Constitutive modelling of Swedish fine grained soils
Modélisation constitutive de sols suédois à grains fins

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ABSTRACT: Constitutive behaviour of Swedish fine grained soils i.e. soft clay and silt, is discussed with respect to both drained and undrained response. Constitutive models based on the theory of elasto-plasticity are suggested for these soils. While volumetric as well as deviatoric hardening are taken into consideration for silt, a rotated yield surface with a mixed hardening is assumed for clay. Explicit as well as implicit methods are used for integration of the constitutive relations. Via an optimization algorithm, constitutive parameters are determined from drained and undrained triaxial tests on clay and silt.

RESUME: Le comportement constitutif de sols suédois à grains fins i.e. des argiles et des silts douces est discuté en tenant compte de la réponse avec drainage et sans drainage. Les modèles constitutifs fondés sur la théorie de l’élasto-plasticité sont suggérés pour ces sols. Alors que des taux d’écrouissage volumétriques ainsi que déviateurs sont considérés dans le cas du silt, une surface d’écoulement tournante avec écrouissage combiné est considérée pour l’argile. Des méthodes implicites et explicites sont utilisées pour intégrer les relations constitutives. Utilisant un algorithme d’optimisation les paramètres constitutifs sont déterminés à partir des tests triaxiaux, avec et sans drainage, sur l’argile et le silt.

1 INTRODUCTION

For solving advanced problems in geotechnical engineering, e.g. slope stability and bearing capacity problems, numerical methods like the finite element method could give valuable information in addition to classical methods. However, in order to make a numerical method applicable to a certain geological technical problem, it must involve a relevant soil model with correct parameter values for the soil to be considered.

Concerning fine-grained soils in Sweden, clay has been extensively investigated, while silt only recently has attracted the same attention. For both soil types, however, there is a lack of proved constitutive models as well as of relevant parameter values for implementation into finite element programs.

In this paper some important constitutive characteristics of Swedish fine-grained soils observed in experiments are briefly reviewed. The constitutive characteristics of silt are simulated by an extended Cam Clay plasticity model accounting for combined strain hardening and dilatancy in addition to contraction. For clay the anisotropy (Ko-consolidation) is accounted for. Implementation of the models in a Constitutive Driver is discussed. Both explicit and implicit integration schemes are utilized. Further, methods based on optimization for determination of material parameters from laboratory tests are presented.

2 BEHAVIOUR OF SILT AND CLAY

Basic behaviour of Swedish fine-grained soils under drained and undrained conditions is by now quite well documented. This holds especially for soft clay, while silt only recently has been more extensively investigated. Here, only some of the most important characteristics for these types of soils are summarized from a constitutive point of view.

2.1 Silt

Recent studies (Axelsson & Yu 1988, Börgesson 1981, Yu 1990) have shown that silt has a dual characteristic of both contractancy and dilatancy, i.e. when sheared, it exhibits contractancy at small strains but dilatancy at larger strains even for normally consolidated samples. Especially in the undrained case, the deviatoric stress can continuously increase which makes it difficult to determine shear strength i.e. (σ1-σ3) max in a conventional way. It is, however, observed that both undrained and drained shear behaviour of silt is related to a failure line (FL) and a critical state line (CSL), in the p-q plane. Hence, it is preferable to use the effective stress analysis for silt. It is also noted that silt has virtually no cohesion. These characteristics can be illustrated by Figs.1 and 2 displaying triaxial tests on silt.

Since capillarity forces create notable physical bonds in a silt skeleton (Yu 1990), its strength is, to a great extent, influenced by mechanical disturbance and by changes in the water content. For example, a great many natural or man-made silt slopes in northern Sweden flow every year during the snow-melting period. Undisturbed natural samples of silt are almost impossible to obtain in the field.

2.2 Soft clay

The existence of an obvious yield surface and of a critical state line, significant anisotropy and a high sensitivity as well as a notable creep effect (not considered here) are some of the most important characteristics of Swedish soft clay (Sällfors 1975, Larsson 1977, Larsson & Sällfors 1981).

