Scientific Computing with Applications in Tribology
-A Course Compendium

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Abstract

Tribology is a multidisciplinary field defined as the science and technology of interacting surfaces in relative motion, and embraces the study of friction, wear and lubrication. A typical tribological interface is the one between the roller and the raceway in a the rolling element bearing, such as the one depicted in Fig. 1. Other examples are the lubricated interfaces in journal- and thrust bearings, cam-mechanisms, between gear teeth and in hydraulic systems. Human joints, contact between teeth during chewing are examples of bio-tribological interfaces. To fully understand the operation of this type of application one has to understand the couplings between the lubricant fluid dynamics, the structural dynamics of the bearing material, the thermodynamical aspects and the resulting chemical reactions. This makes modeling tribological applications an extremely delicate task. Because of the multidisciplinary nature, such theoretical models lead to mathematical descriptions generally in the form of non-linear integro-differential systems of equations. Some of these systems of equations are sufficiently well posed to allow numerical solutions and sometimes even analytical to be carried out, resulting in accurate predictions on performance, and some of these are described herein.
Acknowledgment

Although the compilation of this text is the work solely of the authors, the models and solution procedure presented herein is joint development of many good colleagues and co-authors. Our sincere gratitude is extended towards them all.
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Preface

This compendium comprises models and numerical solution procedure for tribological interfaces. It describes the tribological contact and the classical lubrication regimes. A thorough derivation of the Reynolds equation, governing the fluid pressure, from the Navier-Stokes momentum equations and the continuity equation for conservation of mass, is presented along with its analytical solution for the infinitely wide linear slider bearing.

The compilation of the compendium was conducted by the first author during his tenure as Professor at the Division of Machine Elements, Department of Engineering Sciences and Mathematics, Luleå University of Technology and by the second author during his tenure as a postdoctoral researcher at the same division.

Although the compilation of this text is the work solely of the authors, the models and solution procedure presented herein is joint development of many good colleagues and co-authors. Our sincere gratitude is extended towards them all.
Chapter 1

Introduction

Machines consist of machine elements and their safe and efficient operation relies on carefully designed interfaces between these elements. The functional design of interfaces covers geometry, materials, lubrication and surface topography, and an incorrect design may lead to both lowered efficiency and shortened service life. A misalignment due to the geometrical design could lead to large stress concentrations that in turn may lead to severe damage when mounting, a detrimental wear situation, rapid fatigue during operation, etc. Large stress concentrations also implicitly imply a temperature rise because of the energy dissipation due to plastic deformations. The choice of mating materials is also of great importance, e.g. electrolytic corrosion may drastically reduce service life. Contact fatigue due to low ductility would not only lower the service life but could lead to third body abrasion due to spalling, which in turn could end up lowering the service life of other components. A lubricant serves several crucial objectives; when its main objective is to lower friction, the actions of additives are of concern. If the interface is subjected to excessive wear, the lubricant’s ability to form a separating film becomes even more crucial. In this case, the bulk properties of the lubricant have to be carefully chosen. At some scale, regardless of the surface finish, all real surfaces are rough and their topography influences the contact condition.

As implied above, the design parameters are mutually dependent, i.e. they affect the way others influence the operation of the system. For example, a change in geometry could require another choice of materials that may change the objectives of the lubricant and force the operation into another lubrication regime.

The influence of the aforementioned design parameters, i.e., geometry, materials, lubrication and surface topography on performance has of course been investigated by many researchers in the field both experimentally and numerically. However, because of the multi-disciplinary nature of the field and the complexity of the theoretical models associated with tribological problems, the progress in the development of efficient, still user friendly software has not reached as far as in, e.g., computational structural mechanics and computational fluid dynamics. Moreover, the requirement on the density of the mesh to resolve not only the geometrical part of the tribological contact but also the surface topography is difficult to meet. The material herein is meant to provide understanding of established models and numerical solution procedures that can be used to study behaviour of tribological interfaces. Hopefully, it also inspires and encourages the reader to contribute to further development thereof.
CHAPTER 1. INTRODUCTION
Chapter 2

The Tribological Contact

At start-ups, at stops as well as during operation, most machine elements experiences varying contact conditions. Take, for example, the (axial) tilted pad thrust bearing illustrated in Fig. 2.1. This type of bearing belongs to the class of hydrodynamic fluid film bearings, which are designed for fluid film pressure build-up that separate the rotating and stationary surfaces so that contact less rotation while carrying the load on the shaft. In fact, it is the relative motion of the surfaces, as the lubricant is pulled into the converging geometry between the collar and the pad, that creates the necessary fluid film pressure.

Typically, the collar in a tilting pad thrust bearing is made of steal while the pads have a soft (compliant) facing made of Babbitt (metal alloy) or Teflon® (polytetrafluoroethylene (PTFE)). This means that the littlest direct contact the collar makes with the shaft while rotating, will cause severe wear on the facing surface and it is of crucial importance to have a system that separates the surfaces during initiation of start-up and stop. A common solution is to implement a system that pressurises the supplied lubricant, generating hydrostatic lift. Another example is the piston with its reciprocating motion inside the cylinder of a heavy duty diesel engine, such as the one depicted in Fig. 2.2. In this case, the lubricated ring
interfaces are never seen stationary as they are decelerating from full speed at midstroke to full stop at the dead centres, reversing and then accelerating to reach full speed when back at midstroke again.

![Diagram of piston with rings inside cylinder liner](image)

Figure 2.2: Piston with rings inside cylinder liner. Illustration courtesy Markus Söderfjäll.

Therefore, depending on the application and the operating conditions it is common to characterise the tribological contact by its lubrication regime. The lubricant regimes are often divided into: Boundary Lubrication (BL), Mixed Lubrication (ML) and Full Film Lubrication (FL). In the heavy duty diesel engine, the load that the interface between the compression ring and liner surface see, comes from ring tension and possibly the gas pressure behind the compression ring. During operation, the contact between ring and the liner varies and it is understood that it can be in the full film regime at some parts in the mixed at others and sometimes it may even enter the boundary lubrication regime.

### 2.1 The boundary lubrication regime

In the boundary lubrication (BL) regime, the lubricant’s hydrodynamic action is negligible and the load is carried directly by surface asperities or by surface active additives (a so-called tribofilm). Here, the surface topography is preferably chosen to optimize the frictional behaviour without increasing the rate of wear. To do this, one has to understand how the chemical processes are affected by the actual contact conditions, in terms of e.g. heat generation, pressure peaks, the real area of contact, and vice versa too. This compendium lacks a comprehensive BL-model incorporating all these features. It does, however, present the pure elastic contact mechnanics problem, including the analytical Westergaard and the Hertzian solutions. The linear complementarity problem (LCP) is thoroughly described and it is shown how the numerically exact Lemke algorithm can be applied for its solution. Moreover, the nowadays well-known elasto-plastic contact mechanics model [1–3] with the corresponding numerical approach grounded on a variational formulation, expressed in terms of total complementary potential energy, with acceleration relying on the fast Fourier transform (FFT) [4–6]. This approach has proved to ensure a stable and effective simulation of
(rough) contact mechanics and it can help to increase the understanding of how the surface roughness influences the elastic deflection, the plastic deformation (and plasticity index), the pressure build up and the real area of contact. An in-depth understanding of this connection is required to refine the design of interfaces operating under these circumstances.

As the hydrodynamical action of the lubricant increases, the contact mechanical response becomes less influential in terms of pressure and real contact area, and a transition from the BL- to the ML- regime may therefore occur.

2.2 The mixed lubrication regime

What characterizes the ML regime is that the load is carried by the lubricant’s hydrodynamical action, which may be influenced by the elastic deflection of the surfaces, the tribofilm, directly by surface asperities, or a combination thereof.

This means that the objectives of the surface topography are to support the hydrodynamic action of the lubricant, aid the elastic deflection in rendering a smoother surface, enable bonding of the surface active additives and optimize friction in the contact spots without increasing wear.

Modelling mixed lubrication has turned out to be a true challenge and the models available are built upon assumptions simplifying the physics involved in the transition from the BL- and the FL- regime. As indicated above, a contact mechanics model may be used to indicate a possible transition between the BL- and the ML-regimes. Similarly, modelling performed regarding full-film lubrication has lead to numerical approaches that may be used to increase the understanding of the transition from the FL- to the ML- regimes. One well-known example of an ML-model, is the Luleå mixed lubrication model [3], in which partitioning between lubricant carried load and load carried by direct contact, is determined by the separation. More precisely, when the separation becomes smaller than a chosen measure of the surface roughness height, the lubricant load is alleviated with the amount that the corresponding unlubricated interface would carry at that separation.

2.3 The full-film lubrication regime

When the hydrodynamic action of the lubricant fully separates the surfaces and the load is no longer carried by the contact between the surfaces, the interface enters the full film lubrication (FL) regime. In the FL regime, traction may be reduced by carefully chosen topographies. Even though there is no direct contact, the lubricant pressure may lead to stress concentrations high enough to cause fatigue, likely leading to excessive wear in the form of spalling in highly loaded situations.

This regime is commonly sub-divided into hydrodynamic lubrication (HL) and elastohydrodynamic lubrication (EHL), since the performance is greatly affected by the presence of elastic deflections, i.e., fluid-structure interaction, at the lubricated interface.
2.3.1 Hydrodynamic lubrication

Slider bearings are typical examples of applications that, under certain conditions, operate in the hydrodynamic lubrication (HL) regime where the elastic deformations of the bearing surfaces are sufficiently small to be neglected. For example, the tilting pad thrust bearing, as depicted in Fig. 2.1, exhibits a conformal interface between the pad and the collar and is designed to for operation in the hydrodynamic lubrication regime. Note that the angle of inclination of the pads, which is generally only a fraction of a degree, has been greatly exaggerated in the figure. One problem that arise when modelling conformal interfaces like this one, comes from the large differences in scales. More precisely, the global scale describing the geometry, pad - collar interface, is several orders of magnitude larger than the local scale describing the surface topography/roughness. This situation can be approached by means of homogenisation. This is also a subject discussed herein, see Section 5.8.

2.3.2 Elastohydrodynamic lubrication

Elastohydrodynamic lubrication (EHL) is the type of hydrodynamic lubrication where the fluid-structure interaction (FSI), caused by elastic deformations of the contacting surfaces, plays a major role. This situation may occur when lubricating interacting non-conformal bodies. This leads to highly localised (concentrated) contacts, and it is the lubricant’s piezo-viscous response combined with elastic flattening of surface roughness features facilitate the separation of the interacting surfaces. An example where EHL is typically found, is at the interface between the roller and the raceway in a typical roller bearing, as shown in Fig. 2.3, are most commonly designed to operate in the full-film elastohydrodynamic lubrication regime.

![Figure 2.3: Schematics of a typical rolling element bearing](image)

The apparent contact zone for a rolling bearing is, in general, elliptic in shape. Depending on the design parameters previously mentioned and the actual running conditions, the shape of the ellipse will change. In any case, the contact region is small and the concentrated load implies a severe surface- as well as sub-surface stress condition that may lead to both elastic-
but also plastic deformation. For a bearing in operation, high stresses eventually causes fatigue, which in turn can lead to shortened service life due to, for example, spalling. When the contact is starved of lubricant, or when running conditions do not allow for a hydrodynamic action that fully separates the surfaces, the risk for plastic deformation increases.

If the width of the contact ellipse exceeds the minimum width of the raceway and the roller, the contact will be then truncated and this leads lead to increased stresses in the material. In the case of a contact ellipse which is more than 4 times wider than its length in the rolling direction, the pressure at the centerline in the rolling direction can be approximated to the pressure corresponding to a line contact, Evans et al. [7]. This motivates describing the problem with a two-dimensional instead of a three-dimensional domain. Moreover, it has been shown that the one-dimensional Reynolds equation can give highly accurate estimates of deformations and stresses inside the interface. Still as with most tribological problems, this is a very demanding problem that requires advanced mathematical descriptions as well as highly efficient numerical solution procedures. Homogenisation of roughness, Fast Fourier Transformation (FFT) and multilevel techniques are examples of such. This usually renders quite complex methods that often require end users with rather specialised background.
Chapter 3

Content and Intended Learning Outcomes

The intended learning outcomes are related to modelling and simulation of tribological processes connected to the following topics *Lubricated contacts*, *Dry contacts* and *Wear*. The content include derivations of models, dimensionless formulation, techniques for discretisation, numerical solution procedures and it discusses verification and validation.

3.1 Dry contacts

In relation to modelling and simulation of the dry contact by means of *contact mechanics*, the usage of Fast Fourier Transformation (FFT) will be discussed. It will be shown how it can be used applied in order to accelerate the computation of derivatives and integral equations, and specifically the deflection of linear elastic bodies. The associated complementarity problem and the total complementary potential energy problem will be described, together with two different means of how to numerically solve the contact mechanics model. More precisely, a numerical exact method that finds the solution to the corresponding Linear Complementarity Problem (LCP) in a finite number of pivoting steps and a quadratic programming routine will be explained. A simple way of including plastic deformation will also be presented. In order to verify and validate the results, both the Hertzian contact, for contacting spheres, and Westergaards solution, for harmonic surfaces, are revisited first.

3.2 Lubricated contacts

This part starts with the derivation of the Reynolds equation for the hydrodynamic pressure. The Reynolds equation is a second order Poisson type of differential equation and both the cartesian and the polar form will be presented here. The derivation involves scaling and asymptotic analysis of the Navier-Stokes momentum equations coupled with the continuity equation for mass preservation. The methodology is generic and can be applied in other areas as well. The analytical solution to the one-dimensional Reynolds equation for an infinitely wide bearing is presented. Then it is shown how it can be used to verify the numerical
results obtained with finite difference and finite element based methods for realistic bearing geometries.

Hydrodynamic cavitation is found in various lubrication situations, and without including it in the model the bearing’s load carrying capacity cannot be predicted. The Jacobsson, Floberg and Olsson (JFO) boundary conditions and the Elrod and Adams model will be discussed and use as a basis for the derivation of the state-of-the-art model. In particular, this model addresses the change of the differential equation from elliptic in the fully saturated zones to hyperbolic in the cavitated zones.

Homogenisation is presented here as a means for effective treatment of the roughness of the interacting surfaces. Finally a mixed lubrication model is presented. However, as mixed lubrication involves direct contact between the surfaces, modelling the dry contact is presented on beforehand.

3.3 Wear

Wear does under some circumstances allow for modelling. Here, Archard’s equation is employed, primarily for the modelling of abrasive wear. Archard’s equation is an initial value problem and it is in combination with the contact mechanics model it can be used to predict the material loss in tribological contacts. It is also discussed how the time stepping in subsequent numerical simulation procedure can be adjusted to simulate an adhesive wear processes.
Chapter 4

The Dry Contact

In the previous chapter, we have described different lubrication regimes in which the lubricant has more or less influence. Let us, however, start by considering the case in which no lubricant is present, i.e., the dry contact. In this case, the load with which we press two bodies together will result in a direct contact of their surfaces. If these are rough, the contact will only occur at the top of the highest asperities, leading to a complicated contact pattern. Understanding how much the two surfaces are in contact (i.e., how large is the contact area) as well as how is this area distributed is of importance in many machine elements operating in boundary and mixed lubrication regimes. Indeed, this contact controls friction, wear, contact resistance, leakage, etc. In all this situations, the real operation conditions will probably be such that a complex interaction between a fluid, the surfaces, possibly third bodies and the environment will ultimately decide the overall performance. The dry contact is, of course, a serious simplification of these complex situations. Understanding it is, however, a prerequisite to study the more complex and realistic cases. Moreover it represents a first approximation that already provides for very useful information concerning the functioning of these machine elements. Therefore, this chapter is dedicated to the study of the dry contact.

When studying the contact between two bodies, with rough or smooth surfaces, the boundary element method (BEM) reveals itself as a powerful tool, as it allows us to solve the contact between two bodies by meshing only the interfaces, i.e., the surfaces. This means that we can obtain an approximate solution to the 3D contact between two bodies by means of solving a 2D problem (and the 2D contact by means of solving a 1D problem). This, of course, greatly reduces the computational power and memory needed to simulate the contact. In the (common) case in which the surfaces are rough, a large amount of small elements are needed to obtain a good resolution of the small-scale features. This means that, even with today’s available computational power, a full 3D, finite element based simulation becomes too demanding and BEM is the only viable option. Moreover, because of its simplicity, it facilitates post processing and interpretation of results.

Having motivated its importance, we will in this chapter present the boundary element method. It will then be formulated in its dimensionless form and some analytical solutions will be provided as an example. Thereafter, the discretisation of the problem will be presented and different numerical solution procedures will be introduced.
CHAPTER 4. THE DRY CONTACT

Figure 4.1: Illustration of point loading of a half-space. Top-left showing the half-space in 3D with the point load located at (0, 0, 0), bottom-right, visualising the $x_r x_3$-plane with the rotational symmetric deformation illustrated by the red continuous line. $\rho$ denotes the distance between (0, 0, 0) and $(x_1, x_2, x_3)$ and $r$ is the distance between (0, 0, 0) and $(x_1, x_2, 0)$. The elastic deflection at $(x_1, x_2, 0)$ is given by $u_e$.

4.1 Fundamentals of BEM - The half-space theory

In this section we will describe the fundamentals of BEM, which is based on the half-space theory. Let us then start by defining a half space. Consider the infinite 3D Euclidean space and cut it in half by a plane. Each of the parts will be a half-space. Notice that this will have one boundary (i.e., the plane) but will be infinite in all other directions. We will further assume that this half space is homogeneous and elastic, that the contact is friction free and we will not consider the effect of adhesion. It is important to remember to consider whether these assumptions are reasonable in a given problem, before applying the dimension reduced BEM that will be presented here.

In the subsections below, we present the theoretical backbone for BEM and we start by introducing the relation between a point load and the deformation it causes. Then we generalise this theory to enable the study of the contact between a rigid and an elastic body and thereafter to the contact between two elastic bodies.

4.1.1 The relation between load and deformation

Let us consider a situation such as the one depicted in Fig. 4.1, in which an half space is loaded with a point load at the origin. Let us further assume that the assumptions of linear elasticity holds and that the contact is frictionless and without adhesion between the surfaces. Under these conditions, we can use the Boussinesq solution for the elastic deformation $u_e$ evaluated at the location $(x_1, x_2)$ caused by a point load $P$ applied at the point $(x_1', x_2')$ on
4.1. FUNDAMENTALS OF BEM - THE HALF-SPACE THEORY

the half space, i.e.

\[
u_e(x_1, x_2) = \frac{2(1 - \nu^2)}{\pi E} \frac{P}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}},
\]

(4.1)

where \(E\) and \(\nu\) are the elastic modulus and Poisson’s ratio of the body. From (4.1), it is clear that the deformation is rotational symmetric and inversely proportional to the distance between the point \((x'_1, x'_2)\), where the point load is applied, and the point \((x_1, x_2)\), where the elastic deformation is evaluated. Notice that, as long as we are only interested on the deformations at the surface, the problem has become two-dimensional. For a thorough derivation of this solution, and an extension to more general cases where friction is included, the reader is referred to [8]. It can be noticed in (4.1) that the elastic deformation has a singularity at the point at which load is applied. This clearly non-realistic singularity only arises, however, from considering equally non-realistic point loads. Indeed, it disappears in the more realistic case where the load is distributed over a small area.

In order to compute the deformation caused by an arbitrarily shaped pressure distribution, \(p\), we apply the principle of superposition. More precisely, we consider the pressure distribution to be comprised of a series of point loads, each one acting over a vanishingly small area, i.e. \(P' = p(x'_1, x'_2)\, dA\), where \(dA = dx_1\, dx_2\). The deformation at a point \((x_1, x_2)\) can now be computed by integrating the contribution from all the point loads at points \((x'_1, x'_2)\), i.e.,

\[
u_e(x_1, x_2) = \frac{1 - \nu^2}{\pi E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(x'_1, x'_2)}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}}\, dx'_1\, dx'_2.
\]

(4.2)

The integrals are evaluated from minus- to plus infinity, but it should be noted that it is only necessary to consider the domain where the pressure is positive, as long as we neglect adhesion.

