IMPROVED MINIMIZATION METHODS OF PSEUDO KRONECKER EXPRESSIONS FOR MULTIPLE OUTPUT FUNCTIONS

Per Lindgren*

Division of Computer Engineering
Luleå University of Technology
97187 Luleå, Sweden
pln@sm.luth.se

Rolf Drechsler

Bernd Becker

Institute of Computer Science
Albert-Ludwigs-University
79110 Freiburg im Breisgau, Germany
{drechsle/becker}@informatik.uni-freiburg.de

ABSTRACT

Pseudo Kronecker Expressions (PSDKROs) are a class of AND/EXOR expressions. For a Boolean function with a given variable order the minimal PSDKRO can be derived efficiently using Decision Diagram (DD) techniques. The quality, i.e., the number of products in the expression, of the result is known to be dependent on the variable ordering.

This paper proposes several improvements and enhancements to previous minimization methods. A pruning technique that can be tuned to tradeoff quality for computational resources is presented. By applying dynamic ordering methods, significant improvements to many previously reported results are obtained. Furthermore, a new method for the minimization of multiple output functions is outlined. Experiments on a set of MCNC benchmarks confirm the advantages of the presented algorithms.

1. INTRODUCTION

The use of EXOR gates in the synthesis process reduces the hardware costs in many cases. Additionally, EXOR based circuits often have nice testability properties. In contrast to AND/OR minimization - that in the meantime is well understood - in AND/EXOR minimization several restricted classes are considered, like *Fixed Polarity Reed-Muller Expressions* (FPRMs) and *Kronecker Expressions* (KROs). (For an excellent overview see [7].) This subclasses are of interest, since the minimization of general *Exclusive Sum of Product Expressions* (ESOPs) turned out to be computationally very hard.

As one alternative *Pseudo Kronecker Expressions* (PSDKROs) [7] have been proposed, since they are an interesting compromise: the resulting 2-level forms are of moderate size and additionally the minimization process can be handled within reasonable time bounds using Multi-Place DDs or more recently Ordered Binary DDs (OBDDs) [2] as shown in [3]. Furthermore, for minimization of totally symmetric functions, the algorithm is shown to derive the exact minima in polynomial time [3].

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In this paper we propose a pruning technique for further speeding up the approaches presented in [7, 3]. The method can be tuned to tradeoff quality for CPU and memory resources, producing heuristic results for previously unmanageable problems. The quality of the minimized PSDKRO expression is known to be dependent on the variable order applied during minimization. We employ dynamic variable ordering strategies and give a comparison of quality and efficiency. Furthermore, we have a closer look at the principles how multiple output functions are handled in presented approaches and outline an alternative method relaxing previous decomposition constraints. This method especially becomes interesting if PSDKRO minimization is seen as a preprocessing for general AND/EXOR minimization or multi-level synthesis. Experimental results are reported that demonstrate the efficiency of the proposed algorithms.

2. PSEUDO KRONECKER EXPRESSIONS

In this section we briefly review the essential definitions of *Pseudo Kronecker Expressions* (PSDKROs). (For more details see [7, 3].)

Let f_0 (f_1) denote the *cofactor* of f with respect to $\bar{x}$ (x) and f_2 is defined as $f_2 = f_0 \oplus f_1$, $\oplus$ being the Exclusive OR operation. A Boolean function $B^n \rightarrow B$ can then be represented by one of the following formulae:

$$f = \bar{x}f_0 \oplus xf_1 \quad \text{Shannon } (S) \quad (1)$$

$$f = f_0 \oplus xf_2 \quad \text{positive Davio } (pD) \quad (2)$$

$$f = f_1 \oplus \bar{x}f_2 \quad \text{negative Davio } (nD) \quad (3)$$

If we apply to a function f either S , pD or nD we get two sub-functions. To each sub-function again S , pD or nD can be applied. This is done until constant functions are reached. If we multiply out the resulting expression we get a 2-level AND/EXOR form, called a PSDKRO. In Section 5 we take a closer look at PSDKRO expressions for multiple output functions.

