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Simultaneous Maximum Likelihood Estimation of Time Delay and Time Scaling

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ABSTRACT

In this paper we present a simultaneous maximum likelihood estimator (ML) for time delay and time-scaling in the presence of additive white Gaussian Noise. The Cramér-Rao lower bound for the variance of the estimates is also derived.

The performance of the estimator is evaluated for ultrasound echoes for different time-scalings and different time delays. The performance is compared to the standard cross-correlation estimator.

1. INTRODUCTION

Time-delay estimation is one of the classical estimation problems in the field of signal processing. Finding the time of arrival or position of a known signal waveform corrupted by noise has numerous applications. In additive white Gaussian noise, it is well-known that the standard cross-correlation estimator gives an unbiased maximum likelihood estimate [1]. In this paper we study the case where the signal we are looking for can be both time delayed and time scaled. That is, we assume the following discrete-time signal model:

$$x[n] = s[a(n + n_0)] + e[n] = s[n; \boldsymbol{\theta}] + e[n], \quad (1)$$

where $s[n]$ is the known reference signal, $x[n]$ is a delayed and time-scaled version of $s[n]$, and $e[n]$ is additive white Gaussian noise with zero mean and variance σ^2. The task is to derive an ML estimate of the 2×1 parameter vector $\boldsymbol{\theta} = [\theta_1, \theta_2]^T = [a, n_0]^T$. It should be noted that both the time scaling factor, a, and the time delay, n_0, are allowed to be a continuous.

This differs slightly from what is often assumed in radar and sonar applications. The model in Eq. (1) represents what will happen in presence of Doppler. In most practical cases, however, the signal is either narrowband or the motion of the target is too slow for the effect of time scaling to become a problem [2]. In those cases, the estimation problem is reduced to finding a time and frequency shift [3]. In the paper by Adams *et al.* [4], a standard cross-correlation receiver is modified to compensate for the effect of motion.

In section 2 we derive a maximum likelihood estimator for this, and the corresponding Cramér-Rao lower bound for any unbiased estimator. In section 3 we show with some simulations how the performance of the estimator varies with signal-to-noise ratio (SNR).

2. THEORY

In the derivation of the ML estimator we assume the signal model in Eq. (1).

2.1. The ML Estimator

Assuming that the noise $e[n]$ is additive white Gaussian noise, with zero mean and variance σ^2, the likelihood function of the received signal $x[n]$ of length M, with respect to the parameter vector $\boldsymbol{\theta}$ is

$$f_{\boldsymbol{\theta}}(x) = \prod_{n=0}^{M-1} \left(2\pi\sigma^2\right)^{-1/2} e^{-\frac{1}{\sigma^2}(x[n]-s[n;\boldsymbol{\theta}])^2} = \left(2\pi\sigma^2\right)^{-M/2} e^{-\frac{1}{\sigma^2}\sum_{n=0}^{M-1}(x[n]-s[n;\boldsymbol{\theta}])^2}. \quad (2)$$

The corresponding *log-likelihood function* is

$$\Lambda_{\boldsymbol{\theta}}(\mathbf{x}) = -\frac{M}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{n=0}^{M-1}\left(x[n] - s[n;\boldsymbol{\theta}]\right)^2 \quad (3)$$

Maximizing $\Lambda_{\boldsymbol{\theta}}(\mathbf{x})$ with respect to $\boldsymbol{\theta}$ ($\theta_1 = a$ and $\theta_2 = n_0$) is equivalent to maximizing

$$l(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\sum_{n=0}^{M-1} x[n]\, s[n;\boldsymbol{\theta}] - \frac{1}{2\sigma^2}\sum_{n=0}^{M-1} s^2[n;\boldsymbol{\theta}], \quad (4)$$

that is, twice the cross-correlation between $x[n]$ and $s[n;\boldsymbol{\theta}]$ minus the energy of the scaled and delayed signal.

