Hardy-type inequalities quantum calculus

Serikbol Shaimardan

Mathematics
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2010 Mathematics Subject Classification. 35B27, 76D08

Key words and phrases. Inequalities, Hardy-type inequalities, Riemann-Liouville operator, Integral operator, $q$-analysis, $q$-analog, weights, $h$-calculus, $h$-integral, discrete fractional calculus
Abstract

This PhD thesis deals with fractional Hardy-type inequalities and some new Hardy-type inequalities for the Hardy operator and Riemann-Liouville fractional integral operator and fractional Hardy-type inequalities in quantum calculus, which are given in the frame of $q$-calculus and $h$-calculus.

The thesis contains five papers (papers A, B, C, D and E) and an introduction, which put these papers into a more general frame. In particular, in this introduction we give a brief history of quantum calculus and a short description as basis for the rest of the quantum calculus.

In paper A we study some $q$-analogs of Hardy-type inequalities for the Riemann-Liouville fractional integral operator of order $n \in \mathbb{N}$ and find necessary and sufficient conditions of the validity of these inequalities for all non-negative real functions.

In paper B we define the $q$-analog of the classical Hardy operator and we characterize the $q$-analog of the weighted Hardy inequalities for all possible values of the parameters $p$ and $r$ in $q$-calculus. We also study the corresponding dual results.

In paper C we consider a $q$-analog of the operator $I$ defined by

\[ If(x) := \int_0^x \ln \frac{x}{x-s} \frac{f(s)}{s} ds, \]

which is called the fractional integral operator of infinitesimal order. Moreover, we characterize the $q$-analog of the following Hardy-type
integral inequality:

\[
(1) \quad \left( \int_0^\infty u'(x) \left( \int_0^x t^{\gamma-1} \ln \frac{x}{t} f(t) dt \right)^r dx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p}}, \quad \forall f(\cdot) \geq 0,
\]

where \( u(.) \) is a weight i.e. a measurable function, which is positive a.e. in \((0, \infty)\). In fact, we derive necessary and sufficient conditions for the validity of the \(q\)-analog of the inequality (1) in \(q\)-analysis for the case \(1 < p \leq r < \infty\) and \(\gamma > \frac{1}{p}\). We also consider the problem to find the best constant in the \(q\)-analog of inequality (1).

In paper D we consider that the first power weighted version of Hardy’s inequality which can be rewritten as

\[
\int_0^\infty \left( x^{\alpha-1} \int_0^x \frac{1}{t^\alpha} f(t) dt \right) dx \leq \left( \frac{p}{p-\alpha-1} \right)^p \int_0^\infty f^p(x) dx,
\]

for \( f \geq 0, \ p \geq 1 \) and \( \alpha < p - 1 \), where the constant \( C = \left( \frac{p}{p-\alpha-1} \right)^p \) is sharp. In this paper we prove and discuss some discrete analogues of Hardy-type inequalities in fractional \(h\)-discrete calculus. Moreover, we prove that the corresponding constants are sharp.

In paper E we consider the fractional order Hardy-type inequality in the following form:

\[
\left( \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x-y|^{1+p\alpha}} dx dy \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty |f'(x)|^p x^{(1-\alpha)p} dx \right)^{\frac{1}{p}},
\]

for \( 0 < \alpha < 1 \) and \( 1 < p < \infty \) in fractional \(h\)-discrete calculus, where \( C = \left( \frac{p-\alpha}{(p-\alpha)\alpha} \right)^{\frac{1}{p}} \). A discrete analogue of this inequality, namely a new fractional order Hardy inequality in \(h\)-discrete calculus, is proved and discussed. Moreover, we prove that the same constant is sharp also in this case.
Preface

The main part of this PhD thesis consists of five papers (papers A, B, C, D and E), which deal with a new fractional Hardy inequality and also some new Hardy-type inequalities in quantum calculus. The thesis also contains an introduction, which put these papers to a more general frame.


Remark. Paper [E] is a slightly improved version of the original Report 1 from 2018.
Acknowledgment

First of all I want to express my deep gratitude to my scientific supervisors Professor Lars-Erik Persson (Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden) and Professor Ryskul Oinarov (Eurasian National University, Kazakhstan) for their valuable remarks and attention to my work and their constant support.

Moreover, I thank my other supervisor Professor Peter Wall (Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden), which has given me the possibility to work in this international PhD program and provided me with necessary invitations and related support. My sincere thanks also to Professor Natasha Samko, who has given me several generous and valuable remarks on my introduction and helped me with my Russian-English translations.

I also thank PhD Ainur Temirkhanova for helping me with many practical things.

Moreover, I thank Luleå University of Technology and L. N. Gumilyov Eurasian National University for giving me an opportunity to participate in their partnership program in research and postgraduate education in mathematics. I also thank both universities for financial support which made my studies possible.

Furthermore, I would like to thank everyone at the Department of Mathematics at Luleå University of Technology for helping me in different ways and for the warm and friendly atmosphere.

Finally, and most important, I want to express my deepest gratitude to my wife Aigerim Edenova for all support and understanding.
Introduction

"The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking." - John von Neumann.

Calculus or infinitesimal calculus has a fascinating history. In the 17th century, I. Newton and G. Leibniz independently invented calculus based on the concept of limit (but elements of it have already appeared in ancient Greece). The usual meaning of limit implies that space and time are continuous, and we have maintained that all natural processes happen continuously on smooth curves and surfaces. However, the atomic theory in physics and chemistry in the 19th century paved that the nature process of dividing it into ever smaller parts will terminate in an indivisible or an atom, a part which, lacking proper parts itself, cannot be further divided. In a word, continua are divisible without limit or infinitely divisible. This becomes the origin of developing another type of calculus based on finite difference principle, or calculus without limit which is quantum calculus (the calculus of finite differences was developed at the same time).

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word “quantus” and literally it means how much, in Swedish ”Kvant”). It has two major branches, q-calculus and the h-calculus. And both of them were worked out by P. Cheung and V. Kac [36] in the early twentieth century.
1. Background and further development

$h$-calculus: One of the popular quantum calculus is $h$-calculus. This calculus is the study of the definitions, properties, and applications of related concepts, the fractional calculus and discrete fractional calculus. However, the investigation for fractional calculus was studied already by G. Leibniz after that G. L’Hospital in 1695 asked him: ", what would be the one-half derivative of $x^2$" (see [74]). In 1772, J.L. Lagrange introduced the differential operators of integer order and wrote (see [72]):

$$\frac{d^n}{dx^m} \frac{d^n}{dx^n} y = \frac{d^{n+m}}{dx^{n+m}} y.$$  

In 1819, S.F. Lacroix developed a more mathematical generalizing from a case of integer order [73]. Namely he presented the $n$th derivative in the following form:

$$D^n x^m = \frac{d^n}{dx^n} (x^m) = \frac{m!}{(n+m)!} x^{m-n},$$

with $y = x^m$ and $m, n \in \mathbb{Z}$ such that $m \geq n$. Replacing the factorial symbol by the Gamma function, he developed the formula for the fractional derivative of a power function:

$$D^\alpha x^\beta = \frac{\Gamma (\beta + 1)}{\Gamma (\beta - \alpha + 1)} x^{\beta - \alpha}.$$  

where $\alpha$ and $\beta$ are fractional numbers. Then he gave the example that the derivative of order $\frac{1}{2}$ for $y = x$ is as follows:

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$  

Note that this interesting result of S.F. Lacroix is the same as the Riemann-Liouville definition of a fractional derivative.

However, this above authors did not define derivatives of arbitrary order and they gave no applications or examples. The first application was presented by N.H. Abel [2] in 1823. He applied the fractional calculus in the solution of an integral equation. Abel’s solution was so elegant that it attracted the attention of J. Liouville. In 1832 he took the first step to solve differential equations involving fractional operators (see [75]). Moreover, he gave his definition
of a fractional derivative:

\[ D^\alpha x^a = \frac{(-1)^\alpha \Gamma (a + \alpha)}{\Gamma (a)} x^{-a-\alpha}, a > 0, \]

for \( \alpha \in \mathbb{R} \). He was successful in applying this definition to problems in potential theory. One of the most useful advances in the development of fractional calculus was due to a paper written by B. Riemann during his student days. Seeking to generalize a Taylor series in 1853, he derived a different definition that involved the definite integral in the following form [116]:

\[ D^\alpha f(x) = \frac{1}{\Gamma (\alpha)} \int_c^x (x - t)^{\alpha-1} f(t) dt + \psi(x), a > 0. \]

Because of the ambiguity in the lower limit of integration \( c \), he added to his definition a complementary function \( \psi(x) \).

In the period 1900-1970 a modest amount of published works appeared on the subject of fractional calculus. Some of the contributors were G.H. Hardy and J.E. Littlewood [56], M. Riesz [108], S. Samko [110] and H. Weyl [90].

The theory of discrete fractional calculus is far less developed. It seems as no significant work appeared in this area until some ones primarily devoted to applications of the fractional calculus to ordinary and partial differential equations appeared, with the majority of interest shown only within the past thirty years. In 1974, J.B. Diaz and T. J. Osler [38] introduced a discrete fractional difference operator defined as an infinite series, a generalization of the binomial formula for the \( N \)th-order difference operator \( \Delta^N \) in the following form:

\[ \Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{N}{k} f(x + \alpha + k), \]

More results concerning the process to develop the analogous theory for fractional finite differences was proved by K.S. Miller and B. Ross [87]. In particular, they presented more rules for composing fractional sums. In 2007, this direction was studied by F.M. Atici and P.W. Eloe, which in [20] discussed some of the
properties of this factorial function in the following form:

\[ t^{(\alpha)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha)}, \]

for \( \alpha \in \mathbb{R} \), which is generalization of factorial polynomial:

\[ t^{(n)} = \prod_{j=0}^{n-1} (t - j) = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - n)}, \]

where \( \Gamma \) denotes the special gamma function and if \( t + 1 - j = 0 \) for some \( j \), then we assume the product to be zero. We shall use the convention that division at a pole yields zero.

During the last two decades, the \( h \)-calculus has been successfully applied to several fields within mathematics, see e.g. \([3], [4], [37], [71], [83], [96], [104]\) and \([109]\) the references therein. Finally, we mention that \( h \)-calculus is also important in applied fields such as economics, engineering and physics (see, e.g. \([8], [9], [77], [79], [82], [97]\)).

\( q \)-calculus: \( q \)-calculus is based on the \( q \)-calculus, but this kind of calculus had already been represented by L. Euler \([44]\) in the 18th century. The study of \( q \)-calculus started in 1748 when L. Euler \([44]\) considered the infinite product \( (q; q)_\infty^{-1} = \prod_{k=0}^{\infty} \frac{1}{1 - q^{k+1}}, \) \(|q| < 1\), as a generating function for \( p(n) \) (the partition function \( p(n) \) is the number of ways to write \( n \) as a sum of integers). Furthermore, he discovered the first two \( q \)-exponential functions, a prelude to the \( q \)-binomial theorem (see \([42]\)). One hundred years later the progress of investigation continued under E. Heine, who in 1846 considered a generalization of the hypergeometric \((q\text{-hypergeometric})\) series (see \([46]\)), given by the formula

\[ 2\phi_1 = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n, \quad |z| < 1, \]

where the \( q \)-shifted factorial is defined by

\[ (a; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{m=0}^{n-1} (1 - aq^m), & n \in \mathbb{N}, \end{cases} \]
and the series (2) converges absolutely for $0 < |q| < 1$. When $q \to 1$ we get the Gauss’ series.

The $q$-analog of the gamma function was obtained by J. Thomae [127] and later by F.H. Jackson [58] in the following form:

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty}(1 - q)^{1-x}, \quad 0 < q < 1,$$

for $x \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$.

In 1908 F.H. Jackson [59] (see also [36]) reintroduced the Euler-Jackson $q$-difference operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \in (0, b), \quad q \in \mathbb{C} \setminus \{1\},$$

for $f : [0, b) \to \mathbb{R}$, $0 \leq b < \infty$. It is clear that if $f(x)$ is differentiable, then $\lim_{q \to 1} D_q f(x) = f'(x)$. The $q$-derivative is a discretization of the ordinary derivative and therefore has immediate applications in numerical analysis. At the same time the investigation of the $q$-difference equations theory considered in intensive works especially by C.R. Adams [1], R.D. Carmichael [35] and F.H. Jackson [61] and others.

Moreover, in 1910 F.H. Jackson [60] gave the more general $q$-integral definition

$$\int_0^x f(t)d_q t = (1 - q)x \sum_{k=0}^{\infty} q^k f(q^k x), \quad x \in (0, \infty),$$

and the improper $q$-integral of a function $f(x) : [0, \infty) \to \mathbb{R}$, by the formula

$$\int_0^\infty f(t)d_q t = (1 - q) \sum_{k=-\infty}^{\infty} q^k f(q^k),$$

for $0 < |q| < 1$. Note that the series on the right hand sides of (3) and (4) converge absolutely.

The study the $q$-fractional calculus started from the works of R.P. Agarwal [5], [6], W.A. Al-Salam [11], [12], W.A. Al-Salam and Verma [14]. They introduced several types of fractional $q$-integral operators and fractional $q$-derivatives and here in particular was
defined the fractional $q$-integral of the Riemann-Liouville type $I_{q,\alpha}$ defined by

$$I_{q,\alpha}f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qs/x; q)_{\alpha-1} f(s) d_q s, \quad x > 0,$$

with $q \in \mathbb{C} \setminus \{1\}$, where $(qs/x; q)_\alpha = (qs/x; q)^{\infty}/(q^\alpha qs/x; q)^{\infty}$, $\alpha \in R^+$. 

In 1987, the applications of $q$-calculus in the area of approximation theory was initiated by A. Lupas [76], who first introduced $q$-Bernstein polynomials and the development of this direction has been remarkable, and the most important things can be found in the book [19]. We also mention the work of T. Ernst [42], where he presented applications of the $q$-calculus in many subjects, like umbral calculus, oscillations in $q$-calculus, interpolation theory, quantum groups, quantum algebras, hypergeometric series, complex analysis and particle physics.

Today the interest in this subject has exploded and the $q$-calculus has in the last twenty years served as a bridge between mathematics and physics. The $q$-calculus has numerous applications in various fields of mathematics e.g dynamical systems, number theory, combinatorics, special functions, fractals and also for scientific problems in some applied areas such as computer science, quantum mechanics and quantum physics (see e.g. [18], [23], [42], [43], [45]). Most of the additional information can be found in the work of G. Gasper and M. Rahman [46], where was given simpler proofs of many results (for example: $q$-Clausen’s formula, $q$-orthogonal polynomials, $q$-analogues of various product formulas, etc.) and important applications to other fields (for example: modern algebra, real and complex variables, number theory, etc.). Moreover, for the further development and recent results in $q$-calculus we refer to the books [18], [19], [36] and [42] and the references given therein.

The first results concerning integral inequalities in $q$-calculus were proved in 2004 by H. Gauchman [47] (for example: Steffensen’s, Hermite-Hadamard’s, Iyengar’s and Chebyshev’s inequalities). Later on some further $q$-analogs of the classical inequalities have been proved in the papers [68], [86] and [123]. We also
pronounce the recent book [15] by G.A. Anastassiou, where many important $q$-inequalities are proved and discussed.

An essential part of this PhD thesis is devoted to obtain necessary and sufficient conditions for the validity of Hardy-type inequalities in $q$-calculus. So we focus now our interest on the history and references concerning weighted Hardy-type inequalities.

The theory of Hardy-type inequalities is a wonderful mathematical subject with a proud history and it is very applicable both in mathematics and to problems in many areas outside the mathematical sciences. The investigation of Hardy-type inequalities began in 1915 with the work of G.H. Hardy. In 1925 he proved the following results (see [52]):

**Theorem 0.1.** Let $p > 1$ and \( \{a_n\}_{n=1}^{\infty} \) be a sequence of non-negative real numbers, such that the series \( \sum_{n=1}^{\infty} a_n^p \) converges. Then the inequality

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p,
\]

holds.

**Theorem 0.2.** Let $p > 1$ and $f$ is a non-negative $p$-integrable function on $(0, \infty)$. Then $f$ is integrable over the interval $(0, x)$ for all $x > 0$ and the inequality

\[
\int_{0}^{x} \left( \frac{1}{x} \int_{0}^{x} f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_{0}^{\infty} f^p(x) dx,
\]

holds.

In fact, by applying Theorem 0.2 with step functions we find that Theorem 0.2 implies Theorem 0.1.

The inequalities (6) and (7) are called the *discrete Hardy inequality* and the *continuous Hardy inequality*, respectively. Moreover, the constant \( \left( \frac{p}{p-1} \right)^p \) in both inequalities (6) and (7) is sharp in the sense that it can not be replaced by any smaller number.

Nowadays, a lot of books and articles have been dedicated to the investigation and generalization of Hardy inequalities. The first
book on the Hardy inequality was the book of G.H. Hardy, J.E. Littlewood and G. Pólya [55] in 1934. The first book which was completely devoted to the Hardy inequality, was published in 1990 by B. Opic and A. Kufner [95]. We also mention here the recent book of A. Kufner, L.-E. Persson and N. Samko [69], which is devoted to give a basic overview of weighted Hardy-type inequalities including the most recent developments and open questions (see also [98]). A description of the most important steps in the development of Hardy-type inequalities has been described by A. Kufner, L. Maligranda and L.-E. Persson [70].

In 1928 G.H. Hardy [53] proved the first weighted form of inequality (7) as follows:

Theorem 0.3. Let $p > 1$, $\alpha < p - 1$. Then the inequality

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt\right)^p x^\alpha dx \right)^{\frac{1}{p}} \leq \left(\int_0^\infty \left(\frac{p}{p - \alpha - 1}\right)^p f^p(x) x^\alpha dx\right)^{\frac{1}{p}}$$

holds for all measurable non-negative functions $f$. Moreover, the constant $\left(\frac{p}{p - \alpha - 1}\right)^p$ is the best possible.

It was later on discovered that Theorem 0.3 is not a genuine generalization of Theorem 0.2. In fact, both inequalities (7) and (8) can equivalently be transformed to the same basic Hardy-type inequality (which in turn follows easily from Jensen’s inequality, see [99]).

During the recent decades the inequalities (6) and (7) have been developed to the following forms:

$$\left(\sum_{n=1}^\infty u_n \left(\sum_{k=1}^n a_k\right)^\frac{1}{r}\right)^{\frac{1}{p}} \leq C \left(\sum_{n=1}^\infty |a_n|^p v_n\right)^{\frac{1}{p}},$$

$$\left(\int_a^b \int_a^x f(t)dt \left|u(x)dx\right| \right)^{\frac{1}{r}} \leq C \left(\int_a^b |f(x)|^p v(x)dx\right)^{\frac{1}{p}},$$

respectively, which are called weighted Hardy-type inequalities.

In 1930 G.H. Hardy and J.E. Littlewood [54] studied inequality (10) with parameters $p$ and $r$, $1 < p < r < \infty$. More exactly, for
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the interval \((0, \infty)\), they considered the weight functions \(v(x) \equiv 1\), \(u(x) = x^{s-r}\) with \(s = \frac{r}{p}\) and obtained inequality (10). Moreover, G.A. Bliss found the best constant in this case [31]. Since this result by Bliss from 1930 it has been an open question to find the best constant for other power weights than \(v(x) = 1\). This problem was recently solved by L.E. Persson and S. Samko [100]. The investigation of (10) started for the case \(p = q\) in the papers of P.R. Beesack [25], [26], J. Kadlec and A. Kufner [63], V.R. Portnov [106], V.N. Sedov [111], F.A. Sysoeva [124] and others.

It should be noted that in 1969 G. Talenti [125] and G.A. Tomaselli [128] obtained that the condition

\[
\sup_{t>0} \left( \frac{\infty}{t} \int u(x)dx \right)^{\frac{1}{p}} \left( \frac{t}{0} \int v^{1-p'}(y)dy \right)^{\frac{1}{p'}} < \infty,
\]

is necessary and sufficient for the validity of inequality (10) in the case \(p = r\), where \(p' = \frac{p}{p-1}\). Nowadays this condition is called the Muckenhoupt condition in honour of B. Muckenhoupt, who in 1972 presented a very nice proof of this result in [89] even in a more general form with \(1 \leq p = r < \infty\).

The study of the case with different parameters \(p\) and \(r\) was started by J. S. Bradley in [32]. He considered inequality (10) on \((0, \infty)\) and proved that the condition

\[
\sup_{t>0} \left( \frac{\infty}{t} \int u(x)dx \right)^{\frac{1}{p}} \left( \frac{t}{0} \int v^{1-p'}(y)dy \right)^{\frac{1}{p'}} < \infty,
\]

is necessary for (10) to hold for all \(1 \leq p, r \leq \infty\) and that it is also sufficient for \(1 \leq p \leq r < \infty\).

In the last century weighted Hardy-type inequalities have been intensively studied by several authors e.g. K.F. Andersen and B. Muckenhoupt [16], G. Bennett [29], R.K. Juberg [62], V.M. Kokilashvili [67], V.G. Maz’ya [84], L.-E. Persson and V.D. Stepanov [103], G. Sinnamon [112], [113], G. Sinnamon and V.D. Stepanov [114], V.D. Stepanov [118] and others. Moreover, for more information we refer to the books [30],[55], [69], [70], [88], [95] and to the PhD theses of Z. Abdikalikova [13], A. Abylayeva [10], S. Barza
A more general version of the inequality (10) with non-negative kernel \( K(\cdot, \cdot) \) have been studied by many authors e.g. F. Martin-Reyes and E. Sawyer [81], R. Oinarov [92], [93], L.-E. Persson and V.D. Stepanov [103], D.V. Prokhorov [107], V.D. Stepanov [119] and [121]. In the papers [120] and [122] V.D. Stepanov studied the inequality (10) for the Riemann-Liouville integral operator. These works have given an impulse for further development of such Hardy-type inequalities also in \( q \)-calculus.

The first result related to inequality (9) belongs to K.F. Andersen and H.P. Heinig ([17], Theorem 4.1). In 1985 H.P. Heinig [57] obtained a sufficient condition for the validity of inequality (9). In 1987-1991 G. Bennett [27], [28] and [29] gave a full characterization of the weighted inequality (9), except for the case \( 0 < r < 1 < p < \infty \). The remaining case was characterized by M.S. Braverman and V.D. Stepanov [33] in 1994. C.A. Okpoti [94] in his PhD thesis proved that for the case \( 1 < p \leq r < \infty \) there are even infinite many conditions characterizing (9).

In 2014, the first Hardy-type inequality in \( q \)-calculus was obtained by L. Maligranda, R. Oinarov and L.-E. Persson [78]. They proved that the \( q \)-analog of Theorem 0.3 is the following:

**Theorem 0.4.** Let \( \alpha < \frac{p-1}{p} \). If either \( 1 \leq p < \infty \) and \( f \geq 0 \) or \( p < 0 \) and \( f > 0 \), then the inequality

\[
\int_0^\infty x^p(\alpha-1) \left( \int_0^x t^{-\alpha} f(t)d_q t \right)^{p} d_q x \leq C \int_0^\infty f^p(x)d_q x,
\]

holds with constant

\[
C = \frac{1}{[\frac{p-1}{p} - \alpha]_q^p},
\]

where \( [\alpha]_q = \frac{1-q^{\alpha}}{1-q}, \alpha \in R \).

In the case when \( 0 < p < 1 \) the inequality (11) for \( f > 0 \) holds in the reverse direction with the same constant \( C \). Moreover, in all the three cases the constant \( C \) is the best possible.
**Remark 0.5.** The constant in the inequality (11) is smaller than the one in (8). In fact, if \( \alpha < 1 - 1/p \) with \( p > 1 \) or \( p < 0 \), then
\[
\frac{1}{(p - 1)/p - \alpha}_q < \frac{1}{p - p\alpha - 1},
\]
for \( \alpha > -1/p \). Inequality (12) is reversed if \( \alpha < -1/p \). For \( \alpha = -1/p \) both sides in (12) are equal to 1.

Estimate (12) means that \( (1 - q)/(1 - q^{(1-p)/p-\alpha}) < p/(p\alpha - 1) \) for any \( 0 < q < 1 \), which is true since the function \( h(q) := p(1 - q^{(1-p)/p-\alpha})/(p\alpha - 1) + q - 1 \) has the derivative \( h'(q) = -q^{-1/p-\alpha} + 1 < 0 \) for \( \alpha > -1/p \), and so \( h(q) > h(1) = 0 \).

**Remark 0.6.** From Theorem 0.4 with \( \alpha = 0 \) we obtain the \( q \)-analog of the inequality (7)
\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(t) d_q t \right)^p d_q x \leq \frac{1}{(p - 1)/p}_q \int_0^\infty f^p(x) d_q x, \quad f \geq 0,
\]
if \( p > 1 \) or \( p < 0 \) and \( f > 0 \). Moreover, the constant \( \frac{1}{(p - 1)/p}_q \) is best possible and \( \frac{1}{(p - 1)/p}_q < \left( \frac{p}{p-1} \right)^p \).

Moreover, the following generalization of the inequality (14) with the operator (5) involved was also obtained in [78]:

**Theorem 0.7.** Let \( p > 1 \) and \( \alpha > 0 \). Then the inequality
\[
\int_0^\infty \frac{1}{x^\alpha \Gamma(\alpha)} \left( \int_0^x (x - qt)^{\alpha-1}_q f(t) d_q t \right)^p d_q x \leq C \int_0^\infty f^p(x) d_q x,
\]
holds with best constant
\[
C = \left[ \frac{\Gamma_q(1 - \frac{1}{p})}{\Gamma_q(\alpha + 1 - \frac{1}{p})} \right]^p,
\]
where
\[
(x - qt)^{\alpha-1}_q = x^{\alpha-1}(qt/x; q)_{\alpha-1},
\]
and
\[
(b/a; q)_{\alpha-1} = (qt/x; q)_{\alpha-1} / (q^{\alpha-1}b/a; q)_{\alpha-1}, \quad \alpha \in \mathbb{R}.
\]
Remark 0.8. Up to now there is no sharp discrete analogue of the inequality (8). For examples, the following two inequalities were claimed to hold by G. Bennett ([27, p. 40-41]; see also [28, p. 407]):

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{n^{1-\alpha}} \sum_{k=0}^{n} [k^{\alpha-1} - (k-1)^{\alpha-1}] a_k \right]^p \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right]^p \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

and

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{n^\alpha} \sum_{i=1}^{n} k^{-\alpha} a_k \right]^p \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right]^p \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

whenever \( \alpha > 0, p > 1, \) and \( \alpha p > 1. \) Both the inequalities were proved independently by P. Gao [49, Corollary 3.1-3.2] (see also [50, Theorem 1.1] and [51, Theorem 6.1]) for \( p \geq 1 \) and some special cases of \( \alpha \) (This means that there are still some regions of parameters with no proof of (8)). Moreover, in [78, Theorems 2.1 and 2.3] was got proved an other sharp discrete analogue of the inequality (8) in the following form:

\[
\sum_{n=-\infty}^{\infty} \left[ \frac{1}{q^{n\lambda}} \sum_{k=0}^{n} q^{-k\lambda} a_k \right]^p \leq \frac{1}{(1 - q^\lambda)^p} \sum_{n=-\infty}^{\infty} a_n^p, \quad a_n \geq 0,
\]

and

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{q^{n\lambda}} \sum_{k=0}^{n} q^{-k\lambda} a_k \right]^p \leq \frac{1}{(1 - q^\lambda)^p} \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

for \( 0 < q < 1, p \geq 1 \) and \( \alpha < 1 - 1/p, \) where \( \lambda := 1 - 1/p - \alpha. \)

Remark 0.9. For \( p \geq 1 \) and \( \alpha < 1 - 1/p \) we establish the \( h \)-analogue of inequality (8) in fractional \( h \)-discrete calculus with
sharp constant in the following form (see [101, Theorem 3.1]):

\[
(14) \quad \int_0^\infty \left(x_h^{(\alpha-1)} \int_0^\infty f(t) \delta_h(x) \, d_h t \right)^p \, d_h x \\
\leq \left(\frac{p}{p-\alpha p-1}\right)^p \int_0^\infty f^p(x) \, d_h x, \quad f \geq 0,
\]

which is an other discrete analogue of the inequality (8). By using definitions of \( h \)-integral and factorial function in (14) we find that

\[
\sum_{n=0}^\infty \left(n^{(\alpha-1)} \sum_{k=0}^n \frac{a_k}{k^{(\alpha)}}\right)^p \leq \left(\frac{p}{p-\alpha p-1}\right)^p \sum_{n=0}^\infty a_n^p, \quad a_k \geq 0.
\]

Hence, by \( \lim_{h \to 0} n_h^{(\alpha)} = n^{\alpha} \), we have that the discrete analogue of the inequality (8) which is the more suitable variant other discrete analogue above.

There have been of great interest recently on difference equations in quantum calculus. It is caused by the development of the theory of \( q \)-calculus and \( h \)-calculus and also by its applications, see [1], [7], [19], [20], [22], [71], [104], [109]. Moreover, quantum calculus play increasingly important roles in the modeling of some engineering and science problems, as shown in [34], [36], [38], [40], [42], [43], [48], [65], [66], [85], [110]. It has been established that, in many situations, these models in quantum calculus are more suitable than the analogous models with integer derivatives and limit. See [39], for details. Since the integral inequalities, with explicit estimates of the sharp constants are so important in the study of properties of solutions of differential and integral equations, their finite difference (or discrete) analogues are also useful in the study of properties of solutions of finite difference and fractional difference equations equations in quantum calculus. One of the best known and widely used inequalities in the study of difference equations is Hardy-type inequalities, see e.g [52] and [53]. In this connection we refer to [69] and [70].

In paper A (see also [102]) we study an operator \( I_{q,n} \) of the following form:

\[
I_{q,n} f(x) = \frac{1}{\Gamma_q(n)} \int_0^\infty \mathcal{X}_{(0,x)}(s) K_{n-1}(x, s) f(s) ds,
\]

where \( n \in \mathbb{N} \) and \( K_{n-1}(x, s) = (x - qs)^{n-1} \). The conjugate operator \( I_{q,n}^* \) is defined by

\[
I_{q,n}^* f(s) := \frac{1}{\Gamma_q(n)} \int_0^\infty \mathcal{X}_{(s,\infty)}(x) K_{n-1}(x, s) f(x) dx.
\]

Let \( 1 < r, p \leq \infty \). Then the \( q \)-analog of the Hardy-type inequality for the operator \( I_{q,n} \) is of the following form:

\[
(15) \quad \left( \int_0^\infty u^r(x) (I_{q,n} f(x))^r d_q x \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty v^p(x) f^p(x) d_q x \right)^{\frac{1}{p}}.
\]

The dual inequality of the inequality (15) reads:

\[
(16) \quad \left( \int_0^\infty u^r(x) (I_{q,n}^* f(x))^r d_q x \right)^{\frac{1}{r}} \leq C^* \left( \int_0^\infty v^p(x) f^p(x) d_q x \right)^{\frac{1}{p}}.
\]

For \( 0 \leq m \leq n - 1, m + 1, n \in \mathbb{N} \), we use the following notations:

\[
Q_m^{-1} = \left\{ \int_0^\infty \left[ \int_0^\infty \mathcal{X}_{(0,z)}(s) K_{m}^{(r)}(z, s) v^{-p'}(s) d_q s \right]^{\frac{r}{p(r-1)}} \frac{u^r(x) d_q x}{p-r} \times \left[ \int_0^\infty \mathcal{X}_{(z,\infty)}(x) K_{n-m-1}^{(r)}(x, z) u^r(x) d_q x \right]^{\frac{p}{p-r}} \times D_q \left[ \int_0^\infty \mathcal{X}_{(0,z)}(s) K_{m}^{(p)}(z, s) v^{-p'}(s) d_q s \right]^{\frac{p-r}{p}} \right\}.
\]
\[ Q_{m}^{n-1} = \left\{ \int_{0}^{\infty} \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(s)K_{m}^{r}(z,s)u^{r}(s)d_{q}s \right]^{\frac{r}{p-r}} \right. \]

\[ \times \left[ \int_{0}^{\infty} \mathcal{X}_{[z,\infty)}(x)K_{n-1}^{p'}(z,x)v^{p'}(x)d_{q}x \right]^{\frac{r(p-1)}{p-r}} \]

\[ \times D_{q} \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(s)K_{m}^{r}(z,s)u^{r}(s)d_{q}s \right] \}, \]

\[ H_{m}^{n-1} = \sup_{z>0} \left[ \int_{0}^{\infty} \mathcal{X}_{[z,\infty)}(x)K_{n-1}^{r}(x,z)u^{r}(x)d_{q}x \right]^{\frac{1}{r}} \]

\[ \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(s)K_{m}^{p'}(z,s)v^{p'}(s)d_{q}s \right]^{\frac{1}{p'}} , \]

\[ H_{m}^{n-1} = \sup_{z>0} \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(x)K_{m}^{r}(z,x)u^{r}(x)d_{q}x \right]^{\frac{1}{r}} \]

\[ \int_{0}^{\infty} \mathcal{X}_{[z,\infty)}(x)K_{n-1}^{p'}(z,x)v^{p'}(x)d_{q}x \right]^{\frac{1}{p'}} , \]

\[ A^{+}(z) = \left[ \int_{0}^{\infty} \mathcal{X}_{[z,\infty)}(x)u^{r}(x) \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(t)K_{n-1}^{p'}(x,t)v^{p'}(t)d_{q}t \right]^{\frac{r}{p'}} d_{q}x \right]^{\frac{1}{p'}} , \]

\[ A^{-}(z) = \left[ \int_{0}^{\infty} \mathcal{X}_{(0,z]}(t)v^{p'}(t) \left[ \int_{0}^{\infty} \mathcal{X}_{[z,\infty)}(x)K_{n-1}^{r}(x,t)u^{r}(x)d_{q}x \right]^{\frac{r}{p'}} d_{q}t \right]^{\frac{1}{p'}} , \]
Our main results read:

**Theorem 0.10.** (i) Let $1 < r < p < \infty$. Then the inequality (15) holds if and only if $Q_{n-1} < \infty$. Moreover, $Q_{n-1} \approx C$, where $C$ is the best constant in (15).

