Research Report

Modelling of Damage and its Use in Assessment of a Prestressed Bridge

**Zheng Huang^{1,2} Niklas Grip¹ Natalia Sabourova¹
Niklas Bagge¹ Yongming Tu^{1,2} and Lennart Elfgren¹**

¹ Luleå University of Technology, SE-971 87 Luleå, Sweden,
Niklas.Bagge@ltu.se, Lennart.Elfgren@ltu.se, Niklas.Grip@ltu.se, Natalia.Sabourova@ltu.se.

² School of Civil Engineering, Southeast University, Nanjing, China,
zheng.huang@ltu.se, tuyongming@seu.edu.cn, yongming.tu@ltu.se.

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Division of Structural Engineering
Department of Civil, Environmental and Natural Resources Engineering
Luleå University of Technology, SE- 971 87 LULE

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Niklas Bagge¹, Lennart Elfgren¹, Niklas Grip^{1*},
Zheng Huang², Natalia Sabourova^{1*, **} and Yongming Tu^{2 ††}

¹ Luleå University of Technology, SE-971 87 Luleå, Sweden,

Niklas.Bagge@ltu.se, Lennart.Elfgren@ltu.se, Niklas.Grip@ltu.se, Natalia.Sabourova@ltu.se.

² School of Civil Engineering, Southeast University, Nanjing, China,

zheng.huang@ltu.se, tuyongming@seu.edu.cn, yongming.tu@ltu.se.

Abstract

A 55 years old and 121.5 m long five-span prestressed bridge has been subjected to shear failure test in Kiruna, Sweden. This in-situ test is a desirable test to validate and calibrate the existing nonlinear finite element program for predicting the shear behavior of reinforced and prestressed concrete structures.

Two 3D finite element (FE) models of the Kiruna Bridge are built in commercial software Abaqus, one using shell-elements and one using a combination of shell and beam elements. Predictions obtained from these two models are well consistent with mode shapes and eigenfrequencies computed from acceleration measurements on the bridge before and after loading it to failure. Shear-failure test of this bridge performed by Luleå University of Technology (LTU) is also simulated using the built-in *concrete damage plasticity* (CDP) model in Abaqus. The predicted load-displacement curve is in good agreement with the measurement. Verification of the CDP model is conducted at element and member level with two different damage parameter evolutions. According to the verification, it indicates the damage parameter will affect the predicted shear behavior of reinforced concrete structures and it is not reliable to adopt the CDP model to simulate the shear behavior of reinforced concrete structures based on the present research.

A long term goal is to use the measured mode shapes, eigenfrequencies and FE models for evaluating methods for damage identification. Such methods are important for maintenance of different structures, for extending their life span and for better knowledge of their load carrying capacity. We describe how so-called sparse regularization finite element method updating (FEMU) methods can be used. We then demonstrate some important properties of such methods with simulations on a Kirchhoff plate. For instance, the simulations suggest that both eigenfrequencies and mode shapes should be used for precise localization of the damage.

Keywords: Refined Shell Element Model, Shear Behavior, Concrete Damage Plasticity, Shear Failure Test, Five-Span Prestressed Concrete Bridge, FEM Updating, sparse regularization, Kirchhoff plate.

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1 Introduction

Finite element analysis of concrete structures are now widely employed for research within structural engineering. With the development of computing technology, it is even possible to use this method to assess the existing concrete bridges which are very complex and large in size [SYBM02, PG06, SPG09, PES⁺14]. Regarding the assessment of concrete bridges, it is meaningful to combine the experimental study and the finite element analysis because on one hand, the finite element model can be updated based on the measurements and on the other hand, the validated model can be used to study more detailed behavior of the bridge, which can not be obtained by the test.

Several experiments have been conducted by Luleå University of Technology on the five-span continuous prestressed bridge in Kiruna, Sweden to assess the behavior of this bridge and more importantly, to calibrate and improve the existing methods of assessment of this type of bridges [EBN⁺15, Sus08]. These experiments include: shear-failure test of the FRP (Fiber Reinforced Polymer)-strengthened girders, see Figures 1(a) and 2(a), punching shear-failure test of the bridge deck 2(b), operational modal analysis tests of the bridge before and after the failure test 2(c).

In this paper, two finite element models of the Kiruna Bridge are built. One model is based on shell-elements, which is used for simulating the shear-failure test. The other is built by combined shell and beam elements, which is aimed for damage identification of this bridge. Through modelling the test of a *reinforced concrete* (RC) panel and a RC beam we present a preliminary study on calibrating the damage parameter evolution of the built-in *concrete damage plasticity model* (CDP) in Abaqus, by which the concrete is modelled in simulating the shear-failure test.

This paper is also focusing on damage assessment using acceleration measurements and *finite element model updating* (FEMU). We investigate how certain regularization techniques can be used to give damage identification results that mimic the usually very localized (or sparse) nature of damages in real structures. We describe how such damage identification can be performed only from comparison of eigenfrequencies predicted by the FE model with eigenfrequencies measured on the real structure. Next we demonstrate on a Kirchhoff plate that one drawback with this approach is that symmetries in the structure can prevent exact assessment of the localization and severity of the damage.

The paper is organized as follows. In Section 2, we describe the Kiruna Bridge, the different tests performed on it and the developed FE models as well as the validation of the Abaqus CDP model. In Section 3, we describe some different approaches for damage identification, and, as a first step, demonstrate the use of l_1 -norm sparse regularization on a Kirchhoff plate. We summarize our conclusions and suggestions for future research in Section 4.

2 Kiruna bridge

The Kiruna Bridge was built in 1959, connecting the city center to the mine. Due to the extensive ground deformation and settlement caused by the underground mining operation, this bridge was closed in 2013 and has been demolished in September 2014.

