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$L_p + L_q$ and $L_p \cap L_q$ are not isomorphic for all $1 \le p, q \le \infty$, $p \ne q$

$L_p + L_q$ et $L_p \cap L_q$ ne sont pas isomorphes pour tout $1 \le p, q \le \infty$, $p \ne q$

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ABSTRACT

We prove that if $1 \le p, q \le \infty$, then the spaces $L_p + L_q$ and $L_p \cap L_q$ are isomorphic if and only if $p = q$. In particular, $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic, which is an answer to a question formulated in [2].

RÉSUMÉ

Nous prouvons que si $1 \le p, q \le \infty$, alors les espaces $L_p + L_q$ et $L_p \cap L_q$ sont isomorphes si et seulement si $p = q$. En particulier, $L_2 + L_\infty$ et $L_2 \cap L_\infty$ ne sont pas isomorphes, ce qui est une réponse à une question formulée dans [2].

1. Preliminaries and main result

Isomorphic classification of symmetric spaces is an important problem related to the study of symmetric structures in arbitrary Banach spaces. A number of very interesting and deep results of such a sort is proved in the seminal work of Johnson, Maurey, Schechtman and Tzafriri [9]. In particular, in [9] (see also [12, Section 2.f]) it was shown that the space $L_2 \cap L_p$ for $2 \le p < \infty$ (resp. $L_2 + L_p$ for $1 < p \le 2$) is isomorphic to L_p. A further investigation of various properties of *separable* sums and intersections of L_p-spaces (i.e. with $p < \infty$) was continued by Dilworth in [6] and [7] and by Dilworth and Carothers in [5]. In contrast to that, in the paper [2] we proved that *nonseparable* spaces $L_p + L_\infty$ and $L_p \cap L_\infty$ for all $1 \le p < \infty$ and $p \ne 2$ are not isomorphic. This question was left open for $p = 2$, and this was a motivation to continue this work. Here, we give a solution to this problem and, on the basis of the results of [9] and [2], we prove a more general theorem: $L_p + L_q$ and $L_p \cap L_q$ for all $1 \le p, q \le \infty$ are isomorphic if and only if $p = q$.

In this paper, we use the standard notation from the theory of symmetric spaces (cf. [3], [11] and [12]). Let $L_p(0, \infty)$ be the usual Lebesgue space of p-integrable functions $x(t)$ equipped with the norm

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$$\|x\|_{L_p} = \left(\int_0^\infty |x(t)|^p \, dt \right)^{1/p} \quad (1 \le p < \infty)$$

and $\|x\|_{L_\infty} = \operatorname{ess\,sup}_{t>0}|x(t)|$. For $1 \le p, q \le \infty$, the space $L_p + L_q$ consists of all sums of p-integrable and q-integrable measurable functions on $(0, \infty)$ with the norm defined by

$$\|x\|_{L_p+L_q} := \inf_{x(t)=u(t)+v(t),\, u\in L_p, v\in L_q} \left(\|u\|_{L_p} + \|v\|_{L_q} \right).$$

The space $L_p \cap L_q$ consists of all both p- and q-integrable functions on $(0, \infty)$ with the norm

$$\|x\|_{L_p\cap L_q} := \max\left\{ \|x\|_{L_p}, \|x\|_{L_q} \right\} = \max\left\{ \Big(\int_0^\infty |x(t)|^p \, dt \Big)^{1/p}, \Big(\int_0^\infty |x(t)|^q \, dt \Big)^{1/q} \right\}.$$

$L_p + L_q$ and $L_p \cap L_q$ for all $1 \le p, q \le \infty$ are symmetric Banach spaces (cf. [11, p. 94]). They are separable if and only if both p and q are finite (cf. [11, p. 79] for $p = 1$).