The decompositions are applied with respect to a fixed variable ordering. Note that the choice of the variable ordering in which the decompositions are applied and the choice

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prod psdkro(node f, int prune) {
1  if (f == 0(1)) return 0(1);
2  if (f.prod defined) return f.prod;
3  f0 = cofactor0(f); f1 = cofactor1(f);
4  p0 = psdkro(f0, prune);
5  (P2) → if (p0 ≥ prune) return MAXINT;
6  p1 = psdkro(f1, prune);
7  (P0,P1,P2) → if (min(p0, p1) ≥ prune) return MAXINT;
8  f2 = EXOR(f0, f1);
9  if (memory or time limit reached) return p0 + p1;
10 p2 = psdkro(f2, pf(prune, p0, p1));
11 f.prod = p0 + p1 + p2 - max(p0, p1, p2);
12 return f.prod;
}

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Figure 1: Sketch of the enhanced algorithm

of the decomposition per sub-function largely influences the size of the resulting representation, and may vary from linear to exponential.

Example 1 Consider the parity function $f(x_1, x_2, x_3)$. A minimal (w.r.t. number of products) SOP form is: $f = \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + x_1x_2x_3$ (4 products, as shown by tracing the 1-paths in Figure 2 (a)), while a minimal PSDKRO form is: $f = x_1 \oplus x_2 \oplus x_3$ (3 products, as shown by tracing the 1-paths in Figure 2 (c)). For an n -variable parity function the minimal SOP form requires 2^{n-1} products, while the minimal PSDKRO requires n products.

3. PSDKRO MINIMIZATION

For the implementation of the algorithm we used *Ordered Binary Decision Diagrams* (OBDDs) [2]. The starting point of our algorithm is the OBDD representation of the function that is to be minimized. A 1-path represents a product term with the cost 1.

Starting from the root of the OBDD the graph is recursively traversed towards the terminals $\{0, 1\}$. At each node an EXOR operation is carried out, which by the use of OBDDs can be performed in polynomial time [2]. From the fact that for each of the decomposition formulae from Section 2 only two out of the three possible successors f_0 , f_1 and f_2 are needed to represent the function, we can choose the two least costly in order to minimize the resulting PSDKRO¹.

For each node f the minimal number of product terms needed for the representation as a PSDKRO is stored in the variable $f.prod$. Thus, each node has to be evaluated only once.

Example 2 Consider the algorithm in Figure 1 (without the “prune” related code). Assume that $p_0 = 2$, $p_1 = 2$ and

¹The basic idea of the algorithm is the same as used in [7] for PSDKRO minimization.

Figure 2: Parity function

Figure 3: Pruning of search space

$p_2 = 1$ (lines 4, 6, and 10). The cost of S , pD and nD decompositions are $p_0 + p_1 = 4$, $p_0 + p_2 = 3$ and $p_1 + p_2 = 3$ accordingly. Line 11 computes the least cost 3. Figure 2 (c) shows a PSDKRO DD for the parity function $f(x_1, x_2, x_3)$ using only pD decompositions. This example relates to the decomposition for x_1 .

3.1. An Enhanced Minimization Algorithm

Previous algorithms exhaustively traverse the search space to find optimal decompositions, [7, 3]. We make the following critical observations: The OBDD representation holds an initial (non-optimal) PSDKRO solution having only Shannon decompositions, $(\bar{x}f_0 \oplus xf_1)$ with the cost $p_0 + p_1$. f_2 is a part of the solution iff $p_2 < \max(p_0, p_1)$, i.e., f_2 must be less costly than at least one other cofactor f_0 or f_1 (see Figure 3 (a)). Given a cost limit “prune”, f_2 is a part of the solution iff $p_2 < \text{prune} - \min(p_0, p_1)$, since the cost of a Davio decomposition is $\min(p_0, p_1) + p_2$, which in turn must be less than “prune” (see Figure 3 (b)). Our hands on experiences show that a high cost for f_0 or f_1 in many cases also leads to a high cost for f_2 .

These observations are exploited in Figure 1, lines 5, 7, 9, 10 and the additional pruning parameter. In the following three pruning methods are considered.

P0 By returning *MAXINT* whenever the cost exceeds the cost limit, further “pointless” traversal is prevented, line 7. The pruning function “pf”, line 10, chooses the minimum of (a) and (b) as the cost limit for f_2 , i.e., $\min(\max(p_0, p_1), \text{prune} - \min(p_0, p_1))$.

P1 By the choice of a more aggressive pruning function the algorithm can be tuned to tradeoff quality for computational resources. By choosing $\min(p_0, p_1)$ for condition (a), computation of f_2 is aborted as soon as p_2 equals either p_0 or p_1 . This realizes the idea to traverse f_2 only where Davio decomposition is clearly favorable.