(ii) Let $1 < p \leq r < \infty$. Then the inequality (15) holds if and only if at least one of the conditions $H_{n-1} < \infty$ or $A_q^+ < \infty$ or $A_q^- < \infty$ holds. Moreover, $H_{n-1} \approx A_q^+ \approx A_q^- \approx C$, where $C$ is the best constant in (15).

**Theorem 0.11.** (i) Let $1 < r < p < \infty$. Then the inequality (16) holds if and only if $Q_{n-1} < \infty$. Moreover, $Q_{n-1} \approx C^*$, where $C^*$ is the best constant in (16).

(ii) Let $1 < p \leq r < \infty$. Then the inequality (16) holds if and only if at least one of the conditions $H_{n-1} < \infty$ or $A_q^+ < \infty$ or $A_q^- < \infty$ holds. Moreover, $H_{n-1} \approx A_q^+ \approx A_q^- \approx C$, where $C$ is the best constant in (16).
In paper B (see also [21]) we consider the inequality of the following form:

\[
\left( \int_0^\infty \left( u(x) \int_0^\infty X_{(0,2]}(t)v(t)d_qt \right)^r d_qx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty f^p(x)d_qx \right)^{\frac{1}{p}},
\]

and derive necessary and sufficient conditions (of Muckenhoupt-Bradley type) for the validity of the inequality (17) with all possible positive values of the parameters \( r \) and \( p \). Here

\[
\int_0^\infty X_{(0,2]}(t)f(t)d_qt = (1 - q) \sum_{q^i \leq x} q^i f(q^i).
\]

One main result reads:

**Theorem 0.12.** Let \( 1 < p \leq r < \infty \). Then the inequality (17) holds if and only if

\[
D_1 = \sup_{z > 0} \left[ \int_0^\infty X_{(z,\infty)}(x)u^r(x)d_qx \right]^{\frac{1}{r}} \left[ \int_0^\infty X_{(0,2]}(t)v^p(t)d_qt \right]^{\frac{1}{p}} < \infty
\]

or

\[
D_2 = \sup_{z > 0} \left[ \int_0^\infty X_{(0,z]}(t)v^p(t)d_qt \right]^{-\frac{1}{p}} \left[ \int_0^\infty X_{(0,2]}(x)u^r(x) \int_0^\infty X_{(0,2]}(t)v^p(t)d_qt \right]^r d_qx < \infty
\]
or

\[
D_3 = \sup_{z > 0} \left[ \int_0^\infty \chi_{[z,\infty)}(x) u^r(x) d_q x \right]^{\frac{1}{r}} \frac{1}{r}
\left[ \int_0^\infty \chi_{[z,\infty)}(x) u^r(x) d_q x \right]^{\frac{1}{r}} d_q t < \infty.
\]

Moreover, for the sharp constant in (17) we have that \(C \approx D_1 \approx D_2 \approx D_3\).

We also study the dual inequality of the inequality (17) as follows:

\[
(18)
\left( \int_0^\infty (\int_0^\infty \chi_{(x,\infty)}(x) u(x) d_q x)^{\frac{1}{p'}} \frac{1}{p'} \left[ \int_0^\infty \chi_{[z,\infty)}(x) u^r(x) d_q x \right]^{\frac{1}{p'}} d_q t \right) \leq C \left( \int_0^\infty g^{r'}(t) d_q t \right)^{\frac{1}{r'}},
\]

where

\[
\int_0^\infty \chi_{[x,\infty)}(t) f(t) d_q t = (1 - q) \sum_{q' \geq x} q^i f(q^i).
\]

In this case our main result is the following:

**Theorem 0.13.** Let \(1 < p \leq r < \infty\). Then the inequality (18) holds if and only if

\[
D_1^* = \sup_{z > 0} \left[ \int_0^\infty \chi_{(0,z)}(x) u^r(x) d_q x \right]^{\frac{1}{r}} \frac{1}{r}
\left[ \int_0^\infty \chi_{[z,\infty)}(t) u^{r'}(t) d_q t \right]^{\frac{1}{r'}} < \infty.
\]
or

\[
D^*_2 = \sup_{z>0} \left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(t) v'(t) d_q t \right]^{-\frac{1}{p}} \\
\left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(x) u^r(x) \left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(t) v'(t) d_q t \right] d_q x \right]^{\frac{1}{r}} < \infty
\]

or

\[
D^*_3 = \sup_{z>0} \left[ \int_0^\infty \mathcal{X}_{[0,z]}(x) u^r(x) d_q x \right]^{-\frac{1}{p'}} \\
\left[ \int_0^\infty \mathcal{X}_{[0,z]}(t) v'(t) \left[ \int_0^\infty \mathcal{X}_{[0,z]}(x) u^r(x) d_q x \right] d_q t \right]^{\frac{1}{p'}} < \infty.
\]

Moreover, for the sharp constant in (18) we have that \( C \approx D^*_1 \approx D^*_2 \approx D^*_3 \).

Concerning other possible parameters of \( p \) and \( r \) we have the following complement of Theorem 0.10 (Theorem 0.11):

**Theorem 0.14.** (i). Let \( 0 < p \leq 1, \ p \leq r < \infty \). Then the inequality (17) holds if and only if

\[
D_4 = \sup_{z>0} \left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(x) u^r(x) d_q x \right]^{\frac{1}{p'}} \left[ \int_0^\infty \mathcal{X}_{(qz,z]}(t) v'(t) d_q t \right]^{\frac{1}{p}} < \infty.
\]
(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (17) holds if and only if

$$D_5 = \left[ \int_0^\infty \left[ \int_0^\infty \mathcal{X}_{[0,z]}(t) v'(t) d_q t \right]^{\frac{r(p-1)}{p-r}} \right]^{\frac{1}{r}} \left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(x) u'(x) d_q x \right]^{\frac{1}{p-r}} \frac{\mu}{p} < \infty.$$

(iii). Let $0 < r < p = 1$. Then the inequality (17) is satisfied if and only if

$$D_6 = \left[ \int_0^\infty \sup_{y<z} \left[ \int_0^y \mathcal{X}_{[y,q]}(t) \frac{v(t)}{(1-q)t} d_q t \right]^{\frac{1}{1-r}} \right]^{\frac{r}{1-r}} \left[ \int_0^\infty \mathcal{X}_{[z,\infty)}(x) u'(x) d_q x \right]^{\frac{1}{p-r}} \frac{\mu}{r} < \infty.$$

In all cases (i)-(iii), for the best constant in (17) it yields that $C \approx D_i$, $i = 4, 5, 6$, respectively.

**Theorem 0.15.** (i). Let $0 < p \leq 1$, $p \leq r < \infty$. Then the inequality (18) holds if and only if

$$D_4^* = \sup_{z>0} \left[ \int_0^\infty \mathcal{X}_{[0,z]}(x) u'(x) d_q x \right]^{\frac{1}{p}} \left[ \int_0^\infty \mathcal{X}_{[z,q-1,z]}(t) v'(t) d_q t \right]^{\frac{1}{p}} < \infty.$$
(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (18) holds if and only if

$$D_5^* = \left[ \int_0^\infty \int_0^\infty X_{(0,x)}(x)u^r(x)dx \right]^\frac{r}{p-r} \left[ \int_0^\infty X_{[z,\infty)}(t)v^\varphi(t)dt \right]^\frac{r(p-1)}{p-r} u^r(z)dz < \infty.$$ 

(iii). Let $0 < r < p = 1$. Then the inequality (18) holds if and only if

$$D_6^* = \left[ \int_0^\infty \sup_{y \geq z} \left[ \int_0^y X_{[y,\infty)}(t)v^\varphi(t)dt \right] \right]^\frac{r}{1-r} \left[ \int_0^\infty X_{(0,z)}(x)u^r(x)dx \right]^\frac{1-r}{r} u^r(z)dz < \infty.$$ 

In all cases (i)-(iii), for the best constant in (18) it yields that $C \approx D_i^*$, $i = 4, 5, 6$, respectively.

In paper C (see also [115]) we investigate the inequality of the following form:

$$(19) \left( \int_0^\infty u^r(x) \left( \int_0^x s^{(r-1)} \ln_q \frac{x}{x - qs} f(s)ds \right)^\frac{r}{p} \right)^\frac{1}{r} \leq C \left( \int_0^\infty f^p(s)ds \right)^\frac{1}{p}, \forall f(\cdot) \geq 0,$$

where

$$\ln_q \frac{x}{x - s} := \sum_{j=1}^\infty \frac{(\frac{x}{s})^j}{[j]_q}.$$ 

and $C$ is a positive finite constant independent of $f$. 
Our main results for the inequality (19) read as follows:

**Theorem 0.16.** Let \(1 < p \leq r < \infty, \gamma > \frac{1}{p}\). Then the inequality (19) holds if and only if \(B_1 < \infty\), where

\[
B_1 := \sup_{x > 0} x^{\gamma + \frac{1}{p}} \left[ \int_0^\infty \mathcal{X}_{x, \infty}(t) \frac{u^r(t)}{t^r} \, dq \right]^\frac{1}{r},
\]

Moreover, \(B_1 \approx C\), where \(C\) is best constant in (19).

**Theorem 0.17.** Let \(0 < r < p < \infty, 1 < p < \gamma > \frac{1}{p}\). Then the inequality (19) holds if and only if \(B_2 < \infty\), where

\[
B_2 := \left[ \int_0^\infty \left( \frac{\int_0^\infty \mathcal{X}_{x, \infty}(t) \frac{u^r(t)}{t^r} \, dq}{x^{\gamma + \frac{1}{p}}} \right)^{\frac{p^r}{p - \frac{r}{p}}} \right]^{\frac{r}{p - \frac{r}{p}}} \, dx.
\]

Moreover, \(B_2 \approx C\), where \(C\) is best constant in (19).

**Remark 0.18.** The only \(q\)-analogs of Hardy-type inequalities so far are that in Theorem A and those presented in this PhD thesis. Hence, it remains a great number of open questions.

In paper D (see also [101]) we consider the first power weighted version of Hardy’s inequality (see (8)) in slightly rewritten form. Our first main result is the following \(h\)-integral analogue of the inequality (8) reads for \(1 \leq p < \infty\):

**Theorem 0.19.** Let \(\alpha < \frac{p - 1}{p}\) and \(1 \leq p < \infty\). Then the inequality

\[
\int_0^\infty \left( a_{(\alpha - 1)}^{(h))} \int_0^\infty \frac{f(t)d_h t}{t^{(\alpha)}} \right)^p d_h x \leq \left[ \frac{p}{p - \alpha p - 1} \right]^p \int_0^\infty f^p(x) d_h x,
\]

holds for all \(f \geq 0\). Moreover, the constant \(\left[ \frac{p}{p - \alpha p - 1} \right]^p\) is the best possible in (20).
Our second main result is the following $h$-integral analogue of the reversed form of (7) for $0 < p < 1$.

**Theorem 0.20.** Let $\alpha < \frac{p-1}{p}$ and $0 < p < 1$. Then the inequality

$$
\int_0^\infty f^p(x)dhx \leq \left[ \frac{p - p\alpha - 1}{p} \right]^p \int_0^\infty \left( \delta_h^{(\alpha)} \frac{\int_0^t f(t)dh \delta_h^{(\alpha)}}{t_h^{(\alpha)}} \right)^p dhx,
$$

holds for all $f \geq 0$. Moreover, the constant $\left[ \frac{p - p\alpha - 1}{p} \right]^p$ is the best possible in (21).

**Remark 0.21.** The only $h$-analogs of Hardy-type inequalities so far are that in Theorem A and those presented in this PhD thesis. Hence, it remains a great number of open questions.

I the paper E (see also [117]) we consider the fractional order Hardy-type inequality in the following form:

$$
\left( \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha} p^\alpha} \ dx \ dy \right)^p \leq C \left( \int_0^\infty |f'(x)|^p x^{(1-\alpha)p} \ dx \right)^p
$$

for $0 < \alpha < 1$ and $1 < p < \infty$ in fractional $h$-discrete calculus, where $C = \frac{2^\frac{1}{p} \alpha^{-1}}{(p-p\alpha)^\frac{1}{p}}$.

Concerning this inequality and similar ones we refer to Chapter 5 of the book [69] by A. Kufner, L.-E. Persson and N. Samko. In this case our main result read:
Theorem 0.22. Let $1 < p < \infty$, $0 < \alpha < 1$ and $f(x) = D_h F(x)$. Then the following inequality

\[
\frac{1}{p} \left[ \int_0^\infty \int_0^\infty \left| F(x) - F(y) \right|^p d_h x d_h y \right]^{\frac{1}{p}} \leq C \left[ \int_0^\infty \frac{|f(x)|^p d_h x}{(x + h)^{(\alpha - 1)}} \right]^{\frac{1}{p}},
\]

holds with constant $C = \frac{2^{\frac{1}{p} \alpha - 1}}{(p - \alpha \alpha)^p}$. Moreover, this constant sharp.

Remark 0.23. The inequality (22) is the only fractional order Hardy inequality in $h$-calculus in the literature so also have these are many open questions for further research (s.g. Chapter 5 of the book [69]).
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Paper A
Some new Hardy-type inequalities for Riemann-Liouville fractional $q$-integral operator

Lars-Erik Persson$^{1,2,*}$ and Serikbol Shaimardan$^3$

Abstract

We consider the $q$-analog of the Riemann-Liouville fractional $q$-integral operator of order $n \in \mathbb{N}$. Some new Hardy-type inequalities for this operator are proved and discussed.

MSC: Primary 26D10; 26D15; secondary 33D05; 39A13

Keywords: inequalities; Hardy-type inequalities; Riemann-Liouville operator; integral operator; $q$-calculus; $q$-integral

1 Introduction

In 1910 FH Jackson defined $q$-derivative and definite $q$-integral [1] (see also [2]). It was the starting point of $q$-analysis. Today the interest in the subject has exploded. The $q$-analysis has numerous applications in various fields of mathematics, e.g., dynamical systems, number theory, combinatorics, special functions, fractals and also for scientific problems in some applied areas such as computer science, quantum mechanics and quantum physics (see, e.g., [3–7]). For further development and recent results in $q$-analysis, we refer to the books [2, 3] and [5] and the references given therein. The first results concerning integral inequalities in $q$-analysis were proved in 2004 by Gauchman [8]. Later on some further $q$-analogs of the classical inequalities have been proved (see [9–12]). Moreover, in 2014 Maligranda et al. [13] derived a $q$-analog of the classical Hardy inequality. Further development of Hardy’s original inequality from 1925 (see [14] and [15]) has been enormous. Some of the most important results and applications have been presented and discussed in the books [16, 17] and [18]. Hence, it seems to be a huge new research area to investigate which of these so-called Hardy-type inequalities have their $q$-analogs.

The aim of this paper is to obtain some $q$-analogs of Hardy-type inequalities for the Riemann-Liouville fractional integral operator of order $n \in \mathbb{N}$ and to find necessary and sufficient conditions of the validity of these inequalities for all non-negative real functions (see Theorems 3.1 and 3.2).

The paper is organized as follows. In order not to disturb our discussions later on, some preliminaries are presented in Section 2. The main results can be found in Section 3, while the detailed proofs are given in Section 4.
2 Preliminaries

First we recall some definitions and notations in \( q \)-analysis from the recent books \([2, 3]\) and \([5]\).

Let \( q \in (0, 1) \). Then a \( q \)-real number \([\alpha]_q\) is defined by
\[
[\alpha]_q := \frac{1 - q^\alpha}{1 - q}, \quad \alpha \in \mathbb{R},
\]
where \( \lim_{q \to 1} \frac{1 - q^\alpha}{1 - q} = \alpha \).

The \( q \)-analog of the binomial coefficients is defined by
\[
\begin{align*}
\left[ \begin{array}{c} n \\ k \end{array} \right]_q & := \frac{[n]_q}{[k]_q [n-k]_q}, & n \in \mathbb{N}, \\
\left[ \begin{array}{c} n \\ k \end{array} \right]_q & := \frac{[n]_q}{[n-k]_q [k]_q}, & n \in \mathbb{N}.
\end{align*}
\]

We introduce the \( q \)-analog of a polynomial in the following way:
\[
(x - a)_q^n := \begin{cases} 1 & \text{if } n = 0, \\
(x - a)(x - qa) \cdots (x - q^{n-1}a) & \text{if } n \in \mathbb{N},
\end{cases}
\]
\[
(x - a)_q^{m+n} = (x - a)_q^m (x - q^m a)_q^n, \quad n, m = 0, 1, 2, \ldots
\]

The \( q \)-gamma function \( \Gamma_q \) is defined by
\[
\Gamma_q(n+1) := [n]_q!, \quad n \in \mathbb{N}.
\]

For \( f : [0, b) \to \mathbb{R}, \ 0 < b \leq \infty \), we define the \( q \)-derivative as follows:
\[
D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \in [0, b).
\]

Clearly, if the function \( f(x) \) is differentiable at a point \( x \in (0, 1) \), then \( \lim_{q \to 1} D_q f(x) = f'(x) \).

Let \( 0 < a \leq b < \infty \). The definite \( q \)-integral (also called the \( q \)-Jackson integral) of a function \( f(x) \) is defined by the formulas
\[
\int_0^b f(x) \, d_q x := (1-q)a \sum_{k=0}^{\infty} q^k f(q^k a).
\]

Moreover, the improper \( q \)-integral of a function \( f(x) \) is defined by
\[
\int_0^\infty f(x) \, d_q x := (1-q) \sum_{k=-\infty}^{\infty} q^k f(q^k),
\]
provided that the series on the right-hand sides of (3) and (4) converge absolutely.

Suppose that \( f(x) \) and \( g(x) \) are two functions which are defined on \((0, \infty)\). Then
\[
\int_0^\infty f(x) D_q (g(x)) = \sum_{j=0}^{\infty} f(q^j) (g(q^j) - g(q^{j+1})).
\]
Let $\Omega$ be a subset of $(0, \infty)$ and $\mathcal{X}(t)$ denote the characteristic function of $\Omega$. For all $z$: $0 < z < \infty$, we have that

$$\int_0^\infty \mathcal{X}(t)f(t)\,dt = (1 - q) \sum_{q' \leq z} q'f(q')$$

(6)

and

$$\int_0^\infty \mathcal{X}(z\infty)f(t)\,dt = (1 - q) \sum_{q' \geq z} q'f(q').$$

(7)

Al-Salam (see [19] and also [3]) introduced the fractional $q$-integral of the Riemann-Liouville operator $I_{q,n}$ of order $n \in \mathbb{N}$ by

$$I_{q,n}f(x) := \frac{1}{\Gamma_q(n)} \int_0^x K_{n-1}(x,s)f(s)\,ds,$$

where $K_{n-1}(x,s) = (x - qs)^{n-1}$.

Next we will present a lemma (Lemma 2.1) concerning discrete Hardy-type inequalities which are proved in [20]. In this paper all authors studied inequalities of the form

$$\left( \sum_{j=1}^\infty u_j\left( (S_n f)_{ij} \right)^{\frac{1}{p}} \left( (S_n f)_{ij} \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \leq C \left( \sum_{j=1}^\infty v_j f_{ij} \right)^{\frac{1}{q}}, \quad \forall f \geq 0$$

(8)

for the $n$-multiple discrete Hardy operator with weights of the form

$$(S_n f) := \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \omega_{k_1} \cdots \omega_{k_n} f_1 \cdots f_n$$

where $u = \{u_i\}_{i=1}^\infty$, $v = \{v_i\}_{i=1}^\infty$, $\omega_1 = (\omega_{k_1})_{k_1=1}^\infty$ are positive sequences of real numbers (i.e., weight sequences). She also studied inequality (8) for the operator $S_n^*$ defined by

$$(S_n^* f) := \sum_{j=1}^n f_j A_{n-1,1}(i,j),$$

which is the conjugate to the operator $S_n$, where $A_{n-1,1}(i,j) \equiv 1$ for $n = 1$ and

$$A_{n-1,1}(i,j) = \sum_{k_{n-1,j} = 1}^i \omega_{k_{n-1,j}} \cdots \sum_{k_{n-2,j} = 1}^i \omega_{k_{n-2,j}} \cdots \sum_{k_{1,j} = 1}^i \omega_{k_{1,j}},$$

for $n \geq 2$.

We consider the following Hardy-type inequalities:

$$\left( \sum_{j=-\infty}^\infty u_j\left( (S_n f)_{ij} \right)^{\frac{1}{p}} \right)^{\frac{1}{2}} \leq C \left( \sum_{j=-\infty}^\infty v_j f_{ij} \right)^{\frac{1}{q}}$$

(9)

and

$$\left( \sum_{j=-\infty}^\infty u_j\left( (S_n f)_{ij} \right)^{\frac{1}{p}} \right)^{\frac{1}{2}} \leq C^\ast \left( \sum_{j=-\infty}^\infty v_j f_{ij} \right)^{\frac{1}{q}}.$$ 

(10)
In the sequel, for any $p > 1$, the conjugate number $p'$ is defined by $p' := p/(p - 1)$, and
the considered functions are assumed to be non-negative. Moreover, the symbol $M \ll K$
means that there exists $\alpha > 0$ such that $M \leq \alpha K$, where $\alpha$ is a constant which may depend
only on parameters such as $p, q, r$. Similarly, the case $K \ll M$. If $M \ll K \ll M$, then we
write $M \approx K$.

Lemma 2.1
(i) Let $1 < p \leq r < \infty$ and $n \geq 1$. Then inequality (9) holds if and only if
$\mathfrak{A}(n) = \max_{0 \leq m \leq n-1} \mathfrak{A}_m(n) < \infty$, where

$$\mathfrak{A}_m(n) = \sup_{k \in \mathbb{Z}} \left( \sum_{j=k}^{\infty} \sum_{i=k}^{\infty} A_{i,j} \langle j, \nu \rangle v_j^p \right)^{1/p} \left( \sum_{k=\infty}^{\infty} A_{m-1,m+1}(k, i) u_i^r \right)^{1/r}, \quad n \in \mathbb{N}.$$ 

Moreover, $\mathfrak{A}(n) \approx C$, where $C$ is the best constant in (9).

(ii) Let $1 < p \leq r < \infty$ and $n \geq 1$. Then inequality (10) holds if and only if
$\mathfrak{A}^*(n) = \max_{0 \leq m \leq n-1} \mathfrak{A}_m^*(n) < \infty$, where

$$\mathfrak{A}_{m}^*(n) = \sup_{k \in \mathbb{Z}} \left( \sum_{j=k}^{\infty} A_{i,j}^r \langle j, \nu \rangle v_j^p \right)^{1/p} \left( \sum_{k=\infty}^{\infty} A_{m-1,m+1}(k, i) u_i^r \right)^{1/r}, \quad n \in \mathbb{N}.$$ 

Moreover, $\mathfrak{A}^*(n) \approx C$, where $C$ is the best constant in (10).

We also need the corresponding result for the case $1 < r < p < \infty$, which was proved in
[21].

Lemma 2.2
(i) Let $1 < r < p < \infty$ and $n \geq 1$. Then inequality (9) holds if and only if
$\mathfrak{B}(n) = \max_{0 \leq m \leq n-1} \mathfrak{B}_m(n) < \infty$, where

$$\mathfrak{B}_m(n) = \left( \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} A_{i,j} \langle j, \nu \rangle v_j^p \right)^{1/p} \left( \sum_{k=\infty}^{\infty} A_{m-1,m+1}(k, i) u_i^r \right)^{1/r} \times \Delta^r \left( \sum_{j=k}^{\infty} A_{m-1,m+1}(k, i) v_j^p \right)^{1/p}, \quad \Delta^r E_{ij} = E_{ij} - E_{ij+1}, n \in \mathbb{N}.$$ 

Moreover, $\mathfrak{B}(n) \approx C$, where $C$ is the best constant in (9).

(ii) Let $1 < r < p < \infty$ and $n \geq 1$. Then inequality (10) holds if and only if
$\mathfrak{B}^*(n) = \max_{0 \leq m \leq n-1} \mathfrak{B}_m^*(n) < \infty$, where

$$\mathfrak{B}_{m}^*(n) = \left( \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} A_{i,j}^r \langle j, \nu \rangle v_j^p \right)^{1/p} \left( \sum_{k=\infty}^{\infty} A_{m-1,m+1}(k, i) u_i^r \right)^{1/r} \times \Delta^r \left( \sum_{j=k}^{\infty} A_{m-1,m+1}(k, i) v_j^p \right)^{1/p}, \quad \Delta^r E_{ij} = E_{ij} - E_{ij+1}, n \in \mathbb{N}.$$ 

Moreover, $\mathfrak{B}^*(n) \approx C^*$, where $C^*$ is the best constant in (10).
Let \((a_{ij}^{(n)})\) be a matrix whose elements are non-negative and non-increasing in the second index for all \(i,j: \infty > i \geq j > -\infty\), and the entries of the matrix \(a_{ij}^{(n)}\) satisfy the following (so-called discrete Oinarov condition):

\[
d_{ij}^{(n)} \approx \sum_{\gamma=0}^{n} a_{i,k}^{(\gamma)} d_{k,j}^{\mu_{ij}}, \quad \gamma = 1,2,\ldots,n-1,n \in \mathbb{N} \tag{11}
\]

for all \(\infty > i \geq k \geq j > -\infty\).

**Remark 2.3** Note that the matrices \((d_{ij}^{(m)})\), \(\gamma = 0,1,\ldots,m, m \geq 0\), are arbitrary non-negative matrices which satisfy (11) (see [22]).

Moreover, in [22] necessary and sufficient conditions for inequalities (9) and (10) were proved for matrix operators with a matrix \((a_{ij}^{(n)})\) which satisfies (11). For our purposes we need such characterization on the following form.

**Lemma 2.4**

(i) Let \(1 < p \leq r < \infty\) and the entries of the matrix \((a_{ij}^{(n)})\) satisfy condition (11). Then inequality (9) for the operator \((A f)_i := \sum_{j,i} a_{ij}^{(n)} f_j, f \in Z\), holds if and only if at least one of the conditions \(B_{i}^{+} < \infty\) or \(B_{i}^{-} < \infty\) holds, where

\[
B_{i}^{-} = \sup_{k \in Z} \left( \sum_{i-k}^{\infty} v_{j}^{-p'} \left( \sum_{j=-\infty}^{k} \left( a_{ij}^{(n)} \right)^{p'} u_{j}^{r} \right) \right)^{\frac{1}{p}},
\]

\[
B_{i}^{+} = \sup_{k \in Z} \left( \sum_{i-k}^{\infty} u_{j}^{r} \left( \sum_{j=-\infty}^{k} \left( a_{ij}^{(n)} \right)^{p'} v_{j}^{-p'} \right) \right)^{\frac{1}{p'}}.
\]

Moreover, \(B_{i}^{+} \approx B_{i}^{-} \approx C\), where \(C\) is the best constant in (9).

(ii) Let \(1 < p \leq r < \infty\). Let the entries of the matrix \((a_{ij}^{(n)})\) satisfy condition (11). Then inequality (10) for the operator \((A f)_i := \sum_{j \rightarrow \infty} a_{ij}^{(n)} f_j, f \in Z\), holds if and only if at least one of the conditions \(A_{i}^{+} < \infty\) or \(A_{i}^{-} < \infty\) holds, where

\[
A_{i}^{-} = \sup_{k \in Z} \left( \sum_{i-k}^{\infty} u_{j}^{r} \left( \sum_{j=-\infty}^{k} \left( a_{ij}^{(n)} \right)^{p'} v_{j}^{-p'} \right) \right)^{\frac{1}{p}},
\]

\[
A_{i}^{+} = \sup_{k \in Z} \left( \sum_{i-k}^{\infty} v_{j}^{-p'} \left( \sum_{j=-\infty}^{k} \left( a_{ij}^{(n)} \right)^{p'} u_{j}^{r} \right) \right)^{\frac{1}{p'}}.
\]

Moreover, \(A_{i}^{+} \approx A_{i}^{-} \approx C\), where \(C\) is the best constant in (10).

3 The main results

Let \(1 < r, p \leq \infty\). Then the \(q\)-analog of the two-weighted inequality for the operator \(I_{q,n}\) of the form

\[
\left( \int_{0}^{\infty} u^{q'}(x)(I_{q,n} f(x))^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{0}^{\infty} v^{q}(x)f^{q}(x) dx \right)^{\frac{1}{q}} \tag{12}
\]
has several applications in various fields of science. In the classical analysis two-weighted estimates for the Riemann-Liouville fractional operator were derived by Stepanov for the case with parameters greater than one (see [23, 24]).

We consider the operator $I_{q,p}$ of the following form:

$$I_{q,p}f(x) = \frac{1}{\Gamma_q(n)} \int_0^\infty \lambda_{(0,x]}(s)K_{n-1}(x,s)f(s)\,dq\,s,$$

which is defined for all $x > 0$. Although it does not coincide with the operator $I_{q,p}^*$ (they coincide at the points $x = q^k$, $k \in \mathbb{Z}$), we have the equality

$$\int_0^\infty u'(x)(I_{q,p}f(x))'\,dq\,x = \int_0^\infty u'(x)(I_{q,p}f(x))^p\,dq\,x.$$

Therefore, inequality (12) can be rewritten as

$$\left(\int_0^\infty u'(x)(I_{q,p}f(x))^p\,dq\,x\right)^{1/p} \leq C\left(\int_0^\infty v^p(x)f^p(x)\,dq\,x\right)^{1/p}.$$  \hspace{1cm} (13)

Its conjugate operator $I_{q,p}^*$ can be defined by

$$I_{q,p}^*f(s) := \frac{1}{\Gamma_q(n)} \int_0^\infty \lambda_{[x,\infty)}(s)K_{n-1}(x,s)f(x)\,dq\,x,$$

with the same kernel. The dual inequality of inequality (13) reads as follows:

$$\left(\int_0^\infty u'(x)(I_{q,p}^*f(x))^p\,dq\,x\right)^{1/p} \leq C^*\left(\int_0^\infty v^p(x)f^p(x)\,dq\,x\right)^{1/p},$$

where $C$ and $C^*$ are positive constants independent of $f$ and $u(\cdot)$, $v(\cdot)$ are positive real-valued functions on $(0, \infty)$, i.e., weight functions. In what follows we investigate inequalities (13) and (14).

