direct results

Our next aim is to generalize the direct results in Theorem 1 for the Besov and Triebel–Lizorkin space scales for a continual (fractional) range of the smoothness parameter s . This results in the following theorem:

Theorem 12. *Under the assumptions of Theorem 1, suppose that $1 \leq p \leq \infty$. Then for*

$$m \in \mathbb{N}, \quad 0 < q \leq \infty \quad \text{and} \quad 0 < s < m,$$

it holds that

$$\|f\|_{B_{p,q}^s} \leq c(p, q, s, m) \cdot \left\|\sqrt{f}\right\|_{B_{2p,q}^s}^2.$$

If, moreover, p belongs to the more constrained range $1 < p < \infty$, then it also holds that

$$\|f\|_{F_{p,2}^s} \leq c_1(p)^{1-\theta} \cdot c_2(p, m)^\theta \cdot \left\|\sqrt{f}\right\|_{F_{2p,2}^s}^2,$$

where $\theta = s/m$, $c_1(p)$ is the constant c in (5) in the proof of Theorem 1, and $c_2(p, m)$ is the respective constant in the proof of Theorem 2.

Inverse results

Similar to Theorems 1 and 12, the results of Theorem 2 can also be upgraded using the real and complex interpolation methods.

Theorem 13. *Under the assumptions of Theorem 2, suppose that $1 \leq p \leq \infty$ and ρ and τ are such that $p \leq \min\{\tau, \rho\} \leq \infty$ and $\frac{1}{p} = \frac{1}{\rho} + \frac{1}{\tau}$. Then, for the same assumptions about m, q and s as in Theorem 12, it holds that*

$$\left\| \sqrt{f} \right\|_{B_{p,q}^s} \leq c_1(p, q, s, m, \tau, \rho) \left\| \frac{1}{\sqrt{f}} \right\|_{B_{p,q}^s} \|f\|_{B_{\tau,q}^s},$$

assuming that $1/\sqrt{f} \in B_{\rho,q}^s$. If, moreover, p belongs to the more constrained range $1 < p < \infty$, then it also holds that

$$\left\| \sqrt{f} \right\|_{F_{p,2}^s} \leq c_2(p, s, m, \tau, \rho) \left\| \frac{1}{\sqrt{f}} \right\|_{F_{\rho,2}^s} \|f\|_{F_{\tau,2}^s},$$

assuming that $1/\sqrt{f} \in F_{\rho,2}^s$.

Note that for integer $s = 1, \dots, m$ Theorem 2 is essentially sharper than the interpolation estimate in Theorem 13.

PROOFS

Proof of lemma 1. Because it is easier to estimate $\langle h_1, h_2 \rangle$ when both h_1 and h_2 are non-negative, we show first that under the thresholding assumptions on f bounded (the most common case), $\langle f, h \rangle^2 \sim \langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2$ and $\langle \sqrt{f}, h \rangle^2 \sim \langle \sqrt{f}, h_+ \rangle^2 + \langle \sqrt{f}, h_- \rangle^2$:

$$\begin{aligned} 0 \leq \langle f, h \rangle^2 &= \langle f, h_+ - h_- \rangle^2 = \langle f, h_+ \rangle^2 - 2 \langle f, h_+ \rangle \langle f, h_- \rangle + \langle f, h_- \rangle^2 \\ &\leq 2(\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2), \end{aligned}$$

Here we used that $ab \leq (a^2 + b^2)/2$. For $\sqrt{f}$, the estimate is analogous. Conversely,

$$\begin{aligned} \langle f, h \rangle^2 &= \frac{\langle f, h \rangle^2}{\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2} \cdot (\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) \\ &\geq \frac{d^2}{\|f\|_2^2 (\|h_+\|_2^2 + \|h_-\|_2^2)} \cdot (\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) \\ &= \frac{d^2}{\|f\|_2^2 \cdot 1} \cdot (\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) \geq \frac{d^2}{\|f\|_\infty} (\langle f, h_+ \rangle^2 + \langle f, h_- \rangle^2) \end{aligned}$$

