Simultaneous Time of Flight and Channel Estimation Using a Stochastic Channel Model

Håkan Eriksson[1,2] Per Ödling[1] Per Ola Börjesson[1]

[1] Luleå University of Technology
Div. of Signal Processing
S-951 87 Luleå

[2] University of Karlskrona/Ronneby
Dep. of Signal Processing
S-372 25 Ronneby

Abstract

In this paper we address the problem of estimating the Time-of-Flight of a transmitted signal when the shape of the received waveform is stochastic. Specifically, we examine the case when the transmission system model is stochastic, linear and time discrete, with additive Gaussian noise, and where the transmitted waveform is known to the receiver. The joint estimation is couched in terms of Maximum a Posteriori (MAP) and Maximum Likelihood estimation. When deriving the MAP estimator we assume a priori knowledge of the probability density of the transmission system impulse response. The MAP estimator is then compared to estimators derived using less a priori information and lower order system models.

The ordinary correlation based Time-of-Flight estimator assumes knowledge of the received waveform, that is has a one-dimensional transmission system model. This investigation indicates that a more complex model structure is worthwhile when distortion in excess of low additive noise is present.

1 Introduction

A good example of *Time-of-Flight* estimation is the pulse-echo method when used to measure an unknown distance. Idealized, these applications utilize the assumption that the distance to be measured is proportional to the time it takes for the waveform to travel the path to be measured, the *Time-of-Flight (TOF)*. Various methods have been proposed for estimating the *TOF*. Most methods assume that the received waveform is known, except for a finite set of undetermined parameters. If the *TOF* only corresponds to a time delay of the received waveform, classical delay estimation can be performed using *e.g.* the criteria of *Maximum Likelihood (ML)* [7], *Minimum Mean-Square Error (MMSE)* [3] or *Maximum a Posteriori (MAP)* [2].

When no parameter-dependent signal model is known to be appropriate, the choice of estimation method is more intricate. In the literature on ultrasound, several papers [1, 4, 5, 8] have been concerned with this problem.

Our interest is in methods for modelling the uncertainty in the shape of the received waveform, where the amount of *a priori information* used is a critical issue. In this paper we model the uncertainty in the received signal waveform as an uncertainty in the impulse response of a time discrete linear system, and as an effect of additive noise.

By using a Gaussian assumption on the system statistics, we derive a *MAP* estimator of the *TOF*. A class of estimators using less *a priori* information about the second order statistics, and a system model of lower order, is also derived. These estimators are derived in a matrix formalism using an orthonormal basis for the system model, given by *Singular Value Decomposition*. If the transmitted waveform is narrow band, the ordered singular values produced by the *SVD* will rapidly approach zero, thereby encouraging the model order truncation.

The contents of this paper is as follows. Section 2 presents a stochastic model for the transmission system. In section 3 estimation algorithms for the *TOF* are derived. A numerical example using a narrow band transmitted waveform is given in section 4. Finally, section 5 comments on the result.

2 Model

The received waveform is modelled as a filtered and delayed version of a known waveform, and corrupted by additive noise, see Figure 1.

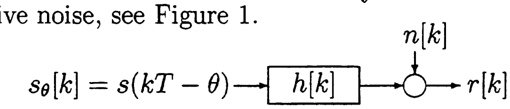

Figure 1: The signal model.

The transmitted waveform $s(\cdot)$ is assumed to be bandlimited in the frequency domain and sampled with the sample rate $\frac{1}{T}$, greater than the Nyquist rate. The parameter θ denotes the *TOF*, and is independent of the system impulse response $h[k]$. The impulse response $h[k]$ represents a stochastic linear system which is time-invariant during the time the waveform passes through it. The uncertainty in the received waveform is contained in the impulse response $h[k]$, the *TOF* θ, and in the measurement noise $n[k]$. The range of the parameter θ is restricted to Ω_θ, usually a continuous interval.

Assume that the system $h[k]$ is causal with length L, and that $s(kT-\theta)=0$ when $k<0$ and $\theta \in \Omega_\theta$. The signal model can then be described in a matrix formalism as:

$$\mathbf{r} = \mathbf{S}_\theta \mathbf{h} + \mathbf{n}, \tag{1}$$

where $\mathbf{r} = [r[0], \ldots, r[M-1]]^T$, $\mathbf{h} = [h[0], \ldots, h[L-1]]^T$, $\mathbf{n} = [n[0], \ldots, n[M-1]]^T$ and

$$\mathbf{S}_\theta = \begin{bmatrix} s_\theta[0] & 0 & \ldots & 0 \\ s_\theta[1] & s_\theta[0] & \ldots & \vdots \\ & & \ddots & 0 \\ \vdots & \vdots & & s_\theta[0] \\ & & & \vdots \\ s_\theta[M-1] & s_\theta[M-2] & \ldots & s_\theta[M-L-2] \end{bmatrix}. \tag{2}$$