The norm in $L_p + L_q$ satisfies the following estimates

$$\Big(\int_0^1 x^*(t)^p \, dt \Big)^{1/p} + \Big(\int_1^\infty x^*(t)^q \, dt \Big)^{1/q} \le \|x\|_{L_p+L_q} \le C_{p,q} \Big(\Big(\int_0^1 x^*(t)^p \, dt \Big)^{1/p} + \Big(\int_1^\infty x^*(t)^q \, dt \Big)^{1/q} \Big)$$

if $1 \le p < q < \infty$, and

$$\Big(\int_0^1 x^*(t)^p \, dt \Big)^{1/p} \le \|x\|_{L_p+L_\infty} \le C_p \Big(\int_0^1 x^*(t)^p \, dt \Big)^{1/p}$$

if $1 \le p < \infty$ (cf. [4, p. 109], [8, Thm. 4.1] and [13, Example 1]). Here, $x^*(t)$ denotes the decreasing rearrangement of $|x(u)|$, that is,

$$x^*(t) = \inf\{\tau > 0\colon m(\{u > 0\colon |x(u)| > \tau\}) < t\}$$

(if $E \subset \mathbb{R}$ is a measurable set, then $m(E)$ is its Lebesgue measure). Note that every measurable function and its decreasing rearrangement are equimeasurable, that is,

$$m(\{u > 0\colon |x(u)| > \tau\}) = m(\{t > 0\colon |x^*(t)| > \tau\})$$

for all $\tau > 0$.

Now, we state the main result of this paper.

Theorem 1. *For every $1 \le p, q \le \infty$ the spaces $L_p + L_q$ and $L_p \cap L_q$ are isomorphic if and only if $p = q$.*

If $\{x_n\}_{n=1}^\infty$ is a sequence from a Banach space X, by $[x_n]$ we denote its closed linear span in X. As usual, the Rademacher functions on $[0, 1]$ are defined as follows: $r_k(t) = \operatorname{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}, t \in [0, 1]$.

2. $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic

Let x be a measurable function on $(0, \infty)$ such that $m(\operatorname{supp} x) \le 1$. Then, clearly, x is equimeasurable with the function $x^* \chi_{[0,1]}$. Therefore, assuming that $x \in L_2$ (resp. $x \in L_\infty$), we have $x \in L_2 + L_\infty$ and $\|x\|_{L_2+L_\infty} = \|x\|_{L_2}$ (resp. $x \in L_2 \cap L_\infty$ and $\|x\|_{L_2\cap L_\infty} = \|x\|_{L_\infty}$).

Theorem 2. *The spaces $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic.*

Proof. On the contrary, assume that T is an isomorphism of $L_2 + L_\infty$ onto $L_2 \cap L_\infty$.

For every $n, k \in \mathbb{N}$ and $i = 1, 2, \ldots, 2^k$, we set

$$\Delta_{k,i}^n = (n - 1 + \frac{i-1}{2^k}, n - 1 + \frac{i}{2^k}],\ u_{k,i}^n := \chi_{\Delta_{k,i}^n},\ v_{k,i}^n := T(u_{k,i}^n).$$

Clearly, $\|u^n_{k,i}\|_{L_2+L_\infty} = 2^{-k/2}$. Therefore, if $x^n_{k,i} = 2^{k/2}u^n_{k,i}$, $y^n_{k,i} = 2^{k/2}v^n_{k,i}$, then $\|x^n_{k,i}\|_{L_2+L_\infty} = 1$ and

$$\|T^{-1}\|^{-1} \le \|y^n_{k,i}\|_{L_2\cap L_\infty} = \max(\|y^n_{k,i}\|_{L_2}, \|y^n_{k,i}\|_{L_\infty}) \le \|T\| \tag{1}$$

for all $n, k \in \mathbb{N}$, $i = 1, 2, \ldots, 2^k$.

At first, we suppose that, for each $k \in \mathbb{N}$, there are $n_k \in \mathbb{N}$ and $1 \le i_k \le 2^k$ such that

$$\|y^{n_k}_{k,i_k}\|_{L_2} \to 0 \text{ as } k \to \infty. \tag{2}$$

Denoting $\alpha_k := x^{n_k}_{k,i_k}$ and $\beta_k := y^{n_k}_{k,i_k}$, observe that $m(\bigcup_{k=1}^\infty \operatorname{supp}\alpha_k) = 1$ and so the sequence $\{\alpha_k\}_{k=1}^\infty$ is isometrically equivalent in $L_2 + L_\infty$ to the unit vector basis of l_2 and $[\alpha_k]$ is a complemented subspace of $L_2 + L_\infty$. Then, since $\beta_k = T(\alpha_k), k = 1, 2, \ldots$, the sequence $\{\beta_k\}_{k=1}^\infty$ is also equivalent in $L_2 \cap L_\infty$ to the unit vector basis of l_2. Moreover, if P is a bounded projection from $L_2 + L_\infty$ onto $[\alpha_k]$, then the operator $Q := TPT^{-1}$ is the bounded projection from $L_2 \cap L_\infty$ onto $[\beta_k]$. Thus, the subspace $[\beta_k]$ is complemented in $L_2 \cap L_\infty$.