Let $N_0 = N \cup \{0\}$. Then, for $0 \leq m \leq n - 1$, $m, n \in N_0$, we use the following notations:

$$Q_{m-1}^{n-1} = \left\{ \int_0^\infty \left( \int_0^\infty \lambda_{(0,z]}(s)K_m^\mu(z,s)v^p(s)\,dq\,s \right)^{\frac{p(n-1)}{p-m}} \right\}^{\frac{p-m}{p}},$$

$$\times \left( \int_0^\infty \lambda_{[z,\infty)}(x)K_{n-m-1}(x,z)u^r(x)\,dq\,x \right)^{\frac{r}{r-m}},$$

$$\times D_q\left( \int_0^\infty \lambda_{(0,z]}(s)K_{m}^\mu(z,s)v^p(s)\,dq\,s \right)^{\frac{p-n}{p}},$$

$$Q_{m}^{n-1} = \left\{ \int_0^\infty \left( \int_0^\infty \lambda_{(0,z]}(s)K_m^\mu(z,s)v^p(s)\,dq\,s \right)^{\frac{p(n-1)}{p-m}} \right\}^{\frac{p-m}{p}},$$

$$\times \left( \int_0^\infty \lambda_{[z,\infty)}(x)K_{n-m-1}(x,z)v^r(x)\,dq\,x \right)^{\frac{r-n}{r-m}},$$

$$\times D_q\left( \int_0^\infty \lambda_{(0,z]}(s)K_{m}^\mu(z,s)u^r(s)\,dq\,s \right)^{\frac{r}{r-m}},$$

$$\times D_q\left( \int_0^\infty \lambda_{[z,\infty)}(x)K_{n-m-1}(x,z)v^p(x)\,dq\,x \right)^{\frac{p-n}{p}},$$

$$\times D_q\left( \int_0^\infty \lambda_{(0,z]}(s)K_{m}^\mu(z,s)u^r(s)\,dq\,s \right)^{\frac{r}{r-m}}.$$
\[ H_m^{-1} = \sup_{z \geq 0} \left( \int_0^\infty X(z,x) u'(x) \, dx \right)^\frac{1}{\alpha} \left( \int_0^\infty X(z,x) v'(x) \, dx \right)^\frac{1}{\beta}, \]

\[ H_m^{-1} = \sup_{z \geq 0} \left( \int_0^\infty X(z,x) u'(x) \, dx \right)^\frac{1}{\alpha} \left( \int_0^\infty X(z,x) v'(x) \, dx \right)^\frac{1}{\beta}, \]

\[ A^+(z) = \left( \int_0^\infty X(z,x) u'(x) \left( \int_0^\infty X(z,x) v'(x) \, dx \right)^\frac{1}{\beta} \right)^\frac{1}{\alpha}, \]

\[ A^-(z) = \left( \int_0^\infty X(z,x) v'(x) \left( \int_0^\infty X(z,x) u'(x) \, dx \right)^\frac{1}{\alpha} \right)^\frac{1}{\beta}, \]

\[ H_{n-1} = \max_{0 \leq k \leq n-1} H_k^{-1}, \quad H_{n-1} = \max_{0 \leq k \leq n-1} H_k^{-1}, \]

\[ A_q^+ = \sup_{z > 0} A^+(z), \quad A_q^- = \sup_{z > 0} A^-(z), \quad A_q^+ = \sup_{z > 0} A^+(z), \quad A_q^- = \sup_{z > 0} A^-(z), \]

\[ Q_n^{-1} = \max_{0 \leq k \leq n-1} Q_k^{-1} \quad \text{and} \quad Q_n^{-1} = \max_{0 \leq k \leq n-1} Q_k^{-1}. \]

Our main results read as follows.

**Theorem 3.1**

(i) Let \( 1 < r < p < \infty \). Then inequality (13) holds if and only if \( Q_{n-1} < \infty \). Moreover, \( Q_{n-1} \approx C \), where \( C \) is the best constant in (13).

(ii) Let \( 1 < p \leq r < \infty \). Then inequality (13) holds if and only if at least one of the conditions \( H_{n-1} < \infty \) or \( A_q^+ < \infty \) or \( A_q^- < \infty \) holds. Moreover, \( H_{n-1} \approx A_q^+ \approx A_q^- \approx C \), where \( C \) is the best constant in (13).

**Theorem 3.2**

(i) Let \( 1 < r < p < \infty \). Then inequality (14) holds if and only if \( Q_{n-1} < \infty \). Moreover, \( Q_{n-1} \approx C^* \), where \( C^* \) is the best constant in (14).

(ii) Let \( 1 < p \leq r < \infty \). Then inequality (14) holds if and only if at least one of the conditions \( H_{n-1} < \infty \) or \( A_q^+ < \infty \) or \( A_q^- < \infty \) holds. Moreover, \( H_{n-1} \approx A_q^+ \approx A_q^- \approx C \), where \( C \) is the best constant in (14).

For the proofs of these results, we need the following lemmata of independent interest.

**Lemma 3.3** Let \( x, t, s : 0 < s \leq t \leq x < \infty \). Then

\[ \max_{0 \leq m \leq n-1} K_{n-m-1}(x,t)K_m(t,s) \leq \sum_{m=0}^{n-1} \left[ \begin{array}{c} n-1 \\ m \end{array} \right] K_{n-m-1}(x,t)K_m(t,s) \]  \hspace{1cm} (15)

for \( m : 0 \leq m \leq n-1, n, m-1 \in \mathbb{N} \) and where \( K_{n-1}(x,s) = (x-s)^{n-1} \).

**Lemma 3.4** Let \( f \) and \( g \) be non-negative functions on \((0, \infty), \alpha, \beta \in \mathbb{R} \) and

\[ I(z) := \left( \int_0^\infty X_0(z,t)^\alpha f(t) \, dt \right)^\frac{1}{\alpha} \left( \int_0^\infty X_{\infty}(x)^\beta g(x) \, dx \right)^\frac{1}{\beta}. \]
Then
\[
\sup_{z>0} I(z) = (1 - q)^{\nu + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{j=k}^{\infty} q^j (q')^j \right) \left( \sum_{k=0}^{\infty} q^j (q')^j \right)^{\beta - r(p-1)} \frac{\alpha}{pr},
\]
where at least one of \(\alpha\) and \(\beta\) is non-negative.

This result was proved in [25], but for the readers’ convenience we will include in Section 4 a proof which is slightly simpler than that in the Russian version given in [25].

**Lemma 3.5** Let \(\alpha, \beta \in \mathbb{R}^+\), \(K(\cdot, \cdot)\) be a non-negative function and
\[
I^+(z) := \left( \int_{0}^{\infty} \mathcal{X}_{[0,z]}(x) g(x) \left( \int_{0}^{\infty} \mathcal{X}_{[0,z]}(t) K(x, t) f(t) \, dt \right)^{\alpha} \, dx \right)^{\beta},
\]
\[
I^-(z) := \left( \int_{0}^{\infty} \mathcal{X}_{[0,z]}(t) f(t) \left( \int_{0}^{\infty} \mathcal{X}_{[0,z]}(x) K(x, t) g(x) \, dx \right)^{\alpha} \, dt \right)^{\beta}.
\]

Then
\[
\sup_{z>0} I^+(z) = \sup_{k \in \mathbb{Z}} \left( 1 - q \right) \sum_{j=-\infty}^{k} q^j (q') \left( 1 - q \right) \sum_{j=-\infty}^{\infty} q^j (q') \right)^{\alpha} \beta,
\]
and
\[
\sup_{z>0} I^-(z) = \sup_{k \in \mathbb{Z}} \left( 1 - q \right) \sum_{j=-\infty}^{k} q^j (q') \left( 1 - q \right) \sum_{j=-\infty}^{\infty} q^j (q') \right)^{\alpha} \beta.
\]

**Lemma 3.6** Let \(Q_m^{n-1}, Q_m^{n-1} < \infty\) for \(0 < m \leq n - 1\). Then
\[
Q_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left( 1 - q \right) \sum_{l=i}^{\infty} q^i q^j K_m^p(q', q') v^{\nu'} (q') \right\}^{\frac{\nu'-1}{\nu'}}
\]
\[
\times \left( 1 - q \right) \sum_{j=-\infty}^{l} q^i q^j K_{n-m-1}^p(q', q') v^{\nu'} (q') \right\}^{\frac{p-1}{\nu'}}
\]
\[
\times \Delta^+ \left( \sum_{n=1}^{\infty} \left( 1 - q \right) q^n K_m^p(q', q') v^{\nu'} (q') \right\}^{\frac{p-1}{\nu'}}
\]
and
\[
Q_m^{n-1} = \left\{ \sum_{i=-\infty}^{\infty} \left( 1 - q \right) \sum_{l=i}^{\infty} q^i q^j K_m^p(q', q') v^{\nu'} (q') \right\}^{\frac{\nu'-1}{\nu'}}
\]
\[
\times \left( 1 - q \right) \sum_{j=-\infty}^{l} q^i q^j K_{n-m-1}^p(q', q') v^{\nu'} (q') \right\}^{\frac{p-1}{\nu'}}
\]
\[
\times \Delta^+ \left( \sum_{n=1}^{\infty} \left( 1 - q \right) q^n K_m^p(q', q') v^{\nu'} (q') \right\}^{\frac{p-1}{\nu'}}
\]
where \(\Delta^+ E_{n,i} = E_{n,i} - E_{n,i+1}\).
4 Proofs

Proof of Lemma 3.3 Let $0 < s \leq t \leq x < \infty$. First we prove the lower estimate. By using (2) we find that

$$K_{n-1}(x, t)K_m(t, s) = (x - qt)_{q}^{n-m-1}(t - qs)_{q}^{m} \leq (x - qs)_{q}^{n-m-1}(x - qs)_{q}^{m} \leq (x - qs)_{q}^{n-m-1}(x - q^{n} s)_{q}^{m} = (x - qs)_{q}^{n-1} = K_{n-1}(x, s)$$

for $0 < s \leq t \leq x < \infty$ and $0 \leq m \leq n - 1$, $m - 1, n \in \mathbb{N}$. Hence,

$$\max_{0 \leq m \leq n-1} K_{n-1}(x, t)K_m(t, s) \leq K_{n-1}(x, s),$$

and the lower estimate in (15) is proved. According to (1) we get that $K_{0}(x, t)K_{0}(t, s) = K_{0}(x, s) \equiv 1$ for $n = 1$. Moreover, we have that

$$K_{1}(x, s) = (x - qs)_{q} < (x - qt)_{q} + (t - qs)_{q} = \sum_{m=0}^{1} \left[ \binom{n}{m} \right] K_{2-m-1}(x, t)K_{m}(t, s)$$

for $n = 2$.

This means that the inequality

$$K_{n-2}(x, s) < \sum_{m=0}^{n-2} \left[ \binom{n-2}{m} \right] K_{n-2-m-1}(x, t)K_{m}(t, s) \quad (19)$$

holds for $n = 3$. Our aim is now to use induction, and we assume that (19) holds for $n = l - 1$, $l \geq 3$, and we will prove that it then holds also for $n = l$.

We use our induction assumption, make some calculations and obvious estimates and find that

$$K_{l-1}(x, s) = K_{l-2}(x, s)(x - q^{l-1} s)$$

$$< \sum_{m=0}^{l-2} \left[ \binom{l-2}{m} \right] K_{l-2-m-1}(x, t)K_{m}(t, s) \left( x - q^{l-1} s \right)$$

$$< \sum_{m=0}^{l-2} \left[ \binom{l-2}{m} \right] K_{l-2-m-1}(x, t)K_{m}(t, s) \left( x - q^{l-m-1} t + q^{l-m-2} t - q^{l-1} s \right)$$

$$= \sum_{m=0}^{l-2} \left[ \binom{l-2}{m} \right] K_{l-2-m-1}(x, t)K_{m}(t, s) \left( x - q^{l-m-1} t \right)$$

$$+ \sum_{m=0}^{l-2} \left[ \binom{l-2}{m} \right] K_{l-2-m-1}(x, t)K_{m}(t, s)q^{l-m-2} (t - q^{n+1} s)$$
\[
\begin{align*}
&= \left[ l - 2 \right]_q \left[ l - 2 \right]_q K_{l-1}(x, t)K_0(t, s) + \sum_{m=1}^{l-2} \left[ l - 2 \right]_q K_{l-m-1}(x, t)K_m(t, s) \\
&\quad + \sum_{m=1}^{l-2} q^{l-m-2} \left[ l - 2 \right]_q K_{l-m-1}(x, t)K_m(t, s) + \left[ \frac{l-2}{l-2} \right]_q K_0(x, t)K_{l-1}(t, s) \\
&= \left[ \begin{array}{c} l - 1 \\ 0 \\ \end{array} \right]_q K_{l-1}(x, t)K_0(t, s) \\
&\quad + \sum_{m=1}^{l-2} \left( q^{l-m-2} \left[ l - 2 \right]_q + \left[ \frac{l-2}{m} \right]_q \right) K_{l-m-1}(x, t)K_m(t, s) \\
&\quad + \left[ \begin{array}{c} l - 1 \\ l - 1 \\ \end{array} \right]_q K_0(x, t)K_l(t, s).
\end{align*}
\]

Since, for any \( m \geq 1 \) \( (q^{l-m-2} \left[ l - 1 \right]_m + \left[ \frac{l-2}{m} \right]_q) \), we get that

\[
K_{l-1}(x, s) < \sum_{m=0}^{l-1} \left[ \begin{array}{c} l - 1 \\ m \\ \end{array} \right]_q K_{l-m-1}(x, t)K_m(t, s).
\]

Hence, (19) holds also with \( n = l \) which, by the induction axiom, means that also the upper estimate in (15) is proved. The proof is complete. \( \square \)

**Proof of Lemma 3.4** From (6) and (7) it follows that

\[
I(z) = (1 - q)\alpha + \beta \left( \sum_{q' \leq z} q'f(q') \right)^\alpha \left( \sum_{q' \geq z} q'g(q') \right)^\beta.
\]

If \( z = q^k \), then, for \( k \in \mathbb{Z} \),

\[
I(z) = (1 - q)\alpha + \beta \left( \sum_{j=k}^{\infty} q'f(q') \right)^\alpha \left( \sum_{j=-\infty}^{k} q'g(q') \right)^\beta.
\]

If \( q^k < z < q^{k-1} \), then, for \( k \in \mathbb{Z} \),

\[
I(z) = (1 - q)\alpha + \beta \left( \sum_{j=k}^{\infty} q'f(q') \right)^\alpha \left( \sum_{j=-\infty}^{k-1} q'g(q') \right)^\beta.
\]

Hence, for \( k \in \mathbb{Z} \) and \( \beta > 0 \), we find that

\[
\sup_{q^k \leq z < q^{k-1}} I(z) = (1 - q)\alpha + \beta \left( \sum_{j=k}^{\infty} q'f(q') \right)^\alpha \left( \sum_{j=-\infty}^{k} q'g(q') \right)^\beta.
\]

Therefore

\[
\sup_{z=0} I(z) = \sup_{k \in \mathbb{Z}} \sup_{q^k \leq z < q^{k-1}} I(z)
= (1 - q)\alpha + \beta \sup_{k \in \mathbb{Z}} \left( \sum_{j=k}^{\infty} q'f(q') \right)^\alpha \left( \sum_{j=-\infty}^{k} q'g(q') \right)^\beta. \tag{20}
\]
We have proved that (16) holds wherever $\beta > 0$.

Next we assume that $\alpha > 0$. Let $q^{k+1} < z < q^k$, $k \in \mathbb{Z}$. Then we get that

$$I(z) = (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{j=k+1}^{\infty} q^{j} f(q^{j}) \right) \left( \sum_{j=-\infty}^{k} q^{j} g(q^{j}) \right)^{\beta},$$

and analogously as above we find that

$$\sup_{q^{k+1} < z \leq q^k} I(z) = (1 - q)^{\alpha + \beta} \left( \sum_{j=k}^{\infty} q^{j} f(q^{j}) \right) \left( \sum_{j=-\infty}^{k} q^{j} g(q^{j}) \right)^{\beta},$$

and (16) holds also for the case $\alpha > 0$. The proof is complete.

**Proof of Lemma 3.5** Let $z = q^k$, $k \in \mathbb{Z}$. By using (6) and (7) we have that

$$I^*(z) = \left(1 - q\right) \sum_{j=-\infty}^{\infty} q^{j} g(q^{j}) \left(1 - q\right) \sum_{i=k+1}^{\infty} q^{i} K(q^i, q^j) \left(1 - q\right) \sum_{i=-\infty}^{k} q^{i} f(q^{i}) \right)^{\alpha} \beta.$$

For the cases $q^{k+1} < z < q^k$ and $q^k < z < q^{k+1}$, $k \in \mathbb{Z}$, we find that

$$I^*(z) = \left(1 - q\right) \sum_{j=-\infty}^{\infty} q^{j} g(q^{j}) \left(1 - q\right) \sum_{i=k+1}^{\infty} q^{i} K(q^i, q^j) \left(1 - q\right) \sum_{i=-\infty}^{k} q^{i} f(q^{i}) \right)^{\alpha} \beta$$

and

$$I^*(z) = \left(1 - q\right) \sum_{j=-\infty}^{\infty} q^{j} g(q^{j}) \left(1 - q\right) \sum_{i=k+1}^{\infty} q^{i} K(q^i, q^j) \left(1 - q\right) \sum_{i=-\infty}^{k} q^{i} f(q^{i}) \right)^{\alpha} \beta,$$

respectively.

Hence, we conclude that

$$\sup_{q^{k+1} < z \leq q^k} I^*(z) = \left(1 - q\right) \sum_{j=-\infty}^{\infty} q^{j} g(q^{j}) \left(1 - q\right) \sum_{i=k+1}^{\infty} q^{i} K(q^i, q^j) \left(1 - q\right) \sum_{i=-\infty}^{k} q^{i} f(q^{i}) \right)^{\alpha} \beta.$$

Since $\sup_{z \in \mathbb{Z}} I^*(z) = \sup_{k \in \mathbb{Z}} \sup_{q^{k+1} < z \leq q^k} I^*(z)$, we find that (17) holds. The identity (18) can be proved in a similar way as (17). The proof is complete.

**Proof of Lemma 3.6** Without loss of generality we may assume that $Q_{m-1}^{\nu-1} < \infty$. By using (5), (6) and (7) we can deduce that

$$Q_{m}^{\nu-1} = \left( \int_{0}^{\infty} \mathcal{X}_{(0,q]}(s) K_{m}^{\nu}(q^i, s) v^{-\nu}(s) ds \right)^{\frac{\mu-1}{\mu-\nu}}$$

$$\times \left( \int_{0}^{\infty} \mathcal{X}_{(q^i,\infty)}(x) K_{\nu-1}^{\mu}(q^i, x) u^{-\mu}(x) dx \right)^{\frac{\mu}{\mu-\nu}}$$

$$\times \left( \int_{0}^{\infty} \mathcal{X}_{(0,q]}(s) K_{m}^{\nu}(q^i, s) v^{-\nu}(s) ds \right).$$

\[
\int_0^\infty \mathcal{X}_{[a,b+1]}(s)K^p_m(q^{(i+1)}s)\nu(s)\,ds \right) \left. \right|_{q=1}^{q=0} \\
= \sum_{i=-\infty}^{\infty} \left(1 - q\right) \sum_{t=i}^\infty q^t K^q_m(q', q') \nu(q') \right) \left. \right|_{q=1}^{q=0} \\
\times \left(1 - q\right) \sum_{j=-\infty}^i q^j K^q_{n-m-1}(q', q') \nu(q') \right) \left. \right|_{q=1}^{q=0} \\
\times \Delta \left(\sum_{n=1}^\infty \left(1 - q\right) q^n K^q_m(q', q') \nu(q') \right) \right|_{q=1}^{q=0},
\]

and the first equality in Lemma 3.6 is proved.

The second inequality can be proved in a similar way, so we leave out the details. The proof is complete. \(\square\)

**Proof of Theorem 3.1**

By using formulas (4) and (6) we find that inequality (13) can be rewritten as

\[
\left(\sum_{k=-\infty}^\infty (1 - q)^{x_1} q^k \nu(q') \left(\sum_{i=j}^\infty q^i f(q') K_{n-1}(q', q') \right) \right)^{\frac{1}{p}} \\
\leq C \left(\sum_{n=1}^\infty (1 - q) q^n f^p(q') \nu(q') \right)^{\frac{1}{p}}. 
\]

(21)

Let

\[ u_i' = (1 - q)^{x_1} q^i \nu(q'), \quad f_i = q^i f(q'), \]
\[ v_j' = (1 - q) q^{(i-j)p} \nu(q'), \quad W_{(i,j)} = K_{n-1}(q', q'). \]

Then we get that inequality (21) can be rewritten as the discrete weighted Hardy-type inequality (see, e.g., [26])

\[
\left(\sum_{j=-\infty}^\infty u_j' \left(\sum_{i=j}^\infty W_{(i,j)} f_i \right) \right)^{\frac{1}{p}} \leq C \left(\sum_{i=-\infty}^\infty v_i' d_i \right)^{\frac{1}{p}}. 
\]

(23)

Hence, inequality (13) is equivalent to inequality (23), where \(W_{(i,j)}\) is the non-negative triangular matrix which has entries \(W_{(i,j)} \geq 0\) for \(j \leq i\) and \(W_{(i,j)} = 0\) for \(j > i\) and is non-decreasing in the first index for all \(i \geq j > -\infty\).

First we will prove that, for \(n \in N\),

\[
(1 - q)^{n-1} \sum_{k_n-1}^i (n-1)_q k_{n-1} \sum_{k_{n-2}}^i (n-2)_q k_{n-2} \cdots \sum_{k_1}^i (1)_q k_1 = W_{(0, i, j)}.
\]

(24)

We will use induction and first we note that \(W_{(0, i, j)} = (q' - q^i)_q^0 \equiv 1\) for \(n = 1\). If \(n = 2\), then

\[
(1 - q)^i \sum_{k_{i+1}}^i q^{k_1} = \sum_{k_{i+1}}^i [1]_q q^{k_1} - q^{k_1+1} = (q' - q^{i+1})_q^1 = W_{(1, i, j)}.
\]
Assume now that formula (24) holds for $n - 1 \in N$, i.e., that

$$(1 - q)_{n-2}^{\alpha-2} \sum_{k_{n-2}+1}^{i} [n-2]q^{k_{n-2}} \sum_{k_{n-3}+k_{n-2}}^{i} [n-3]q^{k_{n-3}} \cdots \sum_{k_{1}+k_{2}}^{i} [1]q^{k_{1}} = W^{(n-1)}(i, j). $$

By using this induction assumption we find that

$$(1 - q)_{n-1}^{\alpha-1} \sum_{k_{n-1}+1}^{i} [n-1]q^{k_{n-1}} \sum_{k_{n-2}+k_{n-1}}^{i} [n-2]q^{k_{n-2}} \cdots \sum_{k_{1}+k_{2}}^{i} [1]q^{k_{1}} = (1 - q)_{n-1}^{\alpha-1} \sum_{k_{n-1}+1}^{i} q^{k_{n-1}} W^{(n-1)}(i, j)$$

$$= (1 - q)^{\alpha-1} \sum_{k_{n-1}+1}^{i} q^{k_{n-1}} (q^{k_{n-1}} - q^{(n-1)})^{\alpha-2}. $$

Since $(q^{(n-1)} - q^{(n-1)+1})^{\alpha-1} - (q^{(n-1)+1} - q^{(n-1)})^{\alpha-1} = (1 - q^{\alpha-1})q^{(n-1)}(q^{(n-1)} - q^{(n-1)+1})^{\alpha-2}$, we get that (24) holds also for $n$. Hence, by the induction axiom, we conclude that (24) holds for each $n \in N$.

Let $w_{m,k} = [m]q(q^{k_{m}} - q^{k_{m+1}})$, $m = 1, 2, 3, \ldots, n - 1$. Then, by using (24), we have that

$$W^{(n)}(i, j) = \sum_{k_{n-1}+1}^{i} W_{n-1,k_{n-1}}^{k_{n-1}} \sum_{k_{n-2}+k_{n-1}}^{i} W_{n-2,k_{n-2}}^{k_{n-2}} \cdots \sum_{k_{1}+k_{2}}^{i} W_{1,k_{1}}^{k_{1}}. \quad (25)$$

Therefore, we see that the matrix operator in (23), defined by

$$(\tilde{S}f)_j := \sum_{i}^{\infty} W^{(n)}(i, j)f_i, \quad j \in Z,$$

is an $n$-multiple discrete Hardy operator with weights (see (25)).

Therefore, Lemma 2.1 and Lemma 2.2 can be used.

(i) Let $1 < r < p < \infty$. Then, based on Lemma 2.2, it follows that inequality (23) holds if and only if $\tilde{Q}_{m}^{-1} = \max_{0 \leq m \leq n-1} \tilde{Q}_{m}^{-1} < \infty$, where

$$\tilde{Q}_{m}^{-1} = \left\{ \sum_{i=-\infty}^{\infty} \left( \sum_{j=0}^{i} (W^{(m+1)}(j, i))^{r'} v_{j}' \right)^{\frac{p}{r'}} \right\}^{\frac{1}{p'}} \times \left( \sum_{k=-\infty}^{i} (W^{(n-m)}(i, k))^{p'} u_{k}' \right)^{\frac{1}{p'}} \times \Delta^{*} \left( \sum_{j=0}^{i} (W^{(n-1)}(j, i))^{p'} v_{j}' \right)^{\frac{1}{p'}}.$$

Since inequality (23) is equivalent to inequality (13), we conclude that the condition $\tilde{Q}_{m}^{-1} < \infty$ is a necessary and sufficient condition for the validity of inequality (13). Moreover, $\tilde{Q}_{m}^{-1} \approx C$. 
By using the definitions (22) in \( \tilde{Q}_{m-1} \), we get that

\[
\tilde{Q}_{m-1}^{-1} = \left( \sum_{q_{-\infty}}^{\infty} (1-q) \sum_{j=1}^{\infty} q^j K_m^p(q', q') v' (q') \right)^{1/p(r-1)}
\]

\[
\times \left( \sum_{k=\infty}^{\infty} (1-q) q^k K_{m-1}^p(q', q') \sum_{j=1}^{\infty} q^j K_{m-1}^p(q', q') v' (q') \right)^{1/r}
\]

\[
\times \left( \sum_{q_{-\infty}}^{\infty} (1-q) q^j K_m^p(q', q') v' (q') \right)^{1/p}.
\]

By using Lemma 3.6, we find that

\[
\tilde{Q}_{m-1}^{-1} = \left( \int_{0}^{\infty} \int_{0}^{\infty} X_{\{0,2\}}(s) K_m^p(z,s) v' (s) d_q s \right)^{1/p(r-1)}
\]

\[
\times \left( \int_{0}^{\infty} X_{\{\infty,0\}}(x) K_{m-1}^p(x,z) u' (x) d_r x \right)^{1/r}
\]

\[
\times D_q \left( \int_{0}^{\infty} X_{\{0,2\}}(s) K_m^p(z,s) v' (s) d_q s \right)^{1/p}.
\]

i.e., that \( \tilde{Q}_{n-1} = Q_{n-1} \). Then we find that inequality (13) holds if and only if \( Q_{n-1} < \infty \). Moreover, \( Q_{n-1} \approx C \), where \( C \) is the best constant in (13). Thus the proof of the statement (i) of Theorem 3.1 is complete.

(ii) Let \( 1 < p \leq r < \infty \). Then from Lemma 2.1 it follows that inequality (23) holds if and only if \( \tilde{H}_{n-1} = \max_{0 \leq m \leq n-1} \tilde{H}_{m-1} < \infty \) holds, where

\[
\tilde{H}_{m-1} = \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} W^{(m+1)}(i,k) \right)^{1/p}
\]

\[
\times \left( \sum_{j=\infty}^{m} W^{(n-m)}(k,j) \right)^{1/r}, \quad n \in \mathbb{N}.
\]

If \( x = q', s = q', t = q^k \), for \( j \leq k \leq i \), then, by Lemma 3.3 and (22), we obtain that (recall that \( W^{(m)}(i,j) = K_{m-1}(q', q') \))

\[
W^{(m)}(i,j) \leq \sum_{m=0}^{n-1} \left[ \begin{array}{c} n - 1 \\ m \end{array} \right] W^{(n-m)}(k,j) W^{(m+1)}(i,k).
\]

(26)

Since, again by Lemma 3.3, \( \max_{0 \leq m \leq n-1} W^{(m+1)}(i,k) W^{(n-m)}(k,j) \leq W^{(m)}(i,j) \), it follows that

\[
W^{(m)}(i,j) \geq h(n) \sum_{m=0}^{n-1} \left[ \begin{array}{c} n - 1 \\ m \end{array} \right] W^{(m+1)}(i,k) W^{(n-m)}(k,j),
\]

(27)

where \( h(n) = \left( \sum_{m=0}^{n-1} \left[ \begin{array}{c} n - 1 \\ m \end{array} \right] q \right)^{-1} \).
According to (26) and (27) we have that
\[ W^{(n)}(i,j) \approx \sum_{m=0}^{n-1} W^{(m)}(i,k) \mathcal{V}^{(m,n)}(k,j), \]
where \( \mathcal{V}^{(m,n)}(k,j) = \left[ \frac{n-1}{q} \right] W^{(n-m-1)}(k,j). \)

Therefore, we have proved that the matrix \((W^{(n)}(i,j))\) in (23) satisfies the Oinarov condition (11) and Lemma 2.4 can be used.

Hence, we have the following necessary and sufficient conditions for the validity of inequality (23):
\[ \tilde{A}^+ = \sup_{k \in \mathbb{Z}} \left( \sum_{j=\infty}^{k} u_j \left( \sum_{i=k}^{\infty} (W^{(n)}(i,j))^p v_i^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} < \infty \]
or
\[ \tilde{A}^- = \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} v_i^p \left( \sum_{j=\infty}^{i} (W^{(n)}(i,j))^{\frac{p}{q}} v_i^{\frac{q}{p}} \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} < \infty. \]

Since, again by Lemma 3.3, \( \max_{0 \leq m \leq n-1} W^{(m+1)}(i,k) W^{(n-m)}(k,j) \leq W^{(n)}(i,j) \), we get that
\[ \tilde{H}_{n-1} = \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} u_i \left( \sum_{j=\infty}^{i} \max_{0 \leq m \leq n-1} \left( W^{(n-m)}(k,j) W^{(m+1)}(i,k) \right)^{\frac{p}{q}} v_i^{\frac{q}{p}} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} \]
\[ \leq \left( \sum_{i=k}^{\infty} (W^{(n)}(i,j))^{\frac{p}{q}} v_i^{\frac{q}{p}} \right)^{\frac{1}{p}} = \tilde{A}^+. \]

Moreover, by (26) we have that
\[ \tilde{A}^+ \leq \sup_{k \in \mathbb{Z}} \left( \sum_{j=\infty}^{k} u_j \left( \sum_{i=k}^{\infty} \left[ \sum_{m=0}^{n-1} \left( \frac{n-1}{m} \right) \right] W^{(n-m)}(k,j) W^{(m+1)}(i,k) \right)^{\frac{p}{q}} v_i^{\frac{q}{p}} \right)^{\frac{1}{p}} \]
\[ \leq \sum_{m=0}^{n-1} \left[ \frac{n-1}{m} \right] \tilde{H}_{n-1} \leq \tilde{H}_{n-1}. \]

Hence, \( \tilde{A}^+ \approx \tilde{H}_{n-1} \). In a similar way it can be proved that \( \tilde{H}_{n-1} \approx \tilde{A}^- \).

Since inequality (13) is equivalent to inequality (23), we get that inequality (13) holds if and only if at least one of the conditions \( \tilde{A}^+ < \infty \) or \( \tilde{A}^- < \infty \), or \( \tilde{H}_{n-1} < \infty \) holds.

Now, using notations (22) in \( \tilde{H}_{n-1}^{-1} \), we obtain that
\[ \tilde{H}_{m-1}^{-1} = (1-q)^{\frac{1}{p}} \sup_{k \in \mathbb{Z}} \left( \sum_{j=\infty}^{k} q_j u_j \left( q_j^{K_{m-1}} \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \]
\[ \times \left( \sum_{i=k}^{\infty} q_i v_i^{\frac{q}{p}} \left( q_i^{K_{m}} \right)^{\frac{q}{p}} \right)^{\frac{1}{p}}. \]
The parameters and functions in $\tilde{H}^{m-1}$ satisfy all the conditions of Lemma 3.4. Therefore, we find that

$$ \tilde{H}^{m-1}_m = \sup_{z > 0} \left( \int_0^\infty \chi_{[z, \infty)}(x) K_{m-1}^r(x, z) u''(x) \, dx \right)^{\frac{1}{2}} \times \left( \int_0^\infty \chi_{[0, 1]}(s) K_m^r(z, s) v'(s) \, ds \right)^{\frac{1}{2}}, $$

i.e., that $\tilde{H}^{m-1}_m = H^{m-1}_m$ and $\tilde{H}_{m-1} = H_{m-1} = \max_{0 \leq m \leq n-1} H^{m-1}_m < \infty$.

In a similar way as above, by using Lemma 3.5 and (22), we get that $A^{+}_q = \tilde{A}^+$ and $A^{-}_q = \tilde{A}^-$.