The covariance matrix of $\mathbf{h}$, $\mathbf{R} = \mathrm{E}\{(\mathbf{h}-\mathbf{h}_0)(\mathbf{h}-\mathbf{h}_0)^T\}$, and $\mathrm{E}\{\mathbf{h}\} = \mathbf{h}_0$ are *a priori* known. The matrix $\mathbf{R}$ is symmetric by definition, and we assume that $\mathbf{R}$ is positive definite such that the factorization $\mathbf{R}^{-\frac{1}{2}}$ and $\mathbf{R}^{\frac{1}{2}}$ exists, with $\mathbf{R}^{\frac{1}{2}}\mathbf{R}^{\frac{1}{2}} = \mathbf{R}$ and $\mathbf{R}^{\frac{1}{2}}\mathbf{R}^{-\frac{1}{2}} = \mathbf{I}$.

We would like to have a system description in terms of a vector of uncorrelated stochastic variables and an orthonormal basis. This can be achieved with the *Singular Value Decomposition (SVD)* of $\mathbf{S}_\theta \mathbf{R}^{\frac{1}{2}} = \mathbf{U}_\theta \mathbf{\Sigma}_\theta \mathbf{V}_\theta^T$ as

$$\mathbf{r} - \mathbf{n} = \mathbf{S}_\theta \mathbf{h} = \mathbf{S}_\theta \mathbf{R}^{\frac{1}{2}} \mathbf{R}^{-\frac{1}{2}} \mathbf{h} = \mathbf{U}_\theta \mathbf{\Sigma}_\theta \mathbf{V}_\theta^T \mathbf{R}^{-\frac{1}{2}} \mathbf{h} = \mathbf{U}_\theta \mathbf{x}, \tag{3}$$

where $\mathbf{U}_\theta$ is an M-by-L matrix. If the range of $\mathbf{S}_\theta$ is a space of dimension γ, we know from the theory of *SVD*, see *e.g.* [6], that the first γ columns of $\mathbf{U}_\theta$ span the same space.

Observe that $\mathbf{R}_\mathbf{x} = \mathrm{E}\{(\mathbf{x}-\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T|\theta\} = \mathbf{\Sigma}_\theta^2$, and $\mathbf{x}_0 = E\{\mathbf{x}|\theta\} = \mathbf{\Sigma}_\theta \mathbf{V}_\theta^T \mathbf{R}^{-\frac{1}{2}} \mathbf{h}_0$ in general is dependent of θ.

3 Time-of-Flight estimation

The estimation problem is to estimate the parameter θ as well as possible by using the observations $\{r[k] \mid k = 0, \ldots, M-1\}$. Assume that $\mathbf{n} \in \mathrm{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$, and that $\mathbf{h} \in \mathrm{N}(\mathbf{h}_0, \mathbf{R})$. If perfect knowledge of the system statistics is available, we can derive the *Maximum a Posteriori (MAP)* estimator of θ as

$$\hat{\theta}(\mathbf{r}) = \arg_\theta \max_{\mathbf{x}, \theta \in \Omega_\theta} \{L(\theta, \mathbf{x})\}, \tag{4}$$

where

$$L(\theta, \mathbf{x}) \triangleq -\frac{||\mathbf{r} - \mathbf{U}_\theta \mathbf{x}||^2}{2\sigma_n^2} - (\mathbf{x}-\mathbf{x}_0)^T \frac{\mathbf{R}_\mathbf{x}^{-1}}{2} (\mathbf{x}-\mathbf{x}_0) \tag{5}$$

is the logarithm of the *a posteriori* density function [7]. To find the maximum of $L(\theta, \mathbf{x})$ with respect to $\mathbf{x}$, $\frac{\partial L(\theta,\mathbf{x})}{\partial \mathbf{x}} = 0$ is solved for a fixed θ. This yields

$$\hat{\mathbf{x}}(\theta) = \left(\mathbf{I} + \sigma_n^2 \mathbf{\Sigma}_\theta^{-2}\right)^{-1} \left(\mathbf{U}_\theta^T \mathbf{r} + \sigma_n^2 \mathbf{\Sigma}_\theta^{-2} \mathbf{x}_0\right). \tag{6}$$

Inserting (5) and (6) into (4), we get

$$\hat{\theta}(\mathbf{r}) = \arg_\theta \max_{\theta \in \Omega_\theta} \{L(\theta, \hat{\mathbf{x}}(\theta))\} \tag{7}$$

and the minimization in equation (4) is reduced to a one-dimensional optimization.