The Kiruna Bridge was a 121.5 m five-span continuous post-tensioned prestressed bridge, see Figure 1(a). The 84.2 m long western part of the bridge was curved with the radius of 500 m, while the eastern part was straight with length of 37.3 m. The bridge had a 5% inclination in longitudinal direction and 2.5% inclination in transverse direction. The superstructure of the bridge consisted of three post-tensioned girders which were 1923 mm in height. The bridge

deck was 15.6 m wide including the edge beam. The thickness of the deck varied along the transverse direction while the height of the girders varied along the longitudinal direction. Six tendons were post-tensioned in the central and eastern segments of the bridge in each girder and four for the western segments. The profile of prestressed tendons is shown in Figure 1(c). Additionally, the girders were reinforced with longitudinal, transverse and shear re-bars. More information about the geometry and reinforcement layout of the Kiruna Bridge can be found in [BNB⁺15]. According to the design drawing, the concrete grade of the superstructure of the Kiruna Bridge was K400 ($f_{cm} = 21.4$ MPa), the steel reinforcement, denoted Ks40, had a yield stress of 410 MPa and a tensile stress of 600 MPa while the corresponding stress for prestressed tendons were 1450 MPa and 1700 MPa, respectively.

Figure 1: (a) Kiruna bridge with loading beam. (b) Adjustment device for correction of the column support position. (c) Geometry of the Kiruna bridge.

2.1 Loading to failure of the Kiruna Bridge

Several experiments have been conducted by Luleå University of Technology on the Kiruna Bridge to assess the behavior of this bridge and more importantly, to calibrate and improve

the existing methods of assessment of this type of bridges [Sus08]. These experiments include: shear-failure test of the FRP (Fiber Reinforced Polymer)-strengthened girders, see Figures 1(a) and 2(a), punching shear-failure test of the bridge deck 2(b), operational modal analysis tests of the bridge before and after the failure test 2(c).

In order to load the girders to shear failure, the flexure capacity of the girders should be strengthened. The central girder was strengthened by attaching the near surface mounted (NSM) CFRP (Carbon Fiber Reinforced Polymer) bars to the bottom while the southern girder was strengthened by CFRP laminates. The northern girder was unstrengthened. The elastic modulus and tensile strength of CFRP bars were 210 GPa and 3300 MPa, respectively. The corresponding values for the laminates were 200 GPa and 2900 MPa, respectively.

In the shear-failure test, load was applied to the mid-span of each girder in span 2-3. Firstly, all the three girders were loaded to 4 MN resulting in total 12 MN. Then, the southern girder was loaded to failure followed by loading the middle girder to failure. The failure modes of both beams were combinations of flexure and shear, including concrete crushing under the load and ultimately stirrup rupture. Extensive shear cracks can be observed at the end of the test as shown in Figure 2(a). After the test of girders, punch shear-failure test was performed on the northern part of the deck resulting in failure pattern shown in Figure 2(b). The details of the strengthening system and the test results can be found in [BNB⁺15].

(a)

(b)

(c)

Figure 2: (a) Shear failure test of the girders. (b) Punching shear failure of the deck. (c) Operational modal analysis test.

2.2 3D finite element models of the Kiruna Bridge

In order to investigate if the finite element method can be employed to precisely simulate the behavior of the Kiruna Bridge, a 3D shell-element model of the Kiruna Bridge was built using the commercial FEM software Abaqus. It is well-known, shell-elements are superior to beam-elements in simulating the nonlinear shear behavior of structures. Moreover, in general, the finite shell-element model is computationally more efficient than the solid-element model of the same structure since smaller number of such elements is required. That is why shell-elements were used in the modelling of the Kiruna Bridge.

As it was mentioned in Section 2, the geometry of the Kiruna Bridge is complex (Figure 1(c)) which affects the model eigenfrequencies and mode shapes. The ordinary steel reinforcement were modelled by smeared re-bars embedded in the shell section and the curved prestressed tendons were simulated by truss-element. As adjustment devices have been installed at the bottom of all columns which enable rotation of the column-end as shown in Figure 1(b), only the three translation degrees of freedom at the column-end are constrained while the rotational ones kept free. In order to assess various behavior of this bridge (e.g. the static behavior subjected to various loads and actions, the dynamic behavior subjected to moving load), the shell-element model was built consistent with the design drawing of the bridge both in geometry and in reinforcement layout, see Figure 3(a).

Figure 3: (a) Geometry of the 3D shell element model. (b) Geometry of the shell-beam element model. (c) Modelling of the prestressed tendons in shell-element model.

The predicted modal data, i.e. eigenfrequencies and mode shapes, of the shell-element model are in good agreement with the measurements as shown in Table 1. This justifies that the stiffness and mass of the bridge are modelled consistently with the real bridge. However, regarding the research on damage identification based on the modal data, the size of shell-element model is still too large (62413 nodes) from the computational point of view. In order to overcome this drawback, a shell-beam-element model was built without much compromise of accuracy. It has 4125 nodes which is only 7% of that of shell-element model and the corresponding predicted results also show good agreement with the measurement, see Table 1.

The shell-element model was also used to simulate the shear-failure test of the Kiruna Bridge. The concrete material was modelled by built-in Abaqus concrete damaged plasticity (CDP) model (ABAQUS 6.10, Abaqus Analysis Users Manual, Section 20.6.3 Concrete damaged plasticity) and the steel was modelled by isotropic plasticity model. The law of damage parameter evolution was defined according to that presented in [PES⁺14]. It should be noted that the material parameters were defined according to the design drawing not the test and the three girders were loaded with equal displacement in the FEM model which is different from the test. Even if there are discrepancies between the model and the real test, consistent predictions of the load-displacement behaviour of this bridge can be obtained as shown in Figure 4. However, based on these results we can not conclude that the considered Abaqus CDP model

Figure 4: Load-displacement curve of the Kiruna Bridge.

is suitable for simulation of the shear behavior of concrete structures which will be discussed in the following section.

2.3 Verification of the Abaqus CDP model

The CDP model, which was probably first introduced in [Kac99] is now used in Abaqus in the following formulation

$$\sigma = (1 - D)E^0 : (\varepsilon - \varepsilon_p) = E : (\varepsilon - \varepsilon_p), \quad (1)$$

where σ is a Cauchy stress tensor, D is a stiffness degradation, also called, damage parameter, E^0 - elastic stiffness tensor of the undamaged material, $E = (1 - D)E^0$ - elastic stiffness tensor of damaged material, ε - strain tensor, ε_p - plastic strain tensor, **introduce** :. In Abaqus this formulation requires

- stress-strain relations $\sigma_c(\varepsilon_c^{in})$ and $\sigma_t(\varepsilon_t^{cr})$ for the uniaxial material behaviors under compressive and tensile loadings, where ε_c^{in} is compressive inelastic strain and ε_t^{cr} is tensile cracking strain.
- damage parameter evolution $D = D(d_c, d_t)$ described by two independent uniaxial degradation variables d_c and d_t under compressive and tensile loadings, respectively.