Hence, we obtain that inequality (13) holds if and only if at least one of the conditions $H_{m-1} < \infty$ or $A^{+}_q < \infty$, or $A^{-}_q < \infty$ holds. Moreover, $A^{+}_q \approx A^{-}_q \approx H_{m-1} \approx C$, where $C$ is the best constant in (13). Also the proof of the statement (ii) of Theorem 3.1 is complete.

**Proof of Theorem 3.2** In a similar way as in the proof of Theorem 3.1, by using (4), (7) and (22), we can prove that we have the following discrete Hardy-type inequality:

$$ \left( \sum_{|j|<\infty} u'_i \left( \sum_{|j|<\infty} W_{q-1}^{(n)}(i,j) f_j \right) \right)^{\frac{1}{2}} \leq C \left( \sum_{|j|<\infty} u'_i a_i \right)^{\frac{1}{2}}, $$

which is equivalent to inequality (14).

(i) Let $1 < r < p < \infty$. By using Lemma 2.2 and Lemma 3.6, we can in a similar way as in the proof of Theorem 3.1(i) derive that inequality (14) holds if and only if $Q_{m-1} < \infty$ holds. Moreover, $Q_{m-1} \approx C^*$, where $C^*$ is the best constant in (14). The proof of part (i) is complete.

(ii) Let $1 < p \leq r < \infty$. By using Lemma 2.4, Lemma 2.1, Lemma 3.4 and Lemma 3.5, we can, analogously as in the proof of the (ii)-part, prove that inequality (14) holds if and only if at least one of the conditions $H_{m-1} < \infty$ or $A^{+}_q < \infty$, or $A^{-}_q < \infty$ holds. Moreover, $H_{m-1} \approx A^{+}_q \approx A^{-}_q \approx C^*$, where $C^*$ is the best constant in (14). The proof of part (ii) is complete. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have an equal level discussed and posed the research questions in this paper. SS is the main author concerning the proofs of the main results and typing of the manuscript. L-EP has put the results into a more general frame in the introduction and instructed how to write the paper in this final form. All authors read and approved the final manuscript.

**Author details**

1. Luleå University of Technology, Luleå, 971 87, Sweden.
2. Narvik University College, P.O. Box 385, Narvik, 8505, Norway.
3. L.N. Gumilyov Eurasian National University, Munaytpasov St. 5, Astana, 010008, Kazakhstan.

**Acknowledgements**

We thank both careful referees and Professor Ryskul Oinarov for generous advice, which have improved the final version of this paper.

**Received: 6 May 2015 Accepted: 7 September 2015 Published online: 24 September 2015**

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Paper B
Abstract. We derive necessary and sufficient conditions (of Muckenhoupt-Bradley type) for the validity of $q$-analogs of $(r,p)$-weighted Hardy-type inequalities for all possible positive values of the parameters $r$ and $p$. We also point out some possibilities to further develop the theory of Hardy-type inequalities in this new direction.

1. Introduction

G. H. Hardy announced in 1920 [17] and finally proved in 1925 [18] (also see [19, p. 240]) his famous inequality

$$
\int_0^{\infty} \left( \frac{1}{x} \int_0^{x} f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) dx, \quad p > 1,
$$

(1.1)

for all non-negative functions $f$ (in the sequel we assume that all functions are non-negative). The constant $\left( \frac{p}{p-1} \right)^p$ in (1.1) is sharp. Since then it has been an enormous activity to develop and apply what is today known as Hardy-type inequalities, see e.g the books [21], [23] and [24] and the references there.

One central problem in this development was to characterize the weights $u(x)$ and $v(x)$ so that the more general Hardy-type inequality

$$
\left( \int_0^{\infty} \left( \int_0^{x} f(t) dt \right)^r u(x) dx \right)^{\frac{1}{r}} \leq C \left( \int_0^{\infty} f^p(x) v(x) dx \right)^{\frac{1}{p}}
$$

(1.2)

holds for some constant $C$ and various parameters $p$ and $r$.

To make our introduction clear we just concentrate on the case $1 \leq p \leq r < \infty$. In this case e.g the following result is well-known:
Proposition A. Let \( 1 < p \leq r < \infty \). Then the inequality (1.2) holds if and only if

\[
A_1 := \sup_{0 < x < \infty} (U(x))^{\frac{1}{r'}} (V(x))^{\frac{1}{p'}} < \infty
\]

or

\[
A_2 := \sup_{0 < x < \infty} \left( \int_0^x u(t) V^r(t) dt \right)^{\frac{1}{r}} V^{-\frac{1}{p'}}(x) < \infty
\]

or

\[
A_3 := \sup_{0 < x < \infty} \left( \int_x^{\infty} v^{1-p'}(t) U^{p'}(t) dt \right)^{\frac{1}{p'}} U^{-\frac{1}{r'}}(x) < \infty,
\]

where \( U(x) = \int_x^{\infty} u(t) dt \), \( V(x) = \int_0^x v^{1-p'}(t) dt \), \( p' = \frac{p}{p-1} \) and \( r' = \frac{r}{r-1} \). Moreover, for the sharp constant in (1.2) we have that \( C \approx A_1 \approx A_2 \approx A_3 \).

Remark 1.1. A nice proof of the condition \( A_1 < \infty \) was given in 1978 by J. S. Bradley [9]. The case \( p = r \) was proved by B. Muckenhoupt [28] already in 1972. The condition \( A_2 < \infty \) was proved in 2002 by L. E. Persson and V. D. Stepanov [30], but was for the case \( p = r \) proved by G. A. Tomaselli [34] already in 1969. The condition \( A_3 < \infty \) is just the dual condition of the condition \( A_2 < \infty \).

In the beginning G. H. Hardy was most occupied with the discrete version of (1.1). The discrete version of (1.2) reads:

\[
\left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} f_k \right)^r u_n \right)^{\frac{1}{r}} \leq C \left( \sum_{n=1}^{\infty} f_n^p \nu_n \right)^{\frac{1}{p}},
\]

where \( u = \{u_n\} \) and \( v = \{v_n\} \) are non-negative weight sequences and the question is to characterize all such weight sequence so that (1.3) holds for an arbitrary non-negative sequence \( f = \{f_n\} \) (in the sequel we assume that the considered sequences are non-negative).

It is interesting that the similar results as that in Proposition A for the discrete case was independently proved by G. Bennett [6] in 1987 (see also [2], [8] and [22, Theorem 7]). It reads:

Proposition B. Let \( 1 < p \leq r < \infty \). Then the inequality (1.3) holds if and only if

\[
B_1 := \sup_{n \in \mathbb{N}} U_n^{\frac{1}{r'}} V_n^{\frac{1}{p'}} < \infty
\]

or

\[
B_2 := \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{n} u_k V_k^r \right)^{\frac{1}{r}} V_n^{-\frac{1}{p'}} < \infty
\]
or

\[ B_3 := \sup_{n \in \mathbb{N}} \left( \sum_{k=n}^{\infty} u_k^{1-p'} U_k^{p'} \right)^{\frac{1}{p'}} U_n^{-\frac{1}{r'}} < \infty, \]

where \( U_n = \sum_{k=n}^{\infty} u_k \) and \( V_n = \sum_{k=1}^{n} u_k^{1-p'} \).

Moreover, for the sharp constant \( C \) in (1.3) it yields that \( C \approx B_1 \approx B_2 \approx B_3 \).

For our purposes we will consider the inequality (1.3) on the following different but equivalent form:

\[
\left( \sum_{n=1}^{\infty} \left( u_n \sum_{k=1}^{n} u_k f_k \right)^{p} \right)^{\frac{1}{p}} \leq C \left( \sum_{n=1}^{\infty} f_n^{p} \right)^{\frac{1}{p}}, \quad (1.4)
\]

with the obvious changes of the conditions \( B_i < \infty, \ i = 1, 2, 3 \).

In 1910, F. H. Jackson defined \( q \)-derivative and definite \( q \)-integral [20] (see also [11]). It was the starting point of \( q \)-analysis. Today the interest in the subject has exploded. The \( q \)-analysis has numerous applications in various fields of mathematics e.g. dynamical systems, number theory, combinatorics, special functions, fractals and also for scientific problems in some applied areas such as computer science, quantum mechanics and quantum physics (see e.g. [3], [5], [12], [13] and [14]). For the further development and recent results in \( q \)-analysis we refer to the books [3], [11] and [12] and the references given therein. The first results concerning integral inequalities in \( q \)-analysis were proved in 2004 by H. Gauchman [15]. Later on some further \( q \)-analogs of the classical inequalities have been proved (see [22], [27], [32] and [33]). We also pronounce the recent book [1] by G.A. Anastassiou, where many important \( q \)-inequalities are proved and discussed. Moreover, in 2014 L. Maligranda, R. Oinarov and L.-E. Persson [26] derived a \( q \)-analog of the classical Hardy inequality (1.1) and some related inequalities. It seems to be a huge new research area to investigate which of these so called Hardy-type inequalities have their \( q \)-analogs.

One main aim in this paper is to prove the \( q \)-analog of the results in Propositions A and B (see our Theorem 3.1). We will also prove the corresponding characterization for other possible values of the parameters \( p \) and \( r \) (see our Theorem 3.3). We also prove the corresponding dual results (see Theorem 3.2 and Theorem 3.4).

Our paper is organized as follows: The main results are stated Section 3 and proved in Section 4. In order not to disturb our discussions there some preliminaries are given in Section 2. In particular, we present some basic facts from \( q \)-analysis and also state Proposition B on a formally more general form namely where \( \sum_{1}^{\infty} \) is replaced by \( \sum_{-\infty}^{\infty} \) (see Proposition 2.2). We also state this result for other parameters which is important for our proof of the Theorem 3.3 (see Proposition 2.3). Finally, in Section 5 we present some remarks and in particular point out the possibility to generalize our results even to modern forms of Propositions A and B, where these three conditions even can be replaced by four scales of conditions (For the continuous case, see the review article [25] and for the discrete case see [29]).
2. Preliminaries

2.1. Some basic facts in $q$-analysis

This subsection gives the definitions and notions of $q$-analysis [11] (see also [12]). Let the function $f$ defined on $(0, b)$, $0 < b \leq \infty$ and $0 < q < 1$. Then

$$D_q f(x) := \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \in (0, b) \quad (2.1)$$

is called the $q$-derivative of the function $f$. This definition was introduced by F. H. Jackson in 1910.

Let $x \in (0, b)$. Then

$$\int_0^x f(t) d_q t := (1 - q)x \sum_{k=0}^{\infty} q^k f(x q^k), \quad (2.2)$$

is called $q$-integral or Jackson integral.

If $b = \infty$ the improper $q$-integral is defined by

$$\int_0^{\infty} f(t) d_q t := (1 - q) \sum_{k=0}^{\infty} q^k f(q^k). \quad (2.3)$$

The integrals (2.2) and (2.3) are meaningful, if the series on the right hand sides converge.

Let $0 < a < b \leq \infty$. Then we have that

$$\int_a^b f(t) d_q t := \int_a^b f(t) d_q t - \int_0^a f(t) d_q t. \quad (2.4)$$

We also need the following fact:

**Proposition 2.1.** Let $k \in \mathbb{Z}$. Then

$$\int_{q^{k+1}}^{\infty} f(t) d_q t = (1 - q) \sum_{j=-\infty}^{k} q^j f(q^j). \quad (2.5)$$

*Proof of Proposition 2.1.* By using (2.2), (2.3) and (2.4) with $b = \infty$, $a = q^{k+1}$ we have that

$$\int_{q^{k+1}}^{\infty} f(t) d_q t = \int_{0}^{q^{k+1}} f(t) d_q t - \int_{0}^{a} f(t) d_q t$$

$$= (1 - q) \sum_{j=-\infty}^{\infty} q^j f(q^j) - (1 - q) \sum_{i=0}^{\infty} q^{i+k+1} f(q^{i+k+1})$$
\[ = (1 - q) \sum_{j=-\infty}^{\infty} q^j f(q^j) - (1 - q) \sum_{i=k+1}^{\infty} q^i f(q^i) \]

\[ = (1 - q) \sum_{j=-\infty}^{k} q^j f(q^j), \]

i.e. (2.5) holds. The proof is complete. \( \square \)

Let \( \Omega \) be a subset of \((0, \infty)\) and \( X_\Omega(t) \) denote the characteristic function of the set \( \Omega \). Let \( z > 0 \). Then from (2.3) we can deduce that

\[ \int_0^\infty X_{[0,z]}(t)f(t)d_qt = (1 - q) \sum_{i=-\infty}^{\infty} q^i X_{[0,z]}(q^i)f(q^i) = (1 - q) \sum_{q^i \leq z} q^i f(q^i), \]  

(2.6)

and

\[ \int_0^\infty X_{[z,\infty)}(t)f(t)d_qt = (1 - q) \sum_{q^i \geq z} q^i f(q^i). \]  

(2.7)

Moreover,

\[ \int_0^\infty X_{[q^k,z]}(t)f(t)d_qt = (1 - q)q^k f(q^k), \]  

(2.8)

for \( q^k \leq z < q^{k-1} \), \( k \in \mathbb{Z} \),

\[ \int_0^\infty X_{[z,q^{-1}z]}(t)f(t)d_qt = (1 - q)q^m f(q^m), \]  

(2.9)

for \( q^{m+1} < z \leq q^m \), \( m \in \mathbb{Z} \).

2.2. An important variant of Proposition B

We consider the inequality:

\[ \left( \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{n} u_k \sum_{i=-\infty}^{\infty} \nu_k f_k \right)^{\frac{1}{r}} \right) \leq C \left( \sum_{n=-\infty}^{\infty} f_n^p \right)^{\frac{1}{p}}, \quad f_n \geq 0. \]  

(2.10)

We need the following formal extension of Proposition B, of independent interest:

PROPOSITION 2.2. Let \( 1 < p \leq r < \infty \). Then the inequality (2.10) holds if and only if

\[ C_1 = \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} u_k^p \right)^{\frac{1}{p}} \left( \sum_{i=-\infty}^{\infty} \nu_i^{p'} \right)^{\frac{1}{p'}} < \infty \]  

(2.11)
or
\[
C_2 = \sup_{n \in \mathbb{Z}} \left( \sum_{i=-\infty}^{n} u_i^{p'} \right)^{-\frac{1}{p}} \left( \sum_{k=-\infty}^{n} u_k^{p'} \left( \sum_{i=-\infty}^{k} u_i^{p'} \right)^r \right)^{\frac{1}{r}} < \infty
\]  
(2.12)

or
\[
C_3 = \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} u_k^{p'} \left( \sum_{i=k}^{\infty} u_i^{p'} \right)^r \right)^{\frac{1}{p'}} < \infty.
\]  
(2.13)

Moreover, for the sharp constant \( C \) in (2.10) it yields that \( C \approx C_1 \approx C_2 \approx C_3 \).

This proposition is even equivalent to Proposition B, which can be seen from the proof below we give for the reader’s convenience.

**Proof of Proposition 2.2.** Let \( \overline{\mathbb{Z}} = \mathbb{Z} \cup \{ + \infty \} \cup \{ - \infty \} \), \( \overline{\mathbb{N}} = \mathbb{N} \cup \{ + \infty \} \). The function \( \varphi : \overline{\mathbb{Z}} \to \mathbb{N} \), given by

\[
\forall n \in \overline{\mathbb{Z}} : \varphi(n) = \begin{cases} 
+\infty & n = +\infty, \\
2n & n > 0, \\
-2n + 3 & n \leq 0, \\
1 & n = -\infty,
\end{cases}
\]
is a bijection.

Therefore, \( \varphi(n) = m, m = 1, 2, \ldots \) and \( \varphi(k) = j, j = 1, 2, \ldots m \), so that

\[
\left( \sum_{n=-\infty}^{\varphi(\infty)} \left( u_n \sum_{k=-\infty}^{n} u_k f_k \right)^r \right)^{\frac{1}{r'}} = \left( \sum_{\varphi(n) = \varphi(\infty)}^{\varphi(\infty)} \left( u_{\varphi(n)} \sum_{\varphi(k) = \varphi(\infty)}^{\varphi(n)} u_{\varphi(k)} f_{\varphi(k)} \right)^r \right)^{\frac{1}{r'}}
\]

\[
= \left( \sum_{m=1}^{\infty} \sum_{j=1}^{m} \tilde{u}_m \tilde{v}_j \tilde{f}_j \right)^{\frac{1}{r}},
\]  
(2.14)

and

\[
\left( \sum_{n=-\infty}^{\varphi(\infty)} f_n^p \right)^{\frac{1}{p'}} = \left( \sum_{\varphi(n) = \varphi(\infty)}^{\varphi(\infty)} f_{\varphi(n)}^p \right)^{\frac{1}{p'}} = \left( \sum_{m=1}^{\infty} \tilde{f}_m^p \right)^{\frac{1}{p'}}
\]  
(2.15)

where \( \tilde{f}_m = f_{\varphi(n)}, \tilde{u}_m = u_{\varphi(n)}, \tilde{v}_j = v_{\varphi(k)} \).

By (2.14) and (2.15), we obtain that (2.10) holds if and only if the inequality

\[
\left( \sum_{m=1}^{\infty} \left( \tilde{u}_m \sum_{j=1}^{m} \tilde{v}_j \tilde{f}_j \right)^r \right)^{\frac{1}{r}} \leq C \left( \sum_{m=1}^{\infty} \tilde{f}_m^p \right)^{\frac{1}{p'}}
\]  
(2.16)

holds.
Let $1 < p \leqslant r < \infty$. By Proposition B we get that the inequality (2.16) holds if and only if

$$B_1 = \sup_{m \in \mathbb{N}} \left( \sum_{j=m}^{\infty} \tilde{u}_j^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^{m} \tilde{v}_i^{p'} \right)^{\frac{1}{p'}} < \infty$$

holds. Moreover, since the function $\varphi^{-1} : \mathbb{N} \to \mathbb{Z}$ is a bijection, we find that

$$B_1 = \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} \left( \sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{1}{p'}} = C_1. \quad (2.17)$$

Hence, according to (2.14), (2.15) and (2.17), we obtain that the inequality (2.10) holds if and only if $C_1 < \infty$. Moreover, by Proposition B we find that $C \approx C_1$, where $C$ is the sharp constant in (2.10).

The proofs of the facts that also $C_2 < \infty$ and $C_3 < \infty$ are necessary and sufficient conditions for the characterization of (2.10), and also that $C \approx C_2 \approx C_3$, are similar so we leave out the details. The proof is complete. \(\square\)

We also need the corresponding result for other cases of possible parameters $p$ and $r$.

**Proposition 2.3.** (i). Let $0 < p \leqslant 1$, $p \leqslant r < \infty$. Then the inequality (2.10) holds if and only if

$$C_4 = \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} v_n < \infty. \quad (2.18)$$

(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (2.10) holds if and only if

$$C_5 = \left( \sum_{n=-\infty}^{\infty} \left( \sum_{i=-\infty}^{n} v_i^{p'} \right)^{\frac{r(p-1)}{p-r}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{p}} u_n^{\frac{r}{p-r}} \right)^{\frac{p-r}{pr}} < \infty. \quad (2.19)$$

(iii). Let $0 < r < p = 1$. Then the inequality (2.10) is satisfied if and only if

$$C_6 = \left( \sum_{n=-\infty}^{\infty} \max_{i \leqslant n} v_i^{\frac{r}{1-r}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{1-r}} u_n^{\frac{r}{1-2r}} \right)^{\frac{1}{1-2r}} < \infty. \quad (2.20)$$

In all cases (i)–(iii) for the best constant in (2.10) it yields that $C \approx B_i$, $i = 4, 5, 6$, respectively.

**Proof of Proposition 2.3.** By using well-known characterizations (see [6], [7], [8], [10], [16] and [21, p. 58]) for the cases (i)–(iii) where $\sum_{i=1}^{\infty}$ is replaced by $\sum_{i=1}^{\infty}$, the proof can be performed exactly as the proof of Proposition 2.2. We leave out the details.
2.3. Some \( q \)-analogos of weighted Hardy-type inequalities

Let \( 0 < r, p \leq \infty \). Then the \( q \)-analog of the discrete Hardy-type inequality of the form (1.4) can be rewritten in the following way:

\[
\left( \int_0^\infty \left( \int_0^x u(t) v(t) f(t) \, dt \right)^r \, dx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty f^p(x) \, dx \right)^{\frac{1}{p}}. \tag{2.21}
\]

By Proposition 2.1 we find that the inequality (2.21) can be rewritten on the following dual form:

\[
\left( \int_0^\infty \left( \int_0^{x} v(t) g(t) \, dt \right)^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_0^\infty g^{r'}(x) \, dx \right)^{\frac{1}{r'}}. \tag{2.22}
\]

We see that the (2.22) lacks some symmetry as in classical analysis.

We consider the operator \( (H_q f)(x) = \int_0^x \mathcal{D}_{(0,x]}(t) v(t) f(t) \, dt \), which is defined for all \( x > 0 \). Although it does not coincide with the operator \( \int_0^x v(t) f(t) \, dt \) (they coincide at the points \( x = q^k, k \in \mathbb{Z} \)) we have the equality

\[
\int_0^\infty \left( \int_0^x u(t) v(t) f(t) \, dt \right)^r \, dx = \int_0^\infty \left( \int_0^\infty \mathcal{D}_{(0,x]}(t) v(t) f(t) \, dt \right)^r \, dx.
\]

Therefore, the inequality (2.21) can be rewritten as

\[
\left( \int_0^\infty \left( u(x) \int_0^\infty \mathcal{D}_{(0,x]}(t) v(t) f(t) \, dt \right)^r \, dx \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty f^p(x) \, dx \right)^{\frac{1}{p}}, \tag{2.23}
\]

which will be called the \( q \)-integral analog of the weighted Hardy-type inequality. The dual inequality of the inequality (2.23) (equivalent of (2.22)) reads:

\[
\left( \int_0^\infty \left( \int_0^\infty \mathcal{D}_{[x,\infty)}(x) u(x) g(x) \, dx \right)^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_0^\infty g^{r'}(t) \, dt \right)^{\frac{1}{r'}}.
\]
3. The main results

Our main result reads:

**Theorem 3.1.** Let $1 < p \leq r < \infty$. Then the inequality (2.23) holds if and only if

$$D_1 = \sup_{z > 0} \left( \int_{\infty}^{0} \mathcal{H}_{\infty}^{-}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} < \infty$$

or

$$D_2 = \sup_{z > 0} \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(t) v^{p'}(t) d_q t \right)^{-\frac{1}{p'}} \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(x) u^r(x) d_q x \right) \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} < \infty$$

or

$$D_3 = \sup_{z > 0} \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(x) u^r(x) d_q x \right)^{-\frac{1}{r}} \left( \int_{0}^{\infty} \mathcal{H}_{\infty}^{-}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} \left( \int_{0}^{\infty} \mathcal{H}_{\infty}^{-}(x) u^r(x) d_q x \right)^{p'} < \infty.$$

Moreover, for the sharp constant in (2.23) we have that $C \approx D_1 \approx D_2 \approx D_3$.

Next, we will consider the corresponding inequality

$$\left( \int_{0}^{\infty} \left( u(x) \int_{0}^{\infty} \mathcal{H}_{\infty}(t) v(t) f(t) d_q t \right)^{r} d_q x \right)^{\frac{1}{r}} \leq C \left( \int_{0}^{\infty} f^p(x) d_q x \right)^{\frac{1}{p}}, \quad (3.1)$$

for the dual operator of $H_q$.

**Theorem 3.2.** Let $1 < p \leq r < \infty$. Then the inequality (3.1) holds if and only if

$$D_1^* = \sup_{z > 0} \left( \int_{0}^{\infty} \mathcal{H}_{0}^{-}(x) u^r(x) d_q x \right)^{\frac{1}{r}} \left( \int_{0}^{\infty} \mathcal{H}_{\infty}^{-}(t) v^{p'}(t) d_q t \right)^{\frac{1}{p'}} < \infty$$

or
\[
D^*_2 = \sup_{z > 0} \left( \int_0^\infty \mathcal{X}_{[z, \infty)}(t) v^{p'}(t) d_q t \right)^{-\frac{1}{p}}
\]
\[
\left( \int_0^\infty \mathcal{X}_{[z, \infty)}(x) u^r(x) \left( \int_0^\infty \mathcal{X}_{[z, \infty)}(t) v^{p'}(t) d_q t \right)^r d_q x \right) < \infty
\]
or
\[
D^*_3 = \sup_{z > 0} \left( \int_0^\infty \mathcal{X}_{(0, z]}(x) u^r(x) d_q x \right)^{-\frac{1}{p'}}
\]
\[
\left( \int_0^\infty \mathcal{X}_{(0, z]}(t) v^{p'}(t) \left( \int_0^\infty \mathcal{X}_{(0, z]}(x) u^r(x) d_q x \right)^{p'} d_q t \right)^{\frac{1}{p'}} < \infty.
\]
Moreover, for the sharp constant in (3.1) we have that \(C \approx D^*_1 \approx D^*_2 \approx D^*_3\).

Concerning other possible parameters of \(p\) and \(r\) we have the following complement of Theorem 3.1:

**Theorem 3.3.** (i) Let \(0 < p \leq 1, \ p \leq r < \infty\). Then the inequality (2.23) holds if and only if
\[
D_4 = \sup_{z > 0} \left( \int_0^\infty \mathcal{X}_{[z, \infty)}(x) u^r(x) d_q x \right) \left( \int_0^\infty \mathcal{X}_{(qz, z]}(t) v^{p'}(t) d_q t \right)^{\frac{1}{r}} < \infty.
\]
(ii) Let \(1 < p < \infty, \ 0 < r < p\). Then the inequality (2.23) holds if and only if
\[
D_5 = \left( \int_0^\infty \left( \int_0^\infty \mathcal{X}_{[0, z]}(t) v^{p'}(t) d_q t \right)^{\frac{r(p-1)}{p-r}} \right) \left( \int_0^\infty \mathcal{X}_{[z, \infty)}(x) u^r(x) d_q x \right)^{\frac{r}{p-r}} \left( \int_0^\infty \mathcal{X}_{[z, \infty)}(z) u^r(z) d_q z \right)^{\frac{p-r}{pr}} < \infty.
\]
(iii). Let $0 < r < p = 1$. Then the inequality (2.23) is satisfied if and only if

$$D_6 = \left( \int_0^\infty \sup_{y < z} \left( \int_0^\infty \mathcal{H}_{(y,z)}(t) \frac{v(t)}{(1 - q)t} dt \right) \right)^{\frac{1}{1 - r}}$$

$$\left( \left( \int_0^\infty \mathcal{H}_{[z,\infty)}(x) u^r(x) dx \right) \left( \int_0^\infty \mathcal{H}_{[\infty,0]}(z) u^r(z) dz \right) \right)^{\frac{1}{r - r}} < \infty.$$  

In all cases (i)–(iii), for the best constant in (2.23) it yields that $C \approx D_i$, $i = 4, 5, 6$, respectively.

Finally, the corresponding complement of Theorem 3.2 reads:

**Theorem 3.4.** (i). Let $0 < p \leq 1$, $p \leq r < \infty$. Then the inequality (3.1) holds if and only if

$$D_4^* = \sup_{z > 0} \left( \int_0^\infty \mathcal{H}_{[0,z]}(x) u^r(x) dx \right)^{\frac{1}{r}} \left( \int_0^\infty \mathcal{H}_{[z,q-1]}(t) v^{p'}(t) dt \right)^{\frac{1}{p'}} < \infty.$$  

(ii). Let $1 < p < \infty$, $0 < r < p$. Then the inequality (3.1) holds if and only if

$$D_5^* = \left( \int_0^\infty \left( \int_0^\infty \mathcal{H}_{[0,z]}(x) u^r(x) dx \right) \right)^{\frac{r}{p - r}} \left( \int_0^\infty \mathcal{H}_{[z,\infty)}(t) v^{p'}(t) dt \right)^{\frac{r(p - 1)}{p - r}} \left( \int_0^\infty \mathcal{H}_{[\infty,0]}(z) u^r(z) dz \right)^{\frac{r}{p - r}} < \infty.$$  

(iii). Let $0 < r < p = 1$. Then the inequality (3.1) holds if and only if

$$D_6^* = \left( \int_0^\infty \sup_{y > z} \left( \int_0^\infty \mathcal{H}_{[y,q-1]}(t) \frac{v(t)}{(1 - q)t} dt \right) \right)^{\frac{1}{1 - r}}$$

$$\left( \left( \int_0^\infty \mathcal{H}_{[0,z]}(x) u^r(x) dx \right) \left( \int_0^\infty \mathcal{H}_{[\infty,0]}(z) u^r(z) dz \right) \right)^{\frac{1}{r - r}} < \infty.$$  

In all cases (i)–(iii), for the best constant in (3.1) it yields that $C \approx D_i^*$, $i = 4, 5, 6$, respectively.
To prove these theorems, we need some Lemmas of independent interest:

**Lemma 3.5.** Let $f$ and $g$ be nonnegative functions and

$$I(z) := \left( \int_0^\infty \mathcal{D}_{(0,z]}(t)f(t)\,dq_t \right)^\alpha \left( \int_0^\infty \mathcal{D}_{(z,\infty)}(x)g(x)\,dx \right)^\beta,$$

for $\alpha, \beta \in \mathbb{R}$, and where at least one of the numbers $\alpha, \beta$ is positive. Then

$$\sup_{z > 0} I(z) = (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^\infty q^i f(q^i) \right)^\alpha \left( \sum_{i=-\infty}^{k} q^i g(q^i) \right)^\beta. \tag{3.2}$$

**Lemma 3.6.** Let $\alpha, \beta \in \mathbb{R}^+$,

$$I^+(z) := \left( \int_0^\infty \mathcal{D}_{(0,z]}(x)f(x)\,dx \right)^\alpha \left( \int_0^\infty \mathcal{D}_{(z,\infty)}(x)g(x)\,dx \right)^\beta,$$

and

$$I^-(z) := \left( \int_0^\infty \mathcal{D}_{[z,\infty)}(x)f(x)\,dx \right)^\alpha \left( \int_0^\infty \mathcal{D}_{(qz,\infty)}(x)g(x)\,dx \right)^\beta.$$

Then

$$\sup_{z > 0} I^+(z) = (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^\infty q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta, \tag{3.3}$$

and

$$\sup_{z > 0} I^-(z) = (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta. \tag{3.4}$$

**Lemma 3.7.** Let $f$, $\varphi$ and $g$ be nonnegative functions. Then

$$D \equiv \int_0^\infty \left( \int_0^\infty \mathcal{D}_{(0,z]}(t)f(t)\,dq_t \right)^\alpha \left( \int_0^\infty \mathcal{D}_{(z,\infty)}(x)g(x)\,dx \right)^\beta \varphi(z)\,dq_z$$

$$= (1 - q)^{\alpha + \beta} \sum_{k=-\infty}^{\infty} \left( \sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left( \sum_{i=k}^{\infty} q^i g(q^i) \right)^\beta q^k \varphi(q^k),$$

for $\alpha, \beta \in \mathbb{R}$. 
Lemma 3.8. Let \( k \in \mathbb{Z}, \alpha \in \mathbb{R} \) and

\[
F(y) := \left( \int_0^{q_k-1} \mathcal{X}_{y, q^{-1}}(t) f(t) dq^t \right)^\alpha.
\]

Then

\[
\sup_{y \geq q^k} F(y) = (1 - q)^\alpha \sup_{i \leq k} \left( q^i f(q^i) \right)^\alpha.
\]

(3.5)

4. Proofs

Proof of Lemma 3.5. From (2.6) and (2.7) it follows that

\[
I(z) = (1 - q)^{\alpha + \beta} \left( \sum_{q^j \leq z} q^j f(q^j) \right)^\alpha \left( \sum_{q^j > z} q^j g(q^j) \right)^\beta.
\]

If \( z = q^k \), then, for \( k \in \mathbb{Z} \),

\[
I(z) = I(q^k) = (1 - q)^{\alpha + \beta} \left( \sum_{j = k}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i = -\infty}^{k-1} q^i g(q^i) \right)^\beta.
\]

If \( q^k < z < q^{k-1} \), then, for \( k \in \mathbb{Z} \),

\[
I(z) = (1 - q)^{\alpha + \beta} \left( \sum_{j = k}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i = -\infty}^{k-1} q^i g(q^i) \right)^\beta.
\]

Hence, for \( k \in \mathbb{Z} \) and \( \beta > 0 \) we find that

\[
\sup_{q^k \leq z < q^{k-1}} I(z) = I(q^k) = (1 - q)^{\alpha + \beta} \left( \sum_{j = k}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i = -\infty}^{k-1} q^i g(q^i) \right)^\beta.
\]

Therefore

\[
\sup_{z > 0} I(z) = \sup_{k \in \mathbb{Z}} \sup_{q^k \leq z < q^{k-1}} I(z)
\]

\[
= (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{j = k}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i = -\infty}^{k-1} q^i g(q^i) \right)^\beta.
\]

We have proved that (3.1) holds wherever \( \beta > 0 \).