Define $\mathbf{p}(\theta) \triangleq \mathbf{y}(\theta) + \sigma_n^2 \mathbf{\Sigma}_\theta^{-1} \mathbf{V}^T \mathbf{R}^{-\frac{1}{2}} \mathbf{h}_0$, where $\mathbf{y}(\theta) \triangleq U_\theta^T \mathbf{r}$, which can be interpreted as the output of a filter bank. Then

$$L'(\theta) = L(\theta, \hat{\mathbf{x}}(\theta)) = \mathbf{p}^T(\theta) \left(\mathbf{I} + \sigma_n^2 \mathbf{\Sigma}_\theta^{-2}\right)^{-1} \mathbf{p}(\theta) \tag{8}$$

where we have omitted additive constants, *c.f.* [7, page 364, Figure 4.74]. Solutions for other *a priori* density functions of $\mathbf{x}$ can be found in the literature, *e.g.* [2]. Several observations can now be made about $L'(\theta)$. If all diagonal elements of $\mathbf{\Sigma}_\theta \to \infty$, *i.e.* that the only information contained in the model is a space of possible impulse responses, equation (8) becomes $L'(\theta) = \mathbf{y}^T(\theta)\mathbf{y}(\theta)$. The estimation problem has then been reduced to determining when the maximum waveform energy in this space arrived. If all diagonal elements of $\mathbf{\Sigma}_\theta \to 0$, there is no uncertainty in the received waveform and $L'(\theta) = \mathbf{r}^T (\mathbf{U}_\theta \mathbf{x}_0)$, resulting in the matched filter receiver.

In an implementation of an estimator according to (7) and (8), the function $\mathbf{y}(\theta)$ has to be calculated for all $\theta \in \Omega_\theta$. This can be done by $(u_i * r)[k] = \{\mathbf{y}((k - M + 1)T - \theta_0)\}_i$, where $*$ denotes time-discrete convolution, and the filter u_i is chosen as the reversed *i:th* column of $\mathbf{U}_{\theta_0}$. Here M must be sufficiently large for $s(kT - \theta_0)$ to have negligible energy outside $k = \{0, 1, \ldots, M-1\}$, compared to the variance of the measurement noise $\mathbf{n}$. Given that $s(\cdot)$ is bandlimited, $\mathbf{y}(\theta)$ is also bandlimited and can be obtained from $\mathbf{y}((k - M + 1)T - \theta_0)$ by interpolation. Furthermore, with the above choice of M, the dependence of $\mathbf{x}_0$ and $\mathbf{\Sigma}_\theta$, evaluated for $\theta = (k - M + 1)T - \theta_0 \in \Omega_\theta$, on k will be weak. This substantially simplifies an implementation, enabling the receiver to be of a correlator - matched filter type.

The ordered singular values λ_i, *i.e.* the diagonal elements of the matrix $\mathbf{\Sigma}_\theta$, are monotonically decreasing. If $s(\cdot)$ is narrow band many of the singular values will be close to zero, and the major part of the expected energy of $\mathbf{h}$ will be contained in the first singular values.

In the absence of perfect *a priori* knowledge of the system statistics an approximation must be made. Here we assume that the choice of signal basis is given, but that we have no knowledge of the singular values. (A discussion of the sensitivity for the choice of signal base is not within the scope of the present paper.) We propose to approximate the singular values as

$$\lambda_i := \begin{cases} C, & i = 1, 2, \ldots, p \leq L \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

thus, to consider only the p most important directions of variation in $\mathbf{h}$. This is then essentially a choice of the model order p. This estimator corresponds to the *Maximum Likelihood* estimator for the case when the true system is of order p, *i.e.* when $L = p$.

4 Estimation example

To illustrate the estimation method presented in Section 3, and to compare the performance of the *MAP*-estimator with the approximated structure suggested in (9), a series of simulations was performed. The transmitted signal was $s(t) = a(t)\cos(2\pi f_0 t)$. The envelope $a(t)$ was bandlimited with Fourier transform $A(f) = 0.5 + 0.5\cos\frac{\pi f}{W}$ when $|f| \leq W$, and zero otherwise. The measurement noise $\mathbf{n} \in \mathrm{N}(\mathbf{0}, \sigma_n^2 \mathbf{I})$ and the *Signal to Noise Ratio (SNR)* was defined as $\mathrm{SNR} \triangleq \mathbf{x}^T\mathbf{x}/\sigma_n^2$, making it a function of a stochastic variable, and thus stochastic itself. For purposes of comparison, the SNR has been normalized to the given values in these simulations.

