When Are the MLSD Respectively the Matched Filter Receiver *Optimal* with Respect to the BER ?

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Abstract—We reconsider the minimum/optimal bit-error probability receiver (OBER) for intersymbol interference channels with Gaussian noise and the reception of finite blocks of bits. We view the OBER as a function with two inputs: the received sequence and an expected signal-to-noise ratio; and one output: the estimated block of bits. Assuming that all sequences are equally probable to be transmitted we prove two results about the behaviour of the OBER. We show that the OBER coincides with the maximum likelihood sequence detector when designed for high signal-to-noise ratios and that it collapses to a matched filter followed by a hard-limiting device for low expected signal-to-noise ratios.

I. A BLOCK TRANSMISSION SYSTEM MODEL

After the introduction of the Viterbi detector as a *Maximum Likelihood Sequence Detector (MLSD)* [3], the optimal, or minimum, bit-error probability receiver (*OBER*) [1], [2], [4] for intersymbol interference (ISI) channels has not been given much attention as a practical receiver. We reconsider the OBER for *block transmission systems*, to gain insight to its properties and its relation to the MLSD.

Consider the transmission of blocks of binary data through a channel with known ISI and additive Gaussian noise at the receiver. Let the vector $\mathbf{b} \in \{-1, +1\}^N$ denote the block of independent bits to be transmitted. We represent the transmission system in matrix notation as:

$$\mathbf{y} = \mathbf{H}\mathbf{b} + \mathbf{n}, \quad (1)$$

where $\mathbf{H}$ is a deterministic and known matrix representing the ISI, the noise $\mathbf{n} \in N(0, \sigma_n^2 \mathbf{I})$ and $\mathbf{y}$ is the $(N + L) \times 1$ stochastic vector observed by the receiver. Further, let $\boldsymbol{\eta}$ denote the outcome of $\mathbf{y}$.

II. THE OPTIMAL BIT-ERROR PROBABILITY RECEIVER

Let us consider the detection of $(\mathbf{b})_{\text{bit } k}$. A geometric interpretation of this binary hypothesis testing is that of choosing the correct halfcube:

$$H_0 : \mathbf{b} \in \mathcal{C}_k^+, \quad H_1 : \mathbf{b} \in \mathcal{C}_k^-, \quad (2)$$

where $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are the halfcubes with $(\mathbf{b})_{\text{bit } k} = +1$ and $(\mathbf{b})_{\text{bit } k} = -1$, respectively. The Bayes decision rule minimizing the probability of detection error is given by [1], [2], [4]

$$\Lambda_k(\mathbf{y}) \triangleq \frac{f_{\mathbf{y}|H_1}(\boldsymbol{\eta}|H_1)}{f_{\mathbf{y}|H_0}(\boldsymbol{\eta}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\Pr\{H_0\}}{\Pr\{H_1\}}, \quad (3)$$

where $\Pr\{H_1\} = 1 - \Pr\{H_0\}$ is the *a priori* probability that H_1 is true, and $f_{\mathbf{y}|H_0}(\cdot)$ and $f_{\mathbf{y}|H_1}(\cdot)$ designate the probability density functions for $\mathbf{y}$ given H_0 and H_1 , respectively.

Proposition 1: Let $\psi(\mathbf{y}|\boldsymbol{\beta})$ denote the conditional density of $\mathbf{y}$ given that the sequence $\boldsymbol{\beta}$ was transmitted (here multi-dimensional Gaussian). Furthermore, let

$$\hat{\mathbf{b}}_{\text{OBER}}(\mathbf{y}) \triangleq \Gamma(\mathbf{y}, \sigma_n) \triangleq \text{sign} \left(\sum_{\boldsymbol{\beta} \in \mathcal{C}_k^+} w(\mathbf{y}, \boldsymbol{\beta}) \boldsymbol{\beta} \right) \quad (4)$$

and $w(\mathbf{y}, \boldsymbol{\beta}) = \psi(\mathbf{y}|\boldsymbol{\beta}) \Pr\{\mathbf{b} = \boldsymbol{\beta}\} - \psi(\mathbf{y} | -\boldsymbol{\beta}) \Pr\{\mathbf{b} = -\boldsymbol{\beta}\}$, where $\Pr\{\mathbf{b} = \boldsymbol{\beta}\}$ is the probability for the sequence $\boldsymbol{\beta} \in \mathcal{C}$ being transmitted. Then $\hat{\mathbf{b}}_{\text{OBER}}(\mathbf{y})$ is the detector of the transmitted bits that minimizes the bit-error probability. ■

Note that (4) represents a parallel block processor structure, *simultaneously* detecting all the individual bits.

As indicated by (4), we find it instructive to view the OBER as a function $\Gamma(\mathbf{y}, \alpha)$ with two inputs, $\mathbf{y}$ and α . The parameter α^2 is the variance the OBER is designed for, and controls the decision regions in $\mathbb{R}^{N+L}$ where $\mathbf{y}$ takes its values. Thus, the OBER depends on the expected SNR and is only optimal when the expected variance α^2 and the true variance σ_n^2 agree.

III. THE BEHAVIOUR OF THE OBER

We will discuss asymptotical properties of the OBER by studying the function $\Gamma(\cdot, \cdot)$ as defined in (4). Assuming that all sequences are equally probable to be transmitted we show that

$$\lim_{\alpha \rightarrow 0} \Gamma(\mathbf{x}, \alpha) = \hat{\mathbf{b}}_{\text{MLSD}}(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{N+L}, \quad (5)$$

and that

$$\lim_{\alpha \rightarrow \infty} \Gamma(\mathbf{x}, \alpha) = \text{sign}(\mathbf{H}^T \mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{N+L}. \quad (6)$$

Equation (5) means that the OBER designed for a high SNR *becomes* the MLSD. It is because of this is that the MLSD will achieve the minimum attainable bit-error probability when used in systems with a high SNR, *cf.* [3]. In equation (6) we find a similar comparison for low SNR between the OBER and the matched filter with hard decisions. If the true SNR is low, the best possible receiver is actually the matched filter receiver as comes to the BER.

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