Next we assume that \( \alpha > 0 \). Let \( q^{k+1} < z < q^k, k \in \mathbb{Z} \). Then we get that

\[
I(z) = (1 - q)^{\alpha + \beta} \sup_{k \in \mathbb{Z}} \left( \sum_{j = k+1}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i = -\infty}^{k} q^i g(q^i) \right)^\beta.
\]
and analogously as above we find that

\[ \sup_{q^{k+1} < z \leq q^k} I(z) = I(q^k) = (1 - q)^{\alpha + \beta} \left( \sum_{j=k}^{\infty} q^j f(q^j) \right)^\alpha \left( \sum_{i=-\infty}^{k} q^i g(q^i) \right)^\beta, \]

and (3.1) holds also for the case \( \alpha > 0 \). The proof is complete. \( \square \)

**Proof of Lemma 3.6.** According to (2.6) and (2.9) we have that

\[ I^+(q^k) = (1 - q)^{\alpha + \beta} \left( \sum_{i=k}^{\infty} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta, \]

for \( z = q^k, \ k \in \mathbb{Z} \), and

\[ I^+(z) = (1 - q)^{\alpha + \beta} \left( \sum_{i=k+1}^{\infty} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta, \]

for \( q^{k+1} < z < q^k, \ k \in \mathbb{Z} \).

Therefore,

\[ \sup_{q^{k+1} < z \leq q^k} I^+(z) = (1 - q)^{\alpha + \beta} \left( \sum_{i=k}^{\infty} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta. \]

Since \( \sup_{z > 0} I^+(z) = \sup_{k \in \mathbb{Z}} \sup_{q^{k+1} < z \leq q^k} I^+(z) \), we conclude that (3.3) holds.

Next, by using (2.7) and (2.8) we find that

\[ I^-(q^k) = (1 - q)^{\alpha + \beta} \left( \sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta, \] (4.1)

for \( z = q^k, \ k \in \mathbb{Z} \), and

\[ I^-(z) = (1 - q)^{\alpha + \beta} \left( \sum_{i=-\infty}^{k-1} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta, \]

for \( q^k < z < q^{k-1}, \ k \in \mathbb{Z} \).

Thus,

\[ \sup_{q^k \leq z < q^{k-1}} I^-(z) = (1 - q)^{\alpha + \beta} \left( \sum_{i=-\infty}^{k} q^i f(q^i) \right)^\alpha \left( q^k g(q^k) \right)^\beta. \]

Since \( \sup_{z > 0} I^-(z) = \sup_{k \in \mathbb{Z}} \sup_{q^k \leq z < q^{k-1}} I^-(z) \), we have that (3.4) holds. The proof is complete. \( \square \)
Proof of Lemma 3.7. By using (2.3), (2.6) and (2.7), we have that

\[
D = (1 - q) \sum_{k=\infty}^{\infty} q^k \left( \int_{0}^{\infty} \mathcal{H}_{[q^k, \infty)}(t) f(t) d_q t \right)^{\alpha} \left( \int_{0}^{\infty} \mathcal{H}_{[0, q^k]}(x) g(x) d_q x \right)^{\beta} \phi(q^k)
\]

\[
= (1 - q)^{\alpha + \beta} \sum_{k=\infty}^{\infty} q^k \left( \sum_{i=\infty}^{k} q^i f(q^i) \right)^{\alpha} \left( \sum_{j=k}^{\infty} q^j g(q^j) \right)^{\beta} \phi(q^k).
\]

The proof is complete. \( \Box \)

Proof of Lemma 3.8. By using (2.9), we get that

\[
F(q^k) = \left( \int_{0}^{\infty} \mathcal{H}_{[q^k, q^{k-1}]}(t) f(t) d_q t \right)^{\alpha} = (1 - q)^{\alpha} \left( q^k f(q^k) \right)^{\alpha}, \tag{4.2}
\]

for \( y = q^k, k \in \mathbb{Z}, \) and

\[
\sup_{y > q^k} F(y) = \sup_{i \leq k} \sup_{q^i < y \leq q^{i-1}} F(y)
\]

\[
= (1 - q)^{\alpha} \sup_{i \leq k} \left( q^{i-1} f(q^{i-1}) \right)^{\alpha}
\]

\[
= (1 - q)^{\alpha} \sup_{i \leq k-1} \left( q^i f(q^i) \right)^{\alpha}, \tag{4.3}
\]

for \( i \leq k \) and \( q^i < y \leq q^{i-1}. \)

From (4.2) and (4.3) it follows that

\[
\sup_{y \geq q^k} F(y) = \max \{ \sup_{y > q^k} F(y), F(q^k) \} = (1 - q)^{\alpha} \sup_{i \leq k} \left( q^i f(q^i) \right)^{\alpha}.
\]

Thus, (3.5) holds so the proof is complete. \( \Box \)

Proof of Theorem 3.2. By using (2.3) and (2.7), we have that

\[
\left( \int_{0}^{\infty} f^p(x) d_q x \right)^{\frac{1}{p}} = (1 - q)^{\frac{1}{p}} \left( \sum_{j=-\infty}^{\infty} q^j f^p(q^j) \right)^{\frac{1}{p}}, \tag{4.4}
\]

and

\[
\left( \int_{0}^{\infty} \left( u(x) \int_{0}^{\infty} \mathcal{H}_{[x, \infty)}(t)v(t) f(t) d_q t \right)^{r} d_q x \right)^{\frac{1}{r}}
\]

\[
= (1 - q)^{\frac{1}{r}} \left( \sum_{j=-\infty}^{\infty} q^j u^r(q^j) \left( \int_{0}^{\infty} \mathcal{H}_{[q^j, \infty)}(t)v(t) f(t) d_q t \right)^{r} \right)^{\frac{1}{r}}
\]
\[
= (1 - q)^{1 + \frac{1}{r}} \left( \sum_{j = -\infty}^{\infty} q^j u^r(q^j) \left( \sum_{q^j > q^i} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}}
\]
\[
= (1 - q)^{1 + \frac{1}{r}} \left( \sum_{j = -\infty}^{\infty} q^j u^r(q^j) \left( \sum_{i = -\infty}^{j} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}}.
\]

By now using (3.1), (4.4) and (4.5) we find that
\[
(1 - q)^{\frac{1}{p} + \frac{1}{r}} \left( \sum_{j = -\infty}^{\infty} q^j \left( u(q^j) \left( \sum_{i = -\infty}^{j} q^i v(q^i) f(q^i) \right)^r \right)^{\frac{1}{r}} \right)^{\frac{1}{p}} \leq C \left( \sum_{j = -\infty}^{\infty} q^j f^p(q^j) \right)^{\frac{1}{p}}.
\]

Let
\[
q^j f^p(q^j) = f_j^p, \quad v_j = q^{\frac{rp}{p'}} v(q^j)(1 - q)^{\frac{1}{p'}}, \quad u_j = (1 - q)^{\frac{1}{r}} q^j u(q^j), \quad j \in \mathbb{Z}.
\]

Then we see that the inequality (3.1) is equivalent to the inequality (2.10). The best constants in inequalities (3.1) and (2.10) are the same.

Since the inequality (3.1) is equivalent to the inequality (2.10) we can use Proposition 2.2 to conclude that the inequality (3.1) holds if and only if at least one of the conditions \( C_1 < \infty \), \( C_2 < \infty \) and \( C_3 < \infty \) holds. Moreover, for the best constant \( C \) in (3.1) it yields that \( C \approx C_1 \approx C_2 \approx C_3 \).

Hence, according to Lemma 3.5 we have that
\[
C_1 = \sup_{n \in \mathbb{Z}} \left( \sum_{k = n}^{\infty} u^r_k \right)^{\frac{1}{r}} \left( \sum_{i = -\infty}^{n} v^p_i \right)^{\frac{1}{p'}}
\]
\[
= (1 - q)^{\frac{1}{r} + \frac{1}{p'}} \sup_{n \in \mathbb{Z}} \left( \sum_{k = n}^{\infty} q^k u^r(q^k) \right)^{\frac{1}{r}} \left( \sum_{i = -\infty}^{n} q^i v^p(q^i) \right)^{\frac{1}{p'}}
\]
\[
= \sup_{n > 0} \left( \int_{x_0^r}^{x_1^r} (x) u^r(t) d_{q^r} \right)^{\frac{1}{r}} \left( \int_{x_0^l}^{x_1^l} (t) v^p(t) d_{q^p} \right)^{\frac{1}{p'}} = D_1^r.
\]

In particular, \( C \approx D_1^r \). Moreover, by arguing as above and using Lemma 3.6 we obtain that \( C_2 \approx D_2^r \) and \( C_3 \approx D_3^r \). Hence, for the best constant \( C \) in (3.1) it yields that \( C \approx D_1^r \approx D_2^r \approx D_3^r \). The proof is complete. \( \square \)

**Proof of Theorem 3.4.** In a similarly way as in the proof of Theorem 3.2, by using (2.3), (2.7) and (4.6), we find that the inequality (2.10) is equivalent to the inequality (3.1).

Since the inequality (3.1) is equivalent to the inequality (2.10) we can use Proposition 2.3 to conclude that the inequality (3.1) holds if and only if the conditions (2.18), (2.19) and (2.20) hold, for considered cases \( 0 < p < 1 \), \( p \leq r \); \( 1 < p < \infty \), \( 0 < r < p \) and \( 0 < r < p = 1 \), respectively.
Next, we prove that the conditions (2.18), (2.19) and (2.20) are equivalent to the conditions $D_4' < \infty$, $D_5' < \infty$ and $D_6' < \infty$, respectively.

By using Lemma 3.6 from (2.18) and (4.6) we obtain that

$$C_4 = \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{1}{r}} v_n = (1 - q) \frac{1}{r} + \frac{1}{r} \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{\infty} q^r u_k^r(q^k) \right)^{\frac{1}{r}} \left( q^n v_p'(q^n) \right)^{\frac{1}{p'}}$$

$$= \sup_{z > 0} \left( \int_0^\infty \mathcal{J}_{(0, z]}(x) u'(x) d_q x \right)^{\frac{1}{r}} \left( \int_0^\infty \mathcal{J}_{[z, -q^{-1} z]}(t) v_p'(t) d_q t \right)^{\frac{1}{p'}} = D_4'.$$

Moreover, by Lemma 3.7 we have that

$$C_5 = \sum_{n=-\infty}^{\infty} \left( \sum_{i=-\infty}^{n} v_i^p \right)^{\frac{r(p-1)}{p-r}} \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{p-r}} u_n^r$$

$$= (1 - q) \frac{r p}{p-r} + \int_{n=-\infty}^{\infty} \left( \sum_{i=-\infty}^{n} q^i v_i^p(q^i) \right)^{\frac{r(p-1)}{p-r}} \left( \sum_{k=n}^{\infty} q^k u_k^r(q^k) \right)^{\frac{r}{p-r}} q^n u_r(q^n)$$

$$= \int_0^\infty \left( \int_0^\infty \mathcal{J}_{(z, \infty)}(t) v_i^p(t) d_q t \right)^{\frac{r(p-1)}{p-r}} \left( \int_0^\infty \mathcal{J}_{[0, z]}(x) u_i^r(x) d_q x \right)^{\frac{r}{p-r}} u_r(z) d_q z = D_5'.$$

Now let $p = 1$ so that $p' = \infty$. Then $v_i = v(q^i)$ in (4.6). By Lemma 3.8 we find that

$$\max_{i \leq n} v_i^r = \left( \max_{i \leq n} v(q^i) \right)^{\frac{r}{r-r}} = \left( 1 - q \right) \max_{i \leq n} \frac{q^i v(q^i)}{(1 - q) q^i}$$

$$= \left( \sup_{y \geq q^n} \int_0^\infty \mathcal{J}_{[y, q^{-1} y]}(t) \frac{v(t)}{(1 - q) t} d_q t \right)^{\frac{r}{r-r}} = \sup_{y \geq q^n} \left( \int_0^\infty \mathcal{J}_{[y, q^{-1} y]}(t) \frac{v(t)}{(1 - q) t} d_q t \right)^{\frac{r}{r-r}}.$$

Therefore,

$$C_6 = \sum_{n=-\infty}^{\infty} \max_{i \leq n} v_i^r \left( \sum_{k=n}^{\infty} u_k^r \right)^{\frac{r}{r-r}} u_n^r$$

$$= \sum_{n=-\infty}^{\infty} q^n \max_{i \leq n} v_i^r \left( 1 - q \right) \sum_{k=n}^{\infty} q^k u_k^r(q^k) u_r(q^n)$$

$$= (1 - q) \sum_{n=-\infty}^{\infty} q^n \sup_{y \geq q^n} \left( \int_0^\infty \mathcal{J}_{[y, q^{-1} y]}(t) \frac{v(t)}{(1 - q) t} d_q t \right)^{\frac{r}{r-r}}$$

$$\times \left( \int_0^\infty \mathcal{J}_{(0, q^n]}(x) u_i^r(x) d_q x \right)^{\frac{r}{r-r}} u_r(q^n).$$
Thus, in all cases (i)–(iii), for the best constant in (3.1) it yields that \( C \approx D_i^* \), \( i = 4, 5, 6 \), respectively. The proof is complete. \( \square \)

**Proof of Theorem 3.1.** As in the proof of Theorem 3.2 we get that the inequality (2.23) is equivalent to the inequality

\[
\left( \sum_{j=-\infty}^{\infty} \left( u_j \sum_{i=-\infty}^{j} v_i f_i \right)^r \right)^{\frac{1}{r}} \leq C \left( \sum_{j=-\infty}^{\infty} f_j^p \right)^{\frac{1}{p}}.
\]

(4.7)

By using standard dual arguments the characterizations similar to those in Proposition 2.2 hold also in this situation (see e.g. [16, p. 59]). Here it is even simpler to just put \( \tilde{u}_i = u_{-i} \), \( \tilde{v}_i = v_{-i} \), \( \tilde{f}_i = f_{-i} \), \( i \in \mathbb{Z} \), and note that then (4.7) reads

\[
\left( \sum_{j=-\infty}^{\infty} \left( \tilde{u}_j \sum_{i=-\infty}^{j} \tilde{v}_i \tilde{f}_i \right)^r \right)^{\frac{1}{r}} \leq C \left( \sum_{j=-\infty}^{\infty} \tilde{f}_j^p \right)^{\frac{1}{p}}.
\]

(4.8)

Now use Proposition 2.2, and find that the inequality (4.8) holds if and only if one of the conditions \( \tilde{C}_i < \infty \), \( 1 \leq i \leq 3 \) holds. Note that here \( \tilde{C}_i \), \( 1 \leq i \leq 3 \), are defined by just in the expressions for \( C_i \) inserting \( \tilde{u}_j \), \( \tilde{v}_j \), \( j \in \mathbb{Z} \). Moreover, for the best constant \( C \) in (4.8) it yields that \( C \approx \tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3 \).

Next, by replacing \( \tilde{u}_j \) and \( \tilde{v}_j \) by \( u_j \) and \( v_j \), \( j \in \mathbb{Z} \), in the expressions \( \tilde{C}_i \), \( 1 \leq i \leq 3 \), respectively, we obtain the corresponding characterizations for the validity of the inequality (4.7). In a similar way as in the proof of Theorem 3.2, from the equivalence of inequalities (2.23) and (4.7) and using Lemma 3.6 we find that the inequality (2.23) holds if and only if \( D_1 < \infty \) or \( D_2 < \infty \) or \( D_3 < \infty \) holds. Moreover, for the best constant \( C \) in (2.23) it yields that \( C \approx D_1 \approx D_2 \approx D_3 \). The proof is complete. \( \square \)

**Proof of Theorem 3.3.** The equivalence between (4.7) and (4.8) holds in the case too. Hence, by arguing exactly as in proof of Theorem 3.1 but using Proposition 2.3 instead of Proposition 2.2 the proof can be done analogously, so we leave out the details.

5. Final remarks

**Remark 5.1.** Assume that \( v(t) = 0 \), \( u(t) = 0 \), \( f(t) = 0 \), \( t > 1 \) and the integrals in the expressions \( D_i \), \( D_i^* \), \( 1 \leq i \leq 6 \) are replaced by the integrals from zero to one and the sets \( [z, \infty) \), \( [z, q^{-1}z) \) are replaced by the sets \( [z, 1] \), \( [z, \min\{q^{-1}z, 1\}] \), respectively. Then, by using Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, we obtain
that the corresponding characterizations for the validity of the inequalities
\[
\left( \int_0^1 \left( u(x) \int_0^1 \mathcal{A}_{[0,1]}(t) v(t) f(t) \,dq_t \right)^r \,dx \right)^{\frac{1}{r}} \leq C \left( \int_0^1 f^p(t) \,dq_t \right)^{\frac{1}{p}},
\]
and
\[
\left( \int_0^1 \left( u(x) \int_0^1 \mathcal{A}_{[x,1]}(t) v(t) f(t) \,dq_t \right)^r \,dx \right)^{\frac{1}{r}} \leq C \left( \int_0^1 f^p(t) \,dq_t \right)^{\frac{1}{p}},
\]
for all parameters \( r \) and \( p \) in these theorems.

**Remark 5.2.** Note that nowadays it is known that the conditions \( B_i < \infty, i = 1,2,3 \), in Proposition B are special cases of more general conditions. More exactly these conditions can be replaced by infinite many conditions, namely the following four scales of conditions (see [29] and also [17, p. 60]):

\[
B_1(s) := \sup_{n \in \mathbb{N}} \left( \sum_{k=n}^n v_k^{1-p'} \right)^{(s-1)/p} \left( \sum_{k=n}^\infty u_k \left( \sum_{m=k}^k v_m^{1-p'} \right) \right)^{r(p-s)/p} < \infty,
\]
for \( s \) satisfying \( 1 < s \leq p' \);

\[
B_2(s) := \sup_{n \in \mathbb{N}} \left( \sum_{k=n}^\infty v_k^{1-p'} \right)^{-s} \left( \sum_{k=n}^\infty u_k \left( \sum_{m=k}^\infty v_m^{1-p'} \right) \right) \left( \sum_{k=n}^\infty u_k \left( \sum_{m=k}^\infty v_m^{1-p'} \right) \right)^{r(1/p-s)}/p' < \infty,
\]
for \( s \) satisfying \( 0 < s \leq \frac{1}{p} \);

\[
B_3(s) := \sup_{n \in \mathbb{N}} \left( \sum_{k=n}^\infty u_k \right)^{-s} \left( \sum_{k=n}^\infty v_k^{1-p'} \right)^{p'(1/p+s)} \left( \sum_{k=n}^\infty u_k \right)^{1/p'} < \infty,
\]
for \( s \) satisfying \( 0 < s \leq \frac{1}{p} \). Note that \( B_1(p) = B_1'(s') = B_2, B_2(\frac{1}{p}) = B_2' \), and \( B_2'(\frac{1}{p'}) = B_3 \).

Our results in Theorems 3.1 and 3.2 can be generalized in a corresponding way namely that the three alternative conditions in these theorems can be replaced by infinite many equivalent conditions.
Remark 5.3. The corresponding alternative conditions for the parameters in Proposition 2.3 are not known except for the continuous case \( r < p, \ p > 1 \) where even four scales of such alternative equivalent conditions are known (see [31]). Hence, at the moment only in this case it seems to be possible to generalize Theorems 3.3 and 3.4 in this direction.

Remark 5.4. Some similar results as those in this paper can found in [4] (in Russian). However, the results in this paper are more complete and putted to a more general frame. The proofs are also different and more precise and clear.

Acknowledgements. This research has been done within the agreement between Luleå University of Technology, Sweden and L. N. Gumilyev Eurasian National University, Kazakhstan. We thank both these universities for financial and other support. We also thank Professors R. Oinarov and V. D. Stepanov for several generous advises which has improved the final version of this paper. The third author was partially supported by project 5495/GF4 of the Scientific Committee of Ministry of Education and Science of the Republic of Kazakhstan and project RFFI 16-31-50042.

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Some new Hardy-type inequalities in $q$-analysis


(Received February 17, 2015)

A. O. Baiarystanov
Eurasian National University
Munaytpasov st., 5, 010008 Astana, Kazakhstan
e-mail: oskar_62@mail.ru

L. E. Persson
Department of Engineering Sciences and Mathematics
Luleå University of Technology
SE-971 87, Luleå, Sweden
and
UiT, The Artic University of Norway
P. O. Box 385, N-8505, Narvik, Norway
e-mail: larserik@ltu.se

S. Shaimardan
Eurasian National University
Munaytpasov st., 5, 010008 Astana, Kazakhstan
e-mail: serikbol-87@yandex.kz

A. Temirkhanova
Eurasian National University
Munaytpasov st., 5, 010008 Astana, Kazakhstan
e-mail: ainura-t@yandex.kz
Paper C
HARDY-TYPE INEQUALITIES FOR THE FRACTIONAL INTEGRAL OPERATOR IN $q$-ANALYSIS

S. Shaimardan

Key words: Hardy-type inequalities, integral operator, $q$-analysis, $q$-integral.

AMS Mathematics Subject Classification: 26D10, 26D15, 33D05, 39A13.

Abstract. We obtain necessary and sufficient conditions for the validity of a certain Hardy-type inequality involving $q$-integrals.

1 Introduction

The $q$-derivative or Jackson’s derivative, is a $q$-analogue of the ordinary derivative. $q$-differentiation is the inverse of Jackson’s $q$-integration. It was introduced by F. H. Jackson [11] (see also [7]). He was the first to develop $q$-analysis. After that many $q$-analogue of classical results and concepts were studied and their applications are investigated.

Concerning recent results on $q$-analysis and its applications we also refer to the recent book by T. Ernst [8]. Some integral inequalities were obtained by H. Gauchman [10]. A Hardy-type inequality in $q$-analysis was recently obtained by L. Maligranda, R. Oinarov and L-E. Persson [13].

In this paper we prove a new Hardy-type inequality in which the Hardy operator is replaced by the $q$-analogue of the infinitesimal fractional operator (see [1] and (1.3) below).

In classical analysis, the hypergeometric function (Gaussian function) is defined for $|z| < 1$ by the power series [9]:

$$2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad \forall \alpha, \beta, \gamma \in \mathbb{C},$$

where $(\alpha)_n$ is the Pochhammer symbol, which is defined by:

$$(\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n > 0.$$

If $B$ denotes the Beta function, then

$$2F_1(\alpha - 1, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{2-\alpha}dx,$$
where $\text{Re}(\gamma) > \text{Re}(\beta) > 0$. When $\beta = \gamma$ we have that

$$2F_1(\alpha - 1, \beta; \beta; z) = (1 - z)^{\alpha - 1}.$$ 

Let $\alpha + \beta < \gamma, \gamma \neq 0, -1, -2, \cdots$. Then the following generalized fractional integral operator was introduced in [14]:

$$I_{\alpha, \beta} f(x) = \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \int_0^x 2F_1\left(\alpha - 1, \beta; \gamma; \frac{s}{x}\right) ds,$$  \hspace{1cm} (1.1)

where $\Gamma(\cdot)$ denotes the Gamma function. If $\beta = \gamma$ then the operator

$$I_{\alpha} f(x) := \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \int_0^x 2F_1\left(\alpha - 1, \beta; \beta; \frac{s}{x}\right) ds,$$

is called the Riemann-Liouville fractional integral operator. When $\gamma = 1, \beta = 2$, we have that

$$\hat{I} f(x) := \lim_{\alpha \to 0} \frac{1}{\Gamma(\alpha)} I_{\alpha}^{1,2} f(x) = \int_0^x \ln \frac{x}{x - s} \frac{f(s)}{s} ds,$$  \hspace{1cm} (1.2)

which is called the infinitesimal fractional integral operator [1].

The purpose of this paper is to find a $q$-analogue of operator (1.2) and to prove a $q$-analogue of the following Hardy-type integral inequality [1]:

$$\left(\int_0^x u'(x) \left(\int_0^x t^{\gamma - 1} \ln \frac{x}{x - t} f(t) dt\right)^r dx\right)^{\frac{1}{r}} \leq C \left(\int_0^\infty f^p(x) dx\right)^{\frac{1}{p}}, \quad \forall f(\cdot) \geq 0,$$   \hspace{1cm} (1.3)

where $C > 0$ is independent of $f$ and $u$ is a positive real valued function on $(0, \infty)$ briefly a weight function. We derive necessary and sufficient conditions for the validity of a $q$-analogue of inequality (1.3) in $q$-analysis for the case $1 < p < \infty, 0 < r < \infty$ and $\gamma > \frac{1}{p}$ (see Theorem 3.1 and Theorem 3.2). We also consider the problem of finding the best constant in a $q$-analogue of inequality (1.3).

The paper is organized as follows: We present some preliminaries in Section 2. The main results and detailed proofs are presented in Section 3.

2 Preliminaries

First we recall definitions and notions of the theory of $q$-analysis, our main references are the books [7], [8] and [9].
Let $0 < q < 1$ be fixed.
For a real number $\alpha \in \mathbb{R}$, the $q$-real number $[\alpha]_q$ is defined by
\[
[\alpha]_q = \frac{1 - q^\alpha}{1 - q}, \quad \alpha \in \mathbb{R}.
\]
It is clear that $\lim_{q \to 1} \frac{1 - q^\alpha}{1 - q} = \alpha$.

The $q$-analogue of the power $(a - b)^k$ is defined by
\[
(a - b)^0_q = 1, \quad k \in \mathbb{N}, \quad (a - b)^k_q = \prod_{i=0}^{k-1} (a - q^i b), \quad \forall \ a, b \in \mathbb{R},
\]
and
\[
(1 - b)^\gamma_q := \frac{(1 - b)_\infty_q}{(1 - q^\alpha b)_\infty_q}, \quad \forall b, \alpha \in \mathbb{R}.
\]
(2.1)
and by using well-known relations this can also be written as
\[
(1 - b)^\gamma_q = \frac{1}{(1 - q^\alpha b)^{-\gamma}}, \quad \forall b, \alpha \in \mathbb{R}.
\]
(2.2)

The $q$-hypergeometric function $\Phi_1$ is defined by ([9]):
\[
\Phi_1 \left[ \begin{array}{ccc} q^\alpha & q^\beta & q^\gamma \\ q & q & q \end{array} ; q : x \right] := \sum_{n=0}^{\infty} \frac{(q^\alpha, q)_n (q^\beta, q)_n (q^\gamma, q)_n}{(q, q)_n (q^\gamma, q)_n} x^n, \quad |x| < 1,
\]
where $(q^\alpha, q)_n = \prod_{i=0}^{n-1} (1 - q^{i+\alpha})$ and $\gamma \neq 0, -1, -2, \ldots$. Moreover, this series converges absolutely and $\lim_{q \to 1} \frac{(q^\alpha, q)_n}{(1-q)^n} = (a)_n$, so
\[
\lim_{q \to 1} \Phi_1 \left[ \begin{array}{ccc} q^\alpha & q^\beta & q^\gamma \\ q & q & q \end{array} ; q : x \right] = \Phi_1 (\alpha, \beta; \gamma; x).
\]

For $f : [0, b] \to \mathbb{R}$, $0 \leq b < \infty$, the $q$-derivative is defined by:
\[
D_q f(x) := \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \in (0, b),
\]
(2.3)
and $D_q f(0) = f'(0)$ provided $f'(0)$ exists. It is clear that if $f(x)$ is differentiable, then
$\lim_{q \to 1} D_q f(x) = f'(x)$.

**Definition 1.** The $q$-Taylor series of $f(x)$ at $x = c$ is defined by
\[
f(x) := \sum_{j=0}^{\infty} \left( D_q^j \right)(c) \frac{(x - c)^j}{[j]_q!},
\]
where \([j]_q! = \left\{ \begin{array}{ll} 1, & \text{if } j = 0, \\
[1]_q \times [2]_q \times \cdots \times [j]_q, & \text{if } j \in \mathbb{N}. \end{array} \right. \)
The definite $q$-integral or the $q$-Jackson integral of a function $f$ is defined by the formula
\[ \int_{0}^{x} f(t) d_q t := (1 - q)x \sum_{k=0}^{\infty} q^k f(q^k x), \quad x \in (0, b), \quad (2.4) \]
and the improper $q$-integral of a function $f(x) : [0, \infty) \rightarrow \mathbb{R}$, is defined by the formula
\[ \int_{0}^{\infty} f(t) d_q t := (1 - q) \sum_{k=-\infty}^{\infty} q^k f(q^k). \quad (2.5) \]

Note that the series in the right hand sides of (2.4) and (2.5) converge absolutely.

**Definition 2.** The function
\[ \Gamma_q(\alpha) := \int_{0}^{\infty} x^{\alpha-1} E_q^{-qx} d_q x, \quad \alpha > 0, \]
is called the $q$-Gamma function, where $E_q^{-qx} = (1 - (1 - q)x)^{-\infty}$.

We have that
\[ \Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha), \]
for any $\alpha > 0$.

**Definition 3.** The function
\[ B_q(\alpha, \beta) := \int_{0}^{1} t^{\alpha-1}(1 - qt)^{\beta-1} d_q t, \quad \alpha, \beta > 0, \]
is called the $q$-Beta function. Note that
\[ B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)}, \]
for $\alpha, \beta > 0$.

Let $\Omega$ be a subset of $(0, \infty)$ and $X_\Omega(t)$ denote the characteristic function of $\Omega$. For all $z > 0$, we have that (see [5]):
\[ \int_{0}^{\infty} X_{[0, z]}(t) f(t) d_q t = (1 - q) \sum_{q^i \leq z} q^i f(q^i), \quad (2.6) \]
\[ \int_{0}^{\infty} X_{[z, \infty)}(t) f(t) d_q t = (1 - q) \sum_{q^i \geq z} q^i f(q^i). \quad (2.7) \]
R.P. Agarwal and W.A. Al-Salam (see [2], [3] and [4]) introduced several types of fractional $q$-integral operators and fractional $q$-derivatives. In particular, they defined the $q$-analogue of the fractional integral operator of the Riemann-Liouville type by

$$I_{q,\alpha}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (1 - \frac{q^s x}{x})^{\alpha-1} f(s) d_q s, \quad \alpha \in \mathbb{R}^+. $$

Using formula (2.2), we can rewrite $I_{q,\alpha}$ as follows:

$$I_{q,\alpha}f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \frac{f(s)}{(1 - q^{\alpha} \frac{s}{x})^{1-\alpha}} d_q s, \quad \alpha \in \mathbb{R}^+. $$

(2.8)

Out next goal is to define a $q$-analogue of the logarithm of the Riemann-Liouville type by

$$\ln_q x = \int_0^x \frac{1}{(1 - q^{\alpha} \frac{s}{x})^{1-\alpha}} d_q s. $$

(2.9)

and

$$\frac{q}{x}^\alpha \ln_q x^{\alpha} = \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{1-\alpha}}, $$

(2.10)

for $\beta, \gamma > 0$, and

$$2\Phi_1 \left[ \begin{array}{c} q^{1-\alpha} q^3 \\ q^7 \end{array} \right] = \frac{1}{B_q(\beta, \gamma)} \int_0^1 \frac{t^{\beta-1} (1 - q^t)^{\gamma-\beta-1}}{(1 - q^{\alpha} \frac{t}{x})^{1-\alpha}} d_q t, $$

(2.11)

for $\beta = \gamma$.

Proof. First we consider equality (2.10). From (2.1) and (2.3), we get that

$$D^1_{q,s} \left( \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{1-\alpha}} \right) = \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{\infty}} - \frac{1}{(1 - q^{\alpha} \frac{1}{x})^{\infty}} \frac{1}{(q - 1)s} $$

$$= \left[ \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{\infty}} - \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{\infty}} \right] \frac{1}{(q - 1)s} $$

$$= \frac{1}{(1 - q^{\alpha} \frac{1}{s})^{\infty}} \left[ \frac{(1 - q^{\alpha} \frac{x}{s})^{\infty}}{(1 - q^{\alpha} \frac{x}{s})^{\infty}} \right] \frac{q^s (q^{1-\alpha} - 1)}{x(q - 1)} $$

$$= \frac{q^s [1 - \alpha]^q}{(1 - q^{\alpha} \frac{x}{s})^{\infty}}. $$

Using this relation and induction, one can easily see that

$$D^j_{q,s} \left( \frac{1}{(1 - q^{\alpha} \frac{x}{s})^{1-\alpha}} \right) \bigg|_{s=0} = \frac{q^{ja}}{x^j} [1 - \alpha]^q [2 - \alpha]^q \cdots [j - \alpha]^q, $$

for any $j \geq 1$. Therefore, we have the $q$-Taylor expansion (see Definition 1)
\[
\frac{1}{(1 - q^a s/x)_q^{1-\alpha}} = \sum_{j=0}^{\infty} \frac{[1 - \alpha]_q [2 - \alpha]_q \cdots [j - \alpha]_q}{[j]_q^\alpha} \left( \frac{q^a s}{x} \right)^j
\]

\[
= \sum_{j=0}^{\infty} \frac{(1 - q^{1-\alpha})_q^j}{(1 - q)_q^j} \left( \frac{q^a s}{x} \right)^j
\]

\[
= 2 \Phi_1 \left[ \frac{q^{1-\alpha} q^a s}{q^\gamma \alpha : x} \right],
\]

and (2.10) is proved.