Another important case can be accomplished by considering a load distributed along a line on surface of a half-space. This exemplifies a situation where it can be assumed that the pressure only varies in one direction and that it the contact is long enough in the other so that the effect of edges can be neglected. In theory, this could be the contact between an infinitely long cylinder pressed against a rigid body, as long as the radius of the cylinder is large enough compared to the width of the contact so that the cylinder can be considered a half-space. This renders a 1D solution, which is of course very advantageous in terms of computational cost. Indeed, for the 2D contact, the resulting pressure-deformation relationship can be expressed in 1D and it can be obtained by starting from

\[
u_e(x) = -\frac{2(1 - \nu^2)}{\pi E} P' \ln |x - x'| + C,
\]

(4.3)

where \(P'\) represents a line load, located at \(x'\) with unit N/m and where \(C\) is determined by choosing arbitrarily a point on the surface as datum for displacements. We remark that also in the case of a line loading the deformation depends solely on the distance between the point \(x'\), where the line load is applied and the location \(x\) where it is evaluated. Again, by
superposition, i.e. \( P' = p(x')dx' \), this results in

\[
ue(x) = -\frac{2(1-\nu^2)}{\pi E} \int_{-\infty}^{\infty} p(x') \ln|\textbf{x} - x'|dx' + C,
\]

(4.4)

for an arbitrarily shaped pressure distribution \( p(x) \). Note again, that it is only necessary to consider the domain where the pressure is positive, as long as we neglect adhesion. For the interested reader, the relation (4.3) is known as the Flamant solution, in the literature and for more details on the derivation of the 1D pressure-deformation relation we refer to [8].

4.1.2 The contact between an elastic body and a rigid flat surface

In the previous section the pressure was regarded as known. In this case, the pressure that causes the elastic body to deform is a priori unknown and results from contact between the elastic and the rigid surfaces. The surface of an elastic body can model, e.g., a smooth ball or a rough surface, as long as it the conditions for the Boussinesq or Flamants solutions apply. We will now see more specifically what conditions that must apply to for (4.2) to be used to model the deformation that arise when a non-flat elastic half space contacts a flat rigid surface. The caveat here is that (4.2) is a model of the deformation resulting from a specific pressure distribution on a perfectly flat half space. It turns out that, to use (4.2) to model the deformation that arise between a non-flat elastic half space and a rigid flat surface, there are two assumptions that must apply. These are

1. The slopes of the surface features are small enough for it to be approximated as being flat. For this to hold true, the ratio between height and spatial distribution of the features must be smaller than 1:10;

2. The contact region is much smaller than the body itself, so that the shape of the bodies far from the contact do not affect the stresses and it can be assumed to be a half-space near the contact region.

Remember that we have also assumed the contact to be friction and adhesion free. These are not necessarily requirements for the use of BEM in general, but the equations presented in the previous section would need to be modified to incorporate such effects. With these assumptions in mind, we can use (4.2) to compute the deformation according to the contact pressure distribution. This, however, is not known a priori. Therefore we need more information. Consider thus the gap \( h \) between the two bodies in contact. It can be computed as

\[
h = g + ue + \delta,
\]

(4.5)

where \( g \) is the original gap (before deformation), \( ue \) is the elastic deformation, given by (4.4) in 1D and (4.2) in 2D and where \( \delta \) is the rigid body movement of the two bodies. Let us also define \( \Omega \) as the domain on which \( g \) is defined. The area of this domain is often referred to as the nominal contact area, \( A_n \). In the context of rough surfaces, this area is often considered the area which appears to be in contact when the bodies are seen macroscopically. The real contact area, here denoted \( A_r \), is often much smaller. Let us denote this domain of contact as \( \Omega_c \). Since there is contact in this domain, it is clear that the gap must be zero and that
4.1. FUNDAMENTALS OF BEM - THE HALF-SPACE THEORY

Figure 4.2: Schema of the deformation of two surfaces under a common pressure distribution (the point load is only used as an example).

the pressure is thus positive. Equally, wherever the gap is positive, there is no contact and thus the pressure must be zero. Also, both the pressure and the gap must be positive. These are complementarity conditions, which in contact mechanics are known as the Kuhn-Tucker conditions, and can be summarised as follows:

\[ h(x) > 0, \quad p(x) = 0, \quad x \notin \Omega_c, \quad (4.6a) \]
\[ h(x) = 0, \quad p(x) > 0, \quad x \in \Omega_c, \quad (4.6b) \]

where \( \Omega_c \) represents the contact regions and ‘\( x \)’ is \((x_1, x_2)\) in the 3D contact (2D problem) and \( x \) in the 2D contact (1D problem). These conditions, together with (4.4) in 1D or (4.2) in 2D and a specified \( \delta \) give a unique solution for the contact mechanics problem. It is often more convenient, however, to specify the total load, \( w \) instead of the rigid body movement \( \delta \). Under stationary conditions, Newton’s first law models the force equilibrium that balances the applied load and the integrated force from the contact pressure distribution, i.e.

\[ w = \int_{-\infty}^{\infty} p(x) \, dx = \int_{\Omega} p(x) \, dx = \int_{\Omega_c} p(x) \, dx. \quad (4.7) \]

4.1.3 The contact between two elastic bodies

Let us generalize the previous case to the contact of two elastic bodies, both of which can have a certain shape (e.g. a ball) and/or have a rough surface. As indicated in Fig. 4.2, both of them would experience the same contact pressure and the deformation will, therefore, have the same shape. If the material properties are different the magnitude of the deformation will, however, not be the equal. The total deformation is clearly the sum of the deformations of the contacting surfaces. Now, the only difference between the deformation of the individual
surfaces, is the material, which will show as a different scaling factor in front of the integrals in (4.4) and (4.2). By denoting the deformation of the two surfaces \( u_e \) and \( v_e \) we can, therefore, formulated the total deformation as

\[
u_e = u_e + v_e = \int_{\Omega} K(|x - x'|)p(x') \, dx',
\]

where

\[
K(|x - x'|) = \begin{cases} 
-2 \ln |x - x'| + C, & \text{in } 2D, \\
\frac{1}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}}, & \text{in } 3D,
\end{cases}
\]

Note that we have replaced the infinite integration limits to an integral over \( \Omega \) assuming that no positive pressure acts outside the domain. The parameter \( E^* \), often referred to as the reduced elastic modulus, is the defined as

\[
\frac{1}{E^*} = \frac{1 - \nu_i^2}{E_i} + \frac{1 - \nu_i^2}{E_i},
\]

where \( \nu_i \) and \( E_i \), \( i = 1, 2 \) denotes the material properties of the two contacting surfaces. Quite frequently one also find \( E'_i = 2E^* \), representing effective material properties. By comparing (4.8) to (4.4) and (4.2), we can see that the contact between any two elastic bodies is equivalent to the contact of an elastic body against a rigid one. We can, therefore, solve them in the same manner. Notice that this only true as long as the two assumptions presented in Section 4.1.2 hold.

We close this section by summarising the BEM formulation of the contact mechanics problem between two elastic bodies under the half-space theory assumptions. It reads,

\[
\begin{align*}
\quad & h(x) > 0, \quad p(x) = 0, \quad x \in \Omega_c, \\
\quad & h(x) = 0, \quad p(x) > 0, \quad x \notin \Omega_c, \\
\quad & h = g + u_e + \delta, \\
\quad & u_e = \int_{\Omega} K(|x - x'|)p(x') \, dx', \\
\quad & w = \int_{\Omega} p(x) \, dx
\end{align*}
\]

In this system the inputs are the total load \( w \), the reduced elastic modulus \( E^* \) and the initial gap \( g \). The dependent variables obtained upon solution of the system are then the pressure distribution \( p \), the elastic deformation \( u_e \), the deformed gap \( h \) and the rigid body movement \( \delta \).

### 4.2 Dimensionless formulation of the contact mechanics problem

In the previous section the BEM for the contact mechanics problem between two elastic bodies was formulated based on the half-space theory. Interpreting this system is not an
4.2. DIMENSIONLESS FORMULATION OF THE CONTACT MECHANICS PROBLEM

An easy task and here we introduce a scaling to transform it into dimensionless form. This may reduce the number of input parameters and thus facilitate numerical analysis and it will help us understand how the input parameters affect the solution. We start the process by introducing the following scaling

\[ X_1 = \frac{x_1}{x_{1r}}, \quad X_2 = \frac{x_2}{x_{2r}}, \quad H = \frac{h}{h_r}, \quad U_e = \frac{u_e}{h_r}, \quad G = \frac{g}{h_r}, \quad \bar{\delta} = \frac{\delta}{h_r}, \quad P = \frac{p}{p_r}. \] (4.12)

Notice that we have scaled all the variables regarding the gap with the same parameter \( h_r \). This is because they all share the same dimension, i.e., the dimension of the gap. Moreover, all of them can be expected to be of the same order of magnitude. Under the scaling proposed, the equations needed to solve the 3D contact mechanics problem, i.e., (4.11), become

\[
\begin{align*}
H(X_1, X_2) &> 0 \quad P(X_1, X_2) = 0, \quad (X_1, X_2) \in \Omega_c, \quad (4.13a) \\
H(X_1, X_2) &> 0 \quad P(X_1, X_2) = 0, \quad (X_1, X_2) \notin \Omega_c, \quad (4.13b) \\
H &= G + U_e + \bar{\delta}, \quad (4.13c) \\
U_e(X_1, X_2) &= \frac{x_r p_r}{h_r} \frac{1}{\pi E^*} \int_{\Omega} \frac{P(X'_1, X'_2)}{\sqrt{(X_1 - X'_1)^2 + \left(\frac{x_2r}{x_{1r}}\right)^2 (X_2 - X'_2)^2}} dX'_1 dX'_2, \quad (4.13d) \\
\frac{w}{p_r x_{1r} x_{2r}} &= \int_{\Omega} P(X_1, X_2) dX_1 dX_2 \quad (4.13e)
\end{align*}
\]

Now, we can freely chose the scaling parameters. We will, however obtain better results if we follow two principles, (1) to eliminate as many input parameters as possible and (2) to scale the non-dimensional variables so as to avoid truncation errors. A first obvious choice concerning the input parameters is to choose \( x_{1r} = x_{2r} = x_r \). Notice that, in most of the cases, both these dimensions are of similar size and thus we also preserve the scaling property. It can sometimes, however, be useful to define different scaling parameters for each direction, e.g. when studying finite EHL line contacts \[9, 10\]. We can also identify the two groups of parameters

\[ \frac{x_r p_r}{h_r} \frac{1}{\pi E^*}; \quad \text{and} \quad \frac{w}{p_r x_r^2}. \] (4.14)

that, for a given dimensionless initial gap \( G \), uniquely determine the solution of the problem, i.e., the contact pressure \( P \) and its distribution and the gap \( H \) between the deformed surfaces. A first option, suitable for arbitrary surface descriptions, is to set both of these parameters to 1. We note that there are now two groups and three reference parameters, of which two belong to the dependent variables \( p \) and \( h \) and the third scales the independent variable \( x \). We can thus choose one reference parameter freely. We can, for example, choose \( x_r = L \), where \( L \) is the size of the nominal contact area. This leads to

\[ p_r = \frac{w}{L^2}; \quad \text{and} \quad \frac{h_r}{L} = \frac{w}{L^2 \pi E^*}, \] (4.15)
which tells us that the scaling for the pressure is around the mean contact pressure and that the ration \( h_r/L \) is very small, as expected. In this case, the equations read

\[
\begin{align*}
H (X_1, X_2) &> 0 \quad P (X_1, X_2) = 0, \quad (X_1, X_2) \in \bar{\Omega}_c, \quad (4.16a) \\
H (X_1, X_2) &< 0 \quad P (X_1, X_2) > 0, \quad (X_1, X_2) \notin \bar{\Omega}_c, \quad (4.16b) \\
H &= G + U_e + \bar{\delta}, \quad (4.16c) \\
U_e (X_1, X_2) &= \int_{\bar{\Omega}} \frac{P (X'_1, X'_2)}{\sqrt{(X_1 - X'_1)^2 + (X_2 - X'_2)^2}} dX'_1 dX'_2, \quad (4.16d) \\
1 &= \int_{\bar{\Omega}} P (X_1, X_2) dX_1 dX_2 \quad (4.16e)
\end{align*}
\]

Notice that in the system posed by (4.16), the only input is introduced through the shape of the initial gap \( G \). Thus, we have extracted important knowledge even before having solved the set of equations. Let us see, for example, what happens when we keep \( w \) constant and stretch the surface, which would result in an increase of \( L \). We can see that if we double \( L \) while halving \( E^* \), \( h_r \) remains constant, which indicates that the deformation will also remain constant. This means that stretching the surface makes its response less stiff. Notice now that the topography of a rough surface, can be described as the sum of many sinusoidal waves, some having long wavelengths and some having shorter ones. We have now seen that the former will flatten easily whereas the latter will require a much larger load.

A very common application of the boundary element method (and one of its few analytical solutions) is that of the Hertzian contact problem. In two dimensions, this problem is simply the application of BEM to the contact of smooth balls. This is reflected in the initial gap, which is

\[
g_H = \frac{x_1^2 + x_2^2}{2Rh_r}, \quad (4.17)
\]

where \( R \) is the combined radius of the balls. In this case, we find another relevant group, namely \( x_r^2/2Rh_r \). This, of course, motivates choosing another scaling. In particular, the following is often chosen,

\[
\frac{x_r^2}{2Rh_r} = \frac{1}{2}, \quad \frac{w}{p_r x_r^2} = \frac{2\pi}{3} \quad \text{and} \quad \frac{x_r p_r}{h_r} \frac{1}{\pi E^*} = \frac{2}{\pi^2}. \quad (4.18)
\]

This leads to \( x_r = a \), where \( a \) is the Hertzian contact radius, \( h_r = a^2/R \) and \( p_r = p_H \), which is the Hertzian pressures. These are given as

\[
p_H = \frac{3w}{2\pi a^2}, \quad a^3 = \frac{3wR}{4E^*}. \quad (4.19)
\]
The equations then read

\begin{align}
H (X_1, X_2) > 0 & \quad P (X_1, X_2) = 0, \quad (X_1, X_2) \in \overline{\Omega_c}, \quad (4.20a) \\
H (X_1, X_2) = 0 & \quad P (X_1, X_2) > 0, \quad (X_1, X_2) \notin \overline{\Omega_c}, \quad (4.20b) \\
H = (X_1^2 + X_2^2) + U_e + \delta, & \quad (4.20c) \\
U_e (X_1, X_2) = \frac{2}{\pi^2} \int_{\Omega} \frac{P (X_1', X_2')}{\sqrt{(X_1 - X_1')^2 + (X_2 - X_2')^2}} dX_1' dX_2', & \quad (4.20d) \\
\frac{2\pi}{3} = \int_{\Omega} P (X_1, X_2) dX_1 dX_2 & \quad (4.20e)
\end{align}

Notice that, in this case, there are no input parameters. This means that there is only one fundamental solution. All contact problems between two balls can thus be seen as a scaling of this fundamental solution. This is also the case in the one-dimensional case, not shown here. Notice that we can also use the non-dimensional parameter groups to infer some relations about this scaling without actually solving the problem. For example, we see that, by doubling \( w \), \( p_H \) is also doubled while \( a \) is increased by a factor \( 2^{1/3} \). Similar relations can be found for all parameters.

Let us finish this discussion with a comment on the 2D contact (1D problem). The same analysis can be carried out, leading to similar conclusions (albeit the hertz parameters are slightly different). There is only one difficulty, concerning \([4,8]\) in the case of a 2D contact. The difficulty occurs when scaling inside the logarithm, which must be done with care. It is in fact related to the constant \( C \), that appears in the 2D contact but is not present in the 3D case. We shall now discuss it. If we include the force balance equation \([4,7]\), i.e.

\[
w = \int_{\Omega} p (x) dx \quad \Leftrightarrow \quad \frac{\int_{\Omega} p (x) dx}{w} = 1. \quad (4.21)
\]

into the reasoning we could write \([4,4]\) as

\[
u_e (x) = -\frac{2 (1 - \nu^2)}{\pi E} \int_{\Omega} p(x') \ln |x - x'| dx' + \frac{C}{w} \int_{\Omega} p(x') dx'
\]

\[
= -\frac{2 (1 - \nu^2)}{\pi E} \int_{\Omega} p(x') \left( \ln |x - x'| + \frac{\pi E}{2 (1 - \nu^2) w} C' \right) dx'
\]

\[
= -\frac{2 (1 - \nu^2)}{\pi E} \int_{\Omega} p(x') \ln \frac{|x - x'|}{d} d x', \quad (4.22)
\]

where \( d \) is a constant (in length units) related the datum of the deformation that satisfies the relation

\[
\ln d = -\frac{\pi E}{2 (1 - \nu^2) w} C. \quad (4.23)
\]

Thus, the dimensionless representation of this equation becomes

\[
U_e (X) = \frac{x_r p_r}{h_r} \frac{2}{\pi E} \int_{\overline{\Omega}} P (X') \ln \frac{|X - X'|}{D} dX'
\]
\[ u_e(X) = \frac{x_r p_r}{h_r} \int_{\Omega} p(X') \ln |X - X'| \, dX' + \frac{2 x_r p_r}{\pi E^*} \int_{\Omega} \ln D \, dX' \]

Let us now denote the second term by \( \bar{\delta}' \) and we have

\[ U_e(X) = \frac{x_r p_r}{h_r} \int_{\Omega} p(X') \ln |X - X'| \, dX' + \bar{\delta}' \] (4.25)

When introducing the non-dimensional deformation into the non-dimensional gap (4.20c), the value of \( \bar{\delta}' \) can be merged with \( \bar{\delta} \), i.e., \( \bar{\delta}' = \bar{\delta} + \bar{\delta}' \). We note that \( \bar{\delta}' \) is a dependent variable in this model which, although related to the rigid body movement, it is no longer equal to it due to the addition of \( \bar{\delta}' \). Lastly, we define \( U_e'(X) := U_e(X) - \bar{\delta}' \) to rewrite the dimensionless deformation equation as

\[ U_e'(X) = \frac{x_r p_r}{h_r} \int_{\Omega} p(X) \ln |X - X'| \, dX' \] (4.26)

### 4.3 Some solutions

In this section, we will give few examples of analytical solution to the contact problem in the context of the boundary element method. As apparent from the form of the equations to be solved, finding analytical solutions is no easy task and thus such solutions only exist for a few particular cases. Here we will first consider the famous theory by Hertz and then consider surfaces that have the shape of a simple sinusoidal wave, which are a conceptual model to understand the behaviour of rough surfaces. Other solutions do exist, usually for two-dimensional contact cases. The solutions are, however, quite complex and would not give the insights that the simpler cases we present here will give us. Therefore we will not consider them here.

#### 4.3.1 Hertz theory

The theory proposed by Hertz [11] is of the first successfully ones in the field of contact mechanics. We will now see that it is, in fact, a particular case of the more general boundary element method. The theory concerns dry non-conformal contacts of elastic bodies, in which the contact occurs in a very small area. These include the ball on ball contact in three dimensions and the cylinder on cylinder contact in two dimensions, but can, in fact, be applied to other non-conformal contacts as well. A key assumption for this theory is that the contact region is much smaller than the radius of curvature and than the bodies themselves, which is a very common characteristic of non-conformal contacts such as ball on ball. Notice, however, that it might not apply in rubber like materials where the deformations can be very large. Another assumption is that the curvature of the surfaces at the contact region is very small compared with the contact size. These two assumptions are equivalent to the
4.3. SOME SOLUTIONS

ones presented in Section 4.1.2 and thus allow us to apply the boundary element method to this problem. Moreover, the contact is assumed to be friction and adhesion free, so that only normal compressive pressures are considered and we can thus use the formulation presented in Section 4.1. With this in mind, let us describe the problem. A conceptual description, that will be useful during this discussion is depicted in Fig. 4.3.

