Analysis of multiple solutions in bifurcation diagrams to avoid unexpected dynamics

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Abstract: In mechanical applications it is essential that unexpected dynamics are avoided. The industry wants to build reliable machines that are not sensitive to initial conditions. Therefore, a simple method has been developed to extract all sets of stable bifurcation diagrams. The method gives a designer a good overview of possible dynamics and thereby the possibility to select a safe operating region. The method is described and demonstrated with a rub-impact rotor. The practical usage of this method is to help the designer to determine if parameter ranges exist where coexistent solutions will appear. Thereby one can design the system to work in parameter ranges where only one acceptable solution exists.

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Keywords: Multiple solutions, dynamics, rub, impact, bifurcation diagrams.

1. Introduction

In industrial applications there are several situations where non-linear vibrations can occur. When designing machines, nonlinear behaviour is normally unwanted. However, these nonlinear phenomena cannot always be avoided. When a system is non-linear it is unsatisfying to present only one possible solution, without considering if other solutions are present. The cell mapping method is one way to find all solutions for a given set of parameters. The method was developed in a series of reports e. g. Hso[1,2], Hsu and Kim[3] and Tongue[4].

The method is useful for low order systems where a global picture of possible dynamic attractors can be obtained. For a single frequency and with all parameters fixed, the cell mapping approach is a powerful tool to analyse all solutions. Though in real life applications, parameters are seldom known with such accuracy that one can be completely satisfied with the method. A small change in the parameters can change the dynamics. If however, the cell mapping approach could be applied to a parameter range, design of systems with known motion in parameter intervals should be possible.

In dynamic analysis on intervals, the bifurcation diagram has become a general tool, Feigenbaum[5]. The diagram gives bifurcation values and also some insight in the type of bifurcations for large parameter ranges. The only disadvantage is that the bifurcation diagram technique only follows one stable solution. Therefore one cannot be sure if other solutions can exist and where.

In this paper a method is suggested, based on the ideas from the cell map and the bifurcation diagram technique. After simulation and analysis of the results, several sets of bifurcation diagrams are extracted. The method is therefore in this paper called “multi-bifurcation diagram”. By use of this method, detailed information about the dynamics of the system can be obtained. When all solutions are known, the designer can hopefully find parameter ranges with acceptable solutions or at least know what kind of problems to expect. The method has been developed in the papers by Aidanpää[6-8] but was not described in detail.

The method is finally demonstrated on a rub-impact Jeffcott rotor with Coulomb friction. Several studies have been performed on the Jeffcott rotor with this kind of rubbing impacts e. g. [9-14]. The model is suitable since several routes to chaos and multiple solutions have been reported.

2. Method

In order to evaluate the dynamics for several stable solutions, a method with ideas from the cell map and bifurcation diagram techniques is used. Hereafter, the method is called multi-bifurcation diagram. For each frequency the dynamic is simulated for a large number of initial conditions (1 in Figure 1). From each initial condition, a set of Poincaré sections is collected after steady state is reached (2 in Figure 1). The resulting Poincaré sections then are plotted in the same way as in the bifurcation diagram technique.

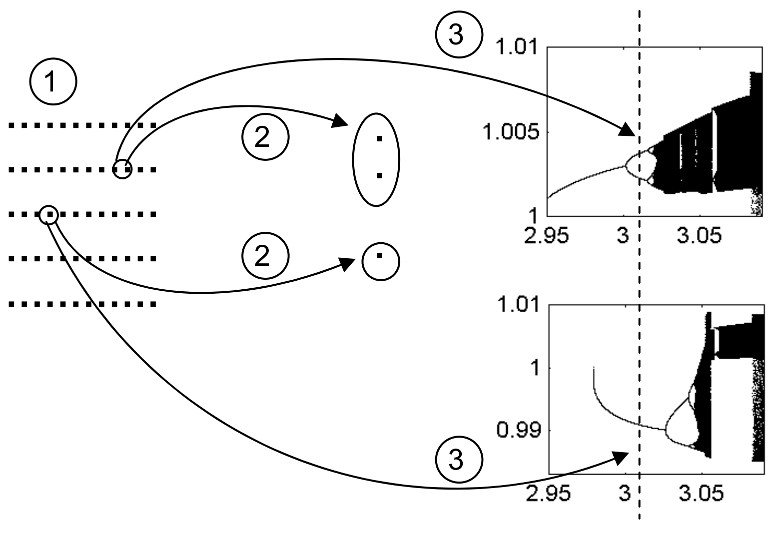

Figure 1. Description of the multi-bifurcation diagram technique. Initial conditions in 1, the corresponding solution from each initial condition plotted in a bifurcation diagram 2 and 3 shows the extracted new bifurcation diagrams.