By using the same arguments as above we see that

\[
\frac{1}{(1 - q^a t/x)_q^{1-\alpha}} = \sum_{n=0}^{\infty} \frac{(1 - q^{1-n})_q^n}{(1 - q)_q^n} \left( \frac{q^a s}{x} \right)^n,
\]

for \( x \geq s, 0 < t \leq 1 \). Therefore

\[
\int_0^1 t^{\beta-1} (1 - qt)_q^{\gamma-\beta-1} \frac{1}{(1 - q^a t/x)_q^{1-\alpha}} dt_0 = \sum_{n=0}^{\infty} \frac{(1 - q^{1-n})_q^n}{(1 - q)_q^n} \left( \frac{q^a s}{x} \right)^n \int_0^1 t^{\beta+n-1} (1 - qt)_q^{\gamma-\beta-1} dt = \sum_{n=0}^{\infty} \frac{(1 - q^{1-n})_q^n}{(1 - q)_q^n} \left( \frac{q^a s}{x} \right)^n \frac{\Gamma_q(\beta+n)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma+\beta)}
\]

\[
= \frac{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma)} \sum_{n=0}^{\infty} \frac{(1 - q^{1-n})_q^n(1 - q^b)_q^n}{(1 - q)_q^n(1 - q^\gamma)_q^n} \left( \frac{q^a s}{x} \right)^n
\]

\[
= B_q(\beta, \gamma) \Phi_1 \left[ \frac{q^{1-\alpha} q^a s}{q^\gamma \alpha : x} \right].
\]

and also (2.9) is proved. 

By Proposition 2.1, the integral (2.8) can be rewritten as

\[
I_{q,\alpha}(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \Phi_1 \left[ \frac{q^{1-\alpha} q^3}{q^\gamma \alpha : x} \right] f(s) ds s, \quad \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}.
\]

More generally, we consider the \( q \)-analogue of \( I_{\alpha,\beta} \) (see (1.1))

\[
I_{\gamma,\alpha,\beta}(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \Phi_1 \left[ \frac{q^{1-\alpha} q^3}{q^\gamma \alpha : x} \right] f(s) ds s, \quad \alpha, \beta, \gamma \in \mathbb{R}^+.
\]

Due to uniform convergence of the series \( \Phi_1 \left[ \frac{q^{1-\alpha} q^3}{q^\gamma \alpha : x} \right] \) for \( 0 < \alpha < 1 \), we get that

\[
\lim_{\alpha \to 0^+} \Phi_1 \left[ \frac{q^{1-\alpha} q^3}{q^\gamma \alpha : x} \right] \frac{s}{x} = \Phi_1 \left[ \frac{q q^3}{q^\gamma \alpha : x} \right] \frac{s}{x}
\]

\[
= \sum_{j=0}^{\infty} \frac{1 - q}{1 - q^j + 1} \frac{s^{j+1}}{x^{j+1}} = \sum_{j=0}^{\infty} \frac{(\frac{s}{x})^j}{[j+1]q} = \sum_{j=0}^{\infty} \frac{(\frac{s}{x})^j}{[j]q},
\]
which is the $q$-analogue of the Taylor series of the function $\ln \frac{x}{x-s}$ with $s < x$.

**Definition 4.** We define the $q$-analogue of the function $\ln \frac{x}{x-s}$, $0 < s < x < \infty$, as follows:

$$\ln_q \frac{x}{x-s} := \sum_{j=1}^{\infty} \frac{(\frac{s}{x})^j}{[j]_q}.$$

**Remark 5.** We define the $q$-analogue of (1.2) as follows:

$$\hat{I}_q f(x) := \int_0^{qx} \ln_q \frac{x}{x-s} \frac{f(s)}{s} ds,$$

which is called the infinitesimal $q$-fractional integral operator.

Observe that:

$$\lim_{q \to 1} \hat{I}_q f(x) = \int_0^x \ln \frac{x}{x-s} \frac{f(s)}{s} ds.$$

Hence, from (2.12) we obtain the $q$-analogue of (1.3) in the following form:

$$\left( \int_0^\infty u^r(x) \left( \hat{I}_q f(x) \right)^r d_q x \right)^{\frac{1}{r}} \leq C \left( \int_0^\infty f^p(t) d_q t \right)^{\frac{1}{p}}, \quad \forall f(\cdot) \geq 0,$$

where $C > 0$ independent of $f$.

In the $q$-integral we are allowed to change variables in the form $x = t \xi$ for $0 < \xi < \infty$ (see [7]). So by making the substitution $t = qs$, and $d_q t = qd_q s$ inequality (2.13) becomes

$$\left( \int_0^\infty u^r(x) \left( \int_0^x s^{\gamma-1} \ln_q \frac{x}{x-qs} \bar{f}(s) d_q s \right)^r d_q x \right)^{\frac{1}{r}} \leq \tilde{C} \left( \int_0^\infty \bar{f}^p(s) d_q s \right)^{\frac{1}{p}}, \quad \forall f(\cdot) \geq 0.$$

where $\bar{f}(s) = f(qs)$, $\tilde{C} = q^{\gamma-\frac{1}{p}} C$.

Since inequality (2.13) holds if and only if inequality (2.14) holds, from now on we will investigate necessary and sufficient conditions the validity of inequality (2.14).

**Notation.** In the sequel, for any $p > 1$ the conjugate number $p'$ is defined by $p' := p/(p-1)$. Moreover, the symbol $M \ll K$ means that there exists $\alpha > 0$ such that $M \leq \alpha K$, where $\alpha$ is a constant which depend only on the numerical parameters such as $p$, $q$, $r$. If $M \ll K \ll M$, then we write $M \approx K$.

For the proof of our main theorems we will need the following well-known discrete weighted Hardy inequality proved by G. Bennett [6] (see also [12], p.58):
Theorem A. Let \( \{u_i\}_{i=1}^\infty \) and \( \{v_j\}_{j=1}^\infty \) be non-negative sequences of real numbers and \( 1 < p \leq r < \infty \). Then the inequality

\[
\left( \sum_{j=-\infty}^{\infty} \left( \sum_{i=j}^{\infty} f_i \right)^{r} u_j^r \right)^{\frac{1}{r}} \leq C \left( \sum_{i=-\infty}^{\infty} v_i^p f_i^p \right)^{\frac{1}{p}}, \quad f \geq 0, \ i \in \mathbb{Z},
\]

with \( C > 0 \) independent of \( f_i, \ i \in \mathbb{Z} \) holds if and only if

\[
\mathcal{B}_1 := \sup_{n \in \mathbb{Z}} \left( \sum_{j=-n}^{n} u_j^r \right)^{\frac{1}{r}} \left( \sum_{i=-\infty}^{\infty} v_i^{-p'} \right)^{\frac{1}{p'}} < \infty, \ p' = \frac{p}{p-1},
\]

Moreover, \( \mathcal{B}_1 \approx C \), where \( C \) is the best constant in (2.15).

Theorem B. Let \( 0 < r < p < \infty \) and \( 1 < p \). Then inequality (2.15) holds if and only if \( \mathcal{B}_2 < \infty \), where

\[
\mathcal{B}_2 := \left( \sum_{k=-\infty}^{\infty} v_k^{-p'} \left( \sum_{i=-\infty}^{\infty} u_i^r \right)^{\frac{p}{p-r}} \left( \sum_{i=k}^{\infty} v_i^{-p'} \right)^{\frac{n(r-1)}{p-r}} \right)^{\frac{n-r}{p'}}.
\]

Moreover, \( \mathcal{B}_2 \approx C \), where \( C \) is the best constant in (2.15).

Also we need the following lemma ([5]):

Lemma A. Let \( f, \varphi \) and \( g \) be nonnegative functions. Then

\[
\int_{0}^{\infty} \left( \int_{0}^{\infty} X_{[\varepsilon, \infty)}(t)f(t)d_qt \right)^{\alpha} \left( \int_{0}^{\infty} X_{[0, \varepsilon]}(x)g(x)dx \right)^{\beta} \varphi(z)d_qz = (1-q)^{\alpha+\beta} \left( \sum_{k=-\infty}^{\infty} q^k f(q^k) \right)^{\alpha} \left( \sum_{j=k}^{\infty} q^j g(q^j) \right)^{\beta} q^k \varphi(q^k),
\]

for \( \alpha, \ \beta \in \mathbb{R} \).

3 Main results

Our main result reads:

Theorem 3.1. Let \( 1 < p \leq r < \infty, \ \gamma > \frac{1}{p} \). Then the inequality

\[
\left( \int_{0}^{\infty} u^r(x) \left( \int_{0}^{x} s^{\gamma-1} \ln_q \frac{x}{x - qs} f(s)d_qs \right)^{r} d_qx \right)^{\frac{1}{r}} \leq C \left( \int_{0}^{\infty} f^p(s)d_qs \right)^{\frac{1}{p}}, \ \forall f(\cdot) \geq 0, \ (3.1)
\]
with \( C > 0 \) independent of \( f \) holds if and only if \( B_1 < \infty \), where
\[
B_1 := \sup_{x > 0} x^{\gamma + \frac{1}{p}} \left( \int_0^\infty \mathcal{X}_{[x, \infty)}(t) \left( \frac{u^r(t)}{t^r} \right)^\frac{1}{r} \right)^{\frac{1}{\gamma}},
\]
Moreover, \( B_1 \approx C \), where \( C \) is the best constant in (3.1).

**Theorem 3.2.** Let \( 0 < r < p < \infty \), \( 1 < p \) and \( \gamma > \frac{1}{p} \). Then the inequality (3.1) holds if and only if \( B_2 < \infty \), where
\[
B_2 := \left( \int_0^\infty x^{\gamma + \frac{1}{p}} \left( \int_0^\infty \mathcal{X}_{[x, \infty)}(t) \left( \frac{u^r(t)}{t^r} \right)^\frac{1}{r} \right)^{\frac{1}{\gamma}} \frac{d_q x}{t} \right)^{\frac{p}{p-r}}.
\]
Moreover, \( B_2 \approx C \), where \( C \) is the best constant in (3.1).

**Remark 6.** By using formulas (2.4) and (2.5) in (3.1) we get that
\[
\left( \sum_{j=-\infty}^{\infty} (1-q)q^j u^r(q^j) \left( \sum_{i=j}^{\infty} (1-q)q^i f(q^i) \ln_q \frac{1}{1-q^{i-j+1}} \right)^{\frac{1}{p}} \right)^{\frac{p}{p-r}} \leq C \left( \sum_{i=-\infty}^{\infty} (1-q)q^i f^p(q^i) \right)^{\frac{1}{p}}.
\]

Let
\[
u_j = (1-q)^{1+p}q^j u^r(q^j), \quad f_i = (1-q)^{1+p}q^i f(q^i), \quad a_{i,j} = \ln_q \frac{1}{1-q^{i-j+1}}.
\]

Then we get that inequality (3.1) is equivalent to the discrete weighted Hardy-type inequality
\[
\left( \sum_{j=-\infty}^{\infty} \nu_j \left( \sum_{i=j}^{\infty} q^{(\gamma - \frac{1}{p})i} f(a_{i,j}) \right)^{\frac{1}{r}} \right)^{\frac{r}{p-r}} \leq C \left( \sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{1}{p}}.
\]

Note that inequality (3.1) holds if and only if inequality (3.3) holds, so we will obtain the desired necessary and sufficient conditions for the validity of inequality (3.3).

Our next Lemmas give a characterization of the discrete Hardy-type inequality (3.3).

**Lemma 3.1.** Let \( 1 < p \leq r < \infty \), \( \gamma > \frac{1}{p} \). Then the inequality (3.3) holds if and only if \( B_1 < \infty \), where
\[
B_1 := \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} q^{(p \gamma + 1)} \right)^{\frac{1}{p}} \left( \sum_{j=-\infty}^{k} q^{-j r} \nu_j \right)^{\frac{1}{r}}.
\]

Moreover, \( B_1 \approx C \), where \( C \) is the best constant in (3.3).
Proof. Necessity. Let us assume that (3.3) holds with some $C > 0$. From (3.2) and Definition 4 we get that $q^{j+1}/q^i \leq a_{i,j}$ for $j \leq i$. Then

$$\sum_{j=-\infty}^{\infty} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma - \frac{1}{p'})} f_i a_{i,j} \right)^r \geq q \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right)^r.$$

Moreover,

$$q \left( \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right) \right)^{\frac{1}{r}} \leq C \left( \sum_{j=-\infty}^{\infty} f_i^p \right)^{\frac{1}{p}}.$$

Hence, by Theorem A we obtain that

$$B_1 \ll C.$$  \hspace{1cm} (3.5)

The proof of the necessity is complete.

Sufficiency. Let $B < \infty$ and $f \geq 0$ be arbitrary. We will show that inequality (3.3) holds.

We consider two cases separately: $0 < q \leq \frac{1}{2}$ and $\frac{1}{2} < q < 1$.

1) Let $0 < q \leq \frac{1}{2}$. Let $j \leq k \leq i$. Then from (3.2) and Definition 4 it follows that

$$a_{j,j} = \ln_q \frac{1}{1-q} \leq \ln_q 2 \text{ (we note that } \ln_q 2 := \sum_{n=1}^{\infty} \frac{2^{-n}}{[n]_q}, \text{ and)}$$

$$q^{-k} q_j - q^{-i} q_{i,j} = q^{-k} \sum_{n=1}^{\infty} \frac{(q/q^i)^n}{[n]_q} - q^{-i} \sum_{n=1}^{\infty} \frac{(qq^i/q^i)^n}{[n]_q}$$

$$= \sum_{n=1}^{\infty} \frac{(q/q^i)^n}{[n]_q} (q^{k(n-1)} - q^{i(n-1)}) \geq 0,$$

i.e.

$$q^{-i} q_{i,j} \leq q^{-k} q_j,$$  \hspace{1cm} (3.6)

for $j \leq k \leq i$.

Thus by (3.6) we have that

$$\sum_{j=-\infty}^{\infty} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma - \frac{1}{p'})} f_i a_{i,j} \right)^r = \sum_{j=-\infty}^{\infty} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} q^{-a_{i,j}} f_i \right)^r$$

$$\leq \sum_{j=-\infty}^{\infty} q^{-jr} a_{j,j}^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right)^r$$

$$\leq (\ln_q 2)^r \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right)^r$$

$$\ll \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right)^r.$$
Hence, by Theorem A we obtain that
\[
\left( \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i \right) \right)^{\frac{1}{r}} \leq B_1 \left( \sum_{j=-\infty}^{\infty} f_i^p \right)^{\frac{1}{p}},
\]
which means that inequality (3.3) is valid and that \( C \ll B_1 \), where \( C \) is the best constant for which (3.3) holds.

2) Let \( \frac{1}{2} < q < 1 \). Then \( \exists i_0 \in N \) such that \( i_0 > 1 \) and \( q^{i_0} \leq \frac{1}{2} < q^{i_0-1} \). We assume that \( Z = \bigcup_{k \in Z} [t_k + 1, t_{k+1}] \) and \( t_{k+1} - t_k = t_0 \). Then the left hand side of (3.3) can be written as
\[
\sum_{j=-\infty}^{\infty} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i a_{i,j} \right)^r = \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i a_{i,j} \right)^r \\
\approx \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r \left( \sum_{i=t_k+1}^{t_{k+2} - 1} q^{i(\gamma + \frac{1}{p'})} f_i a_{i,j} \right)^r \\
+ \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r \left( \sum_{i=t_k+2}^{\infty} q^{i(\gamma + \frac{1}{p'})} f_i a_{i,j} \right)^r
\]
\[= I_1 + I_2. \quad (3.7)\]

To estimate \( I_1 \) we use Hölder's inequality. We find that
\[
I_1 \leq \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r \left( \sum_{i=j}^{\infty} q^{i\gamma + \frac{1}{p'}} a_{i,j}^r \right)^{\frac{p'}{p}} \left( \sum_{i=j}^{\infty} f_i^p \right)^{\frac{1}{p}} \leq \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r \left( \sum_{i=j}^{\infty} q^{i\gamma + \frac{1}{p'}} a_{i,j}^r \right)^{\frac{p'}{p}} \left( \sum_{i=t_k+1}^{\infty} f_i^p \right)^{\frac{1}{p}} \]
\[= \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r q^{j\gamma + \frac{1}{p'}} \left( \sum_{i=0}^{\infty} q^{i\gamma + \frac{1}{p'}} a_{i,0}^r \right)^{\frac{p'}{p}} \left( \sum_{i=t_k+1}^{\infty} f_i^p \right)^{\frac{1}{p}} \]
\[= C_0^{\frac{p'}{p}} \sum_{k} \sum_{j=t_k+1}^{t_{k+1}} u_j^r q^{j\gamma + \frac{1}{p'}} \left( \sum_{i=t_k+1}^{\infty} f_i^p \right)^{\frac{1}{p}}, \quad (3.8)\]
where \( C_0 := \sum_{i=0}^{\infty} q^{i\gamma + \frac{1}{p'}} a_{i,0}^r \).

Since
\[
M := (1 - q) C_0 = \int_0^{1} x^{\gamma + \frac{1}{p'}} \left( \ln_q \frac{1}{1 - qx} \right)^{p'} d_q x < \infty
\]
and
\[
q^{j\gamma + \frac{1}{p'}} \leq q^{(t_k+1)(\gamma + \frac{1}{p'})} = q^{-(t_0-1)(\gamma + \frac{1}{p'})} q^{t_k+1(\gamma + \frac{1}{p'})} \leq 2^{\gamma + \frac{1}{p'}} q^{t_k+1(p'\gamma + 1)\frac{1}{p'}}, \quad (3.9)
\]
for $t_k + 1 \leq j$, we get that

$$I_1 \leq 2^{r(\gamma + \frac{1}{p})} M \tilde{p} \left[ p' \gamma + 1 \right] q \sum_k \left( \sum_{i=t_k+2}^{t_k+2} f_i^p \right)^{\frac{r}{p}} \left( \frac{q^{i+1} \left( p' \gamma + 1 \right)}{1 - q^{p' \gamma + 1}} \right)^{\frac{r}{p}} \sum_{j=-\infty}^{t_k+1} u_j^r q^{-jr}$$

$$\ll \sum_k \left( \sum_{i=t_k+1}^{t_k+2} f_i^p \right)^{\frac{r}{p}} \left( \frac{q^{i+1} \left( p' \gamma + 1 \right)}{1 - q^{p' \gamma + 1}} \right)^{\frac{r}{p}} \sum_{j=-\infty}^{t_k+1} u_j^r q^{-jr}$$

$$= \sum_k \left( \sum_{i=t_k+1}^{t_k+2} f_i^p \right)^{\frac{r}{p}} \left[ \left( \sum_{i=t_k+1}^{t_k+1} q^{jr \left( p' \gamma + 1 \right)} \right)^{\frac{r}{p}} \left( \sum_{j=-\infty}^{t_k+1} u_j^r q^{-jr} \right) \right]^{\frac{1}{r}}$$

$$\ll B_k \left( \sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}. \quad (3.10)$$

Let $j \leq k \leq i$. Then from (3.2) and Definition 4 it follows that

$$q^j a_{i,k} - q^j a_{i,j} \geq q^j (a_{i,k} - a_{i,j}) = q^j \sum_{n=1}^{\infty} \frac{(q^{i+1} / q^j)^n}{[n]_q} - q^j \sum_{n=1}^{\infty} \frac{(q^{i+1} / q^j)^n}{[n]_q}$$

$$= \sum_{n=1}^{\infty} \frac{q^{(i+1)n}}{[n]_q} \left( q^{(k-1)n} - q^{(j-1)n} \right) \geq 0,$$

i.e.

$$q^j a_{i,j} \leq q^k a_{i,k}, \quad (3.11)$$

for $j \leq k \leq i$.

Using (3.6) and (3.11) we find that

$$\frac{1}{q^{-j} a_{i,j}} \leq \frac{1}{d_k - 1} \ln q \frac{1}{1 - q^{k+1} - t_k} = \frac{1}{q^{-a} \ln q} \frac{1}{1 - q^{a}} \leq 2 \ln q,$$

for $j \leq t_k + 1$ and $t_k + 2 \leq i$.

Therefore,

$$I_2 = \sum_k \sum_{j=t_k+1}^{t_k+1} u_j^r \left( \sum_{i=t_k+2}^{t_k+2} q^{(\gamma - 1) / p} f_i a_{i,j} \right)^{\frac{r}{p}}$$

$$= \sum_k \sum_{j=t_k+1}^{t_k+1} q^{-jr} u_j^r \left( \sum_{i=t_k+2}^{t_k+2} q^{(\gamma - 1) / p} \frac{1}{q^{-jr} a_{i,j}} f_i \right)^{\frac{r}{p}}$$

$$\leq \left( 2 \ln q \right) \tilde{p} \sum_k \sum_{j=t_k+1}^{t_k+1} q^{-jr} u_j^r \left( \sum_{i=t_k+1}^{t_k+2} q^{(\gamma - 1) / p} f_i \right)^{\frac{r}{p}}$$

$$\ll \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{(\gamma - 1) / p} f_i \right)^{\frac{r}{p}}.$$
By using Theorem A we have that
\[
I_2 \ll B_1^r \left( \sum_{i=-\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}, \tag{3.12}
\]

Thus, from (3.7), (3.10) and (3.12) it follows that inequality (3.3) is valid and we see that the best constant \( C \) in (3.3) is such that \( C \ll B_1 \), which together with (3.5) gives that \( C \approx B_1 \).

**Lemma 3.2.** Let \( 0 < r < p < \infty \) and \( 1 < p \). Then inequality (3.3) holds if and only if \( B_2 < \infty \), where
\[
B_2 := \left( \sum_{k=1}^{\infty} q^k (p' \gamma + 1) \left( \sum_{i=1}^{k} u_i^r q^{-ir} \right)^{\frac{p-1}{p-r}} \left( \frac{q^i}{p-1} \right) \right)^{\frac{r}{p}}.
\]

Moreover, \( B_2 \approx C \), where \( C \) is the best constant in (3.3).

**Proof.** In a similar way as in the proof of Lemma 3.1, by Theorem B we obtain that inequality (3.3) is valid and that \( C \approx B_2 \), where \( C \) is the best constant for which (3.3) holds for \( 0 < q \leq \frac{1}{2} \).

In case \( \frac{1}{2} < q < 1 \) the necessary part is due to Theorem B. Therefore,
\[
B_2 \ll C. \tag{3.13}
\]

To prove sufficiency we proceed as follows. Applying to (3.8) Hölder’s inequality with the exponents \( \frac{p}{p-r} \) and \( \frac{r}{p} \) we obtain that
\[
I_1 \ll \sum_{k} \left( \sum_{i=t_k+1}^{t_{k+2}} f_i^p \right)^{\frac{r}{p}} q^{t_k+1} (p' \gamma + 1) \left( \sum_{j=-\infty}^{k} u_j^r q^{-jr} \right)^{\frac{p-1}{p-r}} \left( \frac{q^k}{p-1} \right)^{\frac{r}{p}}
\]
\[
\leq \left( \sum_{k} q^{t_k+1} (p' \gamma + 1) \left( \sum_{j=-\infty}^{k} u_j^r q^{-jr} \right)^{\frac{p-1}{p-r}} \right)^{\frac{r}{p}} \left( \sum_{k} \sum_{i=t_k+1}^{t_{k+2}} f_i^p \right)^{\frac{r}{p}}
\]
\[
\ll B_2^{\frac{r}{p}} \left( \sum_{i=\infty}^{\infty} f_i^p \right)^{\frac{r}{p}}.
\]

Since
\[
\tilde{B} := \sum_{i=-\infty}^{\infty} q^i (p' \gamma + 1) \left( \sum_{j=-\infty}^{i} u_j^r q^{-jr} \right)^{\frac{p-1}{p-r}}
\]
\[
\leq \sum_{i=-\infty}^{\infty} q^i (p' \gamma + 1) \left( \frac{q^i}{1 - q^i (p' \gamma + 1)} \right)^{\frac{p-1}{p-r}} \left( \sum_{j=-\infty}^{i} u_j^r q^{-jr} \right)^{\frac{p}{p-r}}
= \tilde{B}_2^{\frac{p-r}{p}},
\]
we have that

$$I_1 \ll B_2^r \left( \sum_{i=-\infty}^{\infty} f_i^p \right) \frac{1}{p}$$

(3.14)

From (3.12) and Theorem B it follows that

$$I_2 \ll \sum_{j=-\infty}^{\infty} q^{-jr} u_j^r \left( \sum_{i=j}^{\infty} q^{i(\gamma + \frac{1}{p})} f_i \right) \leq B_2^r \left( \sum_{i=-\infty}^{\infty} f_i^p \right) \frac{1}{p},$$

(3.15)

Thus, from (3.14) and (3.15) it follows that $C \ll B_2$ which means that the inequality (3.3) is valid, which together with (3.13) gives $B_2 \approx C$.

\[ \square \]

**Lemma 3.3.** Let $\gamma > \frac{1}{p}$, and $B_1 < \infty$. Then

$$\alpha B_1 = \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} q^{i(p'\gamma + 1)} \right)^{1/p} \left( \sum_{j=-\infty}^{k} q^{j(1-r)} u^r(q^j) \right)^{1/r},$$

(3.16)

for $r > 0$, where $\alpha = [p'\gamma + 1]^{-\frac{1}{p}} (1 - q)^{-\frac{1}{r} - \frac{1}{p'}}$.

**Proof.** Let $\gamma > \frac{1}{p}$. By using (2.7) we obtain that

$$I(x) = x^{\gamma + \frac{1}{p}} \left( \int_{0}^{\infty} \lambda_{[x, \infty)}(t) t^{-r} u^r(t) d_q t \right)$$

$$= (1 - q)^{\frac{1}{r}} x^{\gamma + \frac{1}{p}} \left( \sum_{q^j \geq x} q^{(1-r)j} u^r(q^j) \right)^{\frac{1}{r}},$$

for $\forall r > 0$. Then

$$I(x) = (1 - q)^{\frac{1}{p} + \frac{1}{p'}} [p'\gamma + 1] \left( \sum_{i=k}^{\infty} q^{i(p'\gamma + 1)} \right)^{1/p} \left( \sum_{j=-\infty}^{k} q^{(1-r)j} u^r(q^j) \right)^{1/r},$$

for $x = q^k, \forall k \in \mathbb{Z}$. Moreover,

$$I(x) = (1 - q)^{\frac{1}{p} + \frac{1}{p'}} [p'\gamma + 1] \left( \sum_{i=k}^{\infty} q^{i(p'\gamma + 1)} \right)^{1/p} \left( \sum_{j=-\infty}^{k-1} q^{(1-r)j} u^r(q^j) \right)^{1/r},$$

for $q^k < x < q^{k-1}$. Hence

$$\sup_{q^k < x \leq q^{k-1}} I(x) = (1 - q)^{\frac{1}{p} + \frac{1}{p'}} [p'\gamma + 1] \left( \sum_{i=k}^{\infty} q^{i(p'\gamma + 1)} \right)^{1/p} \left( \sum_{j=-\infty}^{k} q^{(1-r)j} u^r(q^j) \right)^{1/r},$$
and
\[ \alpha B_1 = \alpha \sup_{k \in \mathbb{Z}} \sup_{q_k < x \leq q_k - 1} I(x) = \sup_{k \in \mathbb{Z}} \left( \sum_{i=k}^{\infty} q_i^{(p'\gamma + 1)} \right)^{\frac{1}{p'}} \left( \sum_{j=-\infty}^{k} q_j^{(1-r)\gamma} u_j^r(q_j) \right)^{\frac{1}{r}}. \]

We have proved that (3.16) holds. \(\square\)

Next, we prove Theorem 3.1.

**Proof of Theorem 3.1.** First we note that inequality (3.3) is equivalent to inequality (3.1). Moreover, by Lemma 3.1 inequality (3.1) holds if and only if \( B_1 < \infty \). From (3.2) and Lemma 3.3 we have that \( B_1 = \alpha B_1 \), which means that \( B_1 \approx C \) and inequality (3.1) holds if and only if \( B_1 < \infty \). \(\square\)

**Proof of Theorem 3.2.** In a similar way as in the proof of Theorem 3.1, by Lemma 3.2 we have that inequality (3.1) holds if and only if \( B_2 < \infty \). From (3.2) and Lemma A we have that
\[
B_2^{\frac{p}{r}} = \left( 1 - q \right) \sum_{k=-\infty}^{\infty} q_k^{p'\gamma + 1} \left( \sum_{i=-\infty}^{k} q_i^{(p'\gamma + 1)} \right)^{\frac{p}{p'-r}} \times \left( \sum_{i=k}^{\infty} q_i^{(p'\gamma + 1)} \right)^{\frac{p}{p'-r}}
\]
\[= \int_{0}^{\infty} x^{p'\gamma + 1} \left( \int_{0}^{\infty} \mathcal{X}_{[x,\infty)}(t) \frac{u_r^r(t)}{t^r} \frac{p}{p'-r} \left( \int_{0}^{\infty} \mathcal{X}_{[0,x]}(s) \frac{d_q s}{s} \right)^{\frac{p}{p'-r}} d_q s \right)^{\frac{p}{p'-r}} d_q x
\]
\[= \left[ p' \gamma + 1 \right] q^{-\frac{p}{p'-r}} \int_{0}^{\infty} x^{p'\gamma + 1} \left( \int_{0}^{\infty} \mathcal{X}_{[x,\infty)}(t) \frac{u_r^r(t)}{t^r} \frac{p}{p'-r} \right)^{\frac{p}{p'-r}} d_q x
\]
\[\ll B_2,
\]
which means that \( B_2 \approx C \) and inequality (3.1) holds if and only if \( B_2 < \infty \). The proof is complete. \(\square\)

**Acknowledgments**

The author thank Professor Ryskul Oinarov (L.N. Gumilyev Eurasian National University, Kazakhstan) and Lars-Erik Persson (Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden) for good advices which have improved the final version of this paper. This work was supported by Scientific Committee of Ministry of Education and Science of the Republic of Kazakhstan, grant no. 5495/GF4. It was also supported by the Russian Scientific Foundation (project RFFI 16-31-50042).
Some Hardy-type inequalities for the fractional integral operator in q-analysis

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Serikbol Shaimardan
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
2 Satpayev St,
010000 Astana, Kazakhstan
E-mail: serikbol-87@yandex.kz

Received: 20.11.2015
Paper D
Hardy-type inequalities in fractional $h$-discrete calculus

Lars-Erik Persson$^{1,2}$*, Ryskul Oinarov$^3$ and Serikbol Shaimardan$^{1,3}$

Abstract
The first power weighted version of Hardy’s inequality can be rewritten as

$$\int_0^\infty \left( x^{\alpha-1} \int_0^x \frac{1}{t^\alpha} f(t) \, dt \right)^p \, dx \leq \left[ \frac{p}{p-\alpha-1} \right]^p \int_0^\infty f^p(x) \, dx, \quad f \geq 0, \quad p \geq 1, \quad \alpha < p - 1,$$

where the constant $C = \left[ \frac{p}{p-\alpha-1} \right]^p$ is sharp. This inequality holds in the reversed direction when $0 \leq p < 1$. In this paper we prove and discuss some discrete analogues of Hardy-type inequalities in fractional $h$-discrete calculus. Moreover, we prove that the corresponding constants are sharp.

MSC: Primary 39A12; secondary 49J05; 49K05
Keywords: Inequality; Integral operator; $h$-calculus; $h$-integral; Discrete Fractional Calculus

1 Introduction
The theory of fractional $h$-discrete calculus is a rapidly developing area of great interest both from a theoretical and applied point of view. Especially we refer to [1–8] and the references therein. Concerning applications in various fields of mathematics we refer to [9–16] and the references therein. Finally, we mention that $h$-discrete fractional calculus is also important in applied fields such as economics, engineering and physics (see, e.g. [17–22]).