Clay soils display a 'cohesive' strength due to inherent physical-chemical bonds, and, hence, undisturbed samples are easily obtained in the field. Carefully performed triaxial and plane strain tests on such samples have indicated an anisotropic yield surface under in-situ state with Ko often about 0.5, see Fig.3. Oedometer consolidation tests carried out on natural marine deposited samples have given the same result as those by Bjerrum (1967), i.e. the secondary consoli-
3 CONSTITUTIVE MODELS

For modelling the above discussed features of Swedish fine grained soils, two constitutive models within the framework of Cam Clay plasticity have been adopted. The reason for this is that the Cam Clay models, most popular in Soil Mechanics, are able to predict, at least, qualitatively many of the fundamental aspects of soil behaviour. The adopted models, with some extensions of the Cam Clay concept, are here presented in short and with reference to the conventional p-q plane. For both models, associated flow is assumed, i.e. the yield function is equal to the plastic potential function, as in the Cam Clay models.

3.1 Model for silt

The yield surface adopted is

\[ f = q + M \log \frac{p}{p_c} \]  

which is the same form as in the original Cam Clay model. In connection with the works by Nova & Wood (1978), the following hardening law is assumed

\[ dp_c = \frac{1 + e_0}{k} \left( \frac{dP_f}{P_f} + D \cdot \frac{dp}{q} \right) \]  

where \( p_c \) is the preconsolidation pressure, \( q \) the initial void ratio, \( e_0 \) the plastic volumetric and deviatoric strain respectively. In eqn.(2) \( D \) is a constant controlling the

\[ M_f = M + \frac{k}{k} D \]  

which can be used to determine \( D \) from \( M \), the slope of FL. Obviously, the combined strain hardening degenerates to volumetric strain hardening when \( M_f = M \) (resulting in \( D = 0 \)) which is pertinent to the conventional Cam Clay models.
\[ dp_k = \frac{1 + e_0}{\lambda - \kappa} p_k \text{de} \quad dq_k = \frac{1 + e_0}{\lambda - \kappa} \left[ \frac{M - q_k}{p_k} \right] \text{de} \]

where \( q_k \) is a kinematic hardening parameter (or the value of \( q \) at anisotropic consolidation), and \( p_k \) an isotropic hardening parameter. Equation (5) can be regarded as a law of mixed hardening since it accounts both rotation (kinematic hardening) and expansion or contraction (isotropic hardening) of the yield surface. Moreover, it satisfies the conditions at failure, i.e., at failure \( q_k/p_k = M \) and the deviatoric stress does not more increase. Similarly, during steady state consolidation, it ensures continuous expansion of the yield surface only along the \( K_0 \)-line without any rotation as \( q_k/p_k \) becomes a constant. Clearly, for the isotropic case, \( q_k = 0 \) (and \( p_k = p_0 \)) and the model reduces to the conventional isotropic hardening model, i.e. the modified Cam Clay model.

4 MODEL IMPLEMENTATION

4.1 Stress-strain equations

From the plastic flow rule, \( \dot{\varepsilon}^p = \lambda \dot{\varepsilon}_m \), and the consistency condition, \( \dot{f} = 0 \), the elastoplastic-tangential relationship is obtained on the form

\[ \dot{\varepsilon} = \mathbf{D}^p \dot{\varepsilon} = \left[ \mathbf{D}^e - \frac{\mathbf{D}^m \mathbf{m}^T \mathbf{D}^e}{\mathbf{H} + \mathbf{m}^T \mathbf{D}^m} \right] \dot{\varepsilon} \]

(5)

for a strain-driven format (strain control) and on the form

\[ \dot{\varepsilon} = \mathbf{C}^p \dot{\varepsilon} = \left[ \mathbf{C}^e + \frac{\mathbf{m}^T \mathbf{m}}{\mathbf{H}} \right] \dot{\varepsilon} \]

(6)

for a stress-driven format (stress control). Here \( \mathbf{D}^p \) and \( \mathbf{C}^p \) denote the elastic stiffness and compliance matrices respectively while \( \mathbf{m} \) is the gradient of the plastic potential function, \( \mathbf{n} \) the gradient of the yield function and \( \mathbf{H} \) the (generalized) plastic modulus depending on the hardening law. In our case with associated yielding, \( \mathbf{m} = \mathbf{n} \) remains, and \( \mathbf{H} \) takes the form

\[ \mathbf{H} = -\frac{\partial \mathbf{f}}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{\varepsilon}^p} + \frac{\partial \mathbf{p}}{\partial \mathbf{\varepsilon}^p} \frac{\partial \mathbf{f}}{\partial \mathbf{\varepsilon}^p} \]

(7)

for the model of silt and

\[ \mathbf{H} = -\frac{\partial \mathbf{f}}{\partial \mathbf{p}_k} \frac{\partial \mathbf{p}_k}{\partial \mathbf{\varepsilon}^p} + \frac{\partial \mathbf{p}_k}{\partial \mathbf{\varepsilon}^p} \frac{\partial \mathbf{f}}{\partial \mathbf{\varepsilon}^p} \]

(8)

for the model of clay.