The former come from the material test and the later are usually found by "trial and error".

To verify the reliability of different material models a large number of reinforced concrete panels have been tested in University of Toronto [BC87, Vec82] and University of Houston [HM10]. In these tests, the RC (reinforced concrete) panels were designed with a large variety of concrete grades and reinforcement ratios and subjected to various of combination of evenly distributed compression, tension and shear until failure. This experimental data became benchmark tests and has been widely used to calibrate and validate the material model at the element level [OM85, VLSN01, Cer85]. The accepted guideline now according to FIB proposition [MFB⁺08] is that in order to produce reliable simulation of the behavior of concrete structures, the material models of a commercial software need to be calibrated and validated using element level benchmark tests and member level benchmark tests in advance.

2.3.1 Damage parameter evolution

For the validation of the CDP model in this paper we use the following two models of the damage parameter evolution in terms of the evolution of the degradation variables d_c and d_t .

In the first, d_c^1 and d_t^1 were initially proposed in [JL05, Table 2] and then adjusted in [PES⁺14, Table 2]. The latter authors used the defined parameters to make the predicted behavior of the Övik Bridge closer to the measurements. We note here, that the degradation variables evolutions suggested in [PES⁺14] were also used for the prediction of the Kiruna Bridge behavior as it was mentioned in the previous section.

Secondly, we use the degradation variable evolutions as functions of the compressive damage parameter b_c and tensile damage parameter b_t suggested in [BM06] and utilize here the following slightly modified formulas from [BM06]

$$d_c^2(b_c) = \frac{(1 - b_c)\varepsilon_c^{in}}{(1 - b_c)\varepsilon_c^{in} + \sigma_c E_0^{-1}},$$

$$d_t^2(b_t) = \frac{(1 - b_t)\varepsilon_t^{cr}}{(1 - b_t)\varepsilon_t^{cr} + \sigma_t E_0^{-1}}.$$

Moreover, $b_c = \frac{\varepsilon_c^{pl}}{\varepsilon_c^{in}}$ and $b_t = \frac{\varepsilon_t^{pl}}{\varepsilon_t^{cr}}$, where ε_c^{pl} - compressive plastic strain and ε_t^{pl} - tensile plastic strain, while ε_c^{in} and ε_t^{cr} were explained previously. The parameters b_c and b_t can vary from 0 to 1, where 1 means no damage and 0 means total damage. In what it follows we set $b_c = b_t = 0.9$ according to the calibration at the element level, see Section 2.3.2. Figure 5 shows the comparison of these two sets of degradation variables.

Figure 5: (a) Comparison of the compressive degradation variables d_c^1 and d_c^2 . (b) Comparison of the tensile degradation variables d_t^1 and d_t^2 .

2.3.2 Material model validation at the element level

In this paper we verify the CDP model using the RC panel named B1 and defined in [PH95]. The panel is reinforced by orthogonal steel re-bars with reinforcement ratio of 1.193% and 0.596% in the longitudinal and transverse direction, respectively. This panel was subjected to pure shear to failure.

Figure 6(a) shows the comparison of the shear stress-shear strain curve between the simulation and the experiment, where the model CDP-1 is based on the damage parameter $D(d_c^1, d_t^1)$ and CDP-2 is connected to $D(d_c^2, d_t^2)$. Clearly, CDP-1 model cannot reflect the shear stiffening

effect of reinforced concrete due to tension stiffening and aggregates interlock after diagonal cracks emerge. Once cracking occurs, the shear stiffness declines sharply and the predicted stress-strain curve is inconsistent with the experimental result. On the other hand, the CDP-2 model produces more consistent predicted results. It seems that the damage parameter will affect the shear behavior of reinforced concrete which hasn't been mentioned by other researchers before.

Figure 6: (a) Shear stress-strain curve of RC panel B1. (b) Load-displacement curve of RC beam OA1.

2.3.3 Material model validation at the member level

At the member level, one RC beam without shear reinforcement tested in [BS63], denoted OA1, is simulated using model CDP-1 and CDP-2 which are described in the previous section. The details of the beam can be found in [BS63]. All material parameters of concrete are derived from cylinder compressive strength using the expression proposed by Model Code [fib13].

In the first simulation, CDP-1 is employed and the comparison of the prediction and experimental results is shown by the red line in Figure 6(b). Good agreement between the prediction and test can be found in terms of the load-displacement curve. Regarding the crack pattern at peak load, the experiment indicates an inclined crack initiating in the shear span and propagating to the top of the beam caused the final failure as shown in Figure 7(a). However, the predicted result presents a flexure crack initiating near the mid-span of the beam caused the final failure as shown in Figure 7(b) which indicates this model can't simulate the shear cracking behavior of the beam.

Why will these two contradictory conclusions be reached when it comes to simulating this beam? The authors in [SLR12] simulated the behavior of the same beam based on Euler theory which can only take the flexure deformation into account while neglecting the shear deformation. It is shown that even with this assumption the predicted load-displacement curve is still consistent with the experimental results. It indicates most of the deflection at the mid-span is caused by flexure deformation while the shear deformation can be neglected in this case. Namely, CDP-1 model can describe the flexure behavior of reinforced concrete accurately but fail to model the shear behavior (shear cracking).

In the second simulation, CDP-2 is adopted and a good agreement between the prediction and measurement can be found regarding the crack pattern as shown in Figure 7(a) and Figure 7(c). However, the predicted load-displacement curve overestimates the peak load which is illustrated as the yellow line in 6(b).

Figure 7: Crack pattern at peak-load of RC beam OA1. (a) Experiment. (b) CDP-1. (c) CDP-2.

Table 1: Comparison of eigenfrequencies f and damping ratio ξ on the undamaged bridge against the same modal data on the damaged bridge and the frequencies predicted by the FE models. We use the notation “value $\pm$ standard deviation” and $\Delta f \stackrel{\text{def}}{=} \frac{f_{\text{damaged}} - f_{\text{undamaged}}}{f_{\text{undamaged}}} \cdot 100\%$ denotes the relative change of the eigenfrequencies after loading the bridge to failure.