Integral inequalities have always been of great importance for the development of many branches of mathematics and its applications. One typical such example is Hardy-type inequalities, which from the first discoveries of Hardy in the twentieth century now have been developed and applied in an almost unbelievable way, see, e.g., monographs [23] and [24] and the references therein. Let us just mention that in 1928 Hardy [25] proved the following inequality:

$$\int_0^\infty \left( x^{\alpha-1} \int_0^x \frac{1}{t^\alpha} f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-\alpha-1} \right)^p \int_0^\infty f^p(x) \, dx, \quad f \geq 0,$$

for $1 \leq p < \infty$ and $\alpha < p - 1$ and where the constant $\left( \frac{p}{p-\alpha-1} \right)^p$ is best possible. Inequality (1.1) is just a reformulation of the first power weighted generalization of Hardy’s original inequality, which is just (1.1) with $\alpha = 0$ (so that $p > 1$) (see [26] and [27]). Up to now there is

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no sharp discrete analogue of inequality (1.1). For example, the following two inequalities were claimed to hold by Bennett ([28, p. 40–41]; see also [29, p. 407]):

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{n^\alpha} \sum_{k=0}^{n} k^{-\alpha} \right] \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right] \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

and

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{n^\alpha} \sum_{k=0}^{n} k^{-\alpha} \right] \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right] \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

whenever \( \alpha > 0, p > 1, \alpha p > 1 \). Both inequalities were proved independently by Gao [30, Corollary 3.1–3.2] (see also [31, Theorem 1.1] and [32, Theorem 6.1]) for \( p > 1 \) and some special cases of \( \alpha \) (this means that there are still some regions of parameters with no proof of (1.1)). Moreover, in [33, Theorems 2.1 and 2.3] proved another sharp discrete analogue of inequality (1.1) in the following form:

\[
\sum_{n=-\infty}^{\infty} \left[ \frac{1}{n^\alpha} \sum_{k=0}^{n} q^k a_k \right] \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right] \sum_{n=-\infty}^{\infty} a_n^p, \quad a_n \geq 0,
\]

and

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{n^\alpha} \sum_{k=0}^{n} q^k a_k \right] \leq \left[ \frac{1 - \alpha}{p - \alpha p - 1} \right] \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0,
\]

for \( 0 < q < 1, p \geq 1 \) and \( \alpha < 1 - 1/p \), where \( \lambda := 1 - 1/p - \alpha \).

The main aim of this paper is to establish the \( h \)-analogue of the classical Hardy-type inequality (1.1) in fractional \( h \)-discrete calculus with sharp constants which is another discrete analogue of inequality (1.1).

The paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Sect. 2. The main results (see Theorem 3.1 and Theorem 3.2) with the detailed proofs can be found in Sect. 3.

## 2 Preliminaries

We state the some preliminary results of the \( h \)-discrete fractional calculus which will be used throughout this paper.

Let \( h > 0 \) and \( \mathbb{T}_a := \{ a, a + h, a + 2h, \ldots \}, \forall a \in \mathbb{R} \).

**Definition 2.1** (see [34]) Let \( f : \mathbb{T}_a \rightarrow \mathbb{R} \). Then the \( h \)-derivative of the function \( f = f(t) \) is defined by

\[
D_h f(t) := \frac{f(\delta_h(t)) - f(t)}{h}, \quad t \in \mathbb{T}_a,
\]

where \( \delta_h(t) := t + h \).

Let \( fg : \mathbb{T}_a \rightarrow \mathbb{R} \). Then the product rule for \( h \)-differentiation reads (see [34])

\[
D_h(f(x)g(x)) := f(x)D_h g(x) + g(x + h)D_h f(x).
\]
The chain rule formula that we will use in this paper is

$$D_h[x^\gamma(t)] := \gamma \int_0^1 [zx(\delta_h(t)) + (1 - z)x(t)]^{\gamma - 1} dzD_hx(t), \quad \gamma \in \mathbb{R},$$  \hspace{1cm} (2.3)$$

which is a simple consequence of Keller’s chain rule [35, Theorem 1.90]. The integration by parts formula is given by (see [34]) the following.

**Definition 2.2** Let \( f : \mathbb{T}_a \to \mathbb{R} \). Then the \( h \)-integral (\( h \)-difference sum) is given by

$$\int_a^b f(x) \, dh_x := \sum_{k=a/h}^{b/h-1} f(kh)h = \sum_{k=0}^{b/a-1} f(a + kh)h,$$

for \( a, b \in \mathbb{T}_a, b > a \).

**Definition 2.3** We say that a function \( g : \mathbb{T}_a \to \mathbb{R} \), is nonincreasing (respectively, non-decreasing) on \( \mathbb{T}_a \) if and only if \( D_hg(t) \leq 0 \) (respectively, \( D_hg(t) \geq 0 \)) whenever \( x \in \mathbb{T}_a \).

Let \( D_hF(x) = f(x) \). Then \( F(x) \) is called a \( h \)-antiderivative of \( f(x) \) and is denoted by \( \int f(x) \, dh_x \). If \( F(x) \) is a \( h \)-antiderivative of \( f(x) \), for \( a, b \in \mathbb{T}_a, b > a \) we have (see [36])

$$\int_a^b f(x) \, dh_x := F(b) - F(a).$$  \hspace{1cm} (2.4)$$

**Definition 2.4** (see [34]) Let \( t, \alpha \in \mathbb{R} \). Then the \( h \)-fractional function \( \mathcal{I}_h^{(\alpha)} \) is defined by

$$\mathcal{I}_h^{(\alpha)} := h^\alpha \frac{\Gamma\left(\frac{\alpha}{h} + 1\right)}{\Gamma\left(\frac{\alpha}{h} + 1 - \alpha\right)},$$

where \( \Gamma \) is Euler gamma function, \( \frac{\alpha}{h} \notin \{-1, -2, -3, \ldots\} \) and we use the convention that division at a pole yields zero. Note that

$$\lim_{h \to 0} \mathcal{I}_h^{(\alpha)} = t^\alpha.$$

Hence, by (2.1) we find that

$$\mathcal{I}_h^{(\alpha-1)} = \frac{1}{\alpha} D_h[\mathcal{I}_h^{(\alpha)}].$$  \hspace{1cm} (2.5)$$

**Definition 2.5** The function \( f : (0, \infty) \to \mathbb{R} \) is said to be log-convex if \( f(ux + (1 - u)y) \leq f^u(x)f^{1-u}(y) \) holds for all \( x, y \in (0, \infty) \) and \( 0 < u < 1 \).

Next, we will derive some properties of the \( h \)-fractional function, which we need for the proofs of the main results, but which are also of independent interest.

**Proposition 2.6** Let \( t \in \mathbb{T}_0 \). Then, for \( \alpha, \beta \in \mathbb{R} \),

$$\mathcal{I}_h^{(\alpha+\beta)} = \mathcal{I}_h^{(\alpha)} (t - \alpha h)^{\beta}_h,$$  \hspace{1cm} (2.6)$$
\[ t_h^{(p)} \leq [t_h^{(\alpha)}]^p \leq (t + \alpha(p - 1)h)^{(p \alpha)}_h, \quad (2.7) \]

for \( 1 \leq p < \infty \), and

\[ [t_h^{(p \alpha)}]^p \leq t_h^{(p \alpha)}, \quad (2.8) \]

for \( 0 < p < 1 \).

**Proof** By using Definition 2.4 we get

\[ t_h^{(\alpha + \beta)} = h^{\alpha + \beta} \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha - \beta\right)} \]

\[ = h^\alpha \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)} h^\beta \frac{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha - \beta\right)} = t_h^{(\alpha)}(t - \alpha h)^{(\beta)}_h, \]

i.e. (2.6) holds for \( \alpha, \beta \in \mathbb{R} \).

It is well known that the gamma function is log-convex (see, e.g., [37], p. 21). Hence,

\[ [t_h^{(\alpha)}]^p = h^{p \alpha} \left[ \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)} \right]^p \]

\[ = h^{p \alpha} \left[ \frac{\Gamma\left(\frac{1}{h} + 1 + \alpha(p - 1)) + (1 - \frac{1}{p})(\frac{1}{h} + 1 - \alpha)\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)} \right]^p \]

\[ \leq h^{p \alpha} \left[ \frac{\Gamma\left(\frac{1}{h} + 1 + \alpha(p - 1))\Gamma\left(\frac{1}{h} + 1 - \alpha\right)\right)}{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)} \right]^p \]

\[ = h^{p \alpha} \frac{\Gamma\left(\frac{1}{h} + 1 + \alpha(p - 1))}{\Gamma\left(\frac{1}{h} + 1 - \alpha\right)} = (t + \alpha(p - 1)h)^{(p \alpha)}_h \]

and

\[ [t_h^{(\beta)}]^p = h^{p \beta} \left[ \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - \beta\right)} \right]^p \]

\[ = h^{p \beta} \left[ \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left((1 - \frac{1}{p})(\frac{1}{h} + 1) + \frac{1}{p}(\frac{1}{h} + 1 - p\alpha)\right)} \right]^p \]

\[ \geq h^{p \beta} \left[ \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - p\alpha\right)} \right]^p \]

\[ = h^{p \beta} \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - p\alpha\right)} = t_h^{(p \alpha)} \]

so we have proved that (2.7) holds wherever \( 1 \leq p < \infty \). Moreover, for \( 0 < p < 1 \),

\[ t_h^{(p \alpha)} = h^{p \alpha} \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left(\frac{1}{h} + 1 - p\alpha\right)} \]

\[ = h^{p \alpha} \frac{\Gamma\left(\frac{1}{h} + 1\right)}{\Gamma\left((1 - p)(\frac{1}{h} + 1) + p(\frac{1}{h} + 1 - \alpha)\right)} \]
\[
\geq h^{\rho\alpha} \frac{\Gamma\left(\frac{x}{h} + 1\right)}{\Gamma^{(1-p)}\left(\frac{x}{h} + 1\right)\Gamma^{p}\left(\frac{x}{h} + 1 - \alpha\right)} = \left[\frac{h^p}{\Gamma\left(\frac{x}{h} + 1\right)}\right]^p, \\
\]

so we conclude that (2.8) holds for \(0 < p < 1\). The proof is complete. \(\square\)

3 Main results

Our \(h\)-integral analogue of inequality (1.1) reads as follows.

**Theorem 3.1** Let \(\alpha < \frac{p+1}{p}\) and \(1 \leq p < \infty\). Then the inequality

\[
\int_0^\infty \left( x^{(\alpha-1)} \int_0^{h(t)} f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-\alpha p - 1} \right)^p \int_0^\infty f^p(x) \, dx, \quad f \geq 0, \quad (3.1)
\]

holds. Moreover, the constant \(\left( \frac{p}{p-\alpha p - 1} \right)^p\) is the best possible in (3.1).

Our second main result is the following \(h\)-integral analogue of the reversed form of (1.1) for \(0 < p < 1\).

**Theorem 3.2** Let \(\alpha < \frac{p+1}{p}\) and \(0 < p < 1\). Then the inequality

\[
\int_0^\infty f^p(x) \, dx \leq \left( \frac{p - \alpha p}{p} - 1 \right)^p \int_0^\infty \left( x^{(\alpha-1)} \int_0^{h(t)} f(t) \, dt \right)^p \, dx, \quad f \geq 0, \quad (3.2)
\]

holds. Moreover, the constant \(\left( \frac{p - \alpha p}{p} - 1 \right)^p\) is the best possible in (3.2).

To prove Theorem 3.1 we need the following lemma, which is of independent interest.

**Lemma 3.3** Let \(\alpha < \frac{p+1}{p}\), \(p > 1\) and \(\frac{1}{p} + \frac{1}{p'} = 1\). Then the function

\[
\phi(x) := \left[ \left( x - \left( \alpha + \frac{1}{p} \right) h \right)^{\frac{1}{p}} \right]^{\frac{p}{p-1}} \left[ \left( x - \left( \alpha - \frac{1}{p'} \right) h \right)^{\frac{1}{p'}} \right], \quad x \in \mathbb{T}_0,
\]

is nonincreasing on \(\mathbb{T}_0\).

**Proof** Let \(\alpha < \frac{p+1}{p}\) and \(1 \leq p < \infty\). Since \(\Gamma(x) > 0\) for \(x > 0\), and using Definition 2.4, we have

\[
\left( x - \left( \alpha + \frac{1}{p} \right) h \right)^{\frac{1}{p}} = h^{\frac{1}{p}} \frac{\Gamma\left( \frac{x}{h} + \frac{1}{p} - \alpha\right)}{\Gamma\left( \frac{x}{h} + \frac{1}{p} + \frac{1}{p'} - \alpha\right)} > 0
\]

and

\[
\left( x - \left( \alpha - \frac{1}{p'} \right) h \right)^{\frac{1}{p'}} = h^{\frac{1}{p'}} \frac{\Gamma\left( \frac{x}{h} + 1 + \frac{1}{p'} - \alpha\right)}{\Gamma\left( \frac{x}{h} + 1 + \alpha\right)} > 0.
\]
Denote $\xi(x) := (x - (\alpha + \frac{1}{p}) h)^{\left(\frac{1}{p}\right)}$ and $\eta(x) := (x - (\alpha - \frac{1}{p}) h)^{\left(\frac{1}{p}\right)}$. Then by using (2.5) we find that

$$D_h \eta(x) = \frac{(x - (\alpha + \frac{1}{p}) h)^{\left(-\frac{1}{p}\right)}}{p^\prime} \geq 0$$

(3.3)

and

$$D_h \xi(x) = -\frac{(x - (\alpha + \frac{1}{p}) h)^{\left(-\frac{1}{p}\right)}}{p^\prime} \leq 0,$$

(3.4)

From (2.3), (2.6), (3.3) and (3.4) it follows that

$$D_h \left[\xi(x)\right]^\frac{1}{p} = \frac{1}{p^\prime} \int_0^1 \left[z \xi(x + h) + (1 - z) \xi(x)\right]^{\frac{1}{p} - 1} dz D_h \xi(x)$$

$$\leq -\left[\xi(x)\right]^\frac{1}{p} \frac{(x - (\alpha + \frac{1}{p}) h)^{\left(-\frac{1}{p}\right)}}{pp^\prime}$$

$$\leq -\left[\xi(x)\right]^\frac{1}{p} \frac{(x - \alpha h)^{\left(-\frac{1}{p}\right)}}{pp^\prime}$$

(3.5)

and

$$D_h \left[\eta(x)\right]^\frac{1}{p} = \frac{1}{p^\prime} \int_0^1 \left[z \eta(x + h) + z \eta(x)\right]^{\frac{1}{p} - 1} dz D_h \eta(x)$$

$$\leq \left[\eta(x)\right]^\frac{1}{p} \frac{(x - (\alpha - \frac{1}{p}) h)^{\left(-\frac{1}{p}\right)}}{pp^\prime}.$$  

(3.6)

By using the fact that $(x + h - \alpha h)^{\left(\frac{1}{p}\right)}(x - \alpha h)^{\left(-\frac{1}{p}\right)} = 1$, $\eta(x + h) \geq \eta(x)$,

$$\eta(x) \left[(x - \left(\alpha - \frac{1}{p}\right) h)^{\left(-\frac{1}{p}\right)}\right]^{-1} = (x + h - \alpha h)^{\left(\frac{1}{p}\right)},$$

for $x \in T_0$ and (2.2), (3.3), (3.4), (3.5) and (3.6) we obtain

$$D_h(\phi(x)) = \left[\xi(x)\right]^\frac{1}{p} D_h[\eta(x)]^\frac{1}{p} + \left[\eta(x + h)\right]^\frac{1}{p} - \frac{1}{p} D_h[\xi(x)]^\frac{1}{p}$$

$$\leq \left[\xi(x)\right]^\frac{1}{p} \left[\eta(x)\right]^\frac{1}{p} - \left[(x - \left(\alpha + \frac{1}{p}\right) h)^{\left(-\frac{1}{p}\right)} - \eta(x)(x - \alpha h)^{\left(-\frac{1}{p}\right)}\right]$$

$$= \left[\xi(x)\right]^\frac{1}{p} \left[\eta(x)\right]^\frac{1}{p} - \left[(x - \left(\alpha + \frac{1}{p}\right) h)^{\left(-\frac{1}{p}\right)} - 1 + (x + h - \alpha h)^{\left(\frac{1}{p}\right)}(x - \alpha h)^{\left(-\frac{1}{p}\right)}\right]$$

$$\leq 0.$$

Hence, we have proved that the function $\phi(x)$ is nonincreasing on $T_0$ (see Definition 2.4) so the proof is complete. □
Proof of Theorem 3.1 Let $p > 1$. By using Lemma 3.3 and (2.6) in Proposition 2.6 we have

$$x_h^{(\alpha - 1)} = \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$= \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$\times \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$\leq \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \phi(x + h) \tag{3.7}$$

for $t, x \in \mathbb{T}_0 : t \leq x$. Moreover,

$$\frac{\phi(t)}{\phi(x)} = \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p}} \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p'}} \phi(t)$$

$$= \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p}} \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p'}} \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p}} \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p'}}$$

$$\leq \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p}} \left[ (t - \alpha h)^{\alpha - 1} \right]^{\frac{1}{p'}} \phi(t) \tag{3.8}$$

According to (3.7) and (3.8) we have

$$L(f) := \int_0^\infty \left( x_h^{(\alpha - 1)} \int_0^{x_h^{(\alpha)}} \frac{1}{k^\alpha} f(t) dt \right) dx$$

$$\leq \int_0^\infty \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$\times \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$\times \int_0^\infty \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$\times \int_0^\infty \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$

$$= L^p(f).$$

Let $N_0 = \mathbb{N} \cup \{0\}, g = \{g_k\}_{k=1}^\infty \in L_p(N_0), g \geq 0$, and $\|g\|_{L_p} = 1$. Moreover, let $\theta(z)$ be Heaviside’s unit step function ($\theta(z) = 1$ for $z \geq 0$ and $\theta(z) = 0$ for $z < 0$). Then, based on the duality principle in $L_p(N_0)$ and the Hölder inequality, we find that

$$I(f) = \sup_{\|g\|_{L_p} = 1} \sum_{i, k} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p}} \left[ x_h^{(\alpha - 1)} \right]^{\frac{1}{p'}}$$
\[
\times \left[ \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \right] f(kh) \\
\leq \sup_{\|g\|_{L_p^1}} \left( \sum_{i,k} h^{p} \theta(i-k)(ih)^{\left( \frac{1}{p} - \frac{1}{p} \right)} \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \right) f(kh) \\
\times \left( \sum_{i,k} h \theta(i-k)(ih)^{\left( \frac{1}{p} - \frac{1}{p} \right)} \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \right)^{\frac{1}{p}} f(kh) \\
= \sup_{\|g\|_{L_p^1}} \left( g \right) g_{L_p^1}(f).
\] (3.9)

By using Definition 2.3 and combining (2.4), (2.5) and (2.6) we can conclude that

\[ I_1(g) = \sum_{i=0}^{\infty} g_{l_i} h^{p}(ih) \sum_{k=0}^{i} h \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \]
\[ = \sum_{i=0}^{\infty} g_{l_i} h^{p}(ih) \int_{0}^{\infty} D_h \left[ \left( x \left( \alpha + \frac{1}{p} \right) h \right) \left( \alpha - \frac{1}{p} \right) \right] d_l x \]
\[ = \frac{1}{p - \alpha} \sum_{i=1}^{\infty} g_{l_i} h^{p}(ih) \left( \frac{1}{p} - \frac{1}{p} \right) \]
\[ = \frac{1}{p - \alpha} \|g\|_{L_p^1} = \frac{1}{p - \alpha}. \] (3.10)

Furthermore,
\[ I_2(f) = \sum_{i=0}^{\infty} h(ih) \sum_{k=0}^{i} h f_{p}(kh) \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \]
\[ = \sum_{k=0}^{\infty} h f_{p}(kh) \left( kh - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \sum_{i=0}^{\infty} h(ih) \left( \frac{1}{p} - \frac{1}{p} \right) \]
\[ = \frac{1}{p} \int_{0}^{\infty} f_{p}(x) \left( x - \left( \alpha + \frac{1}{p} \right) h \right) \left( \alpha - \frac{1}{p} \right) \int_{\infty}^{\infty} D_h \left[ t^{\left( \frac{1}{p} - \frac{1}{p} \right)} \right] d_t d_l x \]
\[ = \frac{1}{p - \alpha} \int_{0}^{\infty} f_{p}(x) \left( x - \left( \alpha + \frac{1}{p} \right) h \right) \left( \frac{1}{p} - \frac{1}{p} \right) \]
\[ = \frac{1}{p - \alpha} \int_{0}^{\infty} f_{p}(x) d_l x. \] (3.11)

By combining (3.9), (3.10) and (3.11) we obtain
\[ L(f) \leq \left( \frac{p}{p - pa - 1} \right)^{p} \int_{0}^{\infty} f_{p}(x) d_l x, \] (3.12)
i.e. (3.1) holds.
Finally, we will prove that the constant [\(-\frac{p}{p-1}\)] is the best possible in inequality (3.1). Let \(x, a \in \mathbb{T}_0\) be such that \(a < x\), and consider the test function \(f_p(t) = (t^{(\beta)} \chi_{[a, \infty)}(t)) \), \(t > 0\), for \(\beta = -\frac{1}{p} - \varepsilon\).

Then from (2.4), (2.5) and (2.7) it follows that

\[
\int_0^\infty f_p^p(t) \, dt = \int_a^\infty \left[ t^{(\beta)} \right]^p \, dt \leq \int_a^\infty (t + \beta(p - 1)h)^{(p)} \, dt \\
= \frac{1}{p\beta + 1} \int_a^\infty D_h[(t + \beta(p - 1)h)^{(p+1)}] \, dt \\
= \frac{(a + \beta(p - 1)h)^{(p+1)}}{|p\beta + 1|} < \infty.
\]

Since

\[
\left( \int_0^{h(x)} (t - ha)^{(\alpha - 1)} f_p(t) \, dt \right)^p = \left( \int_a^{h(x)} (t - ha)^{(\alpha - 1) + \beta} \, dt \right)^p \\
= \left( \frac{1}{1 - \alpha + \beta} \int_a^{h(x)} D_h[(t - ha)^{(1-\alpha+\beta)}] \, dt \right)^p \\
= \left( \frac{(x + h - ha)^{(1-\alpha+\beta)}}{1 - \alpha + \beta} \left[ 1 - \frac{(a - ha)^{(1-\alpha+\beta)}(x + h - ha)^{(1-\alpha+\beta)}}{(x - ha)^{(1-\alpha+\beta)}} \right] \right)^p \\
\geq \left( \frac{(x + h - ha)^{(1-\alpha+\beta)}}{1 - \alpha + \beta} \right)^p \left[ 1 - p \frac{(a - ha)^{(1-\alpha+\beta)}}{(x - ha)^{(1-\alpha+\beta)}} \right],
\]

we have

\[
L(f_p) \geq \left( \frac{1}{1 - \alpha + \beta} \right)^p \left[ \int_a^\infty \left[ x^{(\alpha-1)} (x + h - ha)^{(1-\alpha+\beta)} \right]^p \, dx \\
- p(a - ha)^{(1-\alpha+\beta)} \int_a^\infty \frac{x^{(\alpha-1)} (x + h - ha)^{(1-\alpha+\beta)} \, dx}{(x - ha)^{(1-\alpha+\beta)}} \right] \\
= \left( \frac{1}{1 - \alpha + \beta} \right)^p \left[ \int_0^\infty f_p^p(x) \, dx - p \int_a^\infty \frac{(a - ha)^{(1-\alpha+\beta)} \, dx}{(x - ha)^{(1-\alpha+\beta)}} \right].
\]  

(3.13)

By using (2.4), (2.5), (2.6) and (2.7) we obtain

\[
\int_a^\infty \frac{(t^{(\beta)})^p \, dt}{(x + h - ha)^{(1-\alpha+\beta)}} \leq \int_a^\infty \frac{(x + \beta(p - 1)h)^{(p)} \, dt}{(x + h - ha)^{(1-\alpha+\beta)}} \\
= \int_a^\infty \frac{(x + \beta(p - 1)h)^{(p+1)} \, dt}{(x + h - ha)^{(1-\alpha+\beta)}} \\
= \frac{\int_a^\infty D_h[(x + \beta(p - 1)h)^{(p-1)+\alpha}) \, dx}{\beta(p - 1) + \alpha} \\
= \frac{1}{|\beta(p - 1) + \alpha|} (a + \beta(p - 1)h)^{(p-1)+\omega)}
\]  

(3.14)
and

\[
(a - hα)^{(1-α+β)}_h = (a - hα)^{-(α)}_h (a - h(pβ + 1))^{(β+1)}_h \int_a^\infty D_h[f^{(β)}_h] \, dt \\
\leq (a - hα)^{-(α)}_h (a - h(pβ + 1))^{(β+1)}_h |βp + 1| \int_a^\infty [f^{(β)}_h] \, dt.
\]

(3.15)

According to (2.6), (3.13), (3.14) and (3.15) we can deduce that

\[
L(f_h) \geq \left( \frac{1}{1-α + β} \right)^p \left[ \int_0^\infty f_h^p (x) \, dx - \theta_p(a) \int_0^\infty f_h^p (x) \, dx \right],
\]

where \(θ_p(a) := \frac{p^{β+1}}{|βp + 1|} (a + β(p - 1)h) \int_a^\infty (a - h(pβ + 1))^{(β+1)}_h \, dt \rightarrow 0, \quad ε \rightarrow 0.\)

Therefore, \(\lim_{ε \rightarrow 0} \frac{L(f_h)}{\int_0^\infty f_h^p (x) \, dx} \geq \lim_{ε \rightarrow 0} (\frac{p}{1-α + β})^p = (\frac{p}{p-α+β})^p,\) which implies that the constant \(\left(\frac{p}{p-α+β}\right)^p\) in (3.1) in sharp.

Let \(p = 1.\) By using Definition 2.3 and (2.5) we get

\[
\int_0^\infty x_h^{(α-1)} \int_0^{h^\prime (x)} \frac{1}{h^\prime (x)} f(t) \, dt \, dx = \sum_{i=0}^\infty h(ih)^{α} \sum_{k=0}^\infty h(kh - αh)^{α} f(kh)
\]

\[
= \sum_{k=0}^\infty h(kh - αh)^{α} f(kh) \sum_{i=k}^\infty h(ih)^{α}
\]

\[
= \int_0^\infty (t - αh)^{α} f(t) \int_t^\infty x_h^{(α-1)} \, dx \, dt
\]

\[
= \frac{1}{α} \int_0^\infty (t - αh)^{α} f(t) \int_t^\infty D_h(x_h^{(α)}) \, dx \, dt
\]

\[
= \frac{1}{α} \int_0^\infty f(t)(t - αh)^{α} x_h^{(α)} \, dt = -\frac{1}{α} \int_0^\infty f(t) \, dt,
\]

which means that (3.1) holds even with equality in this case. The proof is complete. \(\square\)

**Proof of Theorem 3.2.** Let \(0 < p < 1.\) By using (2.4), (2.5) and (2.7) we get

\[
[x_h^{(α-1)}] = \left[ x_h^{(α-1)} \right]^{p-1} x_h^{(α-1)}
\]

\[
= \left[ x_h^{(α-1)} \right]^{p-1} \left[ x_h^{(α-1)} \right]^{\left(1\right)} \left( x + h - (α - \frac{1}{p}) h \right)^{\left(1\right)} h
\]

\[
\geq \left[ x_h^{(α-1)} \right]^{p-1} \left[ x_h^{(α-1)} \right]^{\left(1\right)} \left( x + h - (α - \frac{1}{p}) h \right)^{\left(1\right)} h
\]

\[
\geq \left[ x_h^{(α-1)} \right]^{p-1} \left[ x_h^{(α-1)} \right]^{\left(1\right)} \left( x + h - (α - \frac{1}{p}) h \right)^{\left(1\right)} h
\]

\[
= \left[ x_h^{(α-1)} \right]^{p-1} \left[ x_h^{(α-1)} \right]^{\left(1\right)} h
\]


\[
\left[\left(x - \left(\alpha - \frac{1}{p'}\right) h\right)^{\left(\frac{1}{p'} - \alpha\right)}\right]^{1-p} \left[\left(x - \left(\alpha - \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \leq \left[\left(1 - \frac{1}{p}\right) \int_0^{x_h} \left(t - \left(\alpha + \frac{1}{p}\right) h\right)^{\left(\frac{1}{p'} - \alpha\right)} dt_d t \right]^{1-p} \left[\left(x - \left(\alpha + \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \left[\left(x - \left(\alpha - \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p}
\]
(3.16)

and
\[
\left[\frac{1}{t_h^{\alpha}}\right]^p = \left[\left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{p-1} \frac{1}{t_h^{\alpha}}
\]
\[
= \left[\left(t - \left(\alpha + \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{p-1} \frac{1}{t_h^{\alpha}} \left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}
\]
\[
= \left[\left(t - \left(\alpha + \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{p-1} \frac{1}{t_h^{\alpha}} \left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}
\]
(3.17)

Moreover, by using Definition 2.3, (3.16) and (3.17), and applying the Hölder inequality with powers 1/p and 1/(1 - p), we obtain
\[
\frac{L(f)}{\left(\frac{1}{p} - \alpha\right)^{p}} \geq \int_0^{\infty} x_h^{\frac{1}{p} - 1} \left[\int_0^{x_h} \left(t - \left(\alpha + \frac{1}{p}\right) h\right)^{\left(\frac{1}{p} - \alpha\right)} dt_d t \right] d_h x
\]
\[
= \sum_{k=0}^{\infty} h(kh)_h^{\left(\frac{1}{p} - 1\right)} \left[\sum_{i=0}^{k} h \left(\left(\alpha + \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)} \right]^{1-p} \left[\sum_{i=0}^{k} \frac{f(ih)}{(ih)_h^{\alpha}}\right]^{p}
\]
\[
= \sum_{i=0}^{\infty} h^p(ih) \left[\left(\alpha + \frac{1}{p}\right) h\right]^{\left(\frac{1}{p} - \alpha\right)} \left(\alpha - \frac{1}{p} - 1\right) \left[\frac{1}{(ih)_h^{\alpha}}\right]^{p} \sum_{k=0}^{\infty} h(kh)_h^{\left(\frac{1}{p} - 1\right)}
\]
\[
= \int_0^{\infty} f^p(t) \left(\left(t - \left(\alpha + \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)}\right)^{1-p} \left(\left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}\right)^{1-p} \left(\left(t - \left(\alpha - \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)}\right)^{1-p} \left[\left(t - \left(\alpha - \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \left(\left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}\right)^{1-p}
\]
\[
= \int_0^{\infty} f^p(t) \left[\left(t - \left(\alpha + \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \left[\left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \left[\left(t - \left(\alpha - \frac{1}{p}\right) h\right)_h^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p} \left[\left(t - \alpha h\right)^{\left(\frac{1}{p} - \alpha\right)}\right]^{1-p}
\]
\[
\times \frac{1}{t_h^{\alpha}} \int_0^{\infty} D_h[x_h^{\left(\frac{1}{p} - \alpha\right)}] dt_d x d_h t
\]
\[ = \frac{1}{p' - \alpha} \int_0^\infty f^p(t) \, dt, \]
i.e.
\[ \left[ \frac{1}{p' - \alpha} \right]^p L(f) \geq \int_0^\infty f^p(t) \, dt. \]

Therefore, we deduce that inequality (3.2) holds for all functions \( f \geq 0 \) and the left hand side of (3.2) is finite.