For f we use here that if $f \in L_1 \cap L_\infty$, then $\|f\|_2 < \infty$ and $\|f\|_2^2 \leq \|f\|_1 \|f\|_\infty = \|f\|_\infty$, since $\|f\|_1 = 1$. The claim about $\sqrt{f}$ is obtained analogously with constant of the bound (from below)

$$\frac{d^2}{\|\sqrt{f}\|_2^2 (\|h_+\|_2^2 + \|h_-\|_2^2)} = \frac{d^2}{1 \cdot 1},$$

because $\|\sqrt{f}\|_2 = \|f\|_1 = 1$ and $\|h_+\|_2^2 + \|h_-\|_2^2 = \|\varphi_{j,k}\|_2^2 = 1$. □

It remains to compare $\langle f, h_+ \rangle$, $\langle \sqrt{f}, h_+ \rangle$, $\langle f, h_- \rangle$ and $\langle \sqrt{f}, h_- \rangle$.

Proof of Proposition 1. Because of the non-negativity of f , h_+ and h_- ,

$$0 \leq \langle f, h_+ \rangle = \langle |f|, |h_+| \rangle \leq \|f\|_\infty^{1/2} \langle \sqrt{|f|}, |h_+| \rangle = \|f\|_\infty^{1/2} \langle \sqrt{f}, h_+ \rangle$$

and analogously for $\langle f, h_- \rangle$. □

Proof of Proposition 2.

$$\begin{aligned} \langle \sqrt{f}, h_+ \rangle &= \int_{-\infty}^{\infty} f(x) \frac{h_+(x)}{\sqrt{f(x)} + \delta} dx + \delta \int_{-\infty}^{\infty} \frac{\sqrt{f(x)}}{\sqrt{f(x)} + \delta} h_+(x) dx \\ &\leq \frac{1}{\delta} \int_{-\infty}^{\infty} f(x) h_+(x) dx + \delta \|h_+\|_1 \\ &\leq \frac{1}{\delta} \int_{-\infty}^{\infty} f(x) h_+(x) dx + c\delta 2^{-j/2}, \quad \delta > 0. \end{aligned} \quad (3)$$

For these bounds, we used consecutively that for $\delta > 0$,

$$\sqrt{f(x)} + \delta \geq \delta, \quad \frac{\sqrt{f(x)}}{\sqrt{f(x)} + \delta} \leq 1, \quad h_+(x)h_-(x) = 0 \quad \text{and} \quad h_+ + h_- = \varphi_{j,k},$$

so that

$$\|h_+\|_1 \leq \| |h_+| + |h_-| \|_1 = \|h_+ - h_-\|_1 = \|\varphi_{j,k}\|_1 \leq c2^{-j/2},$$

where $c = \|\varphi\|_\infty \cdot \text{meas}(\text{supp } \varphi)$.

The righthand side of (3) is a function of $\delta \in (0, \infty)$, which tends to $+\infty$ when $\delta \rightarrow 0_+$ or $\delta \rightarrow \infty$. It is easy to see that this function attains its minimum for $\delta = \frac{2^{j/2}}{\sqrt{c}} \cdot \langle f, h_+ \rangle^{1/2}$.