We also list the eigenfrequencies f_{shell} and $f_{\text{shell-beam}}$ for the FE models described in Section 2.2.

Mode nr	f_{shell} (Hz)	$f_{\text{shell-beam}}$ (Hz)	$f_{\text{undamaged}}$ (Hz)	f_{damaged} (Hz)	Δf (%)	$\xi_{\text{undamaged}}$ (%)	ξ_{damaged} (%)
1	1.8172	1.8692	1.91 ± 0.005	$1.851 \pm 0.009\,081$	-3.1	2.275 ± 0.452	$0.965\,1 \pm 0.183\,2$
2	4.2269	4.0998	4.552 ± 0.007	$4.437 \pm 0.000\,017\,86$	-2.59	1.301 ± 0.117	$0.825\,1 \pm 0.0005689$
4	4.7311	4.7456	5.032 ± 0.002	$4.928 \pm 0.000\,000\,558\,8$	-1.0	0.912 ± 0.014	$3.059 \pm 0.000\,019\,43$
7	5.4223	5.4743	5.543 ± 0.001	$5.289 \pm 0.000\,000\,652\,8$	-4.8	$1.309 \pm .029$	$2.46 \pm 0.000\,021\,5$
8	—	—	5.582 ± 0.003	$5.142 \pm 0.013\,04$	-8.6	3.277 ± 0.041	$1.268 \pm 0.223\,1$
9	5.7191	5.7837	5.867 ± 0.008	$5.74 \pm 0.000\,000\,569\,8$	-2.2	2.575 ± 0.238	$1.379 \pm 0.000\,020\,1$
10	6.9174	6.7435	7.092 ± 0.003	$6.882 \pm 0.000\,840\,7$	-3.1	2.968 ± 0.179	$2.271 \pm 0.025\,47$
12	8.2929	8.1370	8.552 ± 0.002	$7.676 \pm 0.000\,000\,619$	-11.4	1.603 ± 0.033	$3.37 \pm 0.000\,014\,87$
16	—	—	13.447 ± 0.059	12.94 ± 1.248	-3.9	1.682 ± 0.549	$0.541 \pm 0.255\,9$

2.4 Acceleration measurements and modal analysis results

Accelerometer measurements of ambient vibrations were performed in May 2014 on the undamaged bridge and twice in August 2014 on the damaged bridge.

Measurements were done with six calibrated [FGS13] Colibrys SF3000L triaxial accelerometers connected with 40–60 m long twisted pair cables to an MGC-Plus data acquisition system using AP801 cards with sample rate 800 Hz. The accelerometers were firmly attached to the bridge with expansion bolts and adjusted to the horizontal plane with three screws. Figure 2 (c) shows the 38 accelerometer locations on the bridge.

Nonlinear trending in the signals was reduced by a smooth padding of the measurements (to reduce discontinuities in the periodized signal) followed by highpass filtering. Measurements that were distorted by malfunctioning electrical power supply were excluded from the analysis. To reduce problems with low signal-to-noise ratio due to nearly no excitation from wind or traffic, we did several hours long measurements and for the damaged bridge, also tried combining two measurement occasions.

Operational modal analysis with all methods available in the software ARTeMIS 4.0 for different combinations of measurement data gave the the eigenfrequencies (f) and damping ratios (ξ) that are summarized in Table 1. We have there restricted to modes with small frequency standard deviation and realistic damping ratio that were found both in the May and August measurements. See [Gri16] for details. The measured eigenfrequencies are lower for the damaged bridge, which also is what to expect from damage theory.

Plotted mode shapes in [Gri16] show that the predicted and measured mode shapes are quite similar for vibration modes 1, 2, 10 and 12 in Table 1. These are the vibration modes that seem most useful for damage identification. For the undamaged bridge, the measured mode shapes computed by ARTeMIS are plotted in Figure 8.

Figure 8: Selected mode shapes for the undamaged bridge.

3 Structural damage identification using FEMU

There exist a lot of methods used for structural damage identification [DFPS96, DFP98, Sin09]. One of the most computationally efficient and recognized is damage detection using sensitivity-based finite element model updating. The finite element model is then initially parameterized by uncertain parameters, which are iteratively updated by some parameter estimation method, usually nonlinear least squares. When the uncertain parameters are updated the derivatives and sometimes even second-order derivatives of the modal data with respect to these parameters are used [Nat88, FM95, Lin01, TMDR02, GST16]. The corresponding matrices are often called sensitivity matrices, which is reflected in the method name. Recently it became more and more popular to use formulation of the sensitivity-based damage identification as a convex problem [BZLO13, Her14] for which there exist special efficient optimization algorithms [BV04]. Furthermore the researchers recognized that damage is a rather local phenomena and started to use sparse regularization in order to reflect this phenomena [BZLO13, Her14, ZX15, WH15]. There exist a number of free open-source Matlab optimization packages that offer all necessary tools to solve such convex sparse regularized problems [cvx, l1m]. In the following sections we briefly introduce these techniques.

Our goal is to apply sparsity together with convexity for the damage identification of the Kiruna Bridge using a SHM finite element model updating package that is developed at Luleå University of Technology and described in more detail in [Gri16]. Most attempts in this direction are applied to simulated data. We also decided to first develop a finite element model of a Kirchhoff plate and investigate the limitations and advantages of these techniques on simulated data, which is the topic of this section.

3.1 Damage parametrization

A discrete linear time-invariant model of structural motion which is used in damage identification process is described by a second-order differential equation:

$$M\ddot{\mathbf{u}}(t) + C\dot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}(t), \quad (2)$$

where the matrices M , C and K are real time-independent square system mass, damping and stiffness matrices of order $d \times d$ with d corresponding to the number of degrees of freedom of the model and $\mathbf{u}(t)$ is a time dependent displacement vector with d entries. Dots represent derivatives with respect to time t and $\mathbf{f}(t)$ is a vector of external forces. Considering the free vibration of an undamped structure, i.e. $\mathbf{f}(t) = \mathbf{0}$ and $C = 0$ and looking for the harmonic solution of Equation (2) in the form $\mathbf{u}(t) = \boldsymbol{\phi}_k e^{j\omega_k t}$ ($j = \sqrt{-1}$), we obtain the following generalized eigenvalue problem

$$(-\lambda_k M + K) \boldsymbol{\phi}_k = \mathbf{0}. \quad (3)$$

Here, $\lambda_k = \omega_k^2 = (2\pi f_k)^2$ and $\boldsymbol{\phi}_k$ are the k^{th} eigenvalue and mass-normalized eigenvector, respectively, whereas f_k is the k^{th} eigenfrequency. From Equation (3) it is easy to see that changes in system matrices M and K cause changes in the modal parameters λ_k and $\boldsymbol{\phi}_k$.