For implementation in a “standard” displacement finite element method, only strain control is needed. However, for implementation in a Constitutive Driver aiming at simulation of laboratory tests, also stress control and even mixed control are of interest. For the special case of undrained behaviour, explicit tangential relationships for stress control and mixed control should be derived. This matter is discussed by Runesson et al. (1990) and Yu & Axelsson (1990).

4.2 Integration of plasticity relations

In the context of a finite element analysis or an analysis with a Constitutive Driver, the rate-formulated stress-strain relations must be numerically integrated for describing the incremental evolution of the stress and strain states. In the Driver used here to simulate experimental tests, both an explicit and an implicit method have been applied. The used explicit method is the standard tangent stiffness method. The stress, \( \sigma^{n+1} = \sigma^n + \Delta \sigma \), predicted by the tangent stiffness relation (6) for a finite strain increment is returned to the yield surface by proportioning, \( \sigma^{n+1} = k \cdot \dot{\varepsilon}^{n+1} \), thus satisfying the yield condition \( f(\sigma^{n+1}, k\dot{\varepsilon}^{n+1}) = 0 \).

Implicit methods can be divided into rate formulation approaches and variational inequality approaches. The latter has extensively been investigated by Runesson et al. (1988 &1990) and Axelsson et al. (1989), for soil plasticity. Here, following the former approach, the rate equation defining the flow rule is taken as the basis for the straightforward application of the Backward Euler method (BE) also called the Closest Point Projection Method (CPPM). From the trial stress increment \( \Delta \sigma^{n+1} \) in a load step (n+1), calculated assuming pure elastic response, the elasto-plastic stress increment is obtained as

\[ \Delta \sigma^{n+1} = \Delta \sigma^{n+1}_t - \Delta \lambda \mathbf{D}^e \mathbf{m}^{n+1} \]

(9)

which, inserted into the updated yield criterion

\[ f^{n+1}(\sigma^n + \Delta \sigma^{n+1}, k^{n+1}) = 0 \]

(10)

yields the scalar-valued function

\[ f(\Delta \lambda^{n+1}) = 0 \]

(12)

for solving the plastic multiplier in the step (n+1). The solution of eqn. (12) is obtained numerically by Newton’s method.

Since the initial estimate of the plastic multiplier strongly affects the overall convergence, the linearized or tangential expressions for \( \Delta \lambda \) in eqn. (6) should be used to start the Newton method.

5 OPTIMAL DETERMINATION OF PARAMETERS

When advanced constitutive models for soils are utilized, e.g. the models suggested above, the corresponding determination of constitutive parameters become difficult and laborious and, in particular, very subjective even for the same test series. Therefore, an efficient optimization method to objectively determine constitutive material parameters from the entire experimental response of interest, including not only conventional tests but also non-conventional tests such as plane strain tests, is needed.

5.1 Principle of optimization

The basic idea of the procedure advocated here is to regard the optimal fitting process for given experimental data as a constrained optimization problem. In this context, the objective function \( \Pi : R^n \rightarrow R \) is simply the sum-of-norms error function defined as

\[ \Pi(x) = \sum_{i=1}^{N} d_i(x) \]

(13)

where \( N \) is the number of observations; \( x \) the material parameter vector (in \( R^m \), \( m \) is the dimension of \( x \)); \( d_i \) the adequate norm which measures the distance between theoretical and experimental results; and \( I \) the \( i \)th data point. The constraints imposed on the optimization problem emanate from physical restrictions placed on the model parameters. These constraints define a feasible domain \( U \subseteq R^m \), which is a convex polygon. The resulting constrained optimization problem is then expressed as
Find:  \( \min \Pi(x) \)  subject to  \( x \in U \)  \( (14) \)

There exist a wide variety of algorithms for solving the standard convex optimization problem in eqn. (14), such as quasi-Newton, Levenberg-Marquardt, Rosenbrock, Powell, and conjugate gradient, see for example Fletcher (1980). All these methods have one feature in common, they can only converge to a local minimum, i.e. the initial guess must be in the neighborhood of the global minimum. Therefore, it must be clearly understood that it is very crucial for the final result to have good initial data. Moreover, general fundamentals of the applied constitutive model must be known a priori in order to succeed with the optimization task. Usually the most effective optimization methods are the gradient based methods, which require not only the evaluation of the objective function, but also its derivatives with respect to all optimization variables. Their efficiency decreases rapidly, when the derivatives cannot be obtained in an analytical way. Therefore a non-gradient based method is here to prefer.