Now, let $\varepsilon_k > 0, k = 1, 2, \ldots$ and $\sum_{k=1}^\infty \varepsilon_k < \infty$ (the choice of these numbers will be specified a little bit later). Thanks to (2), passing to a subsequence (and keeping the notation), we may assume that

$$\|\beta_k\|_{L_2} < \varepsilon_k \text{ and } m\{s > 0 : |\beta_k(s)| > \varepsilon_k\} < \varepsilon_k, k = 1, 2, \ldots$$

(clearly, this subsequence preserves the above properties of the sequence $\{\beta_k\}$). Hence, denoting

$$A_k := \{s > 0 : |\beta_k(s)| > \varepsilon_k\} \text{ and } \gamma_k := \beta_k\chi_{A_k}, k = 1, 2, \ldots,$$

we obtain

$$\|\beta_k - \gamma_k\|_{L_2\cap L_\infty} \le \max\{\|\beta_k\chi_{(0,\infty)\setminus A_k}\|_{L_\infty}, \|\beta_k\|_{L_2}\} \le \varepsilon_k, k = 1, 2, \ldots.$$

Thus, choosing ε_k sufficiently small and taking into account inequalities (1), by the principle of small perturbations (cf. [1, Theorem 1.3.9]), we see that the sequences $\{\beta_k\}$ and $\{\gamma_k\}$ are equivalent in $L_2 \cap L_\infty$ and the subspace $[\gamma_k]$ is complemented (together with $[\beta_k]$) in the latter space.

Denote $A := \bigcup_{k=1}^\infty A_k$. We have $m(A) \le \sum_{k=1}^\infty m(A_k) \le \sum_{k=1}^\infty \varepsilon_k < \infty$ and hence the space

$$(L_2 \cap L_\infty)(A) := \{x \in L_2 \cap L_\infty : \operatorname{supp} x \subset A\}$$

coincide with $L_\infty(A)$ (with equivalence of norms). As a result, $L_\infty(A)$ contains the complemented subspace $[\gamma_k]$, which is isomorphic to l_2. Since this is a contradiction with [1, Theorem 5.6.5], our initial assumption on the existence of a sequence $\{y^{n_k}_{k,i_k}\}_{k=1}^\infty$ satisfying (2) fails.

Thus, there are $c > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y^n_{k_0,i}\|_{L_2} \ge c \text{ for all } n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, 2^{k_0}.$$

Then, by the generalized Parallelogram Law (see [1, Proposition 6.2.9]), we have

$$\int_0^1 \|\sum_{i=1}^{2^{k_0}} r_i(s)y^n_{k_0,i}\|^2_{L_2} ds = \sum_{i=1}^{2^{k_0}} \|y^n_{k_0,i}\|^2_{L_2} \ge c^2\, 2^{k_0}, n \in \mathbb{N},$$

where $r_i = r_i(s)$ are the Rademacher functions. Hence, there exist $\theta^n_i = \pm 1, n = 1, 2, \ldots, i = 1, 2, \ldots, 2^{k_0}$ such that $\|\sum_{i=1}^{2^{k_0}} \theta^n_i y^n_{k_0,i}\|_{L_2} \ge c\, 2^{k_0/2}, n \in \mathbb{N}$, or equivalently $\|\sum_{i=1}^{2^{k_0}} \theta^n_i v^n_{k_0,i}\|_{L_2} \ge c, n \in \mathbb{N}$. So, setting

$$f_n := \sum_{i=1}^{2^{k_0}} \theta^n_i u^n_{k_0,i},\ g_n := \sum_{i=1}^{2^{k_0}} \theta^n_i v^n_{k_0,i},$$

we have

$$\|f_n\|_{L_2+L_\infty} = 1 \text{ and } \|g_n\|_{L_2} \ge c,\ n = 1, 2, \ldots. \tag{3}$$