By analysing the results from the multi-bifurcation diagram different solutions can be found. Since the initial conditions can be traced for each solution, new bifurcation diagrams can be simulated for all solutions. Hence one can plot several sets of bifurcation diagrams which contain several stable solutions of the system (3 in Figure 1). When generating the bifurcation diagrams the system needs to be simulated both for increasing and decreasing frequencies, in order to extract the complete bifurcation diagram. After extracting all solutions, several sets of bifurcation diagrams describing possible solutions to the system become available.

3. Example on a rub impact Jefcott rotor

3.1. ***The model***

The model of the Jeffcott rotor is shown in Figure 2. The mass of the rotor, *2m*, is supported by the shaft with stiffness *k* and damping *c* (not material damping). The rotor is amplitude-limited by the stator which has a diameter 2δ larger than the rotor. The rotor is also subjected to the gravity field *g*.

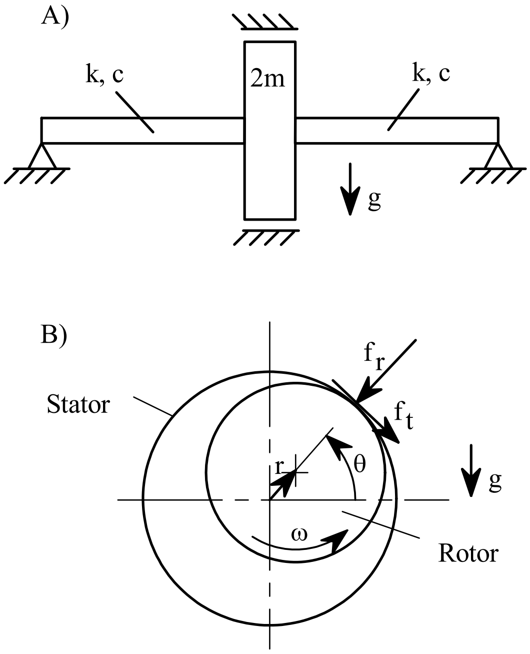

Figure 2. Rub impact model of the Jeffcott rotor.

The origin of the coordinate system is chosen to the centre of the stator according to Figure 2B. The spin speed is ω and the position of the rotor centre is described by the polar coordinates r for the radial displacements and θ for angular displacements. When r exceeds the radial clearance δ the rotor becomes in contact with the stator. This contact is described by a stiffness k_s and frictional coefficient μ which results in the contact forces f_r and f_t.

The equations of motion can then be written

$$
\begin{aligned}
m\ddot{r} + c\dot{r} + \left(k - m\dot{\theta}^2\right)r &= m u \omega^2 \mathbf{cos}(\omega t - \theta) - m g \,\mathbf{sin}(\theta) - f_r \\
m r \ddot{\theta} + \left(2 m \dot{r} + c r\right)\dot{\theta} &= m u \omega^2 \mathbf{sin}(\omega t - \theta) - m g \,\mathbf{cos}(\theta) - f_t
\end{aligned}
\tag{1}
$$

Let R be the radius of the rotor. Then the velocity of the contact point becomes

$$V_c = r\dot{\theta} + R\omega \tag{2}$$

with the slip-contact ($V_c \neq 0$) described by

$$
\left.
\begin{aligned}
f_r &= k_s\left(r - \delta\right) \\
f_t &= \mu f_r \; sign(V_c)
\end{aligned}
\right\}
\quad
\begin{aligned}
&if \; r \geq \delta \\
&otherwise \; f_r = f_t = 0
\end{aligned}
\tag{3}
$$

Stick occurs when V_c is zero and this state is valid as long as the tangential contact force is less than the frictional force. The equation of motion for the stick phase becomes

$$
\begin{aligned}
&\dot{\theta} = -R\omega/r \\
&m\ddot{r} + c\dot{r} + \left(k - m\dot{\theta}^2\right)r = m u \omega^2 \cos\left(\omega t - \theta\right) - m g \sin\left(\theta\right) - k_s\left(r - \delta\right) \\
&valid \;\; while \\
&\left|\left(2 m \dot{r} + c r\right)\dot{\theta} - m u \omega^2 \sin\left(\omega t - \theta\right) + m g \cos\left(\theta\right)\right| < \left|\mu k_s \left(r - \delta\right)\right|
\end{aligned}
\tag{4}
$$