It is very popular to assume that the mass of the undamped structure does not change after the damage is introduced and to update the stiffness matrix by the substructure matrices [Nat88, FM95, Lin01] as follows

$$K(\boldsymbol{\alpha}) = K^0 - \sum_{i=1}^I \alpha_i K_i. \quad (4)$$

Here $K(\boldsymbol{\alpha})$ is the improved stiffness matrix of the parameterized model. K_i is the constant expanded order matrix for the i^{th} element or substructure (group) representing the uncertain model property and location. The resulting widely used dimensionless updating or damage parameters $\alpha_i = \frac{E_i^0 - E_i}{E_i^0}$ come naturally from the simple isotropic damage theory [LD05]. In this theory, the damage is described by a reduction in bending stiffness, as

$$D = \frac{E^0 - E}{E^0}, \quad (5)$$

where E^0 and E is the initial (undamaged) and updated (damaged) elasticity modulus, respectively, and D stands for damage parameter. The matrix K^0 in (4) is then interpreted as the matrix corresponding to the undamaged structure in the content of damage identification. The model is modified only by the updating parameters for the substructure matrices.

Thus, the finite element model is parameterized by

$$K(\boldsymbol{\alpha}) = K^0 - \sum_{i=1}^I \alpha_i K_i, \text{ where } \alpha_i = \frac{E_i^0 - E_i}{E_i^0}, \quad (6)$$

$$K(\boldsymbol{\alpha})\phi_k(\boldsymbol{\alpha}) = \lambda_k(\boldsymbol{\alpha})M\phi_k(\boldsymbol{\alpha}).$$

Clearly, a small value of α_i , or zero in the ideal case, indicates the absence of damage for a particular element or group, positive α_i corresponds to decrease and negative α_i indicates increase of the elasticity modulus for the element or group. A good damage identification method should provide positive α_i for the elements or groups containing damages and $\alpha_i \approx 0$ for the undamaged elements of groups.

Remark 1. The description of damage in terms of reduction in bending stiffness only is more suitable for the simple beam structures. In the case when also torsional components of mode shapes are involved in the measurement data, it is even more advantageous to describe damage by reduction in both bending EI and torsional stiffness GI . In the latter case, one can extend the finite element model parametrization by using similar type of dimensionless parameters as for the elasticity modulus, namely $\alpha_i^G = \frac{G_i^0 - G_i}{G_i^0}$, where G_i^0 and G_i are the torsional shear modulus for the initial and for the updated state, respectively. Thus, the mixed elasticity and shear modulus model parametrization will be

$$K(\boldsymbol{\alpha}) = K^0 - \sum_{i=1}^I \alpha_i^E K_i^E + \alpha_i^G K_i^G, \quad (7)$$

where $\alpha_i^E = \frac{E_i^0 - E_i}{E_i^0}$ and $\alpha_i^G = \frac{G_i^0 - G_i}{G_i^0}$. K_i^E and K_i^G are the nonzero parts of the element or group constant matrix K_i connected to the degrees of freedom responsible for the bending and for the torsional stiffness, respectively. For example, K_i^E is connected to $\{u_x, u_y, u_z, rot_y, rot_z\}$ and K_i^G is connected to $\{rot_x\}$ for the Kiruna Bridge model in Figure 3(b).

3.2 Convex formulation of the optimization problem

In order to solve the parameter estimation problem, we measure the difference between the measured and analytical eigenfrequencies with a *residual*

$$\mathbf{r}(\boldsymbol{\alpha}) = \mathbf{r}_\lambda(\boldsymbol{\alpha}).$$

The residual is a function $\mathbf{r} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with n corresponding to the number of updating parameters and m equal to the number of measured eigenfrequencies. Thus, here we consider structural damage identification based only on the frequency data. In fact, we define the eigenvalue residual as follows

$$(\mathbf{r}_\lambda(\boldsymbol{\alpha}))_j \stackrel{\text{def}}{=} \omega_\lambda(j) \left(1 - \frac{\lambda_j(\boldsymbol{\alpha})}{\lambda_j^{\text{mea}}} \right), \quad j = 1, \dots, m_\lambda, \quad (8)$$

where $\omega_\lambda(j)$ is the j^{th} component of the weighting vector for the eigenvalue residual, λ_j^{mea} is the j^{th} measured eigenvalue, m_λ is the length of this vector and $\boldsymbol{\alpha}$ is the damage index (parameter) vector, which will be defined in short.

At the $(k)^{\text{th}}$ iteration step the free vibration of an undamped structure will be described by (compare with (3))

$$(-\lambda_j^{(k)} M^{(k)} + K^{(k)}) \phi_j^{(k)} = 0. \quad (9)$$

Assuming that the mass remains unchanged for the updated model we have the following relations between two sequential structural properties

$$\begin{aligned} K^{(k+1)} &= K^{(k)} + \Delta K^{(k)} \\ M^{(k+1)} &= M^{(k)} = M \\ \lambda_j^{(k+1)} &= \lambda_j^{(k)} + \Delta \lambda_j^{(k)} \\ \phi_j^{(k+1)} &= \phi_j^{(k)} + \Delta \phi_j^{(k)} \end{aligned} \quad (10)$$