P2 Intrigued by the last observation, we sought an answer whether it might be sufficient to inspect only one of the cofactors to estimate the cost. This question is manifested in line 5, which aborts traversal whenever p_0 reaches the cost limit. In the following *P2*, is used with the exact pruning function from *P0*.

Furthermore as a last effort to reduce the complexity, line 9 computes the cost of a Shannon decomposition, (which is inherent in the OBDD representation as observed above), whenever computational resources are exhausted.

The motivation to apply pruning and heuristic approaches is to save CPU and memory resources, thus enable us to tackle problems for which previous attempts fail due to time or memory limits.

3.2. Literal Minimization

Simultaneous minimization of literal count is straightforward. Whenever we can choose between decomposition types, (all producing the minimal number of products), we choose the decomposition producing the least number of literals, i.e., $f.lit = \min(l_S, l_{pD}, l_{nD})$ corresponding to the literal count of S, pD and nD decompositions, respectively. Further implementation details are left out because of page limitations.

3.3. Complemented Edges

The use of Complemented Edges (CEs) in OBDD packages [1] has several advantages; firstly the number of nodes needed to represent the function is often reduced due to increased overlapping, and secondly further controlling cases for computation can be utilized. The presented algorithm depends heavily on the efficiency of EXOR operations. By the use of complemented edges the result of $f_0 \oplus f_1$ can be directly derived whenever $f_0 = \bar{f}_1$. In Figure 2 all EXOR computations leading to the minimized PSDKRO (c), can be derived in constant time using CEs (b).

4. INFLUENCE OF VARIABLE ORDERING

In [7] the effect of variable ordering was concluded by enumerating all possible orderings for a given function and minimizing the corresponding PSDKROs. However, no method to approach the ordering problem was reported. In general dynamic variable ordering [6] has proven useful to many DD problems and has been applied in [4] to PSDKRO based Maitra term minimization. In this paper we adopt dynamic variable ordering directly to the problem of PSDKRO minimization and present two different approaches.

R1 Move to top strategy. First we compute the cost using the initial ordering of the input variables, e.g., $(x_1, x_2, x_3, \dots, x_n)$, where x_1 is the top variable. Then we try all other variables as the top variable and store the best ordering obtained, e.g., $(x_3, x_1, x_2, \dots, x_n)$. We now repeat the procedure, e.g., with x_3 as the new initial top variable, until no further optimization is obtained or a time limit is reached.

R2 Local variable exchange method. Basically the “sifting” algorithm presented in [6]. Unfortunately the cost cannot be derived from local operations, complete PSDKRO minimization is required for each variable order applied.

5. MULTIPLE OUTPUT FUNCTIONS

In [7] the algorithm for minimizing the PSDKRO for Multiple Output (MO) functions is based on Multi-Place DDs, for which each terminal corresponds to an output vector. However, as used e.g., in [8], the terminals can be encoded by a set of Output Selection (OS) variables, (one variable-node for each output function). This encoding gives us the freedom to change the grouping of output functions by the position of the OS nodes. Moving all OS nodes to the top of the DD allows applying decompositions with respect to each output function separately, while keeping the OS nodes at the bottom of the DD corresponds to the representation used in [7]. In the latter case, decompositions are applied simultaneously to all output functions (analogous to a multi-valued interpretation). Besides these two extremes, OS nodes can be interleaved with the DD nodes. The introduction of output selection variables increases the number of 1-paths (i.e., the cost function), in turn leading to an over-estimation of the number of required products. To correctly account for product sharing of MO functions is to the best of our knowledge trivial only when the OS nodes is at the bottom of the diagram.

6. EXPERIMENTAL RESULTS

In this section we present experimental results for a set of MCNC PLA benchmark functions performed on a *Sun Ultra 1* workstation with 256Mb RAM. (All totally symmetric functions are marked by *.) For the implementation we used the CUDD 2.1.2 BDD-package [9]. The peak memory requirement of the performed experiments was measured to 75Mb. Dynamic reordering methods are implemented by synthesis operations, as internal BDD cost functions do not apply.