We first consider the case of bodies of revolution, such as balls. Around the contact region, the geometry, \( z \), of the body can be described as

\[
z = \frac{x_1^2 + x_2^2}{2R} + \delta, \tag{4.27}
\]

where \( x_1, x_2 \) are the coordinates of the horizontal plane, \( R \) the radius of curvature and \( \delta \) a rigid body displacement. As depicted in Fig. 4.3a, the problem at hand is to compute the contact when two such bodies are pressed against each other with a load \( w \). The first thing to notice is that, as discussed in Section 4.1.3, the problem is equivalent to that of the contact between an equivalent ball and a flat. The radius of this ball, i.e., the equivalent radius, can be found by requiring the resulting initial gap is the same as the original case. It can then easily be seen that this radius is defined as

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{4.28}
\]

Focusing now on this problem, let us see how will the contact look like. It is clear, due to the symmetry of the bodies, that the contact region will be circular. Therefore, we can work in polar coordinates and use the radius \( r = \sqrt{x_1^2 + x_2^2} \) instead of the rectangular coordinates. Moreover the deformation is known at the contact region, i.e.,

\[
u_c(r) = \delta - \frac{1}{2R} r, \quad r \in \Omega_c, \tag{4.29}
\]
where delta is, again some rigid body displacement. This equation comes simply by requiring that the gap is zero at the contact region. The key insight that Hertz had was that this displacements are produced by a pressure of the form

\[
p(r) = p_H \sqrt{1 - \left(\frac{r}{a}\right)^2},
\]

where \(a\) is the radius of the contact region and \(p_H\) is the Hertzian pressure, which is also the maximum contact pressure. In order to match the displacements, the Hertzian pressure must have the value

\[
p_H = \frac{2E^*a}{\pi R}.
\]

The total load, \(w\), that would result in a given contact radius \(a\) can then be found by integrating (4.30) over the contact area, leading to a value of

\[
w = \frac{2}{3}p_H \pi a^2.
\]

Moreover, we can compute the approach between the two bodies as

\[
\delta_H = \frac{\pi a p_H}{2E^*}.
\]

Notice that we here consider the bodies to be originally touching at a single point and \(\delta_H\) measure how much more they approach due to the applied load. Summing up, we can give the relation between the different parameters in their typical form, i.e.,

\[
a = \left(\frac{3wR}{4E^*}\right)^{1/3},
\]

\[
p_H = \left(\frac{3w}{2\pi a^2}\right) = \left(\frac{6wE^2}{\pi^3 R^2}\right)^{1/3},
\]

\[
\delta_H = \frac{a^2}{R} = \left(\frac{9}{16 R E^2}\right)^{1/3}.
\]

Now, recall that we said in Section 4.2 that all Hertzian contact problems collapse to a single solution when considered in a dimensionless form. Let us see that this is, in fact, the case. Recall that the gap is scaled by a factor \(h_r = a^2/R\) whereas the other two dimensions are scaled by \(x_r = a\). Therefore, the initial gap becomes, in non-dimensional form,

\[
G = X_1^2 + X_2^2 + \delta,
\]

where no parameter is present. Similarly, by scaling the contact pressure with \(p_H\), we have

\[
P = \sqrt{1 - R^2},
\]

which, again, does not depend on any parameter. Notice that this equation can be written in the form

\[
P^2 + R^2 = 1.
\]
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This means that the general non-dimensional pressure solution for the Hertzian contact problem is simply as semi-sphere of unitary radius.

Let us now consider briefly the two-dimensional case representing the contact of infinitely long bodies such as cylinders. In this case, the contact will be on a line, symmetric with respect to the centre of the contact. The non-dimensional solution becomes, in this case a semi-circle of unitary radius. This solves the problem completely as any other case can be found via a scaling. Let us therefore solely give a summary of the most common relations in this case

\[
a = \left( \frac{4wR}{\pi E^*} \right)^{1/2},
\]

(4.38a)

\[
p_H = \frac{2w}{\pi a} = \left( \frac{wE^*}{\pi R} \right)^{1/2}.
\]

(4.38b)

Notice that, in this case, \( w \) has the units of force per unit length, (N/m). Notice also that, in this case, the rigid body movement, \( \delta_H \), caused by \( w \) cannot be specified. This is because of a reference point for the deformation cannot be specified, as reflected by the arbitrary constant left in (4.3).

Finally, let us give a comment for the three dimensional case in which the solids are not of revolution. Using appropriate axes, the geometry of these bodies can be described as

\[
z = \frac{x_1}{R'} + \frac{x_2}{R''} + \delta.
\]

(4.39)

Notice that now two radii, \( R' \) and \( R'' \) must be used to describe the surface. In this case the contact region will form an ellipse instead of a circumference. The solution to this problem, however, is not as simple as the case of bodies of revolution, where the cylindrical symmetry could be exploited. We will therefore not give the solution and simply refer the interested reader to, e.g., [8].

4.3.2 Westergaard solution

Another very important although less well known solution is that given by Westergaard [12] for the 2D contact of surfaces whose profile is characterized by a sinusoidal function. This solution was the first to provide a clear insight on the contact of rough surfaces. Without representing a realistic surface topography, the sinusoidal wave is the starting point to understand how roughness behaves and the solution will teach us how varying amplitude and frequency in the roughness will affect the contact behaviour. A representation of the problem and its solution is given in Fig. 4.4. A good description of this problem can also be found in Johnson’s book [8]. We will follow the latter in this description, instead of the more nuanced by also more cumbersome presentation of Westergaard. Let us set the stage by first considering the deformation caused by a sinusoidal pressure, i.e.,

\[
p_{\cos}(x) = p^* \cos \left( \frac{2\pi x}{\lambda} \right),
\]

(4.40)

where \( p^* \) is the amplitude of the pressure wave and \( \lambda \) its wavelength. The deformation caused by this pressure can be computed using (4.9), although the integration process is by no means
Figure 4.4: Representation of the problem posed by Westergaard. The original geometry, depicted at the top, consists of a surface with a sinusoidal waviness. If an average pressure $\bar{p} > p^*$ is applied, the waviness is fully flattened and the pressure is also sinusoidal. On the contrary, if $\bar{p} < p^*$, only partial flattening occurs. The surfaces are then only in contact in patches of size $a$ and a gap is left elsewhere.
trivial. The result is the following:

\[ u_{ecos} (x) = \frac{\lambda}{\pi E^*} p^* \cos \left( \frac{2\pi x}{\lambda} \right) \]  

(4.41)

Notice that this deformation has the same shape as the original pressure, scaled by a factor \( \lambda / (\pi E^*) \). Notice that this scaling factor also reflects what we found in Section 4.2, i.e., that a longer wavelength results in a less stiff surface. We can now use this result to say something about the deformation of a wavy surface. Let us assume that the initial gap between this surface and a flat one can be written as

\[ g_{\cos} (x) = \Delta \left( 1 - \cos \left( \frac{2\pi x}{\lambda} \right) \right), \]

(4.42)

where \( \Delta \) is the amplitude of the wavy surface. By comparing (4.42) with (4.41), we can clearly see that our surface will be flattened completely if \( p^* = \pi E^* \Delta / \lambda \). Of course, pressure must be non-negative, implying that the mean pressure must be greater than \( p^* \) for this to make sense. Therefore, a surface with the gap described in (4.42) will be completely flattened if pressed by a mean pressure \( \bar{p} > p^* \). Moreover, the pressure will have the following form

\[ p_{cos} (x) = \bar{p} + p^* \cos \left( \frac{2\pi x}{\lambda} \right), \quad p^* = \frac{\pi E^* \Delta}{\lambda}, \quad \bar{p} > p^*. \]  

(4.43)

Obviously, whenever \( \bar{p} < p^* \), the equation profile in (4.43) will include negative contact pressure, which is not physical. What will happen in reality is that there will be no full contact. The solution of the equations for BEM in this case can also be found analytically, albeit the process is even more complicated. The result, provided by Westergaard [12] is that when the surfaces are pressed with a mean pressure \( \bar{p} < p^* \), the pressure distribution can be expressed as

\[ p_W (x) = \frac{2\bar{p} \cos \left( \frac{\pi x}{\lambda} \right)}{\sin^2 \psi_a} \sqrt{\sin^2 \psi_a - \sin^2 \psi}, \quad 0 \leq |x| \leq a \]  

(4.44a)

\[ p_W (x) = 0, \quad a \leq |x| \leq \lambda / 2 \]  

(4.44b)

and the deformation as

\[ u_{eW} (x) = \frac{\bar{p} \lambda \cos \left( \frac{\pi x}{\lambda} \right)}{\pi E^* \sin^2 \psi_a} \cos 2\psi, \quad 0 \leq |x| \leq a \]  

(4.45a)

\[ u_{eW} (x) = \frac{\bar{p} \lambda}{\pi E^* \sin^2 \psi_a} \left[ \cos 2\psi + 2 \sin \psi Q - 2 \sin^2 \psi_a \ln \left( \frac{\sin \psi + Q}{\sin \psi_a} \right) \right], \quad a \leq |x| \leq \lambda / 2 \]  

(4.45b)

where \( a \) is the contact width, given by

\[ \frac{2a}{\lambda} = \frac{2}{\pi} \sin^{-1} \left( \frac{\bar{p}}{p^*} \right)^2. \]  

(4.46)

and

\[ Q = \sqrt{\sin^2 \psi - \sin^2 \psi_a}, \quad \psi = \frac{\pi x}{\lambda} \quad \text{and} \quad \psi_a = \frac{\pi a}{\lambda}. \]

Note that one has to be particularly careful with the sign of \( Q \). A schematic on how this solution looks like is depicted in Fig. 4.4b.
4.3.3 Flattening of bi-sinusoidal surfaces

Let us finish with the three dimensional version of the previous one. In this case, we consider an elastic surface described by a bi-sinusoidal function,

\[ z = \Delta \sin \left( 2\pi \frac{x_1}{\lambda_1} \right) \sin \left( 2\pi \frac{x_2}{\lambda_2} \right) + \delta, \]  

(4.47)

facing a flat, rigid surfaces, as depicted in Fig. 4.5. In (4.47), \( \lambda_1 \) and \( \lambda_2 \) are the wavelength in each direction and \( \Delta \) is the amplitude. This case is much harder to solve than the previous two-dimensional case and there is no analytical solution for the partial contact situation. There, only numerical work has been carried out. The interested reader is referred to [13] for a detailed analysis of the numerical solution of the partial contact. We shall here review the solution for the full contact, given by Johnson [14]. Unsurprisingly, the solution has the same form as for the two-dimensional case. Indeed, if mean pressure \( \bar{p} \) is sufficiently large to flat completely the surface, the pressure distribution will be

\[ p = \bar{p} + p^* \sin \left( 2\pi \frac{x_1}{\lambda_1} \right) \sin \left( 2\pi \frac{x_2}{\lambda_2} \right), \quad p^* = \sqrt{2\pi E^*} \frac{\Delta}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \]  

(4.48)

Obviously, this means that \( \bar{p} > p^* \) is the criteria to know weather the flattening is complete or not. Notice that \( p^* \) also has the same structure as in the two-dimensional case. This means that, also in two dimensions, surfaces with longer wave-lengths will be easier to flatten. In this case, however, we need to consider a combination of the wave-lengths in both directions. It is easy to see that if \( \lambda_1 = \lambda_2 \), then \( p^* \) is now equal to the one we obtained in the two-dimensional case.

4.4 Discretisation

The problem set in the previous sections cannot be solved analytically except for few very specific cases, such as the examples given in the previous section. In general, therefore, the
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problem must be solved numerically. For this, it first needs to be discretised. We shall do this in this section, taking one component at a time.

Before we can discretise the set of equations to be solved numerically, we need to specify the computational domain $\Omega$ which we will solve the set of equations on. Here we consider computational domains defined as

$$
\Omega := \begin{cases} 
[a_1, b_1], & \text{in 1D}, \\
[a_1, b_1] \times [a_2, b_2], & \text{in 2D}, 
\end{cases}
$$

(4.49)

where we need to remember that the 1D domain is related to the 2D contact problem and the 2D domain to the 3D contact. Since the apparent contact area is not known a priori, it is not always an easy task to choose the computational domain. One can, however, always use the nominal contact area as a starting point and then decrease it to better match the region required to obtain the wanted accuracy of the solution. It is clear, however, that one must seek a domain such that the pressure is zero at the boundaries (except for periodic domains), as otherwise one can expect the pressure to be non-zero outside the domain. Having the domain, we will start to discretise the different parts of the problem, starting by the domain itself. We will then follow with the gap, the pressure distribution, the equation for deformation, (4.8), and the load balance equation, (4.7). For simplicity we will usually use the 1D case when discretising, giving the specific formulation for the two-dimensional case whenever they are different.

4.4.1 Discretisation of the domain

The domain $\Omega$ can be simply discretised by setting

$$
x_i = a + i\Delta x_i, \quad i = 1, \ldots, N_1 \quad \text{in 1D},
$$

(4.50a)

$$
(x_{ii}, x_{jj}) = (a_1 + i\Delta x_1, a_2 + j\Delta x_2), \quad i = 1, \ldots, N_1, \quad j = 1, \ldots, N_2 \quad \text{in 2D},
$$

(4.50b)

where $\Delta x_i = L_i/(N_i - 1)$, $i = 1, 2$, where $L_i$ is the length of the domain in the $x_i$ direction and $N_i$ is the number of points used to discretise the domain in each direction. Furthermore, $(a_1, a_2)$ defines the South-West corner of the computational domain $\Omega$. Of course, this discretisation, in which $\Delta x_i$ is constant, is the simplest possible and more complicated ones could also be used. For instance [15] used a grid that became finer at the edge of the contact. However, since the contact region is not known a priori, this grid needs to adapt itself as the solution progresses. Therefore, it is much more complex to implement. Moreover, we will see in Section 4.6 that the uniform grid presented in (4.50) allows for using FFT techniques to significantly speed-up the computations.

4.4.2 Discretisation of the gap

The gap is a continuous function. However, we only know its value at points $x_i$, i.e., we know $g_i := g(x_i)$. Similarly the solution for the deformed gap will also be given only at these points, i.e., $h_i := h(x_i)$. As depicted in Fig. 4.6 we do not know how does the gap look like
and we can only assume a certain interpolation (e.g. linear) that will only be correct up to a certain accuracy. This is specially relevant in the case of rough surfaces, where one must be careful to use a sufficiently fine grid to capture the roughness correctly. Notice, for example, that eight points are needed to capture a period of a sinusoidal wave. Using less than that will heavily distort it. Notice, moreover, that even this might be too coarse to obtain good results when using the contact mechanics model. Therefore, it is critical to always perform a convergence test to ensure that the used discretisation is fine enough.

### 4.4.3 Discretisation of the pressure

When deriving (4.2) we described the pressure distribution as the sum of infinite point loads. This, of course, cannot be used in a computer. Therefore, we need to approximate it using the information in only finitely many points. For this, we will define \( p_i := p(x_i) \). In between two grid points, we do not have information of the pressure and thus we need to make an assumption regarding it. As depicted in Fig. 4.7, we assume that the pressure is constant around the point \( x_i \), i.e.,

\[
p(x) = p_i, \quad x \in \left[ x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2} \right], \quad i = 1, \ldots N_1.
\]  

(4.51)

Similarly, the pressure in the two-dimensional case is assumed to be constant in rectangles centred around the grid points \((x_{1i}, x_{2j})\).

### 4.4.4 Discretisation of the deformation equation

Having the pressure discretised, we are ready to formulate discrete equations for the elastic deformation. In the 2D contact (line loading) situation we have seen that due to the parameter \( d \), in (4.22), an additional constant \( \bar{\delta'} \) appeared during the non-dimensionalisation. To remove this complication, we formulated a non-dimensional equation (4.26), in which \( \bar{\delta'} \) is included in \( U'_e \), which we will consider here, albeit in dimensions. Now, since the pressure is
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![Diagram of discretisation of contact pressure]

**Figure 4.7:** Schematics of the discretisation of the contact pressure.

In the intervals defined in (4.51) and zero outside the domain, the integral in (4.26) can, in dimensional form, be computed by taking each interval at a time and summing them up. This means that an approximation can be obtained as

$$ u_e'(x_i) = \frac{2}{\pi E^*} \int_{\Omega} p(x) \ln |x_i - x| \, dx \approx -\frac{2}{\pi E^*} \sum_{k=0}^{N_1-1} p_k \int_{x_k - \Delta x/2}^{x_k + \Delta x/2} \ln |x_i - x| \, dx. $$

This expression can be rewritten in more compact notation by denoting the value of the integral in (4.52) by $C_{ik}$, i.e.

$$ u_e' = -\frac{2}{\pi E^*} \sum_{k=0}^{N_1-1} C_{ik} p_k. $$

For a given pair of points $x_i$ and $x_k$, the coefficient $C_{ik}$ can be interpreted as the deformation at the point $x_i$ caused by a constant unitary pressure applied on the interval with length $\Delta x$ centred at $x_k$. By virtue of the Cauchy principal value, it turns out that $C_{ik}$ can be computed analytically, leading to

$$ C_{ik} = \left( x_i - x_k + \frac{\Delta x}{2} \right) \left( \ln \left| x_i - x_k + \frac{\Delta x}{2} \right| - 1 \right) $$

$$ - \left( x_i - x_k - \frac{\Delta x}{2} \right) \left( \ln \left| x_i - x_k - \frac{\Delta x}{2} \right| - 1 \right). $$

For 3D contact, we obtain

$$ u_{ij} = -\frac{1}{\pi E^*} \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} C_{ijkl} p_{kl}. $$

by using the same reasoning, and we may interpret $C_{ijkl}$ as the deformation at a point $(x_{1i}, x_{2j})$ caused by a unitary pressure applied on a rectangle of size $\Delta x_1 \times \Delta x_2$ centred...
around the point \((x_{1k}, x_{2l})\). In [16], Love obtained a closed form expression for \(C_{ij00}\), i.e., the deformation at \((x_{1i}, x_{2j})\) caused by a unitary pressure applied on a rectangle of size \(\Delta x_1 \times \Delta x_2\) centred around the point \((x_{1k}, x_{2l}) = (0, 0)\). The result is

\[
\pi E^* C_{ij00} = (x_{2j} + \Delta x_2/2) \ln \left(\frac{(x_{1i} + \Delta x_1/2) + \sqrt{(x_{2j} + \Delta x_2/2)^2 + (x_{1i} + \Delta x_1/2)^2}}{(x_{1i} - \Delta x_1/2) + \sqrt{(x_{2j} + \Delta x_2/2)^2 + (x_{1i} - \Delta x_1/2)^2}}\right) \\
+ (x_{2j} - \Delta x_2/2) \ln \left(\frac{(x_{1i} - \Delta x_1/2) + \sqrt{(x_{2j} - \Delta x_2/2)^2 + (x_{1i} - \Delta x_1/2)^2}}{(x_{1i} + \Delta x_1/2) + \sqrt{(x_{2j} - \Delta x_2/2)^2 + (x_{1i} + \Delta x_1/2)^2}}\right) \\
+ (x_{1i} + \Delta x_1/2) \ln \left(\frac{(x_{2j} + \Delta x_2/2) + \sqrt{(x_{1i} + \Delta x_1/2)^2 + (x_{2j} + \Delta x_2/2)^2}}{(x_{2j} - \Delta x_2/2) + \sqrt{(x_{1i} + \Delta x_1/2)^2 + (x_{2j} - \Delta x_2/2)^2}}\right) \\
+ (x_{1i} - \Delta x_1/2) \ln \left(\frac{(x_{2j} - \Delta x_2/2) + \sqrt{(x_{1i} - \Delta x_1/2)^2 + (x_{2j} - \Delta x_2/2)^2}}{(x_{2j} + \Delta x_2/2) + \sqrt{(x_{1i} - \Delta x_1/2)^2 + (x_{2j} + \Delta x_2/2)^2}}\right). 
\]

(4.56)

Notice that \(C_{ijkl}\) can be found simply by translation, i.e., by replacing \(x_{1i}\) by \((x_{1i} - x_{1k})\) and \(x_{2j}\) by \((x_{2j} - x_{2l})\).