Finally, we prove that the constant \( \left[ \frac{p-1}{p} - \alpha \right]^p \) in inequality (3.2) is sharp. Let \( x, a \in T_0 \), be such that \( a < x \), and \( f_{\beta}(t) = i_{\beta}^{(\beta)} x_{[a,\infty)}(t) \), where \( \alpha - 1 < \beta < -\frac{1}{p} \). By using (2.4), (2.5) and (2.8) we find that
\[
\int_0^\infty f_{\beta}(t) \, dt = \int_0^\infty (t_{h}^{(\beta)})^p \, dt \leq \int_0^\infty t_{h}^{(p+1)} \, dt \\
= \frac{1}{p\beta + 1} \int_0^\infty D_{\beta} \left[ t_{h}^{(p+1)} \right] \, dt \\
= \frac{a_{\beta}^{(p+1)}}{|p\beta + 1|} < \infty
\]
and
\[
L(f_{\beta}) = \sum_{i=0}^\infty h \left[ \left( ih \right)^{(\alpha-1)} \sum_{k=0}^{i} h(kh - ah)_{h}^{(\alpha)} f_{\beta}(kh) \right]^p \\
= \sum_{i=0}^{\infty} h \left[ \left( ih \right)^{(\alpha-1)} \sum_{k=0}^{i} h(kh - ah)_{h}^{(\alpha)} f_{\beta}(kh) \right]^p \\
+ \sum_{i=\infty}^{\infty} h \left[ \left( ih \right)^{(\alpha-1)} \sum_{k=0}^{i} h(kh - ah)_{h}^{(\alpha)} f_{\beta}(kh) \right]^p \\
= \int_a^\infty x_{h}^{(\alpha-1)} \int_0^{x_{h}^{(\alpha)}} (t - ah)_{h}^{(\alpha+\beta)} \, dt \, dx \\
= \left[ \frac{1}{1 - \alpha + \beta} \right]^p \int_a^\infty x_{h}^{(\alpha-1)} \int_0^{x_{h}^{(\alpha)}} D_{\beta} \left[ (t - ah)_{h}^{(1+\alpha+\beta)} \right] \, dt \, dx \\
< \left[ \frac{1}{1 - \alpha + \beta} \right]^p \int_a^\infty x_{h}^{(\alpha-1)} \left( x + ah \right)_{h}^{(1+\alpha+\beta)} \, dx \\
= \left[ \frac{1}{1 - \alpha + \beta} \right]^p \int_a^\infty f_{\beta}^{(\alpha)}(x) \, dx = \left( \frac{1}{1 - \alpha + \beta} \right)^p \int_0^\infty f_{\beta}^{(\alpha)}(x) \, dx. \quad (3.18)
\]

From (3.18) it follows that
\[
\sup_{a-1 \geq \beta \geq -1} \frac{\int_0^\infty f_{\beta}(t) \, dt}{L(f_{\beta})} = \sup_{a-1 \geq \beta \geq -1} \left[ 1 - \alpha + \beta \right]^p = \left[ \frac{1}{p' - \alpha} \right]^p ,
\]
and this shows that the constant \( \left[ \frac{p-1}{p} - \alpha \right]^p \) in inequality (3.2) is sharp. The proof is complete. \( \square \)
Now, let us comment which discrete analogue of Hardy inequality we are getting from the Hardy $h$-inequality. Directly from the proof of Theorems 3.1 and 3.2 we obtain the following discrete inequality, which is of independent interest.

**Remark 3.4** On the basis of Definitions 2.4–2.5 we get

$$\sum_{n=0}^{\infty} \left[ \frac{\Gamma\left(\frac{nh}{\pi} + 1\right)}{\Gamma\left(\frac{nh}{\pi} + 2 - \alpha\right)} \sum_{k=0}^{n} \frac{\Gamma\left(\frac{k h}{\pi} + 1 - \alpha\right)}{\Gamma\left(\frac{k h}{\pi} + 1\right)} a_k \right]^p \leq \left( \frac{p}{p - \alpha p - 1} \right)^p \sum_{n=0}^{\infty} a_n^p, \quad a_k \geq 0,$$

for $p \geq 1$ and $\alpha < 1 - 1/p$.

**Acknowledgements**

This work was supported by Scientific Committee of Ministry of Education and Science of the Republic of Kazakhstan, grant no. AP05130975.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have on equal level discussed, posed the research questions and proved the results in this paper. Moreover, all authors have read and approved the final version of this manuscript.

**Author details**

1. Luleå University of Technology, Luleå, Sweden. 2. UiT The Artic University of Norway, Narvik, Norway. 3. L. N. Gumilyev Eurasian National University, Astana, Kazakhstan.

**Publisher’s Note**

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

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25. Hardy, G.H.: Notes on some points in the integral calculus. Messenger Math. 57, 12–16 (1928)
Paper E
Fractional order Hardy-type inequality in fractional $h$-discrete calculus

S. SHAIMARDAN

Eurasian National University
Satpayev str., 2
010008 Astana, Kazakhstan
and
Luleå University of Technology,
97187 Luleå, Sweden
S. Shaimardan. Fractional order Hardy-type inequality in fractional $h$-discrete calculus, Luleå University of Technology, Department of Mathematical sciences, Research Report (2018).

Abstract: We investigate the power weights fractional order Hardy-type inequality in the following form:

$$
\left( \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x - y|^{1+\frac{p\alpha}{p}}} \, dx \, dy \right)^p \leq C \left( \int_0^\infty |f'(x)|^p x^{(1-\alpha)p} \, dx \right)^p
$$

for $0 < \alpha < 1$ and $1 < p < \infty$ in fractional $h$-discrete calculus, where $C = \frac{2^{\frac{1}{p}} \alpha^{-1}}{(p-\alpha p)^\frac{1}{p}}$. For $h$-fractional function we prove a discrete analogue of above inequality in fractional $h$-discrete calculus, is proved and discussed. Moreover, we prove that the same constant is sharp also in this case.


Keywords and Phrases: Fractional Hardy type inequality, $h$-derivative, integral operator, $h$-calculus, $h$-integral, discrete fractional calculus, sharp constant.

Note: This report will be submitted for publication elsewhere.

ISSN: 1400-4003

Luleå University of Technology
Department of Mathematical sciences
SE-971 87 Luleå, SWEDEN
1 Introduction

Fractional $h$-discrete calculus has generated interest in recent years. It is a mathematical subject that has proved to be very useful in applied fields such as economics, engineering and physics (see, e.g. [4], [5], [23], [24], [30]). Concerning applications in various fields of mathematics we refer to [2], [3], [6], [7], [12], [13], [14], [16], [17], [18], [22], [25], [26], [27], [29], [31] and the references therein.

It is well known that integral inequalities play important roles in the research of qualitative as well as quantitative properties of solutions of differential equations, difference equations and dynamic equations. One of the examples is fractional Hardy-type inequalities. In [15], [32], [33], [34] and [35] a series of fractional order Hardy-type inequalities have been presented. We pronounce especially that even Chapter 5 in the new book [20] by A. Kufner, L.-E. Persson and N. Samko is completely devoted to this subject. In particular, it is proved here (see Theorem 5.3) that

$$
\left( \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x-y|^{1+p\alpha}} \, dx \, dy \right)^{1/p} \leq C \left( \int_0^\infty |f'(x)|^p x^{(1-\alpha)p} \right)^{1/p}, \tag{1.1}
$$

for $0 < \alpha < 1$, $1 < p < \infty$, where $C = \frac{\Gamma^{2\alpha-1}}{(p-p\alpha)^{\frac{1}{p}}}$ is the sharp constant.

Moreover, in [1], [9], [10], [21], [28] some discrete Hardy-type inequalities have been established, which can be used as a handy tool in the research of solutions of difference equations. Up to now the discrete analogues of the fractional Hardy-type inequalities are not studied. The main aim of this paper is to establish the $h$-analogue of the fractional Hardy-type inequality (1.1) in fractional $h$-discrete calculus with sharp constants which is a discrete analogue of the inequality (1.1).

The paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Section 2. The main result (see Theorem 3.1) with the detailed proof can be found in Section 3.

2 Preliminaries

First we state some preliminary results of the $h$-discrete fractional calculus, which will be used throughout this paper.
Let $h > 0$ and $\mathbb{T}_a = \{a, a + h, a + 2h, \cdots\}$, $\forall a \in \mathbb{R}$.

**Definition 1.** Let $f : \mathbb{T}_a \to \mathbb{R}$. Then the $h$-derivative of the function $f = f(x)$ is defined by

$$D_h f(t) := \frac{f(\delta_h(t)) - f(t)}{h}, \quad t \in \mathbb{T}_a,$$

where $\delta_h(t) = t + h$.

See e.g. [8]. The chain rule formula that we will use in this paper is

$$D_h \left[ x^\gamma(t) \right] := \gamma \int_0^1 \left[ zx(\delta_h(t)) + (1 - z)x(t) \right]^{\gamma - 1} dz D_h x(t), \quad \gamma \in \mathbb{R},$$

which is a simple consequence of Keller’s chain rule ([11], Theorem 1.90)).

**Definition 2.** Let $f : \mathbb{T}_a \to \mathbb{R}$. Then the $h$-integral ($h$-difference sum) is given by

$$\int_a^b f(x) d_h x := \sum_{k=a/h}^{b/h-1} f(kh) h = \sum_{k=0}^{b-a-h} f(a + kh) h,$$

for $a, b \in \mathbb{T}_a, b > a$.

**Definition 3.** We say that a function $g : \mathbb{T}_a \to \mathbb{R}$, is nonincreasing (respectively, nondecreasing) on $\mathbb{T}_a$ if and only if $D_h g(t) \leq 0$ (respectively, $D_h g(t) \geq 0$) whenever $t \in \mathbb{T}_a$.

Let $D_h F(x) = f(x)$. Then $F(x)$ is called a $h$-antiderivative of $f(x)$ and is denoted by $\int f(x) d_h x$. If $F(x)$ is a $h$-antiderivative of $f(x)$, for $a, b \in \mathbb{T}_a, b > a$, then we have that (see [18]):

$$\int_a^b f(x) d_h x = F(b) - F(a).$$

**Definition 4.** Let $t, \alpha \in \mathbb{R}$. Then the $h$-fractional function $t_h^{(\alpha)}$ is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)}.$$
where $\Gamma$ is Euler gamma function, $\frac{1}{n} \notin \{-1, -2, -3, \cdots\}$ and we use the convention that division at a pole yields zero. Note that

$$\lim_{h \to 0} t_h^{(\alpha)} = t^\alpha.$$  

Hence, by (2.1) we find that

$$t_h^{(\alpha-1)} = \frac{1}{\alpha} D_h \left[ t_h^{(\alpha)} \right].$$  

$$\tag{2.5}$$

$$(a - t - h)_h^{(\alpha-1)} = -\frac{1}{\alpha} D_h \left[ (a - t)_h^{(\alpha)} \right].$$  

$$\tag{2.6}$$

$$\frac{1}{(t + h)_h^{(\alpha+1)}} = -\frac{1}{\alpha} D_h \left[ \frac{1}{t_h^{(\alpha)}} \right].$$  

$$\tag{2.7}$$

$$\frac{1}{(a - t)_h^{(\alpha+1)}} = \frac{1}{\alpha} D_h \left[ \frac{1}{(a - t)_h^{(\alpha)}} \right].$$  

$$\tag{2.8}$$

**Definition 5.** The function $f : (0, \infty) \to \mathbb{R}$ is said to be log-convex if $f(ux + (1 - u)y) \leq f^u(x)f^{1-u}(y)$, holds for all $x, y \in (0, \infty)$ and $0 < u < 1$.

Next, we will derive some properties of the $h$-fractional function, which we need for the proofs of the main results but which are also of independent interest.

**Proposition 2.1.** Let $t \in \mathbb{T}_0$. Then, for $\alpha, \beta \in \mathbb{R}$,

$$t_h^{(\alpha+\beta)} = t_h^{(\alpha)}(t - \alpha h)^{(\beta)};$$  

$$\tag{2.9}$$

$$t_h^{(p\alpha)} \leq \left[ t_h^{(\alpha)} \right]^p \leq (t + \alpha(p - 1)h)_h^{(p\alpha)},$$  

$$\tag{2.10}$$

for $1 \leq p < \infty$.  

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Proof. By using Definition 4 we get that
\[ t_h^{(\alpha+\beta)} = h^{\alpha+\beta} \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha - \beta)} = h^{\alpha} \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)} h^{\beta} \frac{\Gamma(\frac{t}{h} + 1 - \alpha)}{\Gamma(\frac{t}{h} + 1 - \alpha - \beta)} = t_h^{(\alpha)} (t - \alpha h)^{(\beta)}. \]

Therefore, (2.9) holds for \( \alpha, \beta \in \mathbb{R} \).

It’s well known that the gamma function is log-convex (see e.g [19], p 21). Hence,
\[
\left[ t_h^{(\alpha)} \right]^p = h^{p\alpha} \left[ \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)} \right]^p \\
= h^{p\alpha} \left[ \frac{\Gamma\left(\frac{t}{h} + 1 + \alpha(p - 1)\right) + (1 - \frac{1}{p})(\frac{t}{h} + 1 - \alpha)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)} \right]^p \\
\leq h^{p\alpha} \left[ \frac{\Gamma\left(\frac{t}{h} + 1 + \alpha(p - 1)\right)\Gamma^{1 - \frac{1}{p}}(\frac{t}{h} + 1 - \alpha)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)} \right]^p \\
= h^{p\alpha} \frac{\Gamma\left(\frac{t}{h} + 1 + \alpha(p - 1)\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)} \\
= (t + \alpha(p - 1))h^p, \\
\]
and
\[
\left[ t_h^{(\alpha)} \right]^p = h^{p\alpha} \left[ \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)} \right]^p \\
= h^{p\alpha} \left[ \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma((1 - \frac{1}{p})(\frac{t}{h} + 1) + \frac{1}{p}(\frac{t}{h} + 1 - p\alpha))} \right]^p \\
\geq h^{p\alpha} \left[ \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma^{1 - \frac{1}{p}}(\frac{t}{h} + 1)\Gamma^{\frac{1}{p}}(\frac{t}{h} + 1 - p\alpha)} \right]^p \\
= h^{p\alpha} \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - p\alpha)} = t_h^{(p\alpha)}, \\
\]
so we have proved that (2.10) holds whenever \( 1 \leq p < \infty \).

Let \( 1 \leq p \leq q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( f = \{f_i\}_{i=0}^\infty \) be an arbitrary sequence of real numbers. Moreover, suppose that \( \{u_i\}_{i=0}^\infty \), and \( \{v_i\}_{i=0}^\infty \) are
weight sequences, i.e., non-negative sequences. To prove our main result
we use the following result for a standard weighted Hardy inequality, when
\(1 \leq p \leq q < \infty\) (see [1], Theorem 4.1 and e.g. also [20]):

**Theorem B.** Let \(1 \leq p \leq q < \infty\). Then the inequality
\[
\left( \sum_{i=1}^{\infty} \left( \sum_{j=1}^{i} f_{j} \right)^{q} u_{i}^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \left( \sum_{i=1}^{\infty} (f_{i}v_{i})^{p} \right)^{\frac{1}{p}},
\]
holds for all sequences \(f = \{f_{i}\}_{i=0}^{\infty}, f_{i} \geq 0, i \geq 1\), with the best constant
\(C > 0\) if and only if \(B = \sup_{k \geq 1} \left( \sum_{i=k}^{\infty} u_{i}^{q} \right)^{\frac{1}{q}} \left( \sum_{j=1}^{k} v_{j}^{-p'} \right)^{\frac{1}{p'}} \leq \infty\). Moreover,
\(B \leq C \leq p'q^{\frac{1}{p}}B\).

3 The main result

Our main result reads:

**Theorem 3.1.** Let \(1 < p < \infty\), \(0 < \alpha < 1\) and \(f(x) = D_{h}F(x)\). Then the following inequality
\[
\left[ \int_{0}^{\infty} \left( \int_{0}^{\infty} |F(x) - F(y)|^{p} dx dy \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \leq C \left[ \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \frac{x + 3h}{h} \right)^{(\frac{1}{p} + \alpha)} \right)^{p} dx \right]^{\frac{1}{p}}, \tag{3.1}
\]
holds with constant \(C = \frac{2^{\frac{1}{p}+\alpha-1}}{(p-\alpha)a}\). Moreover, this constant sharp.

The next lemma permits to shorten the proof of our main result:

**Lemma 3.1.** Let \(0 < \alpha < 1, 1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{p'} = 1\). Then
\[
B := \sup_{z \in T_{0}} \left( \int_{z}^{\infty} \frac{d_{h}x}{(x + 3h)^{(\frac{1}{p} + \alpha)}} \right)^{\frac{1}{p}} \left( \int_{0}^{\infty} \frac{\delta(t)^{(\frac{1}{p} - 1)}}{\delta(t)^{\alpha} t^{\frac{p}{p'}}} \right)^{\frac{1}{p'}} < \frac{1}{\alpha}. \tag{3.2}
\]
Proof. Let $0 < \alpha < 1$, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Exploiting (2.1) we find that $D_h \left[ x_h^{(\alpha)} \right] = \alpha x_h^{(\alpha-1)} \geq 0$ for $x \in \mathbb{T}_0$. Moreover, in view of Definition 3 we see that $x_h^{(\alpha)} \leq (x')_h^{(\alpha)}$ for $x, x' \in \mathbb{T}_0$ such that $x \leq x'$. Then, according to (2.1), (2.2), (2.7), (2.8) and (2.10), we obtain that

$$D_h \left[ \frac{1}{(x + 2h)_h^{(\alpha)}} \right] = p \int_0^1 \left[ \frac{z}{(x + 3h)_h^{(\alpha)}} + \frac{1 - z}{(x + 2h)_h^{(\alpha)}} \right] dz \cdot D_h \left[ \frac{1}{(x + 2h)_h^{(\alpha)}} \right]$$

$$= -p\alpha \frac{1}{(x + 3h)^{(\alpha+1)}} \int_0^1 \left[ \frac{z}{(x + 3h)_h^{(\alpha)}} + \frac{1 - z}{(x + 2h)_h^{(\alpha)}} \right] dz$$

$$\leq -p\alpha \frac{1}{(x + 3h)^{(\alpha+1)}} \left[ \frac{1}{(x + 3h)_h^{(\alpha)}} \right]^{p-1}$$

and

$$\left[ (x + 3h)^{(\frac{1}{p} + \alpha)} \right] = \left[ (x + 3h)^{(\alpha)} \right]^p \left[ (x + 3h - \alpha h)^{(\frac{1}{p})} \right]^p$$

$$\geq \left[ (x + 3h)^{(\alpha)} \right]^{p-1} \left[ (x + 3h - \alpha h)^{(1)} \right]$$

$$= \left[ (x + 3h)^{(\alpha)} \right]^{p-1} \left[ (x + 3h)^{(\alpha+1)} \right]$$

i.e.

$$\frac{1}{(x + 3h)^{(\frac{1}{p} + \alpha)}} \leq -\frac{1}{p\alpha} D_h \left[ \frac{1}{(x + 2h)^{(\alpha)}} \right]^p.$$

(3.3)

Next we note that, by Definition 3 and (2.5), $i_h^{(\alpha)} \leq (z + h)^{(\alpha)}$, for $t, z \in \mathbb{T}_0$ such that $t \leq z + h$ and then, by applying (2.4), (2.5) and (3.3), we get that

$$B^p \leq \frac{1}{p\alpha} \sup_{z \in \mathbb{T}_0} (z + 2h)^{(\alpha)} \int D_h \left[ \frac{1}{(x + 2h)^{(\alpha)}} \right] d_h x \left[ \int i_h^{(\alpha-1)} d_h t \right]^\frac{p}{p'}$$

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\[
\begin{align*}
\leq & \frac{1}{p\alpha} \sup_{z \in \mathbb{T}_0} (z + 2h\lambda(z)) \int \lim_{h \to 0} \frac{D_h \left[ \frac{1}{(x + 2h\lambda(z))^{\frac{1}{p}}} \right]_{x^+}^{x^-} }{d_h x} \left[ \frac{1}{\alpha} \int_0^\infty D_h \left[ t^{\alpha}_h \right] d_h t \right]^{\frac{1}{p}} \\
& \leq \frac{1}{\alpha^p} \sup_{z \in \mathbb{T}_0} (z + 2h\lambda(z)) \left[ \frac{h}{(z + 2h\lambda(z))^{\frac{1}{p}}} \right]^{\frac{1}{p}} = \frac{1}{\alpha^p} \\
i.e. (3.2) holds so the proof is complete. \quad \square
\end{align*}
\]

**Proof of Theorem 3.1.** By using (2.3) we get that

\[
L(F) := \int_0^\infty \int_0^\infty \frac{|F(x) - F(y)|^p d_h x d_h y}{(|x - y| + 3h_h^{\frac{1}{p} + \alpha})^{p}}
\]

\[
= \sum_{k=0}^\infty \sum_{i=0}^\infty h^2 \left[ \frac{|F(ih) - F(kh)|^p}{(|ih - kh| + 3h_h^{\frac{1}{p} + \alpha})^{p}} \right]
\]

\[
\leq \sum_{k=0}^\infty \sum_{i=0}^k h^2 \left[ \frac{|F(ih) - F(kh)|^p}{(|ih - kh| + 3h_h^{\frac{1}{p} + \alpha})^{p}} \right] + \sum_{k=0}^\infty \sum_{i=k}^\infty h^2 \left[ \frac{|F(ih) - F(kh)|^p}{(|ih - kh| + 3h_h^{\frac{1}{p} + \alpha})^{p}} \right]
\]

\[= 2 \sum_{k=0}^\infty \sum_{i=k}^\infty h^2 \left[ \frac{|F(ih) - F(kh)|^p}{(|ih - kh| + 3h_h^{\frac{1}{p} + \alpha})^{p}} \right]. \quad (3.4)
\]

Let

\[
\tilde{f}_m = h |f(mh)|, \quad \tilde{u}_i = \frac{h^{\frac{1}{p}}}{(|ih - kh| + 3h_h^{\frac{1}{p} + \alpha})^{\frac{1}{p}}},
\]

\[
\tilde{v}_m = \frac{h^{-\frac{1}{p}}}{\left[ (mh - kh)_{\lambda-1} \left( \delta(mh) - kh_h^{\lambda} \right) \right]^{\frac{1}{p}}}
\]

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and \( f(x) = D_h F(x) \). Then, from (2.3) and (3.4) it follows that

\[
L(F) \leq 2 \sum_{k=0}^{\infty} h \sum_{i=k}^{\infty} \left[ \left( |ih - kh| + 3h \right)^{\frac{1}{p} + \alpha} \right]^p \sum_{m=k}^{i-1} h |f(mh)|^p.
\]

Moreover, based on Theorem B we obtain that

\[
\sum_{i=k}^{\infty} \tilde{u}_i^p \left( \sum_{m=k}^{i} \tilde{f}_m \right)^p \leq \tilde{B}_k^p \sum_{m=k}^{\infty} \tilde{f}_m^p \tilde{r}_m^{-p'},
\]

where

\[
\tilde{B}_k^p := \sup_{n \geq k} \left( \sum_{i=n}^{\infty} \tilde{u}_i^q \right)^{\frac{1}{q}} \left( \sum_{j=k}^{n} \tilde{v}_j^{-p'} \right)^{\frac{1}{p'}}.
\]

By combining (3.5) and (3.6) we have that

\[
L(F) \leq 2 \sum_{k=0}^{\infty} h \tilde{B}_k^p \sum_{m=k}^{\infty} \left[ (uh - kh)^{\alpha} \right] \left( \delta(t) - (kh)^{\alpha} \right)^{-\frac{1}{p}} \leq B^p.
\]

Moreover, by using Definition 3 and (2.6) we obtain that

\[
(\delta(mh) - t)^{(\alpha - 1)} \leq (mh - t)^{(\alpha - 1)},
\]

for \( t \in T_0 \).

Hence, in view of (2.1), (2.2) and (2.8) we get that
\[ \begin{align*}
D_{h,t} \left[ \frac{1}{(\delta(mh) - t)_{h}^{(a-1)}} \right]^{p} &= p \int_{0}^{1} \left[ \frac{z}{(mh - t)_{h}^{(a-1)}} + \frac{(1 - z)}{(\delta(mh) - t)_{h}^{(a-1)}} \right]^{p-1} d\zeta \\
& \times D_{h,t} \left[ \frac{1}{(\delta(mh) - t)_{h}^{(a-1)}} \right]^{p-1} \\
& = \frac{p(\alpha - 1)}{(\delta(mh) - t)_{h}^{(a-1)}} \int_{0}^{1} \left[ \frac{z}{(mh - t)_{h}^{(a-1)}} \right]^{p-1} d\zeta \\
& + \frac{(1 - z)}{(\delta(mh) - t)_{h}^{(a-1)}} \int_{0}^{1} \left[ \frac{z}{(mh - t)_{h}^{(a-1)}} \right]^{p-1} d\zeta \\
& \leq \frac{p(\alpha - 1)}{(\delta(mh) - t)_{h}^{(a-1)}} \left[ \frac{1}{(mh - t)_{h}^{(a-1)}} \right]^{p-1}.
\end{align*} \]

Consequently,
\[
\frac{1}{(\delta(mh) - t)_{h}^{(a)}} \left[ \frac{1}{(mh - t)_{h}^{(a-1)}} \right]^{p-1} \leq \frac{1}{p(\alpha - 1)} D_{h,t} \left[ \frac{1}{(\delta(mh) - t)_{h}^{(a-1)}} \right]^{p}. \tag{3.8}
\]

Thus, by now using Lemma 3.1 and (3.7) and (3.8), we obtain that
\[
L(F) \leq 2B^{p} \sum_{k=0}^{\infty} h \sum_{m=k}^{\infty} \frac{h |f(mh)|^{p}}{(mh - kh)_{h}^{(a-1)}(\delta(mh) - kh)_{h}^{(a)}} \\
\leq 2B^{p} \sum_{m=0}^{\infty} h |f(mh)|^{p} \sum_{k=0}^{m} \frac{h}{(mh - kh)_{h}^{(a-1)}(\delta(mh) - kh)_{h}^{(a)}} \\
\leq \frac{2\alpha^{p}}{p(\alpha - 1)} \sum_{m=0}^{\infty} h |f(mh)|^{p} \int_{0}^{\delta(mh)} D_{h,t} \left[ \frac{1}{(\delta(mh) - t)_{h}^{(a-1)}} \right]^{p} d_{h,t}
\]

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\[
\leq \frac{2\alpha^{-p}}{p(1-\alpha)} \sum_{m=0}^{\infty} h \left( mh + h \right)^{(\alpha-1)} \left( mh + h \right)^{p} \\
\leq \frac{2\alpha^{-p}}{p(1-\alpha)} \int_{0}^{\infty} |f(x)|^{p} \frac{d_{h,x}}{(x + h)^{(\alpha-1)}},
\]

which means that inequality (3.1) holds.

Finally, we will show that the constant \( \frac{2^{\frac{1}{p}}}{(p\alpha^{-p})^{\frac{1}{p}}} \) in (3.1) sharp. Let \( x, y, a \in \mathbb{T}_0 \) such that \( y \leq a \leq x - 4h \). By Definition 3 we obtain that

\[
\left( x - y + 2h - \alpha h + \frac{1}{p'} h \right) \leq \left( x - y + 3h - \alpha h \right)_{h}^{(1)}, \\
\left( x - y + 2h - (\alpha - 1)h \right)_{h}^{(1)} \leq \left( x + 4h - \alpha h \right)_{h}^{(1)}.
\]

Then, by using (2.2), (2.8), (2.9) and (2.10) we find that

\[
\left[ |x - y + 3h|_{h}^{\left( \frac{1}{p} + \alpha \right)} \right]^{p} = \left[ (x - y + 3h)_{h}^{(\alpha-1)} \right]^{p} \left[ (x - y + 2h - \alpha h)_{h}^{(\frac{1}{p})} \right]^{p} \\
\times \left[ (x - y + 3h - (\alpha - 1)h)_{h}^{(1)} \right]^{p} \\
\leq \left[ (x - y + 3h)_{h}^{(\alpha-1)} \right]^{p-1} \\
\times \left[ (x - y + 2h)_{h}^{(\alpha-1)} (x - y + 2h - \alpha h + 1/p'h)_{h}^{(1)} \right]^{p} \\
\times \left[ (x - y + 3h - (\alpha - 1)h)_{h}^{(1)} \right]^{p} \\
\leq \left[ (x - y + 3h)_{h}^{(\alpha-1)} \right]^{p-1} (x - y + 3h)_{h}^{(\alpha)} \\
\times \left[ (2x - a + 4h - \alpha h)_{h}^{(1)} \right]^{p}
\]

and

\[
D_{h,y} \left[ \frac{1}{(x - y + 3h)_{h}^{(\alpha-1)}} \right]^{p} = p \int_{0}^{1} \left[ \frac{z}{(x - y + 2h)_{h}^{(\alpha-1)}} + \frac{1 - z}{(x - y + 3h)_{h}^{(\alpha-1)}} \right]^{p-1} dz \\
\times D_{h,y} \left[ \frac{1}{(x - y + 3h)_{h}^{(\alpha-1)}} \right].
\]

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\[ = \int_0^1 \left[ \frac{z}{(x - y + 2h)^{\alpha h}} + \frac{1 - z}{(x - y + 3h)^{\alpha h}} \right]^{p-1} dz \times \frac{p(\alpha - 1)}{(x - y + 3h)^{(\alpha)}} \geq - \left[ \frac{1}{(x - y + 3h)^{(\alpha)}} \right]^{p-1} \frac{p(1 - \alpha)}{(x - y + 3h)^{(\alpha)}} \times \left\{ (2x - a + 4h - \alpha h)^{(1)} \right\}^p \left[ (x - a)^{(1)} \right]^p. \]

Therefore,
\[ \left[ \frac{1}{(x - y + 3h)^{(\alpha)}} \right]^p \geq - \left[ \frac{(x - a)^{(1)}_h}{p(1 - \alpha)} \right]^{-p} D_h \left[ \frac{1}{(x - y + 3h)^{(\alpha)}} \right]^p. \quad (3.9) \]

Assume now on the contrary that there exists a constant \( C < \frac{2^\frac{1}{p} \alpha^{-1}}{(p \alpha - p)^{\frac{1}{p}}} \) such that (3.1) holds for all measurable functions where the right hand side is finite. We now consider the test function
\[ f_0 := \chi_{[a, a']}(t) (t - a - h + \alpha h)^{(\alpha - 1)}_h, \]
for \( a' \in T_0 \) such that \( x \leq a' \). Then, by using (2.4), (2.5) and (2.9) we can deduce that
\[ |F(x) - F(y)|^p = \left| \int_a^x (t - a - h + \alpha h)^{(\alpha - 1)}_h \, dt \right|^p = \frac{1}{\alpha^p} \left| \int_a^x D_h \left[ (t - a - h + \alpha h)^{(\alpha)}_h \right] \, dt \right|^p = \frac{1}{\alpha^p} \left( (x - a - h + \alpha h)^{(\alpha)}_h \right)^p = \frac{1}{\alpha^p} \left( (x - a - h + \alpha h)^{(\alpha - 1)} \right)^p \left[ (t - a)^{(1)} \right]^p, \quad (3.10) \]
where \((-h + \alpha h)^{(a)}_h = h^{\alpha} \Gamma(a) / \Gamma(0) = 0\) and
\[
\int_0^\infty \frac{f^p_0(x)dh}{(x + h)^{(a-1)}_h} \leq \int_a^\alpha \frac{[x - a - h + \alpha h]^{(a-1)}_h]^p}{[(x + h)^{(1-a)}_h]^p} dh < \infty.
\]
By combining (2.3) and (3.4) we obtain that
\[
L(F) := \int_0^\infty \int_0^x \frac{|F(x) - F(y)|^p dh_x dh_y}{(|x - y| + 3h)^{(1+\alpha)}_h} \leq \int_0^\infty \int_0^x \frac{|F(x) - F(y)|^p dh_x dh_y}{(|x - y| + 3h)^{(1+\alpha)}_h} < \infty.
\]
From (3.9) and (3.10) it follows that
\[
I_1 \geq \int_a^{a'} \left[ \frac{F(x) - F(y)}{|x - y| + 3h} \right]^p dh_x dh_y \geq \frac{\alpha^{-p}}{p(1-\alpha)} \int_a^{a'} \left[ (x - a - h + \alpha h)^{(a-1)}_h \right]^p dh_x dh_y \geq \frac{\alpha^{-p}}{p(1-\alpha)} \int_0^\infty f^p_0(x) dx,
\]
where \(\frac{1}{(h)^{a-1}} = \frac{\Gamma(a-1)}{\Gamma(0)} = 0\).

In the same way we can deduce that
\[
I_2 \geq \frac{\alpha^{-p}}{p(1-\alpha)} \int_0^\infty \frac{f^p_0(x)dx}{(x + h)^{(1-a)}_h}.
\]
By now using (3.11), (3.12) and (3.13) we obtain that

\[ C^h \geq \frac{L(F)}{\int_0^\infty \frac{f_\alpha^p(x) d_x x}{[(x+h)^{\alpha-1}]^p}} = \frac{2\alpha^{-p}}{p(1 - \alpha)}, \]

which contradicts our assumption so we conclude that the constant \( \frac{2\alpha^{-1}}{(p-\alpha\alpha)^p} \)
in (3.1) sharp. The proof is complete.

**References**


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