Hence the minimal possible righthand side of (3) is $2\sqrt{c}2^{-j/4} \langle f, h_+ \rangle^{1/2}$. From the above estimates, the general upper bound for $\langle \sqrt{f}, \varphi_{j,k} \rangle$ follows easily. The special upper bound when $f|_{\text{supp } \varphi_{j,k}} \geq c_f > 0$ is easy: $\langle \sqrt{f}, h_+ \rangle \leq \frac{1}{\sqrt{c_f}} \langle f, h_+ \rangle$, from where the proof is easily completed. □

Proof of Theorem 1. When $m = 0$, it is easy to check that

$$\|f\|_{L_p} = \left\| \sqrt{f} \right\|_{L_{2p}}^2 \quad \text{for } 0 < p \leq \infty. \quad (4)$$

In fact, (4) is trivial for $p = \infty$ and for $0 < p < \infty$ we have

$$\|f\|_{L_p} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} = \left(\int_{-\infty}^{\infty} |\sqrt{f(x)}|^{2p} dx \right)^{\frac{2}{2p}} = \left\| \sqrt{f} \right\|_{L_{2p}}^2$$

If $0 < p < \infty$, $m = 1, 2, \dots$ and $g(x) = \sqrt{f(x)}$, then by the Leibniz rule,

$$\begin{aligned} \|f\|_{W_p^m} &= \|g\|_{L_{2p}}^2 + \|(g^2)^m\|_{L_p} = \|g\|_{L_{2p}}^2 + \left\| \sum_{\mu=0}^m \binom{m}{\mu} g^{(m-\mu)} g^{(\mu)} \right\|_{L_p} \\ &\leq c(p, m) \sum_{\mu=0}^m \binom{m}{\mu} \|g^{(m-\mu)} g^{(\mu)}\|_{L_p} = R_1, \end{aligned}$$

where $c(p, m) = 1$ for $p \in [1, \infty)$ (when L_p is normed, and not just quasi-normed). In the integral in $\|g^{(m-\mu)} g^{(\mu)}\|_{L_p}$ apply the Cauchy–Bounyakowskii-Schwarz inequality and then the discrete Cauchy–Bounyakowskii-Schwarz inequality:

$$\begin{aligned} R_1 &\leq c(p, m) \sum_{\mu=0}^m \binom{m}{\mu} \|g^{(m-\mu)}\|_{L_{2p}} \|g^{(\mu)}\|_{L_{2p}} \\ &\leq c(p, m) \left(\sum_{\mu=0}^m \binom{m}{\mu} \|g^{(m-\mu)}\|_{L_{2p}}^2 \right)^{1/2} \left(\sum_{\mu=0}^m \binom{m}{\mu} \|g^{(\mu)}\|_{L_{2p}}^2 \right)^{1/2} \\ &\leq c_1(p, m) \sum_{\mu=0}^m \|g^{(\mu)}\|_{L_{2p}}^2 \leq c_2(p, m) \|g\|_{W_{2p}^m}^2 \end{aligned}$$

Here we also used well-known embedding inequalities for the intermediate derivatives of g (see, e.g., [11]). The case $p = \infty$ is analogous, but simpler. Plugging in $g(x) = \sqrt{f(x)}$, we get

$$\|f\|_{W_p^m} \leq c \left\| \sqrt{f} \right\|_{W_{2p}^m}^2, \quad 0 < p \leq \infty. \quad (5)$$

□

Proof of Theorem 2. After differentiating once, we get

$$(\sqrt{f})' = \frac{1}{2} \cdot \frac{f'}{\sqrt{f}}$$

Now it is possible to repeat, mutatis mutandis, several steps in the proof of the direct result in Theorem 1: applying the Leibniz rule to the product $f' \cdot \frac{1}{\sqrt{f}}$, followed by application of the general form of the Hölder inequality

$$\|f_1 f_2\|_p \leq \|f_1\|_q \|f_2\|_r \quad \text{for} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r},$$

after which the proof is completed again by applying respective embedding inequalities for the intermediate derivatives of f and $\frac{1}{\sqrt{f}}$. □

Outline of proof of theorems 3–6. For appropriate values of the respective parameters corresponding to each of these theorems, [7, Corollary p. 143, (25), (26), Theorem p. 145–146 and Corollary p. 146] are used. (see also [12]). □

Proof of Theorem 7.