2.2. The Cramér-Rao Lower Bound

Now, let us determine a lower bound for the covariance of any unbiased estimate of the parameter vector $\boldsymbol{\theta}$. This is given by the Cramér-Rao lower bound [1], which states that the covariance matrix

$$\mathbf{C} = \begin{bmatrix} Var(\theta_1) & Cov(\theta_1,\theta_2) \\ Cov(\theta_1,\theta_2) & Var(\theta_2) \end{bmatrix} \geq \mathbf{J}^{-1}, \quad (5)$$

where $\mathbf{J}$ is the Fisher information matrix which elements, J_{ij}, are given by

$$J_{ij} = -E\left\{\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln f_{\boldsymbol{\theta}}(\mathbf{X})\right\} = -E\left\{\frac{\partial^2}{\partial\theta_i\partial\theta_j} l(\boldsymbol{\theta})\right\}. \quad (6)$$

Calculating the inverse of the Fisher information matrix we find that

$$Var\{a\} = Var\{\theta_1\} \geq \frac{\sigma^2 B_{11}}{A_{11}A_{22} - A_{12}^2} \quad (7)$$

$$Var\{n_0\} = Var\{\theta_2\} \geq \frac{\sigma^2 B_{22}}{A_{11}A_{22} - A_{12}^2}, \quad (8)$$

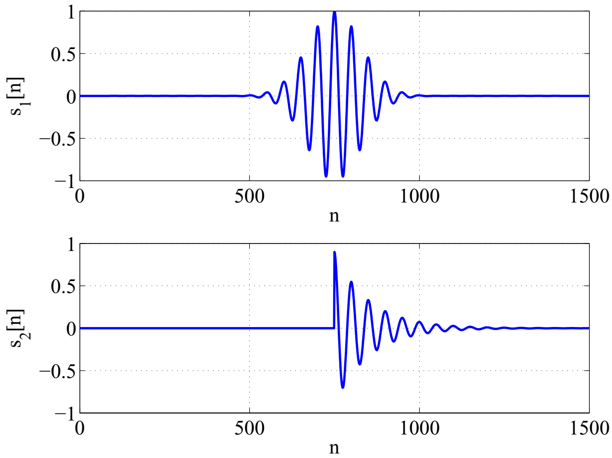

Figure 1: Signal waveforms, $s_1[n]$ and $s_2[n]$, used in simulations 1 and 2, respectively.

where

$$A_{11} = \sum_{n=0}^{M-1}(n+n_0)^2\left(s'[n;\boldsymbol{\theta}]\right)^2 \quad (9)$$

$$A_{22} = a^2\sum_{n=0}^{M-1}\left(s'[n;\boldsymbol{\theta}]\right)^2 \quad (10)$$

$$A_{12} = a\sum_{n=0}^{M-1}(n+n_0)\left(s'[n;\boldsymbol{\theta}]\right)^2 \quad (11)$$

$$B_{11} = a^2\sum_{n=0}^{M-1}\left(s'[n;\boldsymbol{\theta}]\right)^2 \quad (12)$$

$$B_{22} = \sum_{n=0}^{M-1}(n+n_0)^2\left(s'[n;\boldsymbol{\theta}]\right)^2. \quad (13)$$

3. SIMULATIONS

In order to evaluate the performance of the estimators, we set up the following tests:

- Evaluate the performance of the joint estimator for a symmetric input signal.
- Evaluate the performance of the joint estimator for a non-symmetric input signal.
- Evaluate the sensitivity to time scaling of the standard cross-correlation time delay estimator.

The motivation for testing the symmetric and non-symmetric signals is that the symmetry of the resulting cost function is affected by the time scaling. This has effect on the estimator being biased or not.

In all simulations, the time delay, $n_0 = 100$ samples, the sampling time was 0.02 s, and the time scaling, $a = 3$.