From the mass normalization $\phi_j^{(k)T} M \phi_j^{(k)} = I$ and (9), we have that $\phi_j^{(k)T} K^{(k)} \phi_j^{(k)} = \lambda_j^{(k)}$. Thus substitution of equations (10) into (9) ($k \rightarrow k+1$) and premultiplying it with $\frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \phi_j^{(k+1)T}$ gives

$$\begin{aligned} 0 &= \frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \phi_j^{(k+1)T} (-\lambda_j^{(k+1)} M + K^{(k+1)}) \phi_j^{(k+1)} \\ &= \frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \left(-\lambda_j^{(k+1)} + \phi_j^{(k+1)T} (K^{(k)} + \Delta K^{(k)}) \phi_j^{(k+1)} \right) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \phi_j^{(k)T} \Delta K^{(k)} \phi_j^{(k)} &= \\ &= \omega_\lambda(j) \left(\frac{\lambda_j^{(k+1)}}{\lambda_j^{\text{mea}}} - \frac{\phi_j^{(k+1)T} K^{(k)} \phi_j^{(k+1)}}{\lambda_j^{\text{mea}}} - 2 \frac{\phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}} - \frac{\Delta \phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}} \right) \\ &= \omega_\lambda(j) \left(\frac{\lambda_j^{(k+1)}}{\lambda_j^{\text{mea}}} - \frac{\lambda_j^{(k)}}{\lambda_j^{\text{mea}}} - 2 \frac{\phi_j^{(k)T} K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}} - \frac{\Delta \phi_j^{(k)T} K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}} \right) \end{aligned}$$

$$-2 \frac{\phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}} - \frac{\Delta \phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)}}{\lambda_j^{\text{mea}}}$$

For any $j = 1, \dots, m_\lambda$ we assume that $\lambda_j^{(k+1)} - \lambda_j^{(k)} \gg 2\phi_j^{(k)T} K^{(k)} \Delta \phi_j^{(k)} + \Delta \phi_j^{(k)T} K^{(k)} \Delta \phi_j^{(k)} + 2\phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)} + \Delta \phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)} = \phi_j^{(k)T} K^{(k)} \Delta \phi_j^{(k)} + \phi_j^{(k+1)T} K^{(k)} \Delta \phi_j^{(k)} + \phi_j^{(k)T} \Delta K^{(k)} \Delta \phi_j^{(k)} + \phi_j^{(k+1)T} \Delta K^{(k)} \Delta \phi_j^{(k)}$, which gives the approximation

$$\frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \phi_j^{(k)T} \Delta K^{(k)} \phi_j^{(k)} \approx \omega_\lambda(j) \left(\frac{\lambda_j^{(k+1)}}{\lambda_j^{\text{mea}}} - \frac{\lambda_j^{(k)}}{\lambda_j^{\text{mea}}} + 1 - 1 \right)$$

or using Equation (8)

$$- \frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \phi_j^{(k)T} \Delta K^{(k)} \phi_j^{(k)} \approx \mathbf{r}_\lambda^{(k+1)} - \mathbf{r}_\lambda^{(k)}. \quad (11)$$

The initial stiffness matrix or the matrix corresponding to the undamaged structure in the context of damage identification can be represented by assembling the element or group stiffness matrices as follows

$$K^{(0)} = \sum_{i=1}^I K_i^{(0)} = \sum_{i=1}^I E_i^{(0)} \left[\frac{K_i^{(0)}}{E_i^{(0)}} \right], \quad (12)$$

where $K_i^{(0)}$ is the constant expanded order matrix for the i^{th} element or substructure (group) representing the uncertain model property and location, $E_i^{(0)}$ is the corresponding element or group initial elasticity modulus.

Assume, that for any iteration step $k \in \mathbb{N}$ the updated matrix $K^{(k)}$ can be written as a linear combination of the matrices $\frac{K_i^{(0)}}{E_i^{(0)}}$. Then, we can define

$$K^{(k)} \stackrel{\text{def}}{=} \sum_{i=1}^I E_i^{(k)} \left[\frac{K_i^{(0)}}{E_i^{(0)}} \right] \stackrel{\text{def}}{=} \sum_{i=1}^I K_i^{(k)}, \quad (13)$$

where $E_i^{(k)}$ is the updated elasticity modulus for the i^{th} element or group. Consequently,

$$\begin{aligned} \Delta K^{(k)} &= K^{(k+1)} - K^{(k)} = \sum_{i=1}^I K_i^{(k+1)} - K_i^{(k)} = \sum_{i=1}^I \left(\frac{E_i^{(k+1)} - E_i^{(k)}}{E_i^{(0)}} \right) K_i^{(0)} \\ &= - \sum_{i=1}^I \left(\frac{E_i^{(0)} - E_i^{(k+1)}}{E_i^{(0)}} - \frac{E_i^{(0)} - E_i^{(k)}}{E_i^{(0)}} \right) K_i^{(0)} = - \sum_{i=1}^I \Delta \alpha_i^{(k)} K_i^{(0)}, \end{aligned} \quad (14)$$

where $\Delta \alpha_i^{(k)} = \alpha_i^{(k+1)} - \alpha_i^{(k)}$ for $i = 1, \dots, I$ and $\alpha_i^{(k)} = \frac{E_i^{(0)} - E_i^{(k)}}{E_i^{(0)}}$ are the entries of the damage index (parameter) vector $\boldsymbol{\alpha}$ (compare with (6)). Note also that Equations (12) and (13) give

$$K^{(k)} = K^0 - \sum_{i=1}^I \alpha_i^{(k)} K_i^{(0)}. \quad (15)$$

Inserting (14) into (11) we get

$$S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} = \mathbf{r}_\lambda^{(k+1)} - \mathbf{r}_\lambda^{(k)}, \quad (16)$$

where $S^{(k)} = S(\boldsymbol{\alpha}^{(k)})$ is the matrix with elements

$$S_{ji}^{(k)} = \frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \boldsymbol{\phi}_j^{(k)T} K_i^{(0)} \boldsymbol{\phi}_j^{(k)}. \quad (17)$$

On the other hand, using the following Fox-Kapoor formula [FK68] for the mass-normalized eigenvector established for Equation (9) and Equation (15) we get

$$\frac{\partial \lambda_j^{(k)}}{\partial \alpha_i^{(k)}} = \boldsymbol{\phi}_j^{(k)T} \frac{\partial K^{(k)}}{\partial \alpha_i^{(k)}} \boldsymbol{\phi}_j^{(k)} = -\boldsymbol{\phi}_j^{(k)T} K_i^{(0)} \boldsymbol{\phi}_j^{(k)}.$$

This inserted in (17) and (8) gives that

$$S_{ji}^{(k)} = -\frac{\omega_\lambda(j)}{\lambda_j^{\text{mea}}} \frac{\partial \lambda_j^{(k)}}{\partial \alpha_i^{(k)}} = \frac{\partial (\mathbf{r}_\lambda)_j^{(k)}}{\partial \alpha_i^{(k)}}.$$

The matrix S for the partial derivatives of residuals with respect to the updating parameters is also known as the sensitivity matrix.