$$\begin{aligned}
\left\| (\sqrt{f})^{(m)} \right\|_{B_{p,q}^{s-m}} &= \left\| \left(\frac{f}{\sqrt{f}} \right)^{(m)} \right\|_{B_{p,q}^{s-m}} = \left\| \sum_{\mu=0}^m \binom{m}{\mu} \left(\frac{1}{\sqrt{f}} \right)^{(\mu)} f^{(m-\mu)} \right\|_{B_{p,q}^{s-m}} \\
&\leq c_m \sum_{\mu=0}^m \left\| \left(\frac{1}{\sqrt{f}} \right)^{(\mu)} f^{(m-\mu)} \right\|_{B_{p,q}^{s-m}} \quad ([7, \text{p. 143, Remark 1}]) \\
&\leq c_m \sum_{\mu=0}^m \left\| \left(\frac{1}{\sqrt{f}} \right)^{(\mu)} \right\|_{B_{p,q}^{s-m}} \cdot \left\| f^{(m-\mu)} \right\|_{B_{p,q}^{s-m}} \quad ([7, \text{p. 113, Remark 2}]) \\
&\leq c_m \sum_{\mu=0}^m \left\| \left(\frac{1}{\sqrt{f}} \right)^{(\mu)} \right\|_{B_{p,q}^{s-m}} \cdot \|f\|_{B_{p,q}^{s-\mu}} \quad (B_{p,q}^s \hookrightarrow B_{p,q}^{s-\mu} \text{ for } \mu \geq 0.) \\
&\leq c_m \sum_{\mu=0}^m \left\| \left(\frac{1}{\sqrt{f}} \right)^{(\mu)} \right\|_{B_{p,q}^{s-m}} \cdot \|f\|_{B_{p,q}^s}
\end{aligned}$$

□

Outline of proof of Theorem 8. The proof is analogous to the proof of Theorem 7, but with [7, p. 143, (25)] used instead of [7, p. 143, Remark 1]. □

Outline of proof of Theorem 9. The proof is analogous to the ones of theorems 7 and 8, but [7, p. 143, (26)] is being utilized in the respective place. □

Outline of proof of Theorem 10. The proof is analogous to the proofs of theorems 7–9, using the theorem in [7, Section 2.8.3, p. 145–146]. □

Outline of proof of Theorem 11. The proof is analogous to the proofs of theorems 7–10, using the corollary and Remark 2 in [7, Section 2.8.3, p. 146]. □

Proof of Theorem 12. Consider the bilinear mapping $T(h_1, h_2) = h_1(x)h_2(x)$. It can be proved analogously to (4) and (5) that T is a bounded bilinear operator: $L_{2p} \oplus L_{2p} \longrightarrow L_p$ and $W_{2p}^m \oplus W_{2p}^m \longrightarrow W_p^m$ ($\oplus$ denotes direct sum of spaces), i.e.,

$$\|T(h_1, h_2)\|_{L_p} = \|h_1 h_2\|_{L_p} \leq c(p) \|h_1\|_{L_{2p}} \|h_2\|_{L_{2p}} \quad (6)$$

and

$$\|T(h_1, h_2)\|_{W_p^m} = \|h_1 h_2\|_{W_p^m} \leq c(p, m) \|h_1\|_{W_{2p}^m} \|h_2\|_{W_{2p}^m}. \quad (7)$$

Interpolating between (6) and (7) yields (see [13, p. 96])

a) with the real interpolation method $K_{\theta,q}$, $0 < \theta < 1$, $0 < q \leq \infty$: $\theta = s/m$, $0 < s < m$

$$\|T(h_1, h_2)\|_{B_{p,q}^s} \leq c \cdot c(p)^{1-\theta} \cdot c(p, m)^\theta \cdot \|h_1\|_{B_{2p,q}^s} \cdot \|h_2\|_{B_{2p,q}^s}, \quad 1 \leq p \leq \infty. \quad (8)$$

b) with the complex interpolation method $C_{[\theta]}$, $0 < \theta < 1$: $\theta = s/m$, $0 < s < m$

$$\|T(h_1, h_2)\|_{F_{p,2}^s} \leq c(p)^{1-\theta} \cdot c(p, m)^\theta \cdot \|h_1\|_{F_{2p,q}^s} \cdot \|h_2\|_{F_{2p,q}^s}, \quad 1 < p < \infty. \quad (9)$$