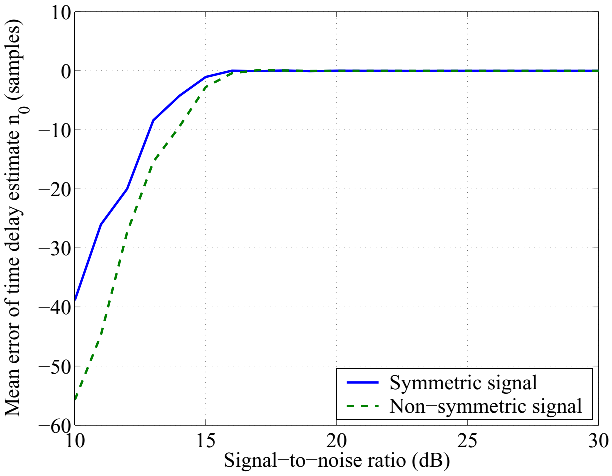

Figure 2: Mean error (ME) of the time delay estimate, as function of SNR. The solid and dashed line show the mean error for the symmetric and the non-symmetric signal, respectively.

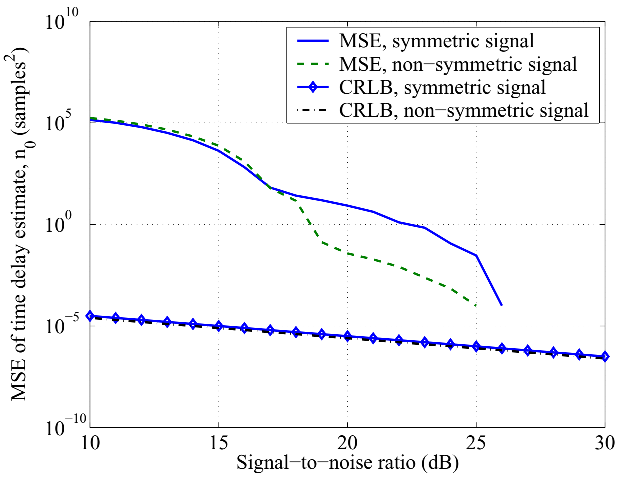

Figure 4: Mean-square error (MSE) of the time delay estimate, as function of SNR. The solid and dashed line show the mean error for the symmetric and the non-symmetric signal, respectively.

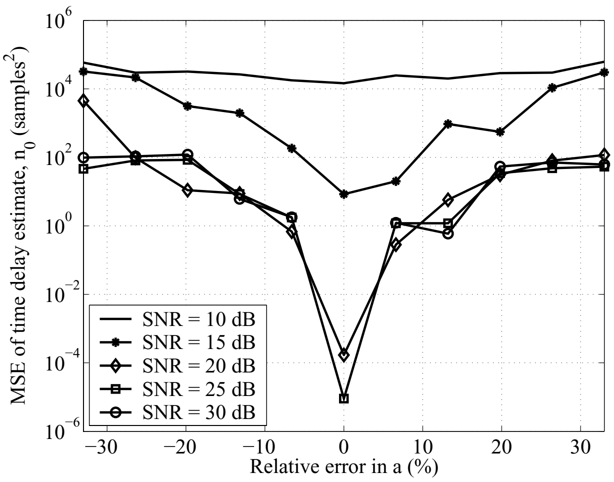

Figure 3: Sensitivity of the cross-correlation time delay estimator to variations in time scaling, for the non-symmetric signal. The figure shows the logarithm of the MSE as a function of scale factor error, for different SNR:s.