5.2 Adopted method

The Rosenbrock’s method, a direct search method, is utilized. In this case, the optimization strategy does not require derivatives and is only based on the computed values of the objective function. A special combined Euclidean norm is chosen as the objective function

\[
\Pi = m \cdot D_{\max} + \sum_{l=1}^{m} D_l
\]  \( (15) \)

where \( m \) is the number of representative tests, and \( D_l \) is the norm for test \( l \)

\[
D_l = w \left( \frac{1}{k+1} \sum_{j=1}^{k} d_j + d_i \right)
\]  \( (16) \)

In eqn. (16) \( k \) is the number of experimental points, \( d \) the Euclidean distance between experimental and predicted values, and the subscript \( t \) the termination point up to which a theoretical prediction follows e.g. a maximum stress in stress control. The weight factor, \( w \), characterizes the importance of a test. In the case of principle stresses or strains occurring, \( d \) is computed as

\[
d = \left[ \sum_{i=1}^{3} (\sigma_i^b - \sigma_i^a)^2 + \sum_{i=1}^{3} (\varepsilon_i^b - \varepsilon_i^a)^2 \right]^{1/2}
\]  \( (17) \)

where \( a \) and \( b \) denote the closest point on the theoretical curve and the experimental point respectively. Since both stresses and strains are often involved in soil testing, the normalized stresses and strains are used in eqn. (17).

While the objective function has been defined, the remaining problem is to search for its minimum. In this method a vector base is defined in the m-dimensional vector space which is determined by the number of optimization variables. Starting from the initial point with an initial step size, the program searches for any improvement of the objective function in all base directions, and when the search is successful, an increase in the step size in the particular direction is produced. If the search is unsuccessful, the program will search in the opposite direction using a smaller step size during the next loop. This procedure is repeated until there is no improvement after the optimization step.

6 COMPARISON WITH EXPERIMENTS

Based on the presented integration approaches, the two suggested models have been implemented in a Constitutive Driver (Yu & Axelsson 1990) to simulate experimental tests and to perform parametric sensitivity studies. In connection with this work, the above optimization method is used for determining the constitutive parameters for the samples of silt obtained from Piteå, northern Sweden, and of clay from Bäckebol at the west coast of southern Sweden. All the experimental results are obtained from conventional and \( K_0 \)-triaxial testing and, hence, the optimization is restricted to principal stresses or strains. The optimized parameters together with other known parameters are given below.

for silt

\[ G=30 \text{ MPa}, \quad \kappa=0.0038, \quad \lambda=0.014, \quad M=1.215, \quad D=0.392 \]

for clay

\[ G=10 \text{ MPa}, \quad \kappa=0.05, \quad \lambda=0.14, \quad M=1.17, \quad e_0=1.05 \]

The constitutive models together with the optimized parameters are utilized for predicting the response for other loading paths than those used for the parameter determination. For instance, predictions for extension tests and \( p' \)-constant tests are made. Comparison of these predictions with corresponding experimental results are given in Figs. 6, 7, 8 and 9 demonstrating that these models, to a large degree, capture the main discussed behaviour for Swedish fine grained soils. For instance, the effective stress path of silt under undrained compression is satisfactorily predicted, and the calculated stress - strain relation of soft clay agrees well with the corresponding experimental data.

![Fig. 6 Comparison of test results with predictions for silt](image)

![Fig. 7 Comparison of test results with predictions for silt](image)
7. CONCLUSIONS

Based on the discussions of drained and undrained response of Swedish fine-grained soils, i.e. soft clay and silt, two constitutive models within the concept of the Cam Clay are suggested. Volumetric as well as deviatoric hardening are accounted for and a rotated yield surface is assumed in conjunction with dilatancy and initial anisotropy. In order to implement those models in a Constitutive Driver and also in a finite element program, both explicit and implicit integration approaches are presented. A general optimization method with a direct search strategy is found quite useful for properly determining constitutive parameters. Comparison of model predictions and experimental results shows importances of the discussed issues in the paper.

REFERENCES