Using Equation (16) the residual can be linearized as follows

$$\mathbf{r}_\lambda^{(k+1)} = S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} + \mathbf{r}_\lambda^{(k)}.$$

To find the updating parameters we minimize $1/2 \|\mathbf{r}(\boldsymbol{\alpha})\|_2^2$ by solving in each iteration the linearized minimization problem

$$\min_{\Delta \boldsymbol{\alpha}^{(k)} \in \mathbb{R}^n: l \leq \Delta \boldsymbol{\alpha}^{(k)} \leq u} \frac{1}{2} \|S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} + \mathbf{r}_\lambda^{(k)}\|_2^2 \quad (18)$$

for $\Delta \boldsymbol{\alpha}^{(k)}$. Then, in each iteration step the updating parameter vector is updated as $\boldsymbol{\alpha}^{(k+1)} = \boldsymbol{\alpha}^{(k)} + \Delta \boldsymbol{\alpha}^{(k)}$ (see also [WPP09, Eq. 19]).

The minimization problem (18) is a convex problem. Namely, the set $\Omega = \{\Delta \boldsymbol{\alpha}^{(k)} : S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} = -\mathbf{r}_\lambda^{(k)}\}$ is convex. In fact, at each iteration step the matrix $S^{(k)}$ is defined at the previous step and thus it is considered as being constant. Thus, for any $t \in [0, 1]$ and $\Delta \boldsymbol{\alpha}_1^{(k)}, \Delta \boldsymbol{\alpha}_2^{(k)} \in \Omega$ we have $S^{(k)}(t\Delta \boldsymbol{\alpha}_1^{(k)} + (1-t)\Delta \boldsymbol{\alpha}_2^{(k)}) = tS^{(k)}\Delta \boldsymbol{\alpha}_1^{(k)} + (1-t)S^{(k)}\Delta \boldsymbol{\alpha}_2^{(k)} = -t\mathbf{r}_\lambda^{(k)} - (1-t)\mathbf{r}_\lambda^{(k)} = -\mathbf{r}_\lambda^{(k)}$ and therefore $t\Delta \boldsymbol{\alpha}_1^{(k)} + (1-t)\Delta \boldsymbol{\alpha}_2^{(k)} \in \Omega$.

3.3 Problem regularization

In the presence of noise in the measured observations, the estimated parameters found by an iterative method (18) can have a pronounced tendency to form an oscillating pattern that makes it difficult to localize and quantify the damage [GST16, Figures 12 and 13]. A standard solution of this problem is to use a regularization technique

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^n: l \leq \boldsymbol{\alpha} \leq u} \frac{1}{2} \|S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} + \mathbf{r}_\lambda^{(k)}\|_2^2 + \lambda R(\Delta \boldsymbol{\alpha}^{(k)}), \quad (19)$$

where λ and R are the regularization parameter and the regularization function, respectively. The regularization function describes the properties of the expected solution, for example, measure of smoothness, sparsity, etc. Below we describe two regularization methods.

3.3.1 l_2 -norm regularization or Tikhonov regularization

Tikhonov or l_2 -norm regularization belongs to traditional and most used regularization method [FM95, WPP09]. It smooths the solution significantly and thus results in the solution vector full of nonzero elements [BV04].

$$\min_{\Delta \boldsymbol{\alpha}^{(k)} \in \mathbb{R}^n: \mathbf{l} \leq \Delta \boldsymbol{\alpha}^{(k)} \leq \mathbf{u}} \frac{1}{2} \|S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} + \mathbf{r}_\lambda^{(k)}\|_2^2 + \beta \|\Gamma \Delta \boldsymbol{\alpha}^{(k)}\|_2^2. \quad (20)$$

The problem (20) has a unique minimum-norm solution, which can be given in a close-form, see [Ela10]. It can be used with identity or finite difference matrix Γ .

3.3.2 Sparse regularization with l_1 -norm

The nature of the damage is quite local and sometimes is compared with mathematical δ function. So the damage is associated only with few locations on a structure and thus the damaged elements are sparse compared to all the elements used in the model of the structure. The most simple and intuitive measure of sparsity of vector $\mathbf{x}$ as a solution of the underdetermined system of linear equations $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m \times n}$ for $m < n$, is by counting the number of nonzero entries in it or using, so-called, l_0 -”norm”

$$\|\mathbf{x}\|_0 = \#\{i : x_i \neq 0\}.$$

It is not really a norm, since it does not satisfies the homogeneity property.

The l_0 -norm regularization problem belongs to the class of combinatorial problems, which are computationally difficult [Ela10]. That is why for simplicity its closest convex relaxation l_1 -norm is used in regularization instead

$$\min_{\Delta \boldsymbol{\alpha}^{(k)} \in \mathbb{R}^n: \mathbf{l} \leq \Delta \boldsymbol{\alpha}^{(k)} \leq \mathbf{u}} \|S^{(k)} \Delta \boldsymbol{\alpha}^{(k)} + \mathbf{r}_\lambda^{(k)}\|_2^2 + \beta \|\Delta \boldsymbol{\alpha}^{(k)}\|_1, \quad (21)$$

where $\|\Delta \boldsymbol{\alpha}^{(k)}\|_1$ is often called sparsifying term. Regularization with l_1 -norm leads to sparse solution with only few nonzero elements [BV04]. There are different technical sufficient conditions under which the solution of the l_1 -norm regularization coincide with the solution of the l_0 -norm regularization, and thus is guaranteed to be optimally sparse. See, for instance, [TKS13], for a lengthy discussion and further references. We will see examples of sparse but not optimally sparse solutions in next section, as discussed in Section 3.4.2.