Moreover, by the definition of the norm in $L_2 + L_\infty$ and the fact that

$$\Big|\sum_{n=1}^m f_n\Big| = \Big|\sum_{n=1}^m \sum_{i=1}^{2^{k_0}} \theta^n_i u^n_{k_0,i}\Big| = \sum_{n=1}^m \sum_{i=1}^{2^{k_0}} \chi_{\Delta^n_{k_0,i}} = \chi_{(0,m]},$$

we obtain

$$\Big\|\sum_{n=1}^{m} f_n\Big\|_{L_2+L_\infty} = \|f_1\|_{L_2} = 1,\ m = 1, 2, \ldots. \tag{4}$$

On the other hand, since $\{f_n\}$ is an 1-unconditional sequence in $L_2 + L_\infty$, for each $t \in [0, 1]$ we have

$$\Big\|\sum_{n=1}^{m} f_n\Big\|^2_{L_2+L_\infty} = \Big\|\sum_{n=1}^{m} r_n(t)\, f_n\Big\|^2_{L_2+L_\infty} \ge \frac{1}{\|T\|^2}\Big\|\sum_{n=1}^{m} r_n(t)\, g_n\Big\|^2_{L_2\cap L_\infty}.$$

Integrating this inequality, by the generalized Parallelogram Law and (3), we obtain

$$\begin{aligned}\Big\|\sum_{n=1}^{m} f_n\Big\|^2_{L_2+L_\infty} &\ge \frac{1}{\|T\|^2}\int_0^1 \Big\|\sum_{n=1}^{m} r_n(t)\, g_n\Big\|^2_{L_2\cap L_\infty} \mathrm{d}t \ge \frac{1}{\|T\|^2}\int_0^1 \Big\|\sum_{n=1}^{m} r_n(t)\, g_n\Big\|^2_{L_2} \mathrm{d}t \\ &= \frac{1}{\|T\|^2}\sum_{n=1}^{m}\|g_n\|^2_{L_2} \ge \Big(\frac{c}{\|T\|}\Big)^2 \cdot m, \quad m = 1, 2, \ldots.\end{aligned}$$

Since the latter inequality contradicts (4), the proof is completed. □

Remark 1. Using the same arguments as in the proof of the above theorem, we can show that the spaces $L_p + L_\infty$ and $L_p \cap L_\infty$ are not isomorphic for every $1 \le p < \infty$. This gives a new proof of Theorem 1 from [2]. However, note that in the latter paper (see Theorems 3 and 5), it is proved the stronger result, saying that the space $L_p \cap L_\infty$, $p \ne 2$, does not contain any complemented subspace isomorphic to $L_p(0, 1)$.

3. $L_p + L_q$ and $L_p \cap L_q$ are not isomorphic for $1 < p, q < \infty$, $p \ne q$

Both spaces $L_p + L_q$ and $L_p \cap L_q$ for all $1 \le p, q \le \infty$ are special cases of Orlicz spaces on $(0, \infty)$.

A function $M\colon [0, \infty) \to [0, \infty]$ is called a *Young function* (or *Orlicz function* if it is finite-valued) if M is convex, non-decreasing with $M(0) = 0$; we assume also that $\lim_{u\to 0+} M(u) = M(0) = 0$ and $\lim_{u\to\infty} M(u) = \infty$.

The *Orlicz space* $L_M = L_M(I)$ with $I = (0, 1)$ or $I = (0, \infty)$ generated by the Young function M is defined as

$$L_M(I) = \{x \text{ measurable on } I\colon\ \rho_M(x/\lambda) < \infty \text{ for some } \lambda = \lambda(x) > 0\},$$

where $\rho_M(x) := \int_I M(|x(t)|)\,\mathrm{d}t$. It is a Banach space with the *Luxemburg–Nakano norm*

$$\|x\|_{L_M} = \inf\{\lambda > 0 : \rho_M(x/\lambda) \le 1\}$$

and is a symmetric space on I (cf. [3], [10–15]). Special cases of Orlicz spaces on $I = (0, \infty)$ are the following (cf. [14, pp. 98–100]):

(a) for $1 \le p, q < \infty$, let $M(u) = \max(u^p, u^q)$, then $L_M = L_p \cap L_q$;

(b) for $1 \le p < \infty$, let

$$M(u) = \begin{cases} u^p & \text{if } 0 \le u \le 1, \\ \infty & \text{if } 1 < u < \infty, \end{cases} \quad \text{then } L_M = L_p \cap L_\infty;$$