Let $\omega_n = \sqrt{k/m}$ and introducing the non-dimensional quantities

$$
\begin{aligned}
&\hat{r} = \frac{r}{\delta}, \quad \hat{R} = \frac{R}{\delta}, \quad \hat{\omega} = \frac{\omega}{\omega_n}, \quad \hat{t} = t\,\omega_n, \quad \hat{u} = \frac{u}{\delta}, \\
&\varsigma = \frac{c}{2 m \omega_n}, \quad \hat{g} = \frac{g}{\omega_n^2 \delta}, \quad \hat{f}_r = \frac{f_r}{m \omega_n^2 \delta}, \quad \hat{f}_t = \frac{f_t}{m \omega_n^2 \delta}, \\
&\hat{\omega}_s = \frac{k_s}{m \omega_n^2}, \quad \hat{V}_c = \frac{V_c}{\delta\, \omega_n}
\end{aligned}
\tag{5}
$$

The non-dimensional equations of motions then become

$$\begin{aligned} &\hat{r}'' + 2\varsigma\,\hat{r}' + \left(1-\theta'^2\right)\hat{r} = \hat{u}\,\hat{\omega}^2\cos\left(\hat{\omega}\hat{t}-\theta\right) - \hat{g}\sin(\theta) - \hat{f}_r \\ &\hat{r}\,\theta'' + 2\left(\hat{r}' + \varsigma\,\hat{r}\right)\theta' = \hat{u}\,\hat{\omega}^2\sin\left(\hat{\omega}\hat{t}-\theta\right) - \hat{g}\cos(\theta) - \hat{f}_t \end{aligned} \tag{6}$$

with the slip contact ($\hat{V}_c \neq 0$) described by

$$\left.\begin{aligned} \hat{f}_r &= \hat{\omega}_s\left(\hat{r}-1\right) \\ \hat{f}_t &= \mu\,\hat{f}_r\ sign(\hat{V}_c) \end{aligned}\right\} \quad \begin{aligned} &if \quad \hat{r} \geq 1 \\ &otherwise\ \hat{f}_r = \hat{f}_t = 0 \end{aligned} \tag{7}$$

The equation of motion for the stick phase ($\hat{V}_c = 0$) becomes

$$\begin{aligned} &\theta' = -\hat{R}\,\hat{\omega}/\hat{r} \\ &\hat{r}'' + 2\,\varsigma\,\hat{r}' + \left(1-\theta'^2\right)\hat{r} = \hat{u}\,\hat{\omega}^2\cos\left(\hat{\omega}\,\hat{t}-\theta\right) - \hat{g}\sin(\theta) - \hat{\omega}_s\left(\hat{r}-1\right) \\ &valid\ \ while \\ &\left|\,2\left(\hat{r}' + \varsigma\,\hat{r}\right)\theta' - \hat{u}\,\hat{\omega}^2\sin\left(\hat{\omega}\,\hat{t}-\theta\right) + \hat{g}\cos(\theta)\right| < \left|\,\mu\,\hat{\omega}_s\left(\hat{r}-1\right)\right| \end{aligned} \tag{8}$$

The model (Eqs 6 to 8) is now complete for analysis and simulations.

3.2. ***Results***

From Equation (5) it is clear that an initial rub is possible when $\hat{g} \geq 1$. A system is selected with $\hat{u}=0.125$, $\zeta=0.6$, $\hat{g}=1.962$, $\mu=0.2$ and $\hat{\omega}_s=240$. This system is similar to the one suggested in Chu and Zhang[10]. The bifurcation diagram for increasing frequency $\hat{\omega}$ is shown in Figure 3.

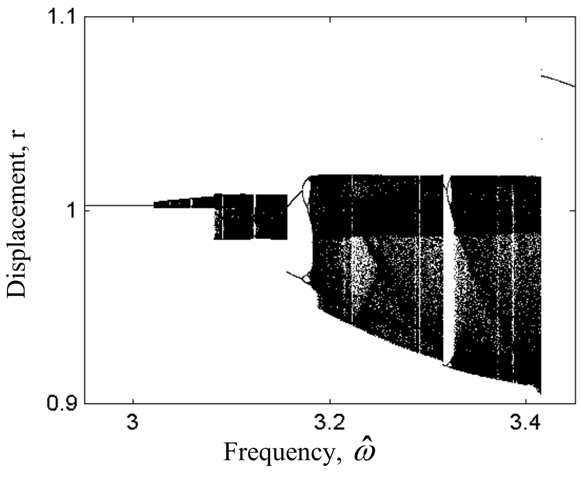

Figure 3. Bifurcation diagram of the system

The system has a period one solution up to $\hat{\omega} \approx 3$. Then a region of long-periodic or chaotic motion appears which ends with a sudden jump to a period one solution. In Figure 3 one can observe that the periodic solution suddenly jumps in to long-periodic motions and as the frequency increases several crises and jump phenomena occur.