Plugging in $h_1(x) = h_2(x) = \sqrt{f(x)}$ in (8) and (9) yields the result of Theorem 12. $\square$

Outline of proof of Theorem 13. The proof is analogous to the proof of Theorem 12, with the following modifications: the bilinear operator $T(h_1, h_2) = h_1 h_2$ is shown to act boundedly from $L_p \oplus L_\tau$ to L_p and from $W_p^m \oplus W_\tau^m$ to W_p^m ; the index $2p$ in (6) and (7) is replaced by respective indices p and τ ; the choice $h_1 = h_2 = \sqrt{f}$ is replaced by $h_1 = 1/\sqrt{f}$, $h_2 = f$. $\square$

CONCLUDING REMARKS

There exist reasonable connections between Besov regularity of f and of $\sqrt{f}$. For example, if $2 \leq p \leq \infty$, $0 < q \leq \infty$, $s > 0$, then $\sqrt{f} \in B_{pq}^s(\mathbb{R})$ implies $f \in B_{p/2,q}^s(\mathbb{R})$; moreover, if $0 < p \leq \infty$ and either $1 < q \leq \infty$ and $s > 1/p$ or $0 < q \leq 1$ and $s > 1/p$ then $\sqrt{f} \in B_{pq}^s(\mathbb{R})$ implies $f \in B_{pq}^s(\mathbb{R})$. In the opposite direction, if $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $s_1 > s > 0$, and if f is bounded away from zero on $[T_1, T_2]$ and $\frac{1}{\sqrt{f}} \in B_{\infty\infty}^{s_1}[T_1, T_2]$ then $f \in B_{pq}^s[T_1, T_2]$ implies $\sqrt{f} \in B_{pq}^s[T_1, T_2]$. These assertions were formulated without proofs in [1, Remark 4.4]; their proofs are provided in the present paper.

These correspondences between the regularity of f and $\sqrt{f}$ show that the constrained estimates in [1] attains comparable asymptotic rates as the unconstrained estimator in [4], while obeying the constraints about positivity and equality of its integral to 1. This observation is in line also with the conclusions in [3] about other classes of constrained estimators.

It should be noted also that the results in Theorems 3-11 based on pointwise multipliers can also be upgraded to results with variable metric indices. The idea of these upgrades is similar to the proofs of Theorems 12 and 13, but their proofs are more technically involved, and the range of the metric parameters p and q is broader than in Theorems 12 and 13.

Another theoretical application of the direct and inverse bounds obtained in this paper is based on the fact that the positive cone in Triebel-Lizorkin spaces $F_{p,q}^s$ does not depend on the second metric index q [14]. This fact implies, for example in relevance to Theorem 12, that if for the nonnegative function f , $\sqrt{f} \in F_{2p,p}^s$ for every $p \in (0, +\infty]$, then $f \in F_{p,\tau}^s$ for every $\tau \in (0, +\infty]$. Similarly, in the reverse situation, Theorem 13 implies for the nonnegative function f that if $1/\sqrt{f} \in F_{\rho,\alpha}^s$ for any $\alpha \in (0, +\infty]$ and if $f \in F_{\tau\beta}^s$ for any $\beta \in (0, +\infty]$ and if $\frac{1}{p} + \frac{1}{\tau} = \frac{1}{\rho}$, then $\sqrt{f} \in F_{\rho,\gamma}^s$ for every $\gamma \in (0, +\infty]$.

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