To conclude this section, let us discuss the shape of the matrix-vector representation of (4.54). We notice that \(u_{ei} = u_e(x_i)\) depends on the distance between the points \(x_i\) and \(x_k\), \(|x_i - x_k|\), rather than on both \(x_i\) and \(x_k\), individually. Since we use a regular grid to discretise our domain, \(C_{ik}\) depends on \(|i - k|\), rather than on both \(i\) and \(k\), individually. We can thus define a new coefficient \(K\) as

\[
K_r = C_{ik} \quad \forall i, k : |i - k| = r. 
\]

(4.57)

With this coefficient, we can see more clearly the structure of the matrix-vector representation of (4.54). As an example, for the case \(N = 4\) it looks like

\[
\begin{bmatrix}
u_0 \\ u_1 \\ u_2 \\ u_3
\end{bmatrix} =
\begin{bmatrix}
K_0 & K_1 & K_2 & K_3 \\
K_1 & K_0 & K_1 & K_2 \\
K_2 & K_1 & K_0 & K_1 \\
K_3 & K_2 & K_1 & K_0
\end{bmatrix}
\begin{bmatrix}
p_0 \\ p_1 \\ p_2 \\ p_3
\end{bmatrix}. 
\]

(4.58)

Clearly, the matrix has a particular symmetry, in which the value only depends on the distance from the main diagonal. This type of matrix is called a Toeplitz matrix. As we shall see in Section 4.6, this type of matrices is closely related to discrete convolutions. A similar reasoning could be used to describe the symmetries of the matrix for the two-dimensional case. We shall, however, not discuss the two-dimensional version as it becomes more cumbersome and, most importantly, we will see in Section 4.6 that we actually do not need it.

### 4.4.5 Discretisation of the load-balance equation

The last piece to discretise is the load balance equation (4.7). Again, we start from the discretisation of the pressure given in Section 4.4.3. The total load can thus be approximated
by simply adding the area of each rectangle in Fig. 4.7, i.e., in the one-dimensional case,

\[ w = \int p(x) \, dx \approx \Delta x \sum_{i=0}^{N-1} p_i. \]  

(4.59)

The extension to the two-dimensional case is trivial.

Notice that, by assuming that the pressure varies linearly between grid points, the approximation of the integral in (4.7) would be, following the trapezoidal rule,

\[ w = \int p(x) \, dx \approx \frac{\Delta x}{2} \sum_{i=1}^{N-1} (p_{i-1} + p_i) = \Delta x \sum_{i=0}^{N-1} p_i, \]  

(4.60)

which means that our approximation is of first order instead of the zeroth order approximation that would result from approximating the pressure as constant between two grid points.

### 4.5 Solving the CM problem as a complementarity problem. Lemke’s algorithm

In the previous section, the problem has been posed in a discrete manner. We have, however, not yet discussed how solve it. The solution is not trivial since neither the pressure nor the deformation are known a priory. In this section we thus present a method to obtain the solution. For this, let us write the full discrete problem in the following manner,

\[ h = Kp + g, \]  

(4.61a)

\[ h \cdot p = 0, \]  

(4.61b)

\[ h \geq 0, \quad p \geq 0. \]  

(4.61c)

This problem is, of course, very close to a discrete version of (4.11). The only major difference is that, in this case, the load \( w \) is not taken as an input. Instead, \( \delta \), which has been included in the initial gap \( g \), is now the input. One can thus see that \( g \) must now be negative at some points and that the resulting pressure will be such that the surfaces are pushed away to prevent penetration. With this clarification, it is clear that the first equation corresponds to (4.11c) and (4.11d) whereas the latter two correspond to (4.11a) and (4.11b).

The reason for writing (4.61) in this particular manner is that it has the form of a Linear Complementarity Problem. These types of problems are very common and therefore several algorithms to solve them are available. We can therefore make use of them to solve our contact mechanics problem. In this section, we will introduce one of them, i.e., Lemke’s algorithm. Note, however, that one can find other, more efficient algorithms in the literature.

Lemke’s algorithm is a pivoting type algorithm whose main advantage is that it reaches the exact solution after a finite number of steps. Its disadvantage, however, is that the number of pivots can be quite large and thus it can be slow. Moreover, we are forced to operate with the full matrix \( K \), which requires a lot of memory. A description and justification of Lemke’s algorithm can be found in \[17\]. We will here only present a short description of it.

To initialize the algorithm, we construct the a table in the form \( A = [I, -K, -1]g \), where \( I \) is the identity matrix and \(-1\) is a column vector with all values equal to \(-1\). An example
Table 4.1: Example of the initialization of Lemke’s algorithm for vectors $h$ and $p$ of length 2 and $K = [2 - 1; -11]$ and $g = [-3; 1]$

<table>
<thead>
<tr>
<th></th>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$p_0$</th>
<th>$p_1$</th>
<th>$\eta$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

of this table is given in Table 4.1. The variables to which the identity matrix correspond to are termed basic while the other ones are referred to as non-basic. At the initialization stage, all $h_i$ are basic while all $p_i$ are non-basic. We have also added an extra dummy variable $\eta$ to give us sufficient flexibility. Now, it is clear that when $g$ is positive we can simply set all the non-basic variables to zero and the problem is solved. If it is not, the key of the algorithm is to perform a change of base to change the position of the Identity matrix by pivoting until the resulting $g$ is positive. Note, however, that we must proceed not only until $g$ is positive but also until the variable $\eta$ is non-basic at the same time, since it is just a dummy variable. From the initial state, the algorithm consists of the following steps:

1. **Select the pivoting variable**: for the first pivoting, the chosen variable is $\eta$. After that complementarity is used to select it. If $p_i$ stopped being basic in the previous iteration, we select $h_i$ and vice-versa. This is done to ensure that for all pairs $h_i$ and $p_i$ one of them is basic and the other is non-basic.

2. **Select the pivoting row**: for this we compute the ratios between the values in the column under the pivoting variable and the column of $g$. We then select the row with the smallest, non-negative ratio. This is done to ensure that the column in $g$ remains positive after the first pivoting.

3. **Pivoting**: for all rows $R_k$, do a pivoting by setting $R_k^{\text{new}} = R_k^{\text{old}}a_{ij} - R_i a_{jk}$, where $j$ is the index for the pivoting variable, $i$ is the index of the pivoting row and $a_{kj}$ is the element of the matrix $A$ in the row $k$ and the column $j$. Notice that both $a$ and these indexes do not refer to the components of the vectors $p$ and $h$ but to the matrix $A = [I, -K, -1|g]$.

4. **Normalizing**: for all rows, divide by the number necessary to ensure that the basic variables have a one in their columns, to ensure that an identity matrix is present.

5. **Test the solution**: if the column of $g$ is composed of positive values and $\eta$ is non-basic, terminate. Otherwise, go back to point 1.

With this algorithm, a solution to (4.61) can be found. Then, the total load can be computed by integrating the pressure following the discretization in Section 4.4.5. Notice that, by shifting the initial gap $g$ up and down, the load can be decreased and increased.

### 4.6 Acceleration via FFT

When looking at the algorithm presented in Section 4.5 it is clear that the main burden comes from pivoting in such a large and full matrix. Although one can use other methods...
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\[ u_e(x) = \int_{-\infty}^{\infty} k(x - x') p(x') dx' \]

\[ F[k](\omega) = \frac{2}{E^* \omega} \]

\[ u_i = DF^{-1}[DF[k](\omega_i)DF[p_i]] \]

\[ DFT-CC method, large error \]

\[ DFT-DC method, preferred one \]

Figure 4.8: A representation of the different ways in which the Fourier transform and the Discrete Fourier Transform have been used to speed up the computation of the deformation.

which might be more efficient in terms of total number of iterations, the burden posed by the matrix \( K \) will still remain. Indeed, computing the deformation using the matrix-vector multiplication in (4.58) is going to be extremely expensive, as \( K \) is of size \( N^2 \) in one dimension and \( N_1^2 N_2^2 \) in two-dimensions. Moreover, it is a full matrix, meaning that it has no zeros that would allow for simplification. It is thus clear that this operation will require a lot of memory and computational power, especially for three-dimensional contacts. Therefore, before introducing a more efficient algorithm, as we will do in Section 4.7, we need to find a manner to compute the deformation in a more efficient way. For this, we can use the fact that \( K \) has a very particular structure, which can be used to significantly lower the requirements both in terms of memory and in terms of computational time. In this section, we will recognize the computation of deformation as a convolution, which will allow us to use the Fast Fourier Transform (FFT) to simplify the computations. To set the stage, however, let us begin discussing the continuous case, i.e., the continuous convolution. After this, we will discuss the discrete convolution, which will be more practical in use. A summary of the different paths that we will consider is given in Fig. 4.8. Throughout this chapter, we will consider only the 1D problem modelling 2D contacts. However, the 2D Fourier transform is obtained by simply applying the 1D Fourier Transform in each direction. Therefore, everything we consider here is directly applicable to the 3D contacts as well. A definition of the Fourier Transform and some of its properties can be found in the Appendix.

4.6.1 Continuous convolution method

As stated previously, the computation of the deformation can be written as

\[ u_e(x) = \int_{-\infty}^{\infty} K(x - x') p(x') dx'. \] (4.62)
By comparing it to (A.27), this equation can be identified to be the convolution between the kernel $K$ and the pressure $p$. A common way to interpret a convolution is to note that it becomes larger the more correlated the two functions are. Indeed, we can see that the more $K$ and $p$ overlap, i.e., the closer the point $x$ is to a peak of $p(x)$, the larger the deformation is. Having established the deformation as a convolution, we can make use of the Convolution Theorem, presented in Section [A.4]. This states that, since $u_e$ is computed as the convolution of $K$ and $p$, we can write

$$\mathcal{F}\{u_e\} = \mathcal{F}\{K\} \mathcal{F}\{p\},$$

(4.63)

where $\mathcal{F}\{\cdot\}$ indicates the Fourier transform. To make good use of (4.63), we need to find an expression for $\mathcal{F}\{K\}$. Although we could simply try to compute the Fourier transform of $K$ directly, we can also make use of result we presented in Section 4.3.2. It states that the deformation caused by a sinusoidal wave is also a sinusoidal wave. More precisely,

$$p(x) = a \sin(\omega x) \rightarrow u_e(x) = \frac{2}{E^* \omega} a \sin(\omega x),$$

(4.64)

where $\omega$ is the angular frequency. By comparing the pressure and the deformation, we can clearly see that

$$\mathcal{F}\{K\}(\omega) = \frac{2}{E^* \omega}.$$

(4.65)

With this, we can construct the first method to compute deformation, as represented by the blue path in Fig. 4.8. In this method, we first start by taking the Fourier transform of the pressure. Then, this is multiplied by $\mathcal{F}\{K\}$. Finally, the inverse Fourier transform can be used to recover the deformation. Of course, this method is not particularly useful as computing $\mathcal{F}\{p\}$ analytically is not a trivial task. Because of this, the method is of limited use. We can use it, however, to generalize the results obtained in Section 4.3.2 and Section 4.3.3 concerning the flattening of sinusoidal surfaces. Consider a one-dimensional profile, which can be expressed on the following form:

$$g = \bar{g} + \sum_{k=1}^{n} \Delta_i \sin(\omega_i x),$$

(4.66)

where $\bar{g}$ is the average initial gap. If we want to completely flatten this profile, we clearly need a deformation of the form

$$u_e = \delta - \sum_{k=1}^{n} \Delta_i \sin(\omega_i x).$$

(4.67)

Following (4.63), we can easily see, that the pressure needed for such purpose is

$$p = \bar{p} + \sum_{k=1}^{n} \frac{2\Delta_i}{E^* \omega} \sin(\omega_i x),$$

(4.68)

which follows from (4.64) and the linearity of the Fourier transform. Moreover, the pressure needed to completely flatten the profile is that required to ensure that the pressure in (4.68) is always non-negative, i.e.,

$$\bar{p} \geq \sum_{k=1}^{n} \frac{2\Delta_i}{E^* \omega}.$$ (4.69)

A similar result can be obtained for bi-sinusoidal surfaces.
4.6.2 DFT - Continuous convolution method

As we saw, the continuous convolution can facilitate the computation of deformation in some cases. Despite that, it is not always easy to obtain analytically the required Fourier transforms. A reasonable approach, taken, for example, by [18], is thus to perform them numerically, which leads to the Discrete Fourier Transform - Continuous convolution method (DFT-CC), depicted in red in Fig. 4.8. In essence, this approach applies the same treatment as we did to obtain (4.68) to general pressure profiles, which may not fully flatten the surface. For this, we start by taking the Discrete Fourier transform of the (discrete) pressure distribution for which we want to compute the deformation, leading to the following expression for the pressure distribution

\[ p_i := p(x_i) \approx \frac{a_0}{2} + \sum_{n=1}^{N/2} a_n \cos (\omega_n x_i) + \sum_{n=1}^{N/2} b_n \sin (\omega_n x_i) , \]  

(4.70)

where \( \omega_n \) is a discretisation of the continuous frequency \( \omega \), i.e.,

\[ \omega_n = n \Delta \omega, \quad \Delta \omega = \frac{2\pi}{L}, \quad n = 0, \ldots, N, \]  

(4.71)

where \( L \) is the length of the domain. The continuous convolution method presented in the previous section can now be applied to (4.70). Note that this method is equivalent to computing the deformation as

\[ u_e = \mathcal{D}^{-1} \left\{ \frac{2}{E^* \omega_n} \mathcal{D} \{ p \} \right\} , \]  

(4.72)

where \( \omega_n \) should be interpreted as a series presented in (4.71) and \( \mathcal{D} \{ \cdot \} \) and \( \mathcal{D}^{-1} \{ \cdot \} \) stand for the Discrete Fourier Transform and its inverse.

This method is nowadays out of use, since it introduces a larger error in the computation of \( u_e \). This error comes from the representation of the pressure in (4.70). Indeed, the method will only give accurate results as long as (4.70) is a good approximation to the pressure. This means that the frequency components of \( p \) must be well represented in intervals of size \( \Delta \omega \) and that frequencies above \( \omega_{N/2} \) are not contributing significantly. We can now ask ourselves whether this is a common situation. By realizing that (4.70) has the same form as (4.68), we can see that it corresponds to the complete flatting of a surface and the method would then be accurate in such a case. In other cases, however, the DFT-CC method can cause large errors. As indicated by Liu and Wang [19], the error is caused by aliasing. Indeed, the pressure distribution corresponding to a partial contact will have a sharp knee at the edge of the contact (see e.g. Fig. 4.3 and Fig. 4.4). Because of this, its Fourier transform will be composed of infinitely many frequency components, which do not decay rapidly. These components will then merge with the ones below \( \omega_{N/2} \) thus causing the error.

A way to increase the accuracy of the DFT-CC method is to reduce the effect of aliasing. For this, two options are available. The first one is to have a finer discretisation, which increases the amount of frequencies considered. The other is to increase \( L \), which reduces \( \Delta \omega \) and, again, increases the number of frequencies considered as long as \( \Delta x \) is kept constant. As depicted in Fig. 4.9 the convergence of this approaches is, however, quite poor, and thus
4.6.3 DFT - Discrete Circular Convolution method (periodic surfaces)

As we have seen in the previous section, the DFT-CC method is not very accurate. As discussed, aliasing is the responsible for this inaccuracy. Ultimately, the problem can be tracked to the mixing between the use of the Discrete Fourier Transform (DFT) to describe the pressure and the Continuous convolution theorem to analyse the deformation caused by this pressure. To avoid this issue, one should therefore consider directly the discrete version of the problem. The resulting method is known as the DFT-Discrete Convolution method.
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Figure 4.10: Representation of the operations to be performed for a convolution, (4.74) (left), and to compute the deformation, (4.73) (right). To compute, e.g., \( g_0 \), one needs to multiply the values of \( f \) with the values of \( h \) below and then add the results. The arrows indicate the change in \( h \) or \( k \) to advance one index in \( g \) or \( u_e \).

Remember that the discrete equation for deformation, derived in Section 4.4.4, is

\[
 u_n = -\sum_{k=0}^{N-1} C_{n,k} p_k = -\frac{2}{\pi E} \sum_{k=0}^{N-1} C_{|n-k|} p_k, \quad n = 0, 1, ..., N - 1. \quad (4.73)
\]

Now, this equation is not directly a convolution, although it is fairly close. As described in Section A.5, a circular convolution is defined for periodic functions and has the form

\[
 g_n = h \ast f = \sum_{i=0}^{N-1} h_{n-i} f_i \quad n = 0, 1, ..., N - 1, \quad (4.74)
\]

where the negative indexes in \( h \) must be understood through the periodicity of the function. The difference between this two operations is shown in Fig. 4.10. An important difference is, of course, that the functions in (4.74) are periodic whereas those in (4.73) are not. The study of the deformation of periodic surfaces is, however, relevant for the study of roughness. Let us therefore modify (4.73) to account for that. For this, we define the deformation as

\[
 u_n = -\sum_{k=0}^{N-1} K_{n-k} p_k, \quad n = 0, 1, ..., N - 1, \quad (4.75)
\]

where the \( K \) coefficients are defined as

\[
 K = \left[ C_0, C_1, ..., C_{N/2-1}, C_{N/2}, C_{N/2-1}, ..., C_1 \right]. \quad (4.76)
\]

A graphical interpretation of the meaning of this coefficients can be found in Fig. 4.11.
Figure 4.11: Representation of the computation of $u_{e_i}$. To compute it, one needs to multiply the pressure $p$ by the corresponding $K$ (or $C$) below and then add the results together. The arrows indicate the change in $K$ (or $C$) to advance one index in $u_{e_i}$. It is clear that $K$ behaves like a convolution while the index of $C$ reflects the distance to the index in $u_{e_i}$, accounting for periodicity. The dashed box marks one period.
that the only thing we have done is to compute the distance allowing for periodicity, which is nicely captured by our repeated and mirrored coefficients in (4.76). For example, the point $N - 1$ is actually at a distance 1 from the point 0 and thus the last coefficient in (4.76) is $C_1$. This new deformation equation is in the form of a circular convolution and both the pressure and the kernel $K$ are now considered periodic. Therefore, we can apply the discrete convolution theorem to write

$$\mathcal{D} \{u_e\} = \mathcal{D} \{K\} \mathcal{D} \{p\}.$$  \hspace{2cm} (4.77)

As shown in Section A.5 this relation is exact, i.e., it will give the same solution for $u_e$ as (4.75). This means that the only error will come from the discretisation itself, as shown in Fig. 4.12 for a slightly different case.

Without having to worry about errors introduced by the model, we can take full advantage of the better performance of this method as compared to the matrix-vector multiplication. Memory-wise we gain a lot since the kernel $K$ is represented only as a vector of size $N$, instead of a matrix of size $N^2$ (for the 2D contact). In computational terms, the cost to compute the deformation using the DFT-DC method comes from the DFT. Thankfully, there is an algorithm, called Fast Fourier transform (FFT) which has a complexity $N \log N$. This is much quicker than the complexity of the matrix-vector multiplication, which is of order $N^2$. Notice that the complexity must be understood as the rate at which the time increases as a function of $N$. The time for the DFT-DC is then proportional to $N \log N$ while the one for the matrix-vector multiplication is $N^2$. This means that for sufficiently large $N$, the DFT-DC method will always by much faster.