In a first set of experiments (see Table 1) we show the effect of pruning and heuristic minimization under the variable ordering given by the PLAs. The results obtained by re-implementing the algorithm for MO PSDKRO minimization from [3] are marked *MO*. Columns *P0* and *P1* show the results for the exact and aggressive pruning functions,

name	in/out	Number of Products			CPU Time in Seconds			
		MO/PO	P1	P2	MO	P0	P1	P2
5xp1	7/10	47	51	47	0.01	0.01	0.01	0.01
add6	12/ 7	132	133	132	0.03	0.03	0.03	0.03
bc0	26/11	180	221	180	5.34	2.79	0.81	1.34
co14*	14/ 1	14	14	14	0.01	0.01	0.01	0.01
duke2	22/29	108	157	108	25.49	23.88	9.19	17.09
in2	19/10	117	128	117	2.59	2.49	1.13	2.00
in7	26/10	42	82	42	0.17	0.17	0.16	0.16
inc	7/ 9	31	33	31	0.01	0.01	0.01	0.01
intb	15/ 7	500	1153	500	0.75	0.72	0.59	0.69
misex3	14/14	754	1199	754	5.11	4.29	2.92	3.94
rd53*	5/ 3	20	27	20	0.01	0.01	0.01	0.01
rd73*	7/ 3	63	79	63	0.01	0.01	0.01	0.01
rd84*	8/ 4	107	134	107	0.01	0.01	0.01	0.01
sao2	10/ 4	41	63	41	0.01	0.01	0.01	0.01
t481	16/ 1	13	17	13	0.01	0.01	0.01	0.01
tial	14/ 8	939	1231	939	1.30	1.27	1.21	1.18
vg2	25/ 8	293	401	295	1.54	1.54	1.45	1.38
x6dn	39/ 5	104	161	104	5.90	1.68	0.44	0.38

Table 1: Pruning experiments

name	in/out	Number of Products			CPU Time in Seconds		
		R1	R2	SO	R1	R2	SO
5xp1	7/10	43	42	51	0.05	0.13	0.01
add6	12/ 7	132	132	132	0.32	1.97	0.03
bc0	26/11	175	174	299	43.93	289.58	0.51
co14*	14/ 1	-	-	14	-	-	0.01
duke2	22/29	91	85	153	920.43	3949.75	0.08
in2	19/10	117	117	138	18.83	139.98	0.09
in7	26/10	36	37	51	2.06	34.66	0.02
inc	7/ 9	31	30	47	0.03	0.11	0.00
intb	15/ 7	402	401	390	33.98	62.89	0.21
misex3	14/14	686	638	929	71.89	478.56	0.39
rd53*	5/ 3	-	-	20	-	-	0.01
rd73*	7/ 3	-	-	55	-	-	0.01
rd84*	8/ 4	-	-	90	-	-	0.01
sao2	10/ 4	33	33	48	0.10	0.30	0.01
t481	16/ 1	13	13	13	0.02	0.25	0.01
tial	14/ 8	728	820	752	39.99	104.72	0.36
vg2	25/ 8	250	216	233	75.25	324.79	0.21
x6dn	39/ 5	101	98	118	29.61	196.35	0.14

Table 2: Ordering experiments

respectively. The latter is clearly more efficient while compromising the quality. Under the assumption that inspecting a single cofactor is enough for cost estimation $P2$ efficiently obtains optimal results for the given set of benchmarks except for the near optimal result for $vg2$.

In the second set of experiments the results for different variable reordering methods are compared (see Table 2). Columns $R1$ and $R2$ show the effect of dynamic reordering methods applied to the MO minimization method. To speed up the cost calculation heuristic $P2$ is used. In general better results are obtained from the time consuming sifting method $R2$. The results show significant improvements to many previously reported results MO in Table 1. Columns SO show the results from Single Output (SO) minimization, (corresponding to OS nodes at top, Section 5), using the PLA variable orderings. During the minimization the number of products required is overestimated, thus leading to non-optimal solutions. However, common products are counted only once in the reported results. Compared to MO in Table 1, SO minimization shows quality improvements in many cases. For some benchmarks (e.g., $intb$, $tial$ and $vg2$) SO even compares to results obtained by variable reordering. Furthermore, for the benchmark set SO minimization consumes significantly less computational resources than MO minimization.

Based on these promising results it is focus of current work to integrate the PSDKRO minimizer in an EXOR based synthesis tool. Another open problem is the optimization of incompletely specified functions. First results can be found in [5].

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