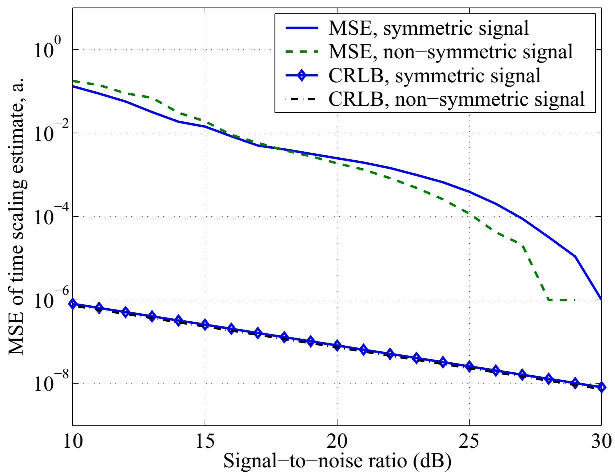

Figure 5: Mean-square error (MSE) of the time scaling estimate, as function of SNR. The solid and dashed line show the mean error for the symmetric and the non-symmetric signal, respectively.

3.1. Implementation

The estimator was implemented in MATLAB. Since no closed expression is available for the estimated parameters, a two-dimensional search over the parameters is necessary. Another problem arises because the score function in Eq. (4) is not convex. To assure that algorithm does not converge to a local maximum, a sparse search is first made in order to get close to the global maximum. After this, a numerical 2D-maximization algorithm was used. In this case, a Nelder-Mead simplex (direct search) method [5].

3.2. Case 1: Symmetric Signal

For a symmetric signal $s[n]$, the cross-correlation (first term of Eq. (4) is symmetric with respect to the time delay, n_0, regardless of the scaling parameter, a. For small scalings, compared to the signal duration, the maximum also coincides with the true parameter value. It is therefore expected that the time delay estimation should be less sensitive to an error in a if the signal waveform is symmetric. For a symmetric score function, and knowing that the noise is AWGN with zero mean, the resulting estimator of n_0 should be unbiased, for small differences in a.

3.3. Case 2: Non-Symmetric Signal

If a non-symmetric signal is cross-correlated with a time scaled version of itself, the resulting cross-correlation function is no longer symmetric. To test if this affects the performance of the parameter estimation, the lower signal in Fig. 1 was used.

3.4. Results

Fig. 2 shows the mean error of the time delay estimate, for a symmetric and non-symmetric signal respectively. It is clear that for low SNR the estimator is biased, but appears to be asymptotically unbiased. Fig. 3 shows the performance of a standard cross-correlation time delay estimator when the time scaling is neglected, for a non-symmetric signal. From the simulations it is clear an incorrect scaling results in a severe performance-loss for the time delay estimation. The sensitivity appears to be higher for a non-symmetric signal, although the effect is significant also for the symmetric case. Fig. 4 shows the mean-square error (MSE) for the time delay estimate, n_0, for a the symmetric and the non-symmetric pulses in Fig. 1. Fig. 5 shows the corresponding MSE:s for the time scaling estimate.

4. DISCUSSION

From the simulations we note that the estimator is biased for low SNR. This means the Cramér-Rao bound is not valid other than asymptotically. We also note from the simulations that the estimator does not reach the bound even for an SNR of 30 dB. The time delay estimation (c.f. Fig. 4) appears to become zero already at 27 dB. This is an artefact most likely due to the numerical optimization routine used in the simulation. The true time delay (100 samples) is a multiple of the step size in the algorithm, which causes it to terminate with zero error. For non-rational time delays, it is expected that the variance reaches the lower bound, rather than zero.

Another complication is the need to scale the reference signal. In the simulations, we had access to an analytical expression for the signals in Fig. 1. In a practical case, however, An interpolation and re-sampling scheme might be necessary. This increases the complexity of the estimator significantly. This problem was also recognized in a continuous-time case, by Knapp and Carter [6].

5. CONCLUSIONS

In this paper we have derived a simultaneous maximum likelihood estimator for time scaling and time delay in additive white Gaussian noise. We have also derived the Cramér-Rao lower bound for the variances of the parameter estimates. For the two signal waveforms, one symmetric and one non-symmetric, the simulations show that we obtain asymptotically unbiased parameter estimates, that approaches the bound for high SNR.

6. REFERENCES

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