Now, the estimators were simulated for $f_0 = 1/10$, $M = 400$, $\theta_0 = \theta = 200$, and with a sampling rate $f_s = \frac{1}{T} = 1$. The system statistics was $\mathbf{h} \in \mathrm{N}(\mathbf{0}, \mathbf{R})$, with the covariance as $\mathbf{R} = \mathrm{diag}\{1, 1/2, \ldots, 1/L\}$, and L, the length of $\mathbf{h}$, equalled 10. The performances of the estimators were evaluated in terms of the sample mean squared error, calculated for $N = 5000$ realizations. The results are presented in Figure 2.

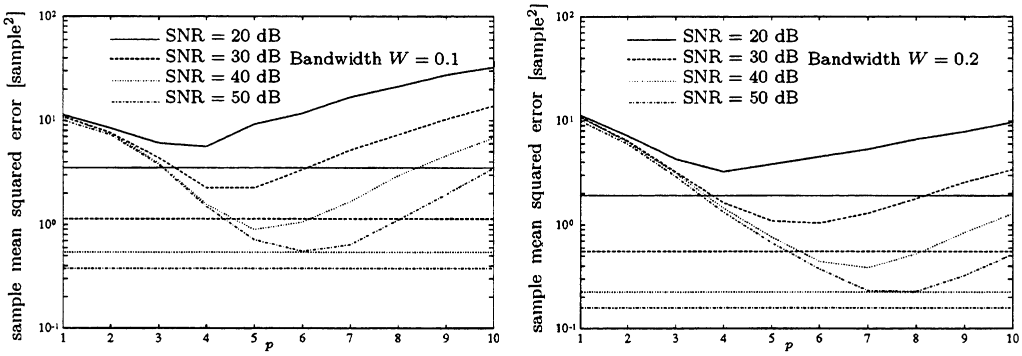

Figure 2: The graphs has the sample mean squared error, $1/N\Sigma_{n=1}^{N}(\theta - \hat{\theta}(\mathbf{r}_n))^2$, on the y-axis, and truncation order, p, of the approximated estimator, suggested by (7), (8) and (9), on the x-axis. There are performance plots of the approximated estimator and of the *MAP* estimator (appearing as horizontal lines) for various SNRs and bandwidths.

5 Discussion and Conclusions

Modelling uncertainties in a received waveform as a Gaussian variation of the transmission system impulse response seems to be a tractable method for *TOF* estimation, especially when the *SNR* is high.

Using a low model order p, *e.g.* assuming a known signal shape, can give large systematic errors. Increasing the model order decreases the systematic errors but amplifies the effects of the additive noise, which is illustrated by the estimation example in section 4.

The optimum choice of model order depends on the SNR and on the bandwidth of the transmitted waveform, but the optimum is most often flat.

The performances of these algorithms will be evaluated using an ultrasonic A-scan pulse-echo measurement system.

References

[1] B. Barshan and R. Kuc. Active sonar for obstacle localization using envelope shape information. In *Proc.1991 Int. conf. on Acoustics, Speech and Signal Processing*, pages 1273–1276, Toronto, May 1991.

[2] Per Ola Börjesson, Olle Pahlm, Leif Sörnmo, and Mats-Erik Nygårds. Adaptive QRS detection based on maximum a posteriori estimation. *IEEE Transactions on Biomedical Engineering*, 29(5):341–351, 1982.

[3] J. N. Bradley and R. L. Kirlin. Delay estimation by expected value. *IEEE Trans. on Acoustics, Speech, and Signal Processing*, 32(1):19–27, 1984.

[4] Håkan Eriksson, Per Ola Börjesson, and Nils-Gunnar Holmer. Modelling of waveform deformation and time-of-flight estimation, with application to the ultrasonic pulse-echo method. In *Proceedings of the third int. symposium on signal processing and its applications*, pages 188–191, Gold Coast, Australia, August 1992.

[5] M. Ragozzino. Analysis of the error in measurement of ultrasound speed in tissue due to waveform deformation by frequency-dependent attenuation. *Ultrasonics*, 12:135–138, May 1981.

[6] Gilbert Strang. *Linear Algebra and its Applications.* Harcourt Brace Jovanovich, Inc., San Diego, 1986.

[7] Harry L. van Trees. *Detection, Estimation, and Modulation Theory.* Wiley, New York, 1968.

[8] Vim A. Verhoef, Marius J. T. M. Clostermans, and Johan M. Thijssen. Diffraction and dispersion effects on the estimation of ultrasound attenuation and velocity in biological tissues. *IEEE Transactions on Biomedical Engineering*, 32(7):521–529, 1985.