Before finishing, let us make it clear that the method as presented here is only valid to compute the deformation caused by a periodic pressure distribution. Therefore, the surfaces studied must be periodic as well. One might argue that there is no such thing as a periodic surface, which is true. However, if we want to study the effect of surface roughness, it is not a bad approximation to assume that the roughness is periodic. This approximation will work fine as long as the wavelength of the roughness is much smaller than the overall contact size. This situation is encountered in conformal contacts, which are found in many applications, e.g., journal bearings. Notice that this approximation is also made in the Westergaard solution, discussed in Section 4.3.2.

4.6.4 DFT - Discrete Linear Convolution method (non-periodic surfaces)

As we discussed in the previous section, the DFT-DC method is very advantageous in terms of memory and computational time. We therefore want to be able to use it for all contact problems, including the non-conformal contacts (e.g. the hertzian contact problem) in which the assumption of periodicity cannot be made. As discussed in Section A.5, it is possible to transform linear convolutions of non-periodic functions into circular convolutions of periodic functions by extending the domain. The same can be done for (4.73). In order to see how to do it, let us consider the computation of $u_{e0}$, $u_{e1}$ and $u_{eN/2-1}$ in Fig. 4.11. We notice that the coefficients of the pressures $p_0$ to $p_{N/2}$ are correct in our non-periodic computation. By correct we here mean that the index of the coefficient $C$ they are multiplied with matches
with $|i - k|$. It can easily be seen that this holds for all $u_{ei}$ with $i < N/2$. The problem arises when considering pressure points with index larger than $N/2$. For those points, the distance is too short. This would not be a problem, however, if all those pressures would be zero. In that case, the computation in Fig. 4.11 would be equal to the desired one. Following this thoughts, one can come up with the following strategy, presented, e.g., in [4],

1. Start with a discrete pressure vector $Np$ and a vector of coefficients $NC$, both of size $N$

2. Create a vector of coefficients of size $2N$ as $2N K = [C_0, C_1, ..., C_{N-1}, 0, C_{N-1}, ... , C_1]$

3. Create another pressure vector $2N p$, of size $2N$, via zero-padding, i.e., $2N p_i = N p_i$ for $i = 0, 1, ..., N - 1$ and $2N p_i = 0$ for $i = N, N + 1, ..., 2N - 1$

4. Compute the deformation $2N u_e = DF^{-1} \{ DF \{ 2N k \} DF \{ 2N p \} \}$.

5. The vector $2N u_e$ is now too long, but we only need to keep the initial values, i.e., $u_{ei} = 2N u_{ei}$ for $i = 0, 1, ..., N - 1$.

It is easy to see that this zero-padding procedure leads to a convolution which is exactly equal to (4.58). Therefore, we keep all the benefits of the DFT-DC method, even for non-periodic signals. The price to pay is that the domain needs to be doubled. With this doubling, however, we obtain an exact results and thus no further extension of the domain is ever needed. The accuracy of the method is depicted in Fig. 4.12 where the deformation caused by a Hertzian pressure is computed using the vector-multiplication method as well as the non-periodic DFT-DC method. Notice that even when an extremely coarse mesh is used, both methods coincide. Indeed, the DFT-DC method is numerically exact to the vector-multiplication method and the error only comes from the discretisation itself.

Figure 4.12: Comparison between the deformation caused by a Hertzian pressure (plot in black as a reference) as computed using the matrix-vector multiplication method and using the DC-FFT method. The computation is plotted for two discretisations, i.e., $N = 2^3$ and $N = 2^8$. 

![Graph showing comparison between deformation caused by Hertzian pressure using matrix-vector multiplication and DC-FFT methods.](image)
4.7 Solving the CM problem using a variational principle

Let us now see a more efficient way to solve the contact mechanics problem, as compared to Lemke’s algorithm (see Section 4.5), which exploits the acceleration via FFT we described in Section 4.6. By this, we reduce both the computational burden caused by the matrix-vector multiplication but also save memory, as the full matrix need not be stored. The method presented is based on a variational principle, which is an approach to mechanics in which we seek the solution through the minimization of a function. A typical example is to find the position of minimum total energy to identify the equilibrium position of a system. In our case we will use, instead, the variational complementary energy. This concept is similar to that of energy, but related to the principle of virtual work. A simple way to interpret this complementary energy is that if energy is the work done by a force while moving a certain distance, the complementary energy is done when a distance is moved under a certain force. It is more convenient to use the latter because, at the contact locations we actually known the distance moved (i.e., the deformation): it is equal to the initial gap. In the contact mechanics problem, the total complementary energy can be expressed as an integral, in which the complementary energy at each point is added. In particular, it has the form

\[ V^* = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{2} p u_e - pg \right] dx_1 dx_2. \tag{4.78} \]

The principle of minimum complementary energy guaranties us that \( V^* \) is minimal at the actual contact configuration. We can thus use this to solve the contact mechanics problem. To do so, we start by discretising (4.78). Using the discretisation presented in Section 4.4 and expressing the deformation as a function of pressure through (4.58), we can write

\[ V^* \approx \frac{1}{2} p^T K p - p^T g, \tag{4.79} \]

where \( p \) and \( g \) are vectors and \( K \) is the matrix in (4.58).

The minimization problem posed can be recognized as a quadratic optimisation problem, for which many algorithms exist. We shall here only focus on a simple one that will lead us to the solution. I simply consists on following the gradient of \( V^* \) towards the minimum. The gradient of \( V^* \) is

\[ \nabla V^* = K p - g = u_e - g. \tag{4.80} \]

Notice that we do not need to know the value of \( V^* \) but only its gradient. For this, we need to compute \( u_e \), which can be done through the fast DFT-DC technique presented in Section 4.6.4. A simple algorithm would then require to start with a given pressure distribution, e.g., a constant one. Then compute the deformation and \( \nabla V^* \) and then update the pressure until convergence is reached. The pressure is updated via

\[ p^{new} = p^{old} + r \nabla V^* = p^{old} + r (u_e - g), \tag{4.81} \]

where \( r \) is a relaxation factor. To define it, one could use several techniques, such as the conjugate gradient method used in [21] but one could also simply define it as a constant.
and then reduce it if convergence is too slow. Note that, while updating the pressure, we have another constraint, i.e., the total load \( w \) is also specified. We thus need to take it into account as well. For this, after every update on the pressure, this is modified to comply with the load balance constrain. This can be done by shifting the pressure up or down a certain value \( \alpha \). An overview of the overall algorithm resulting from these considerations is given in Fig. 4.13.

Let us conclude with a comment on the constraints imposed by the Kuhn-Tucker complementary conditions, (4.11a) and (4.11b). We have not mention them so far in the context of this solver. These are, however, implicitly enforced by the minimization of the complementary energy. Indeed, at its minimum, \( \nabla V^* \) is zero. Note, however, that this means that, for the points in contact, \( u_e = g \) and thus \( h = 0 \). We therefore only need to be careful to avoid negative pressures at each step of the iteration procedure.

### 4.8 Plastic deformation

It is clear that introducing plastic deformation in the context of the Boundary Element Method is not going to be easy. This is because we used very early the linear behaviour of elasticity and the resulting principle of superposition. Remember that we did that to describe the deformation of a given pressure distribution, given in (4.2), as the sum of the deformation caused by many point loads, given in (4.1). Plasticity is, however, not linear and thus does not allow for such convenient treatment. In this section we will, however, describe a simple method to include plastic deformation, which provides for a good first approximation to the
4.8. PLASTIC DEFORMATION

The idea behind the approach presented here is that, when the contact pressure at a given point exceeds a certain value, this point is allowed to flow freely towards the contact plane without any further increase in the pressure. This maximum allowed value for the pressure is defined as the hardness of the softer material, $H$, which can be computed as $H = 2.8 \sigma_Y$, where $\sigma_Y$ is the yield limit of the material \[8\]. The system defining the contact mechanics then becomes

\begin{align*}
  h(x) > 0, & \quad p(x) = 0, \quad x \in \Omega_c, \quad (4.82a) \\
  h(x) = 0, & \quad 0 < p(x) \leq H, \quad x \notin \Omega_c, \quad (4.82b) \\
  h = g + u_e + u_p + \delta, & \quad (4.82c) \\
  u_e = \int_{\Omega} K(|x - x'|)p(x', x') \, dx', & \quad (4.82d) \\
  W = \int_{\Omega} p(x) \, dx. & \quad (4.82e)
\end{align*}

In order to solve this problem, we discretise it in the same manner as the elastic case. Then, it can be solved by the variational principle method presented in Section 4.7, considering only the elastic deformation. Two modifications must be made. First, when updating the pressure, this must be truncated to ensure $p \leq H$. We must also consider that the points for which $p = H$ have a non-zero plastic deformation $u_p$. Therefore, $g \neq u_e$ and thus $\nabla V^* \neq 0$ even if they are in contact. Because of that, these points must be excluded from the convergence criterion. Besides these two changes, the algorithm to solve the problem is the same. The plastic deformation can be then computed once the solution is obtained. Indeed, the elastic deformation at points with $p = H$ will be too small to ensure that $h = 0$. Therefore, the plastic deformation $u_p$ can be computed from (4.82c) by enforcing $h = 0$. 
Chapter 5

The Lubricated Contact

In his celebrated paper \[22\], Reynolds presented an analysis of hydrodynamic flows in thin gaps. Examples of such are the gap between the rolling element and the raceway in a bearing, between the contacting surfaces in a seal, between the eye and the contact lens, in our joints and when a water film is generated between a car’s tyre and the road surface. Fig. 5.1 illustrates the typical schematics of a flow domain in lubrication, where \( h_0 \ll l_0 \) and \( h_0 \ll b_0 \).

![Figure 5.1: Schematic illustration of a thin gap - not to scale, between the two impermeable surfaces \( h_u \) and \( h_l \), moving relatively to each other with the velocities \( u_u \) and \( u_l \), respectively.](image)

The parameter \( h_0 \) is a reference measure of the gap between the surfaces \( h_u \) and \( h_l \) and \( l_0 \) and \( b_0 \) are reference measures for the size in the \( x \)- and \( y \)-directions, respectively. These relevant lengths, \( l_0 \) and \( b_0 \) are often chosen as the macroscopic dimensions of the machine elements. For instance, if studying a journal bearing, \( l_0 \) would be the journal circumference (= \( 2\pi R \)) and \( b_0 \) could be its width. Notice, however, similarly as we saw in Chapter 4, this condition also requires that the surface has small slopes. Otherwise one would be forced to choose \( l_0 \) and \( b_0 \) as the relevant wavelengths of the roughness, which would be close to \( h_0 \). Again, a factor 1:10, which is commonly encountered is sufficient. Where the thin film approximation
holds, i.e., where \(h_0 \ll l_0\) and \(h_0 \ll b_0\). Reynolds’ dimension reduced model for the flow in the thin gap between the impermeable surfaces, can normally be applied instead of the full set of the Navier-Stokes equations.

As we will soon see, the model resulting from applying the thin film approximation to the Navier-Stokes equations provides for a two-dimensional partial differential equation for the fluid pressure. The 3D velocity field can then easily obtained from its solution. In turn, this gives a clear computational advantage, as only two dimensions, instead of three, are to be considered and meshed. It is the main reason behind that the Reynolds equation is widely used and frequently found in the literature. In the majority of the available publications concerning lubrication, some of which are listed in the review article [23], the Reynolds equation is presented as

\[
\frac{\partial (\rho h)}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\rho h^3}{12 \mu} \frac{\partial p}{\partial x} - \frac{1}{2} \frac{\rho h}{\partial x} (u_u + u_l) \right) + \frac{\partial}{\partial y} \left( \frac{\rho h^3}{12 \mu} \frac{\partial p}{\partial y} - \frac{1}{2} \frac{\rho h}{\partial y} (v_u + v_l) \right),
\]

(5.1)

where \(h = h_u - h_l\) is the (thin) gap between the surfaces, \(p\) is the fluid pressure and \(\rho\) and \(\mu\) are the density and the viscosity of the fluid. Moreover, the velocities of the upper and the lower surfaces are \(u_u = (u_u, v_u, w_u)\) and \(u_l = (u_l, v_l, w_l)\), respectively.

Common examples of this use of Reynolds equation can be found in the field of hydrodynamic lubrication of thrust- and journal bearings, elastohydrodynamic lubrication of rolling element bearings, gears and cams, and leakage in seals, where (5.1) is often used together with semi-empirical expressions for density- and viscosity-pressure relationships. One must, however, take precaution before applying (5.1) to study the lubrication performance of the aforementioned devices. The reason for this, comes from the the fact that the 2D partial differential equation for the fluid pressure, \(p\), that Reynolds derived is based on the assumption that the fluid film is very thin, but also that the fluid was an oil, which he regarded as incompressible and also as “nearly iso-viscous”, to use his own wording. Moreover, he assumed that the flow was free from eddies and that the forces arising from weight and inertia were altogether small compared with the stresses arising from viscosity. The interested reader is referred to [24] for a mathematically rigorous approach to derive the time dependent Reynolds equation under these conditions. Now, please note that using in (5.1) with density and viscosity modeled as temperature and pressure dependent, would violate the aforementioned assumptions. This means that, although (5.1) is often a very good approximation, it is important to understand the limits of its applicability. In order to do so, one needs to understand how the dimension reduced Reynolds equation can be derived from the (more general) Navier-Stokes equations.

In the following section, a simplistic scaling and straight forward analysis will be applied to obtain a reduced form of the Navier-Stokes equations. More precisely, the reduced equations for the classical lubrication approximation, leading to the well-known Reynolds equation for incompressible and iso-viscous fluids will be presented. After that, the same approach will be applied without assuming that the fluid is incompressible, in order to investigate when is the use of Reynolds equation justified.
5.1 Navier-Stokes system of equations

The derivation of the Reynolds equation given herein, is an elaborate presentation of the scaling and asymptotic analysis of the Navier-Stokes equations for flows in thin gaps. This is quite commonly referred to as the thin film approximation. We start by introducing the continuity equation describing conservation of mass and the Navier-Stokes momentum equations, for compressible and viscous flow in 3D, i.e.,

\[ 0 = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z}, \]

\[ \rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} - 2 \frac{\partial}{3 \partial x} \left( \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + 2 \frac{\partial}{\partial x} \left( \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right), \]

\[ \rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} - 2 \frac{\partial}{3 \partial y} \left( \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right), \]

\[ \rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} - 2 \frac{\partial}{3 \partial z} \left( \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( \mu \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + 2 \frac{\partial}{\partial z} \left( \mu \frac{\partial w}{\partial z} \right), \]

where

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \]

This system of equations is defined on the 3D domain \( \Omega^3 \) confined between the surfaces \( h_u \) and \( h_l \), with \( \Omega = \{(x,y) : 0 < x < l_0 \land 0 < y < b_0 \} \) being its projection onto the xy-plane, see Fig. 5.1. The fluid pressure \( p \) and the fluid velocity \( u = (u,v,w) \) are the primary dependent variables and the time \( t \) and the spatial coordinates \( x, y \) and \( z \) are the independent variables. The acceleration \( g = (g_x,g_y,g_z) \) due to body forces, is input data. The density \( \rho \) and viscosity \( \mu \) are fluid properties which may or may not depend on the pressure or other variables. The precise constitutive relations between \( \rho \) and \( \mu \) and the other variables must be considered before the asymptotic analysis is carried out. These constitutive relations are typically semi-empricial expressions that are representative within a given parameter range. Other, important, input data to this problem are related to the domain \( \Omega^3 \). That is, the topography of the surfaces given by \( h_u \) and \( h_l \), the velocity of the surfaces \( u_u \) and \( u_l \), the spatial size \( l_0 \) and \( b_0 \) and boundary conditions for pressure and/or velocity at the \( xz- \) and \( yz- \) faces at \( x = 0 \land l_0 \) and \( y = 0 \land b_0 \).

The specific form of the Navier-Stokes equations presented in (5.2), is based on the Stokes’ assumption that the bulk viscosity is zero, which is an assumption that is justified for ideal gases and for monatomic liquids, see e.g. [25]. We highlight that, although a fluid (in general) exhibits both compressibility and nonlinear viscosity- pressure and temperature dependence, the analysis presented herein is restricted to iso-thermal conditions.
In the following we will use the fact that fluid film is very thin relative to the other dimensions, in order to derive simplified forms of (5.2). More precisely, this will be done by scaling and dimensional analysis.

5.1.1 Scaling of the Navier-Stokes equations

We will now cast the system into non-dimensional form, by employing a set of scaling parameters, similar to what was used in [26]. More precisely,

\[
X = \frac{x}{l_0}, \quad Y = \frac{y}{b_0}, \quad Z = \frac{z}{h_0}, \quad T = \frac{t}{t_0},
\]

\[
\bar{u} = \frac{u}{u_0}, \quad \bar{v} = \frac{v}{v_0}, \quad \bar{w} = \frac{w}{w_0},
\]

\[
\bar{\mu} = \frac{\mu}{\mu_0}, \quad \bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{p} = \frac{p}{p_0}.
\]

(5.3)

By means of (5.3) the non-dimensional form of (5.2a) becomes

\[
\frac{\rho_0}{l_0} \frac{\partial \bar{p}}{\partial T} + \rho_0 \frac{u_0}{l_0} \frac{\partial (\bar{\rho} \bar{u})}{\partial X} + \rho_0 \frac{v_0}{b_0} \frac{\partial (\bar{\rho} \bar{v})}{\partial Y} + \rho_0 \frac{w_0}{h_0} \frac{\partial (\bar{\rho} \bar{w})}{\partial Z} = 0.
\]

(5.4)

Let us stop here to clarify our purpose for performing this scaling, which is differ from the purpose we had when we applied the scaling to the contact mechanics problem in Chapter 4. Here, our objective is to derive a simplified system obeying the thin film approximation, which can be used to simulate the flow in narrow interfaces, such as the ones typically found in tribological contacts. A key aspect is therefore to ascertain that the scaling parameters are of the same order of magnitude as the scaled variables. For example, \( p_0 \) should be similar to the mean or maximum value of \( p \). To assure this, we require that the non-dimensional variables, e.g. \( \bar{p} \), are of \( O(1) \). This will enable an order of magnitude analysis, comparing the size of all the different terms in the equations. Let us now turn the attention to the continuity equation (5.4). In particular, in the last term in \( \partial (\bar{\rho} \bar{w})/\partial Z \) is of \( O(1) \) which implies that the whole term is of order \( \rho_0 w_0/h_0 \), provided that we have chosen representative values for \( \rho_0, w_0 \) and \( h_0 \). If this is the case, then we can directly compare this term with the others by considering only the factors composed of the scaling parameters. From this comparison, we can clearly see if there are terms which are much smaller than the others, and thus if we can safely neglect them and take a step in the direction towards obtaining a simplified system. A relevant difficulty is, indeed, to find the right scaling parameters. For some variables, this might be easy. For example, \( h_0 \) should be similar to the mean separation between the surfaces. Other parameters, mainly those related to dependent variables such as the pressure or the density, it is not to obvious which value should be taken. As we shall soon see, other arguments, more complex than the one we used to define \( h_0 \), can be used.