In Figure 4 the results of the simulations are shown for the multi-bifurcation diagram

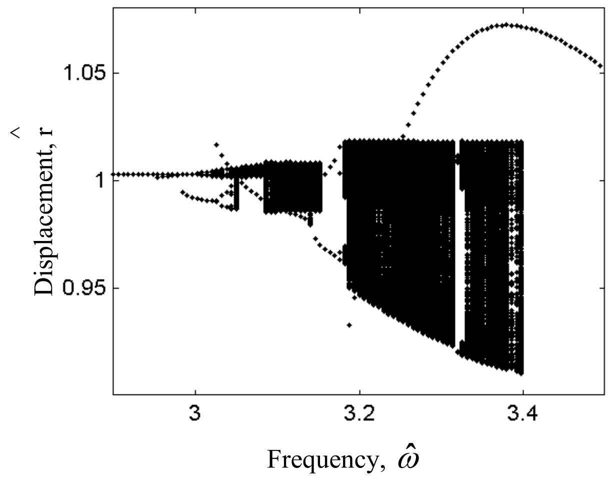

Figure 4. The multi-bifurcation diagram.

Comparing Figure 3 with Figure 4 shows clear differences. By analysing the Poincaré section one can separate the different solutions. Since the initial conditions are known for each solution one can perform new simulations of each attractor. The first point of each new solution is taken as initial conditions for new bifurcation diagrams. From these initial points the bifurcation diagrams are generated for increasing and decreasing frequencies. The decreasing frequency simulation is only performed in order to find the exact position of the first appearance of the new attractor. In the interval $\hat{\omega}$ *=[2.95, 3.4],* five different attractors can be found. In Figure 5 four of them are shown. In the upper left position the solution is shown for the same bifurcation diagram as in Figure 3 (solution A). One can again observe the sudden jump from a period one solution to a long periodic solution at $\hat{\omega} \approx 3.02$. In the upper right position (solution B) a bifurcation diagram with a period doubling sequence in to the long-periodic or chaotic motion is found. One can observe that the sudden jump in solution A goes to the same attractor as in solution B. In the lower left position solution C is shown. Here another period doubling sequence is found in the same interval.

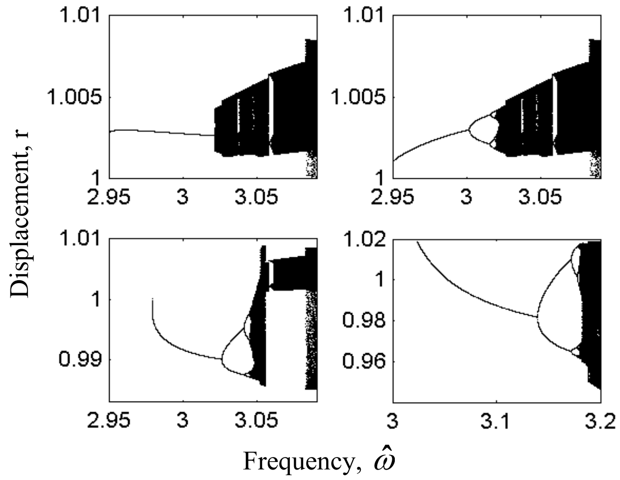

Figure 5. Bifurcation diagrams extracted from the solutions found in the multi-bifurcation method.

It is interesting to note that the solution jumps back to solution A and B after $\hat{\omega} \approx 3.05$. In the lower right corner solution C is shown. Here a third period doubling sequence is found. Even a fifth attractor exist (solution E) starting at $\hat{\omega} \approx 3.18$. This is the initiation of the period one solution which can be observed after $\hat{\omega} \approx 3.4$ in Figure3. This solution can also be observed as the period 1 curve at the top of Figure 4.

7. Conclusions

In this paper a simple method (multi-bifurcation diagram) is presented to extract sets of bifurcation diagrams. The method is then demonstrated on a rub-impact jeffcot rotor. It is found that five different bifurcation diagrams exist for the studied interval. However, large regions are found where only one solution exist. By detecting regions containing a single solution, the method can be used to design a system with low risk of unwanted dynamics.

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