(c) for $1 \le p, q < \infty$, let $M(u) = \min(u^p, u^q)$, then M is not a convex function on $[0, \infty)$, but $M_0(u) = \int_0^u \frac{M(t)}{t}\mathrm{d}t$ is convex and $M(u/2) \le M_0(u) \le M(u)$ for all $u > 0$, which gives $L_M = L_{M_0} = L_p + L_q$;

(d) for $1 \le p < \infty$, let

$$M(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1, \\ u^p - 1 & \text{if } 1 < u < \infty, \end{cases} \quad \text{then } L_M = L_p + L_\infty.$$

A Young (Orlicz) function M satisfies the Δ_2-*condition* if $0 < M(u) < \infty$ for $u > 0$ and there exists a constant $C \ge 1$ such that $M(2u) \le CM(u)$ for all $u > 0$. An Orlicz space $L_M(0, \infty)$ is separable if and only if M satisfies the Δ_2-condition (cf. [10, pp. 107–110], [14, Thm. 4.2 (b)], [15, p. 88]). With each Young function M one can associate another convex function M^*, i.e. the *complementary function* to M, which is defined by $M^*(v) = \sup_{u>0}[uv - M(u)]$ for $v \ge 0$. Then M^* is also a Young function and $M^{**} = M$. An Orlicz space $L_M(0, \infty)$ is reflexive if and only if M and M^* satisfy the Δ_2-condition (cf. [14, Thm. 9.3], [15, p. 112]).

Theorem 3. *Let M and N be two Orlicz functions on $[0, \infty)$ such that both spaces $L_M(0, \infty)$ and $L_N(0, \infty)$ are reflexive. Suppose that $L_M(0, \infty)$ and $L_N(0, \infty)$ are isomorphic. Then, the functions M and N are equivalent for $u \ge 1$, that is, there are constants $a, b > 0$ such that $aM(u) \le N(u) \le bM(u)$ for all $u \ge 1$.*

Proof. If both functions M and N are equivalent to the function u^2 for $u \geq 1$, then nothing has to be proved. So, suppose that the function M is not equivalent to u^2. Then, clearly, $L_M(0,1)$ is a complemented subspace of $L_M(0,\infty)$ and $L_M(0,1)$ is different from $L_2(0,1)$, even up to an equivalent renorming. By hypothesis, $L_N(0,\infty)$ contains a complemented subspace isomorphic to $L_M(0,1)$. Then, by [12, Corollary 2.e.14(ii)] (see also [9, Thm. 7.1]) $L_M(0,1) = L_N(0,1)$ up to equivalent norm. This implies that M and N are equivalent for $u \geq 1$ (cf. [10, Thm. 8.1], [14, Thm. 3.4]). $\square$

Corollary 1. *Let* $1 < p, q < \infty$, $p \neq q$, *then* $(L_p + L_q)(0,\infty)$ *and* $(L_p \cap L_q)(0,\infty)$ *are not isomorphic.*

Proof. For such p, q, the Orlicz spaces $(L_p + L_q)(0,\infty)$ and $(L_p \cap L_q)(0,\infty)$ are reflexive, and are generated by the Orlicz functions $M(u) = \min(u^p, u^q)$ and $N(u) = \max(u^p, u^q)$ respectively, which are not equivalent for $u \geq 1$ whenever $p \neq q$. Thus, by Theorem 3, these spaces cannot be isomorphic. $\square$

4. Proof of Theorem 1

Proof of Theorem 1. We consider four cases.

(a) For $p \in [1,2) \cup (2,\infty)$ and $q = \infty$, it was proved in [2, Theorem 1].

(b) For $p = 2$ and $q = \infty$, it is proved in Theorem 2.

(c) Let $p = 1$ and $1 < q < \infty$. If we assume that $L_1 + L_q$ and $L_1 \cap L_q$ are isomorphic, then the dual spaces will be also isomorphic. The dual spaces are $(L_1 + L_q)^* = L_{q'} \cap L_\infty$ and $(L_1 \cap L_q)^* = L_{q'} + L_\infty$, where $1/q + 1/q' = 1$. By (a) and (b), the spaces $L_{q'} + L_\infty$ and $L_{q'} \cap L_\infty$ are not isomorphic, thus their preduals cannot be isomorphic.

(d) For $1 < p, q < \infty$, $p \neq q$, the desired result follows from Corollary 1, and the proof is completed. $\square$

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