Coming back to (5.4), the current objective is to analyse the asymptotic behaviour of (5.2) as \( h_0/l_0 \) goes to zero. We therefore introduce the notation

\[
\varepsilon = \frac{h_0}{l_0}.
\]

(5.5)

In this notation, (5.4) becomes

\[
\frac{l_0}{t_0} \frac{\partial \bar{p}}{\partial T} + \frac{\varepsilon}{u_0} \frac{\partial (\bar{\rho} \bar{u}_\varepsilon)}{\partial X} + \frac{\varepsilon}{b_0 v_0} \frac{\partial (\bar{\rho} \bar{v}_\varepsilon)}{\partial Y} + \frac{w_0}{u_0} \frac{\partial (\bar{\rho} \bar{w}_\varepsilon)}{\partial Z} = 0.
\]

(5.6)
where the subscript $\varepsilon$ indicate dependent variables’ parameterisation in $\varepsilon$. At this point, it is realised that (5.6) have little if any meaning if not at least the three last terms are of the same order. In case not, $\bar{\rho}_\varepsilon \bar{w}_\varepsilon$ would not depend on $z$, which is unreasonable for most types of flow situations, e.g. for an incompressible fluid confined in a (narrow) converging gap. This is the motivation for the scaling $w_0 = \varepsilon u_0$ and without loss of generality we also chose

$$v_0 = u_0, \quad b_0 = l_0, \quad t_0 = l_0 / u_0.$$  \hspace{1cm} (5.7)

In this scaling, the continuity equation (5.6) become

$$\frac{\partial \bar{\rho}_\varepsilon}{\partial T} + \frac{\partial (\bar{\rho}_\varepsilon \bar{u}_\varepsilon)}{\partial X} + \frac{\partial (\bar{\rho}_\varepsilon \bar{v}_\varepsilon)}{\partial Y} + \frac{\partial (\bar{\rho}_\varepsilon \bar{w}_\varepsilon)}{\partial Z} = 0,$$  \hspace{1cm} (5.8)

and further we can simplify, (5.2b)-(5.2d) to

$$\varepsilon^2 \rho_0 u_0^2 \left[ \frac{\bar{\rho}_\varepsilon}{l_0} \left( \frac{\partial \bar{u}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Z} \right) \right] = \varepsilon^2 \rho_0 \bar{g}_x - \varepsilon^2 \rho_0 \frac{\partial \bar{\rho}_\varepsilon}{\partial X} + \varepsilon^2 \frac{2 \mu_0 u_0}{3 l_0^2} \frac{\partial}{\partial X} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial Y} + \frac{\partial \bar{v}_\varepsilon}{\partial X} \right) \right) + \frac{\mu_0 u_0}{l_0^2} \frac{\partial}{\partial Y} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right) \right),$$  \hspace{1cm} (5.9)

and

$$\varepsilon^2 \rho_0 u_0^2 \left[ \frac{\bar{\rho}_\varepsilon}{l_0} \left( \frac{\partial \bar{v}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Z} \right) \right] = \varepsilon^2 \rho_0 \bar{g}_y - \varepsilon^2 \rho_0 \frac{\partial \bar{\rho}_\varepsilon}{\partial Y} + \varepsilon^2 \frac{2 \mu_0 u_0}{3 l_0^2} \frac{\partial}{\partial Y} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} \right) \right) + \frac{\mu_0 u_0}{l_0^2} \frac{\partial}{\partial Y} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{w}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) \right),$$  \hspace{1cm} (5.10)

and

$$\varepsilon^2 \rho_0 u_0^2 \left[ \frac{\bar{\rho}_\varepsilon}{l_0} \left( \frac{\partial \bar{w}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right] = \varepsilon \rho_0 \bar{g}_z - \rho_0 \frac{\partial \bar{\rho}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{2 \mu_0 u_0}{3 l_0^2} \frac{\partial}{\partial Z} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right) + \frac{\mu_0 u_0}{l_0^2} \frac{\partial}{\partial Y} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) \right) + \frac{\mu_0 u_0}{l_0^2} \frac{\partial}{\partial Y} \left( \varepsilon \bar{\mu}_\varepsilon \left( \frac{\partial \bar{w}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) \right),$$  \hspace{1cm} (5.11)

Multiplication of (5.9) - (5.11) with

$$\frac{l_0^2}{\mu_0 u_0}$$
leads to

\[
\varepsilon^2 \rho_0 u_0 l_0 \bar{\rho}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \rho_0 l_0^2 \bar{\rho}_\varepsilon g_x - \varepsilon^2 p l_0 \frac{\partial \bar{p}_\varepsilon}{\partial X} + \\
-\varepsilon^2 \frac{2}{3} \frac{\partial}{\partial X} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right) + 2\varepsilon^2 \frac{\partial}{\partial X} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right) \right) + \\
+\varepsilon^2 \frac{\partial}{\partial Y} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial Y} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} \right) \right) + \frac{\partial}{\partial Z} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right),
\]

\[
\varepsilon^2 \rho_0 u_0 l_0 \bar{\rho}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \rho_0 l_0^2 \bar{\rho}_\varepsilon g_y - \varepsilon^2 p l_0 \frac{\partial \bar{p}_\varepsilon}{\partial Y} + \\
-\frac{2}{3} \varepsilon^2 \frac{\partial}{\partial X} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right) + \varepsilon^2 \frac{\partial}{\partial X} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right) \right) + \\
+2\varepsilon^2 \frac{\partial}{\partial Y} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial Y} \right) + \frac{\partial}{\partial Z} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) \right),
\]

and

\[
\varepsilon^2 \rho_0 u_0 l_0 \bar{\rho}_\varepsilon \left( \frac{\partial \bar{w}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) = \varepsilon \rho_0 l_0 \bar{\rho}_\varepsilon g_z - \frac{p l_0}{\mu_0 u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z} + \\
-\frac{2}{3} \frac{\partial}{\partial Z} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \right) + \frac{\partial}{\partial X} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right) \right) + \\
+\frac{\partial}{\partial Y} \left( \bar{\mu}_\varepsilon \left( \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) \right) + 2\frac{\partial}{\partial Z} \left( \bar{\mu}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right).
\]

We note the Reynolds number defined as

\[
\mathcal{R} := \frac{\rho_0 u_0 l_0}{\mu_0},
\]

appears in the left hand sides of \eqref{eq:5.12}-\eqref{eq:5.14}.

The importance of incorporating the precise constitutive relationship \( \mu(p) \) before carrying out the asymptotic analysis, rather than after, was highlighted in \[27\,29\]. A similar consideration can be made for the case of compressible fluids. The underlying reason for this is that otherwise an educated estimate for \( \rho_0 \) and \( \mu_0 \) cannot be made, since both \( \rho \) and \( \eta \) depend on the pressure. As an example of this, the derivation of a model for a gas lubricated system, need to incorporate the well-known linear constitutive relation between density and pressure for an ideal gas, i.e., \( \rho = kp \), where \( k \) is a constant, before the asymptotic analysis is carried out. In the following we will thus perform the asymptotic analysis for different cases. We will start with the original case studied by Reynolds, i.e., considering an incompressible and
iso-viscous fluid. Then, we will consider compressible gasses in general to show that it is not always possible to reach a reduced system. We will then provide the reduced system for two cases where it is possible, i.e., the ideal gas model and fluid for which density increases with pressure following a power law. We will show, in particular, that different compressibility relations (e.g., the ideal gas model) can render a different reduced system as \( \varepsilon \) goes to zero as compared to the incompressible and iso-viscous case.

### 5.1.2 Incompressible and iso-viscous flow

We hereby with incompressible mean that the density is constant and without loss of generality we denote \( \rho = \rho_a \). Similarly, by iso-viscous we mean that the viscosity is constant and we define \( \mu = \mu_a \). Clearly, in this case the density and viscosity are no longer dependent variables and we can choose \( \bar{\rho}_\varepsilon = 1 \) and \( \bar{\mu}_\varepsilon = 1 \) and that the continuity equation (5.8) reduces to

\[
\frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} = 0.
\]

(5.16)

Further, by using (5.16) it follows that (5.12)-(5.14) become

\[
\varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \left( \frac{\partial \bar{u}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \frac{\rho_a l_0^2}{\mu_a u_0} g_x - \varepsilon^2 \frac{p_0 l_0}{\mu_a u_0} \frac{\partial \bar{p}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{w}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{u}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{v}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X}.
\]

(5.17)

\[
\varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \left( \frac{\partial \bar{v}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \frac{\rho_a l_0^2}{\mu_a u_0} g_y - \varepsilon^2 \frac{p_0 l_0}{\mu_a u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Y} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{v}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{v}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{v}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \varepsilon^2 \frac{\partial \bar{v}_\varepsilon}{\partial Z}.
\]

(5.18)

and

\[
\varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \left( \frac{\partial \bar{w}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \frac{\rho_a l_0^2}{\mu_a u_0} g_z - \varepsilon^2 \frac{p_0 l_0}{\mu_a u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{w}_\varepsilon}{\partial X} + \varepsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} \frac{\partial \bar{w}_\varepsilon}{\partial Y} + 2 \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \frac{\partial \bar{w}_\varepsilon}{\partial Y} + 2 \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Z}.
\]

(5.19)

In the following we will investigate different ways of obtaining dimension reduced models for the flow of this particular type of fluid.

Without considering the physics of the flow that a reduced model would reflect, one could
just neglect the terms of order $\varepsilon$ and higher in (5.17)-(5.19). This leads to the system

\[
\frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} = 0, \tag{5.20a}
\]

\[
0 = \frac{\partial^2 \bar{u}_\varepsilon}{\partial Z^2}, \tag{5.20b}
\]

\[
0 = \frac{\partial^2 \bar{v}_\varepsilon}{\partial Z^2}, \tag{5.20c}
\]

\[
\frac{\partial^2 \bar{u}_\varepsilon}{\partial X \partial Z} + \frac{\partial^2 \bar{v}_\varepsilon}{\partial Y \partial Z} + 2 \frac{\partial^2 \bar{w}_\varepsilon}{\partial Z^2} = \frac{p_0 l_0}{\mu_a u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z}, \tag{5.20d}
\]

which after using (5.16) reads

\[
\frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} = 0, \tag{5.21a}
\]

\[
0 = \frac{\partial^2 \bar{u}_\varepsilon}{\partial Z^2}, \tag{5.21b}
\]

\[
0 = \frac{\partial^2 \bar{v}_\varepsilon}{\partial Z^2}, \tag{5.21c}
\]

\[
\frac{\partial^2 \bar{w}_\varepsilon}{\partial Z^2} = \frac{p_0 l_0}{\mu_a u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z}. \tag{5.21d}
\]

From (5.21b) and (5.21c) we see that this model implies that $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ varies linearly with $z$, as in the case with moving parallel plates, in other words, plain Couette type of flow. Of course, it is not the case that all thin film flows are Couette flows since this would not create any pressure built up and lubrication would not work. To obtain a more general case, we must realize that we do not know a priori the order of magnitude of $p_0$. In order to model more general flow situations the pressure have to scale with $\varepsilon$. Let us elaborate on this by assuming that $p_0 \propto \varepsilon^q$. If $q < -2$, then what would remain after neglecting terms of $O(\varepsilon)$ and higher, would be $\nabla p = 0$ which implies plain Poiseuille type of flow. This flow is characteristic of static surfaces and would thus also lead to unrealistic results in general. On the other hand, if $q > -2$, then only the first part of the last terms in (5.17) and (5.18) would remain and that, again, would imply plain Couette type of flow. This motivates the scaling $p_0 = \varepsilon^{-2} \mu_a u_0 / l_0$, which leads to that the system (5.17)-(5.19) is reduced to

\[
\frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} = 0, \tag{5.22a}
\]

\[
\frac{\partial \bar{p}_\varepsilon}{\partial X} = \frac{\partial^2 \bar{u}_\varepsilon}{\partial Z^2}, \tag{5.22b}
\]

\[
\frac{\partial \bar{p}_\varepsilon}{\partial Y} = \frac{\partial^2 \bar{v}_\varepsilon}{\partial Z^2}, \tag{5.22c}
\]

\[
\frac{\partial \bar{p}_\varepsilon}{\partial Z} = 0, \tag{5.22d}
\]

when terms of order $\varepsilon$ and higher are neglected. Note that this is actually the classical set of lubrication equations as obtained by Reynolds in his original derivation. In dimensions, it
5.1. NAVIER-STOKES SYSTEM OF EQUATIONS

reads

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \]  
(5.23a)

\[ \frac{\partial p}{\partial x} = \mu_a \frac{\partial^2 u}{\partial z^2}, \]  
(5.23b)

\[ \frac{\partial p}{\partial y} = \mu_a \frac{\partial^2 v}{\partial z^2}, \]  
(5.23c)

\[ \frac{\partial p}{\partial z} = 0. \]  
(5.23d)

This system can be integrated to lead to the 2D Reynolds equation, as we will see in Section 5.2. Note that the fact that \( p \) is constant across the film, which is an important ingredient for the integration, is not an assumption but is obtained in (5.23d) as a result of the asymptotic analysis.

Remember that one underlying assumption for arriving at (5.23) is that \( p \) scales with \( \varepsilon^{-2} \). Actually, the assumption of incompressibility also played a role as otherwise \( \rho_0 \) would have depended on \( \varepsilon \) as well, through the relation between pressure and density. It is also important to remember that even though the explicit time dependence has disappeared from the original set of equations, the dependent variables are still functions of time. For example, the domain may change with time as in the case with moving surfaces.

5.1.3 Compressible and iso-viscous flow

Let us now consider the case of compressible fluids. As we shall see, in this case, the Navier-Stokes equations do not, in general, permit the application of the same type asymptotic analysis that leads to the system (5.23) and ultimately, see Section 5.2, to Reynolds equation.

In this section we present a reduced form of the system (5.8), (5.12)-(5.14), governing the flow of compressible and iso-viscous fluids. That is, a class fluids exhibiting different constitutive equations for the density-pressure relationship and with constant viscosity. We start by considering a general density-pressure relationship and then we apply the same type of asymptotic analysis, that was carried out in Section 5.1.2 for the incompressible and iso-viscous case. We will see that we cannot arrive far unless a given model for compressibility is specified. We will then give the resulting reduced systems for the particular case of ideal gases, for which density and pressure are proportional, as well as other models in which density follows a power law.

An arbitrary density-pressure relationship

In Section 5.1.2 we learned that, in order to obtain a reduced system resembling the physics of a lubricated conjunction in a relevant way, it was necessary to introduce an \( \varepsilon \)-dependence of the pressure scaling parameter \( p_0 \). Note that the Reynolds number \( R \) contains scaling parameters for the density \( \rho \) and the viscosity \( \mu \). In the incompressible and iso-viscous case, both these are constants and the Reynolds number does not scale with \( \varepsilon \). Under the assumption of compressible flow the density is a dependent variable. In this case, the constitutive equation will, as we will see, determine the \( \varepsilon \)-dependence of the density.
Let us now introduce the following density-pressure relationship to describe the compressibility for a large class of fluids

\[ \rho = \rho_a f(p), \quad (5.24) \]

where \( f(p) \) is a positive and strictly increasing function and \( \rho_a \) is the density at the ambient pressure \( p_a \). In non-dimensional form this relationship can be stated as

\[ \bar{\rho} = \frac{\rho_a}{\rho_0} F(\bar{p}), \quad (5.25) \]

where \( F(\bar{p}) = f(p_0\bar{p}) \). For instance, an ideal gas satisfies \( f(p) \) with the function

\[ f(p) = \frac{M}{\rho_a RT} p, \quad (5.26) \]

where \( M \) is the molar mass in \( \text{kg/mol} \), \( T \) is the temperature in K and \( R = 8.314 \text{ J/(mol K)} \) is the universal gas constant. Another example is a fluid obeying the constant bulk-modulus type of compressibility, for which

\[ f(p) = e^{(p-p_a)/\beta}, \quad (5.27) \]

where \( \beta \) is the so-called bulk-modulus of the lubricant. We note that,

\[ \nabla \rho = \rho_a f'(p) \nabla p, \quad \frac{\partial \rho}{\partial t} = \rho_a f'(p) \frac{\partial p}{\partial t} \]

and in non-dimensional form we have

\[ \nabla \bar{\rho} = \frac{\rho_a}{\rho_0} F'(\bar{p}) \nabla \bar{p}, \quad \frac{\partial \bar{\rho}}{\partial T} = \frac{\rho_a}{\rho_0} F'(\bar{p}) \frac{\partial \bar{p}}{\partial T}. \]

Under the assumption that the fluid is iso-viscous and that its density-pressure relationship is given by \( (5.24) \), the dimensionless continuity equation \( (5.8) \) reads

\[ F'(\bar{p}_e) \frac{\partial \bar{p}_e}{\partial T} + F(\bar{p}_e) \left( \frac{\partial \bar{u}_e}{\partial X} + \frac{\partial \bar{v}_e}{\partial Y} + \frac{\partial \bar{w}_e}{\partial Z} \right) + F'(\bar{p}_e) \left( \bar{u}_e \frac{\partial \bar{p}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{p}_e}{\partial Y} + \bar{w}_e \frac{\partial \bar{p}_e}{\partial Z} \right) = 0, \quad (5.28) \]

where \( F'(\bar{p}_e) = dF(\bar{p}_e)/d\bar{p}_e \). Moreover, the momentum equations, i.e., \( (5.12)-(5.14) \) read

\[ \epsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} F(\bar{p}_e) \left( \frac{\partial \bar{u}_e}{\partial T} + \bar{u}_e \frac{\partial \bar{u}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{u}_e}{\partial Y} + \bar{w}_e \frac{\partial \bar{u}_e}{\partial Z} \right) = \epsilon^2 \frac{\rho_a l_0^2}{u_0 \mu_a} F(\bar{p}_e) g_x - \]

\[ \epsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} F(\bar{p}_e) \frac{\partial \bar{p}_e}{\partial X} - \epsilon^2 \frac{2}{3} \frac{\partial}{\partial X} \left( \frac{\partial \bar{u}_e}{\partial X} + \frac{\partial \bar{v}_e}{\partial Y} + \frac{\partial \bar{w}_e}{\partial Z} \right) + \]

\[ 2 \epsilon^2 \frac{\partial^2 \bar{u}_e}{\partial X^2} + \epsilon^2 \frac{\partial}{\partial Y} \left( \frac{\partial \bar{u}_e}{\partial Y} + \frac{\partial \bar{v}_e}{\partial Y} + \frac{\partial \bar{w}_e}{\partial Z} \right) + \]

\[ \epsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} F(\bar{p}_e) \left( \frac{\partial \bar{v}_e}{\partial T} + \bar{u}_e \frac{\partial \bar{v}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{v}_e}{\partial Y} + \bar{w}_e \frac{\partial \bar{v}_e}{\partial Z} \right) = \epsilon^2 \frac{\rho_a l_0^2}{u_0 \mu_a} F(\bar{p}_e) g_y - \]

\[ \epsilon^2 \frac{\rho_a u_0 l_0}{\mu_a} F(\bar{p}_e) \frac{\partial \bar{p}_e}{\partial Y} - \frac{2}{3} \epsilon^2 \frac{\partial}{\partial Y} \left( \frac{\partial \bar{u}_e}{\partial X} + \frac{\partial \bar{v}_e}{\partial Y} + \frac{\partial \bar{w}_e}{\partial Z} \right) + \]

\[ \epsilon^2 \frac{\partial}{\partial X} \left( \frac{\partial \bar{v}_e}{\partial X} + \frac{\partial \bar{w}_e}{\partial X} \right) + 2 \epsilon^2 \frac{\partial^2 \bar{v}_e}{\partial Y^2} + \epsilon^2 \frac{\partial}{\partial Z} \left( \frac{\partial \bar{v}_e}{\partial Z} + \frac{\partial \bar{w}_e}{\partial Z} \right) \]

\[ \epsilon^2 \frac{\partial}{\partial X} \left( \frac{\partial \bar{u}_e}{\partial X} + \frac{\partial \bar{v}_e}{\partial X} \right) + 2 \epsilon^2 \frac{\partial^2 \bar{u}_e}{\partial Y^2} + \epsilon^2 \frac{\partial}{\partial Z} \left( \frac{\partial \bar{u}_e}{\partial Z} + \frac{\partial \bar{v}_e}{\partial Z} \right). \]
and
\[ \varepsilon^2 \frac{\rho_0 u_0 l_0}{\mu_a} F(\bar{p}_\varepsilon) \left( \frac{\partial \bar{w}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) = \varepsilon \frac{\rho_0 l_0}{u_0 \mu_a} F(\bar{p}_\varepsilon) \mathcal{g}_z - \]
\[ \frac{p_0 l_0}{u_0 \mu_a} \frac{\partial \bar{p}_\varepsilon}{\partial Z} - 2 \frac{\partial}{\partial X} \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \]
\[ \frac{\partial}{\partial X} \left( \frac{\partial \bar{u}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right) + \frac{\partial}{\partial Y} \left( \frac{\partial \bar{v}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial Y} \right) + 2 \frac{\partial^2 \bar{w}_\varepsilon}{\partial Z^2}. \]  

By comparing the incompressible and iso-viscous set of equations (5.16)-(5.19) with the compressible and iso-viscous set of equations (5.28)-(5.31) we clearly see that the asymptotic behaviour is different, if not \( f \equiv 1 \). We further stress that we cannot proceed with the analysis without assuming a specific form of \( F(p) \).