3.4 Simulation results for a damage on a Kirchhoff plate

3.4.1 Kirchhoff plate

We test the regularization methods on a square plate with size $1 \times 1 \times 0.01$ m (c.f. [Her14]). The initial elastic modulus for all elements is set to 200 GPa. The model is built using shell elements with 4 nodes each and 6 degrees of freedom: three translational and 3 rotational. The size of each finite element is 0.05×0.05 m, thus the model contains 400 elements. The plate is fixed on all sides. The numbering of the elements is show in Matrix (22). Plate is built as assembly of parts which is tested in the framework of the SHM finite element model updating package [Gri16] and cvx open-source code [cvx].

Figure 9: Damage 10% at element no. 211 with reduction in elasticity modulus from 200 *GPa* to 180 *GPa*. No noise is added. Boundary conditions for $\Delta\alpha$ is 0 to 80%. l_1 -norm regularization with $\lambda = 0.000001$ and identity pairing of eigenfrequencies.

(a) True damage location.

(b) First 3 eigenfrequencies. $\alpha(211) = \alpha(190) = \alpha(210) = \alpha(191) \approx 2.11\%$, Total damage $D(\alpha_i > 0.001) \approx 8.43\%$ (4 elements).

(c) First 7 eigenfrequencies. $\alpha(211) = \alpha(190) \approx 3.63\%$, $\alpha(210) = \alpha(191) \approx 1.38\%$. Total damage $D(\alpha_i > 0.001) \approx 10.02\%$ (4 elements).

(d) First 8 eigenfrequencies. $\alpha(211) = \alpha(190) \approx 4.6\%$, $\alpha(210) = \alpha(191) \approx 1.38\%$. Total damage $D(\alpha_i > 0.001) \approx 10.04\%$ (4 elements).

(e) First 9 eigenfrequencies. $\alpha(211) = \alpha(190) \approx 4.95\%$, $\alpha(210) = \alpha(191) \approx 0.07\%$. Total damage $D(\alpha_i > 0.001) \approx 9.90\%$ (2 elements).

(f) First 10 eigenfrequencies. $\alpha(211) = \alpha(190) \approx 4.85\%$, $\alpha(210) = \alpha(191) \approx 0.17\%$. Total damage $D(\alpha_i > 0.001) \approx 10.04\%$ (2 elements).

3.4.2 Simulation results

The plate elements have the numbering

$$\begin{array}{cccccc}
 381 & 382 & \dots & 390 & 391 & \dots & 399 & 400 \\
 361 & 362 & \dots & 370 & 371 & \dots & 379 & 380 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 201 & 202 & \dots & 210 & 211 & \dots & 219 & 210 \\
 181 & 182 & \dots & 190 & 191 & \dots & 199 & 200 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 21 & 22 & \dots & 30 & 31 & \dots & 39 & 40 \\
 1 & 2 & \dots & 10 & 11 & \dots & 19 & 20
 \end{array} \tag{22}$$

Figure 9 (a) shows a damage at element 211. For symmetry reasons, this damage gives exactly the same vibration mode eigenfrequencies and residual vector as an identical damage in element 210, 190 or 191. In fact, from a physical point of view, it is just the same plate with the same boundary conditions rotated 90, 180 or 270 degrees. Thus there is no way for a damage identification method to tell these four damages apart only from a comparison of eigenfrequencies. At best, if the l_1 -norm regularization (21) gives an optimally sparse solution

Figure 10: Identical settings as in Figure 9 except for using l_2 -norm regularization. (a) First 3 eigenfrequencies. $\alpha(211) = \alpha(190) = \alpha(210) = \alpha(191) \approx 1.61\%$. Total damage $D(\alpha_i > 0.01) \approx 6.43\%$ (4 elements). (b) First 10 eigenfrequencies. $\alpha(211) = \alpha(190) \approx 2.64\%$, $\alpha(210) = \alpha(191) \approx 2.24\%$. Total damage $D(\alpha_i > 0.01) \approx 9.85\%$ (4 elements).

Figure 11: Two parallel cracks with 10% stiffness reduction at each damaged element. The same λ and pairing as in Figure 9. (a) True damage location in elements no. 54, 75, 92, 96, 113, 134. (b) l_1 -norm regularization, first 20 eigenfrequencies. Total damage $D(\alpha_i > 0.001) \approx 55.31\%$. (c) l_2 -norm regularization, first 20 eigenfrequencies. Total damage $D(\alpha_i > 0.001) \approx 60.45\%$.

with only one nonzero element, it will indicate a damage in *one* of the elements 190, 191, 210 and 211 (with 25 % chance of picking the right one). In Figure 9 (b)–(f), we see that as the number of eigenfrequencies used in the residual vector increase from 3 to 10, the location of the indicated damage is narrowed down from four to two of the elements 190, 191, 210 and 211.

Figure 10 (a) and (b) shows the corresponding results for l_2 -norm regularization. As expected, we see that it gives a more smoothed and less sparse solution than the l_1 -norm regularization.

Figure 11 (a) shows a damage that resembles two parallel cracks. In (b), we see that l_1 -norm regularization gives damage identification with very roughly the right localization, as well as asymmetrically placed "ghost damage" for the same reasons as explained above. In Figure 11 (c), finally, we see that l_2 -norm regularization again gives a less sparse and more smooth solution.

4 Conclusion and future work

In conclusion, in order to verify the capability of a material model to simulate the shear behavior of reinforced concrete structures, not only the load-displacement curve but also the local reaction such as strain distribution and crack pattern should be compared with the member level benchmark test. Based on the present research, it is not reliable to adopt the CDP model to simulate the shear behavior of reinforced concrete structures because consistent predictions of both load-displacement curve and crack pattern compared to the measurement can't be obtained. Further research on calibrating the damage parameter evolution of this model should be performed.

In our test of damage identification using l_1 -norm regularization on the Kirchhoff plate, we got more sparse solution than with l_2 -norm regularization, but still not optimally sparse. For an optimally sparse solution, we suggest to extend the residual to also contain a comparison of predicted and measured mode shapes. Then a next step can be to try applying the same sparse regularization on larger and more complicated structures, such as the Kiruna Bridge. We explained shortly in Section 2.4 how measurements and modal analysis on that bridge were performed before and after loading that bridge to failure, and found at least four mode shapes suitable for the damage identification.

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