**Ideal gas flow**

For the flow of an ideal gas, the constitutive density-pressure compressibility relationship is governed by the ideal gas law, viz.
\[ \rho = \frac{M}{RT} p, \]  

We also recall that from (5.25) this means that
\[ F(\bar{p}_\varepsilon) = \frac{p_0}{\mu_a RT} \bar{p}_\varepsilon. \]

Under these assumptions the continuity equation, i.e. (5.28), becomes
\[ \frac{\partial \bar{p}_\varepsilon}{\partial T} + \bar{p}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \bar{u}_\varepsilon \frac{\partial \bar{p}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{p}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{p}_\varepsilon}{\partial Z} = 0. \]  

Moreover, (5.29)-(5.31) becomes
\[ \varepsilon^2 \frac{u_0 l_0 p_0}{\mu_0} \frac{M}{RT} \bar{p}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Z} \right) = \varepsilon^2 \frac{l_0^2 p_0}{\mu_0 u_0} \frac{M}{RT} \bar{p}_\varepsilon \mathcal{g}_z - \]
\[ \varepsilon^2 \frac{u_0 l_0 p_0}{\mu_0} \frac{M}{RT} \bar{p}_\varepsilon \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \]
\[ 2 \varepsilon^2 \frac{\partial^2 \bar{u}_\varepsilon}{\partial X^2} + \varepsilon^2 \frac{\partial^2 \bar{u}_\varepsilon}{\partial Y^2} + \frac{\partial}{\partial Z} \left( \frac{\partial \bar{u}_\varepsilon}{\partial Z} + \varepsilon^2 \frac{\partial \bar{w}_\varepsilon}{\partial X} \right), \]
\[ \varepsilon^2 \frac{u_0 l_0 p_0}{\mu_0} M \frac{\partial \bar{v}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Z} = \varepsilon^2 \frac{l_0^2 p_0}{\mu_0 u_0} M \frac{\partial \bar{p}_\varepsilon}{\partial T} \bar{g}_y - \varepsilon^2 \frac{l_0 p_0}{\mu_0 u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Y} - \frac{2 \varepsilon}{3} \frac{\partial}{\partial Y} \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \varepsilon^2 \frac{l_0 p_0}{\mu_0 u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z} - \frac{2 \varepsilon}{3} \frac{\partial}{\partial Z} \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) \]

and

\[ \varepsilon^2 \frac{u_0 l_0 p_0}{\mu_0} M \frac{\partial \bar{w}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{w}_\varepsilon}{\partial Z} = \varepsilon \frac{l_0 p_0}{\mu_0 u_0} M \frac{\partial \bar{p}_\varepsilon}{\partial T} \bar{g}_z - \frac{2 \varepsilon}{3} \frac{\partial}{\partial Z} \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \varepsilon^2 \frac{l_0 p_0}{\mu_0 u_0} \frac{\partial \bar{p}_\varepsilon}{\partial Z} - \frac{2 \varepsilon}{3} \frac{\partial}{\partial Z} \left( \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \right) + \frac{2 \varepsilon^2}{3} \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial \bar{w}_\varepsilon}{\partial Z} \]

If we neglect all the terms of order \( \varepsilon \) and higher, then we will only capture the Couette flow situation and this does not change if we only neglect \( \varepsilon^2 \) and higher. Once again, we arrive at the conclusion that the scaling parameter \( p_0 \) must be a function of \( \varepsilon \), for the reduced model to be able to capture flow situations encountered in lubrication.

In view of (5.34)-(5.35) it is realised that, if the pressure is not scaled proportionally to \( \varepsilon^{-2} \), then the only remaining type of flow the reduced model would be able to capture is again of the Couette type. As before, we use the scaling \( p_0 = \varepsilon^{-2} \mu_0 u_0 / l_0 \), which after neglecting terms of order \( \varepsilon^2 \) and higher, reduces (5.34)-(5.35) to

\[ u_0^2 M \frac{\partial \bar{u}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial Z} = M \frac{\partial \bar{p}_\varepsilon}{\partial T} \bar{g}_x - \frac{\partial \bar{u}_\varepsilon}{\partial X} + \frac{\partial^2 \bar{u}_\varepsilon}{\partial Z^2} \]

and

\[ u_0^2 M \frac{\partial \bar{v}_\varepsilon}{\partial T} + \bar{u}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial X} + \bar{v}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \bar{w}_\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial Z} = M \frac{\partial \bar{p}_\varepsilon}{\partial T} \bar{g}_y - \frac{\partial \bar{v}_\varepsilon}{\partial Y} + \frac{\partial^2 \bar{v}_\varepsilon}{\partial Z^2} \]

and

\[ \frac{\partial \bar{p}_\varepsilon}{\partial Z} = 0 \]

As in the analysis in Section 5.1.2 for incompressible and iso-viscous fluids, the \( z \)-dependence of the pressure solution vanishes as \( \varepsilon \) goes to zero, although the full resulting system is not as simplistic. Summing up, we have derived the following lubrication approximation for an
ideal gas, i.e.,

\[
\frac{\partial \bar{p}_e}{\partial T} + \bar{u}_e \frac{\partial \bar{p}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{p}_e}{\partial Y} + \bar{w}_e \left( \frac{\partial \bar{u}_e}{\partial X} + \frac{\partial \bar{v}_e}{\partial Y} + \frac{\partial \bar{w}_e}{\partial Z} \right) = 0. \tag{5.40a}
\]

\[
u_0^2 \frac{M}{RT} \bar{p}_e \left( \frac{\partial \bar{u}_e}{\partial T} + \bar{u}_e \frac{\partial \bar{u}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{u}_e}{\partial Y} + \bar{w}_e \frac{\partial \bar{u}_e}{\partial Z} \right) = \frac{M}{RT} \bar{p}_e g_x - \frac{\partial \bar{p}_e}{\partial X} + \frac{\partial^2 \bar{u}_e}{\partial Z^2}, \tag{5.40b}
\]

\[
u_0^2 \frac{M}{RT} \bar{p}_e \left( \frac{\partial \bar{v}_e}{\partial T} + \bar{u}_e \frac{\partial \bar{v}_e}{\partial X} + \bar{v}_e \frac{\partial \bar{v}_e}{\partial Y} + \bar{w}_e \frac{\partial \bar{v}_e}{\partial Z} \right) = \frac{M}{RT} \bar{p}_e g_y - \frac{\partial \bar{p}_e}{\partial Y} + \frac{\partial^2 \bar{v}_e}{\partial Z^2}, \tag{5.40c}
\]

\[
\frac{\partial \bar{p}_e}{\partial Z} = 0, \tag{5.40d}
\]

or in dimensions

\[
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0. \tag{5.41a}
\]

\[
\frac{M}{RT} p \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \frac{M}{RT} pg_x - \frac{\partial p}{\partial x} + \mu_a \frac{\partial^2 u}{\partial z^2}, \tag{5.41b}
\]

\[
\frac{M}{RT} p \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \frac{M}{RT} pg_y - \frac{\partial p}{\partial y} + \mu_a \frac{\partial^2 v}{\partial z^2}, \tag{5.41c}
\]

\[
\frac{\partial p}{\partial z} = 0. \tag{5.41d}
\]

From the reduced model (5.41) it is clear that, the corresponding lubrication equations for iso-viscous flow of an ideal gas are not at the same as the system for an iso-viscous and incompressible fluid (5.23).

**Other types of compressibility**

Consider the class of iso-viscous fluids exhibiting compressibility of the type

\[
\rho = \rho_a \left( \frac{p}{p_a} \right)^{1-\alpha/2}, \tag{5.42}
\]

where \( p_a \) is the pressure for which \( \rho = \rho_a \). Then, from (5.24) we have that

\[
f(p) = \left( \frac{p}{p_a} \right)^{1-\alpha/2} \tag{5.43}
\]

and from (5.25) we have

\[
F(\bar{p}_e) = \left( \frac{p_0 \bar{p}_e}{p_a} \right)^{1-\alpha/2}. \tag{5.44}
\]

so that a modified Reynolds number, which quantifies the effect of inertia, can be expressed as

\[
\mathcal{R}^\alpha_{\varepsilon} := \varepsilon^2 \rho_a u_0 l_0 \frac{p_0 \bar{p}_e}{\mu_a} \left( \frac{p_0 \bar{p}_e}{p_a} \right)^{1-\alpha/2}. \tag{5.45}
\]
The characteristic behaviours correspond to three different cases, i.e. \( \alpha = 0, \alpha > 0 \) and \( \alpha < 0 \). The case \( \alpha = 0 \) corresponds to the ideal gas type of compressibility. Following the same analysis as done above, it can be seen that when \( \alpha > 0 \) it is possible to obtain a dimension reduced system. Indeed, for the system (5.29)-(5.31) to govern a more general flow situation, after neglecting terms of order \( \varepsilon \) and higher, than the plain Couette one, we must have that \( p_0 \propto \varepsilon^{-2} \). From (5.45) it is then clear that if \( p_0 \propto \varepsilon^{-2} \), then \( R_\varepsilon^\alpha \propto \varepsilon^\alpha \) and as long as \( \alpha > 0 \) the terms governing inertia may be neglected for small \( \varepsilon \). Thus this will lead to the Reynolds equation (5.1). An \( \alpha < 0 \) complicates the asymptotic analysis and we leave further investigations outside this work.

### 5.2 The Reynolds equation

We will now derive the classical Reynolds equation, which is a single equation for the pressure \( p \) in, for an incompressible and iso-viscous fluid. This derivation is found in Section 5.2.1. In Section 5.2.2, the case of compressible fluids will be considered. In particular, the Reynolds equation in the form in which is commonly found in the literature will be presented.

#### 5.2.1 Incompressible and iso-viscous fluids

The derivation of the classical Reynolds equation for incompressible and iso-viscous starts from the reduced form of the 3D Navier-Stokes system of equations (5.23), repeated and re-numbered as (5.46), viz.

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \quad (5.46a) \\
\frac{\partial p}{\partial x} &= \mu_a \frac{\partial^2 u}{\partial z^2}, \quad (5.46b) \\
\frac{\partial p}{\partial y} &= \mu_a \frac{\partial^2 v}{\partial z^2}, \quad (5.46c) \\
\frac{\partial p}{\partial z} &= 0, \quad (5.46d)
\end{align*}
\]

for the readers convenience. First we note that (5.46d) implies that \( p \) does not depend on \( z \), hence we may integrate (5.46b) and (5.46c) twice with respect to \( z \). This leads to

\[
\begin{align*}
u(x, y, z, t) &= 1 \left( \frac{\partial \partial z^2}{2 \mu_a} \right) + A(x, y, t)z + B(x, y, t), \quad (5.47a) \\
v(x, y, z, t) &= 1 \left( \frac{\partial \partial z^2}{2 \mu_a} \right) + C(x, y, t)z + D(x, y, t), \quad (5.47b)
\end{align*}
\]

where, the functions \( A, B, C \) and \( D \) may be found via boundary conditions on the surfaces. Indeed, let us consider the case when it can be assumed that the fluid stick to the surfaces, i.e. no-slip boundary conditions. This means that the velocity boundary condition at the lower surface is

\[
U_l = \begin{bmatrix} u(x, y, h_l(x, y, t), t) \\ v(x, y, h_l(x, y, t), t) \\ w(x, y, h_l(x, y, t), t) \end{bmatrix} = \begin{bmatrix} u_l(x, y, t) \\ v_l(x, y, t) \\ \frac{\partial h_l}{\partial t}(x, y, t) \end{bmatrix}, \quad (5.48)
\]
and at the upper surface it is

\[
U_u = \begin{bmatrix}
u(x, y, h_u(x, y, t), t) \\
v(x, y, h_u(x, y, t), t) \\
w(x, y, h_u(x, y, t), t)
\end{bmatrix} = \begin{bmatrix}
u(x, y, t) \\
v(x, y, t) \\
\frac{\partial h_u}{\partial t}(x, y, t)
\end{bmatrix},
\]

where \(u_l, v_l, u_u\) and \(v_u\) are explicitly given by the problem the model is applied to. Regarding the boundary conditions for \(w\), these depend indirectly on the changes in the gap geometry due to the movement \(u_l, v_l, u_u\) and \(v_u\) of the surfaces. However, they may also be related to an enforced squeeze motion. One example of where there’s a combination of these two actions can be found in the piston ring - cylinder liner lubrication situation. More precisely, as the piston moves between the dead centres it gives the ring a unidirectional sinusoidal motion to the piston ring and \(w\) changes mainly due to the geometry change induced by the sliding motion. However as the ring comes to a stop and then changes direction, at one of the dead centres in the cylinder, the ring tension and possible also the gas pressure forces the ring vertically against the cylinder. In a hydraulic cylinder, the fluid is squeezed between the piston head and the bottom of the cylinder and the boundary condition for \(w\) would be an explicit specification of the speed of the piston would be given as input to the model.

The derivation of the Reynolds equation now proceeds by addressing the boundary condition at the lower surface \((5.48)\), which together with \((5.47)\) gives

\[
u(x, y, h_l, t) = \frac{1}{2\mu_a} \frac{\partial p}{\partial x} h_l^2 + Ah_l + B = u_l,
\]

\[
v(x, y, h_l, t) = \frac{1}{2\mu_a} \frac{\partial p}{\partial y} h_l^2 + Ch_l + D = v_l
\]

and for the upper surface we obtain

\[
u(x, y, h_u, t) = \frac{1}{2\mu_a} \frac{\partial p}{\partial x} h_u^2 + Ah_u + B = u_u,
\]

\[
v(x, y, h_u, t) = \frac{1}{2\mu_a} \frac{\partial p}{\partial y} h_u^2 + Ch_u + D = v_u.
\]

It is straightforward to find \(A, B, C\) and \(D\) by solving the system \((5.50)-(5.51)\). Inserting the result into \((5.47)\) gives

\[
u = \begin{bmatrix}
u(x, y, z, t) \\
v(x, y, z, t)
\end{bmatrix} = \begin{bmatrix}
\frac{(z - h_l)(z - h_u)}{2\mu_a} \frac{\partial p}{\partial x} + \frac{(u_u - u_l)}{h_u - h_l} (z - h_l) + u_l \\
\frac{(z - h_l)(z - h_u)}{2\mu_a} \frac{\partial p}{\partial y} + \frac{(v_u - v_l)}{h_u - h_l} (z - h_l) + v_l
\end{bmatrix},
\]

The final step in the derivation is to integrate the continuity equation \((5.23a)\) from \(z = h_l\) to \(z = h_u\), i.e.

\[
\int_{h_l}^{h_u} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0.
\]
It follows from straight forward calculations that

\[
\int_{h_l}^{h_u} \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \left( -\frac{h^3}{12\mu_a} \frac{\partial p}{\partial x} + \frac{(u_u + u_l)h}{2} \right) \quad (5.54a)
\]
\[
\int_{h_l}^{h_u} \frac{\partial v}{\partial y} dz = \frac{\partial}{\partial y} \left( -\frac{h^3}{12\mu_a} \frac{\partial p}{\partial y} + \frac{(v_u + v_l)h}{2} \right) \quad (5.54b)
\]
\[
\int_{h_l}^{h_u} \frac{\partial w}{\partial z} dz = w(x,y,h_u,t) - w(x,y,h_l,t) = \frac{\partial h}{\partial t} \quad (5.54c)
\]

where \( h = h_u - h_l \). Indeed, we have obtained the well-known Reynolds equation for an incompressible and iso-viscous fluid;

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{h^3}{12\mu_a} \frac{\partial p}{\partial x} - \frac{(u_u + u_l)h}{2} \right) + \frac{\partial}{\partial y} \left( \frac{h^3}{12\mu_a} \frac{\partial p}{\partial y} - \frac{(v_u + v_l)h}{2} \right). \quad (5.55)
\]

In matrix form (5.55) become

\[
\frac{\partial h}{\partial t} = \nabla_x \cdot \left( \frac{h^3}{12\mu_a} \nabla_x p - \frac{u_s}{2} h \right), \quad (5.56)
\]

where

\[
\frac{u_s}{2} = \frac{1}{2} \begin{bmatrix} u_u + u_l \\ v_u + v_l \end{bmatrix}. \quad (5.57)
\]

For the sake of completeness, the polar form of (5.56) will be presented. The polar form can be used to simulate the film formation and pressure build-up in e.g. tilting pad thrust bearings such as the one depicted in 2.1. Indeed,

\[
\frac{\partial h}{\partial t} = \nabla_x \cdot \left( \frac{h^3}{12\mu_a} \begin{bmatrix} x_2 & 0 \\ 0 & 1/x_2 \end{bmatrix} \nabla_x p - \frac{\omega}{2} \begin{bmatrix} x_2 \\ 0 \end{bmatrix} h \right), \quad (5.58)
\]

where \( x_1 \) is the angular coordinate, \( x_2 \) the radial coordinate and \( \omega \) is the angular speed of the rotating surface. A schematic illustration of the pressure distribution over the segments of a tilting pad thrust bearing is visualised in Fig. 5.2 by means of a colour height map, where blue corresponds to low and yellow to high pressure. The pressure solution for one segment is obtained by solving the polar-coordinate form of the Reynolds (5.58), assuming that each of the six segments are identical. The film thickness \( h \) models the gap between the rotating collar and the inclined stationary surface of the bearing segment and \( \omega \) is the angular speed of the assumedly perfectly flat collar, which rotates in the counter clock wise direction. A corresponding FDM scheme is presented in Section 5.6.2.

The Reynolds equation represents continuity of flow and below it will be formulated in terms of the flow \( q \). For incompressible flow we have

\[
q(x,y) = \begin{bmatrix} q_x(x,y) \\ q_y(x,y) \end{bmatrix} = \frac{u_s}{2} h - \frac{h^3}{12\mu_a} \nabla_x p \quad (5.59)
\]

in Cartesian coordinates. Obviously, \( q \) represents volumetric flow, in units of \( \text{m}^3/\text{s} \). This means that the Reynolds equation (5.56) can be written as

\[
\frac{\partial h}{\partial t} + \nabla_x \cdot q = 0. \quad (5.60)
\]
5.2. THE REYNOLDS EQUATION

Figure 5.2: A schematic illustration of a polar coordinate system \((x_1, x_2)\) applied to model a tilting pad thrust bearing, such as the one in Fig. 2.1. Colour map depicts the pressure distribution from low = blue to high = yellow. The opposing (assumedly perfectly flat) collar rotates in the counter clockwise direction, with angular velocity \(\omega\).

5.2.2 Compressible and piezo-viscous fluids

In the literature, there are two common ways of deriving the Reynolds equation. One approach starts from the momentum equations directly and the density-pressure relationship is not considered during the asymptotic analysis. In the second approach, it is assumed that the inertial effects may be neglected even though the characteristic density and pressure are related, through an expression of the compressibility. Both these approaches leads to the same expression for the velocity field as in (5.52). After this, mass-conservation is considered. More precisely, the continuity equation (5.2a) is integrated with respect to \(z\) (across the fluid film, from \(h_l\) to \(h_u\)) viz.

\[
0 = \int_{h_l}^{h_u} \left( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) dz, \tag{5.61}
\]

where it is assumed that density is independent of \(z\), i.e. \(\rho = \rho(x, y)\). This leads to a Reynolds equation including a density which is not necessarily constant. Indeed, one can, in this way, a Reynolds’ type of equation (5.1) can be derived, implying that both the density and the viscosity do not need to be constants and thus may depend on the pressure. In matrix form, it can be written as

\[
\frac{\partial (\rho h)}{\partial t} = \nabla_x \left( \frac{\rho h^3}{12\mu} \nabla_x p - \frac{u_s}{2} \rho h \right). \tag{5.62}
\]

The Cartesian expression for the mass flow \(q\), with units kg/s, is in this case

\[
q(x, y) = \begin{bmatrix} q_x(x, y) \\ q_y(x, y) \end{bmatrix} = \frac{u_s}{2} \rho(p) h - \frac{\rho(p) h^3}{12\mu_a} \nabla_x p. \tag{5.63}
\]

This means that the Reynolds equation (5.62) can be written as

\[
\frac{\partial (\rho(p) h)}{\partial t} + \nabla_x \cdot q = 0. \tag{5.64}
\]
Note that in both the aforementioned approaches, the density is implicitly assumed to be constant. More precisely, both of them build on the fact that the characteristic density and pressure are NOT related. Strictly speaking this is inconsistent and, although in many situations it is of minor influence, it will have a major impact in some other cases, as will be discussed in Section 5.5. In addition, one finds various derivations where the viscosity-pressure relation is also disregarded during the asymptotic analysis of the momentum equations, which also leads to a similar inconsistency.

Note that in his original derivation [22], Reynolds considers the viscosity to be “nearly constant”. He also assumes that “the forces arising from weight and inertia are altogether small compared with the stresses arising from viscosity”. Under these assumptions, which are correct for an incompressible and iso-viscous fluid, all the terms involving the density in the momentum equations may be omitted and he arrives at (5.55). As we have seen previously, however, not all the compressibility models lead to a system such that it can be integrated to obtain a Reynolds type equation in the form of (5.62). We have seen, indeed, that this is only the case of sub-linear density-pressure relations. In the case of other models, such as the ideal gas flow, this is not possible in general. We will see, however, that even in this case a Reynolds equation in the form of (5.62) could be a good approximation provided that the velocity is low enough.

Although we have not discussed it in detail here, a similar caution must be had with piezoviscous fluids, i.e., those in which viscosity varies with pressure. An explicit example with pressure-dependent viscosity is given in [27], where they employ Barus’ viscosity-pressure relationship to elucidate upon the inconsistency in applying the Reynolds equation to elastohydrodynamically lubricated applications.

We conclude this section with a final remark. In the case with stationary surfaces, the non-linear equation (5.1) was studied in [30], for arbitrary density-pressure relationships (in fact piezo-viscosity was also considered). In particular, they obtained a transformation that maps (5.1) into a linear equation which can readily be solved numerically. This means that the solution to (5.1) can be easily obtained by inverse transformation.

### 5.2.3 Load carrying capacity and friction force

In the context of Tribology, there are two quantities of relevance that can be obtained from the result of Reynolds equation. These are the load the load carrying capacity (LCC) and the friction force/torque. In this section, expressions for these will be given. The LCC for a bearing is straight-forwardly computed from the fluid pressure distribution (given that consideration to hydrodynamic cavitation, to be considered in Section 5.7, has been taken). Given that the pressurised domain is $\Omega$, the load carrying capacity equals normal force $F_p$ supported by the fluid film and reads

$$F_p = \int_{\Omega} p(x) \, dx.$$  \hspace{1cm} (5.65)

The friction force $\mathbf{F}_f$ is a vector valued function computed from the shear stresses $\tau_x$ and $\tau_y$, which induce resistance against the motion the surface dragging the fluid into the lubricated conjunction has. For a Newtonian fluid, the constitutive relationships between shear stress
and shear strain rate reads

\[
\begin{bmatrix}
\tau_x \\
\tau_y
\end{bmatrix} = \mu \frac{\partial u}{\partial z} = \mu \frac{\partial}{\partial z} \begin{bmatrix} u \\ v \end{bmatrix}
\] (5.66)

and the friction force can be obtained by solving the corresponding Reynolds equation for the fluid pressure, then equating the pressure gradient so that the shear stresses can be obtained through (5.66). Indeed

\[
F_f = -\int_{\Omega} \mu \frac{\partial u}{\partial z} \bigg|_{z=h, z'=h} \, dx.
\] (5.67)

Let us now make a change of variables \( z' = z - h_t \), and introduce the notation \( h = h_u - h_t \), which if we considers iso-viscous flow leads to a simplified form of (5.67). That is,

\[
F_f = -\int_{\Omega} \mu a \frac{\partial u}{\partial z'} \bigg|_{z'=0, z'=h} \, dx = -\int_{\Omega} \mu a \left( \begin{bmatrix} z'(z' - h) \frac{\partial p}{\partial x} + (u_u - u_l) \frac{z'}{h} + u_t \\ \frac{z'(z' - h)}{2\mu_a} \frac{\partial p}{\partial y} + (v_u - v_l) \frac{z'}{h} + v_t \end{bmatrix} \right) \bigg|_{z'=0, z'=h} \, dx.
\] (5.68)

Evaluated at the lower surface \((z = 0)\) we have

\[
F_f(z = 0) = -\int_{\Omega} \left[ -\frac{h}{2} \frac{\partial p}{\partial x} + \frac{\mu_a(u_u - u_l)}{h} \right] \, dx
\] (5.69)

and evaluated at the upper surface \((z = h)\) we have

\[
F_f(z = h) = -\int_{\Omega} \left[ \frac{h}{2} \frac{\partial p}{\partial x} + \frac{\mu_a(u_u - u_l)}{h} \right] \, dx.
\] (5.70)

For a rotating device the frictional resistance is related to torque \( T_f \). The expression (5.71) is the friction torque evaluated at the rotating lower surface, assuming that it has angular velocity \( \omega \) and that the upper surface is stationary,

\[
T_f = -\int_{x_1} \left( -\frac{h(x_1, x_2)}{2x_1} \frac{\partial p}{\partial x_2} - \frac{\mu \omega x_1}{h(x_1, x_2)} \right) x_1 dx_1 dx_1,
\] (5.71)

where \( x_1 \) and \( x_2 \) are the radial and angular dimensions in polar coordinates.
5.2.4 Force balance and rigid body separation

In the real application, it is often the load that is applied \( f_{\text{app}} \) on a given system, e.g. a bearing. It is therefore convenient to treat is as the input instead of a given rigid body separation, \( h_0 \). In general, the thin gap between two surfaces can be expressed as

\[
h(x, y) = h_0 + h_1(x, y), \tag{5.72}
\]

where \( h_1 \) contains the shape of the surfaces and \( h_0 \) controls the average separation by shifting the shape up and down. In a case where the load is applied, the two surfaces will get closer or further away depending on the magnitude of the load applied. This calls, therefore, for an additional condition, i.e. a force equilibrium equation balancing the applied load \( f_{\text{app}} \) and the load carried by the fluid film \( f_p \). Quite generally, this is exactly Newton’s first law, applied in \( z \)-direction, viz.

\[
f_{\text{app}}(t) - \int p(x, y) \, dx \, dy = m \frac{d^2 h_0}{dt^2}, \tag{5.73}
\]

where it is understood that \( f_{\text{app}} \) is posed as load \((\text{N}/\text{m})\) in the 3D case and as load per length unit \((\text{N}/\text{m})\) in 2D cases. Moreover, \( m \) is the systems mass of inertia. The right hand side of (5.73) describes the acceleration of the surfaces towards or away each other. In the stationary case, this equation reads

\[
f_{\text{app}} - \int p(x, y) \, dx \, dy = 0. \tag{5.74}
\]

In this case, there is no explicit appearance of \( h_0 \). This parameter will, however, influence the pressure distribution. Therefore (5.74) will still determine the static separation between the surfaces through \( h_0 \).

5.3 Solution to the 1D Reynolds equation

In few cases, it is possible to obtain an explicit expression for the fluid pressure solution to the Reynolds equation. We will here consider one such case. In particular, a sliding bearing such as the one depicted in Fig. 5.3 will be considered. We here assume that the bearing is infinitely long and thus the edges of the bearing (in \( y \)-direction, perpendicular to the paper in Fig. 5.3) do not have any effect. Because of this, there is no dependency on \( y \) and the problem can be considered two-dimensional. Moreover, since the Reynolds equation reduces the dimensionality of the problem, we can study this bearing using the one-dimensional stationary form of the Reynolds equation (5.55). More precisely, a model bearing with a stationary upper surface is and the lower is moving with the speed \( u_l \), i.e., \( u(x, h) = 0 \) and \( u(x, 0) = u_l \). The mathematical description of the geometry of this bearing is done through the film thickness function \( h \) and it is given by

\[
h(x) = h_L - \frac{h_L - h_T}{l} x, \tag{5.75}
\]

where \( h_L \) and \( h_T \) are the leading and trailing edge gap heights and \( l \) the length of the bearing. We will introduce the inclination parameter \( k \), which is used to describe \( h_L \) as a scaling of
5.3. SOLUTION TO THE 1D REYNOLDS EQUATION

Figure 5.3: An infinitely wide linear slider bearing.

$h_T$, i.e., $h_L = (1 + k)h_T$. In terms of $h_T$ and $k$ (5.75) becomes

$$h(x) = h_T \left(1 + k - \frac{k}{l}x\right), \quad (5.76)$$

5.3.1 The pressure distribution for a "D linear slider bearing"

Let us start by computing the pressure distribution in the bearing. The one-dimensional stationary form of the Reynolds equation (5.55), for the slider in Fig. 5.3, can be written as

$$\frac{d}{dx} \left( \frac{h^3}{12\mu_a} \frac{dp}{dx} \right) = \frac{u_l}{2} \frac{dh}{dx}, \quad 0 \leq x \leq l, \quad (5.77)$$

with (pressure) boundary conditions

$$p(0) = p_T \quad \text{and} \quad p(l) = p_L.$$ \hspace{1cm} (5.78)

Note that if $p_L = p_T = p_a$, where $p_a$ is some ambient pressure, then the solution to (5.77) can be found by solving it with homogeneous Dirichlet boundary conditions $p_L = p_T = 0$ and then just adding $p_a$ to the resulting solution. In the following we will solve (5.77) for $p_L = p_T = 0$. Obviously, (5.77) can be integrated once, without making any other assumption on the geometry except that $h = h(x)$ is a continuous function. Indeed,

$$\frac{h^3}{12\mu_a} \frac{dp}{dx} = \frac{u_l}{2} h + C^*, \quad (5.79)$$

where $C^*$ is a constant of integration that we later on will use the boundary conditions (5.78) to determine. Since film thickness in reality is a strictly positive quantity, i.e., $h > 0$, we can divide with both sides of (5.79) with

$$\frac{h^3}{12\mu_a}$$
to obtain
\[ \frac{dp}{dx} = \frac{6\mu_a u_l}{h^2} + \frac{C}{h^3}. \] (5.80)

At this stage it is realised that consideration to the actual geometrical description need to be made. Since (5.75) is linear in \( x \) it permits integration using the following rule

\[ \int \frac{1}{(a + bx)^n} \, dx = \begin{cases} 
\frac{1}{b} \ln (a + bx), & n = 1, \\
-\frac{1}{b(n-1)} \frac{1}{(a + bx)^{n-1}}, & n > 1,
\end{cases} \] (5.81)

for constants \( a \) and \( b \) such that \( a + bx > 0 \). Using (5.81) and choosing \( a = h_T(1 + k) \) and \( b = -h_Tk/l \)

\[ p(x) = \frac{6\mu_a u_l}{kh_T} \frac{1}{h(x)^2} + \frac{l}{2kh_T h(x)^3} + D. \] (5.82)

From the boundary conditions, i.e. \( p_L = p_T = 0 \), it follows that

\[ C = -12h_T\mu_a u_l \frac{1 + k}{2 + k} \quad \text{and} \quad D = -\frac{6\mu_a u_l}{kh_T^2} \frac{1}{2 + k}. \] (5.83)

In turn this means that (5.82) becomes

\[ p(x) = \frac{6\mu_a u_l}{kh_T} \frac{1}{h(x)} - \frac{6\mu_a u_l}{k} \frac{1 + k}{2 + k} \frac{1}{h(x)^2} - \frac{6\mu_a u_l}{kh_T^2} \frac{1}{2 + k} = \]

\[ = \frac{6\mu_a u_l}{h_T^2} \left[ \frac{1}{k} \left( \frac{1}{1 + k - k \frac{x}{l}} - \frac{1 + k}{2 + k} \frac{1}{\left(1 + k - k \frac{x}{l}\right)^2} - \frac{1}{2 + k} \right) \right], \] (5.84)

which is the explicit presentation of the pressure solution to the 1D Reynolds equation for incompressible and iso-viscous flow that we set out to deduce. This way of expressing the solution divides it into the pressure scaling factor

\[ \frac{6\mu_a u_l}{h_T^2}, \] (5.85)

also known as the \textit{bearing number}; and

\[ \frac{1}{k} \left( \frac{1}{1 + k - k \frac{x}{l}} - \frac{1 + k}{2 + k} \frac{1}{\left(1 + k - k \frac{x}{l}\right)^2} - \frac{1}{2 + k} \right), \] (5.86)

being a rational expression describing the shape of the pressure distribution. Figure 5.4 depicts the shape of the pressure distribution, according to (5.86), for the value \( k = 1.2 \) of the inclination parameter.
5.3. **SOLUTION TO THE 1D REYNOLDS EQUATION**

![Image](image-url)

Figure 5.4: The shape of the pressure distribution according to (5.86), for the inclination parameter \( k = 1.2 \).

Having an explicit expression for the fluid pressure means that we can obtain an explicit expression for the velocity field from the two-dimensional representations of (5.52) and (5.46a). Adapted to the 2D slider bearing problem at hand, these can be formulated as

\[
\frac{d^2p}{dx^2} = 6\mu_a u_l \left( \frac{2}{k} \frac{1}{(1 + k - k \frac{x}{l})^3} - \frac{6}{k^2} \frac{1}{2 + k} \frac{1}{(1 + k - k \frac{x}{l})^4} \right)
\]

(5.90)
By inserting (5.89) into (5.87) we get

\[ u(x, z) = u_l \left( \frac{3z(z - h(x))}{h^2(x)} \left( 1 - \frac{1 + k}{2 + k h(x)} \right) - \frac{z}{h(x)} + 1 \right), \]  

(5.91)

after some algebraic operations. The velocity field for the 2D slider with inclination parameter \( k = 3 \) is depicted in Fig. 5.5. A few things to be noted here are i) It is quite clear that the velocity at the lower surface (moving in the \( x \)-direction) is \( u_l \) and that the upper surface is stationary \( (u(x, h) = 0) \), ii) There is a vortex at the leading edge side, where fluid is flowing out from the conjunction, i.e. \( u < 0 \) there, and iii) the velocity at the trailing edge side exceed \( u_l \). All of this is directly coupled to continuity of flow and for this problem the expression (5.59) for volume flow reads

\[ q(x) = u_l \frac{h}{2} - \frac{h^3}{12 \mu_a} \frac{dp}{dx}. \]

(5.92)

To obtain \( w \) from (5.88), we will first find the partial derivative \( \partial u/\partial x \) and to do this we restart from (5.91). This means that

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( u_l \left( \frac{3z(z - h(x))}{h^2(x)} \left( 1 - \frac{1 + k}{2 + k h(x)} \right) - \frac{z}{h(x)} + 1 \right) \right) =
\]

\[
= u_l \frac{\partial}{\partial x} \left( \frac{3z^2}{h^2(x)} - \frac{1 + k}{2 + k h^3(x)} + \frac{3z}{h(x)} + \frac{1 + k}{2 + k h^2(x)} - \frac{z}{h(x)} + 1 \right) =
\]

\[
= u_l \frac{kh_T}{l} \left( \frac{2z}{h^2(x)} - 2 \left( 3z^2 - 6zh_T \frac{1 + k}{2 + k} \right) \frac{1}{h^3(x)} - 18z^2 \frac{1 + k}{2 + k h^4(x)} \right) =
\]

\[
= u_l \frac{kh_T}{l} \frac{1}{h^2(x)} \left( 2z - 2 \left( 3z^2 - 6zh_T \frac{1 + k}{2 + k} \right) \frac{1}{h(x)} - 18 \frac{1 + k}{2 + k h^2(x)} \right).
\]
It is now possible to obtain an explicit expression for \( w \), by integrating (5.88) once with respect to \( z \). That is,

\[
w(x, z) = -u_l \frac{kh_T}{l} \frac{z^2}{h^2(x)} \left( 2 - 2 \left( z - 3h_T \frac{1 + k}{2 + k} \right) \frac{1}{h(x)} - 6 \frac{1 + k}{2 + k} \frac{zh_T}{h^2(x)} \right) + C_w(x). \]

For impermeable surfaces the boundary conditions \( w(x, 0) = 0 \) and \( w(x, h(x)) = 0 \) must hold. For the former to hold \( C_w \equiv 0 \). Thus

\[
w(x, z) = -u_l \frac{kh_T}{l} \frac{z^2}{h^2(x)} \left( 2 - 2 \left( z - 3h_T \frac{1 + k}{2 + k} \right) \frac{1}{h(x)} - 6 \frac{1 + k}{2 + k} \frac{zh_T}{h^2(x)} \right), \quad (5.93)
\]

from which we also see that the boundary condition at the upper surface, i.e. \( w(x, h(x)) = 0 \), holds. This vertical velocity component is depicted in Fig. 5.6, for the inclination parameter \( k = 3 \). Note here that the velocity \( w \) in the \( z \)-direction, in dimensions, is significantly smaller

Figure 5.6: The velocity field \( w/(uh_T/l) \) according to (5.93), for the inclination parameter \( k = 3 \). (\( h_T/l \) than than the velocity \( u \) in the \( x \)-direction.

### 5.3.2 LCC for the 2D linear slider bearing

For the 2D slider bearing the LCC can be presented as an explicit expression, i.e.

\[
f_p = \int_0^l p(x) \, dx = \frac{6\mu_a u_l}{h_T^2} \int_0^l \frac{1}{k} \frac{1}{1 + k - k \frac{x}{l}} - \frac{11 + k}{k^2 + k} \left( \frac{1}{2 + k} \left( 1 + k - k \frac{x}{l} \right)^2 - \frac{1}{2 + k} \right) \, dx =
\]
\[ \frac{f_p}{(6\mu_au_l^2/h_T^2)} = \frac{6\mu_au_l}{h_T^2} \int_0^l \left( -\frac{1}{k^2} \ln \left( 1 + k - k \frac{x}{l} \right) + \frac{1}{k^2} \ln \left( 1 + k - k \frac{x}{l} \right) \right) dx. \]

The friction force for the 2D linear slider bearing

For the 2D slider bearing, the lower surface is moving with velocity \( u_l \) and the upper surface is stationary. This means that we should evaluate the friction force as the force required to keep the lower surface in motion with velocity \( u_l \). As discussed in Section 5.2.3, this corresponds to the integral of the shear stresses exerted by fluid on the lower surface \( (z = 0) \) and in this particular case (5.69) becomes

\[ f_f(0) = -\int_0^l \frac{h(x)}{2} \frac{dp}{dx} - \mu_au_l \frac{h(x)}{h_T} \]  

Note that the \( f_f \) is force per length unit, i.e., it has the unit N/m. Inserting the explicit expression for \( \frac{dp}{dx} \) given by (5.89) means that

\[ f_f(0) = \int_0^l \frac{h_T}{2} \left( \frac{1}{1 + k - k \frac{x}{l}} \right)^2 - \frac{1 + k}{2 + k} \left( \frac{1}{1 + k - k \frac{x}{l}} \right)^3 \]  

Figure 5.7 depicts the dimensionless load carrying capacity, i.e., \( f_p/(6\mu_au_l^2/h_T^2) \) as a function of the inclination parameter \( k \). From this graph it is clear that there is an optimum bearing design, which occurs for \( k \approx 1.2 \).
5.3. SOLUTION TO THE 1D REYNOLDS EQUATION

\[ + \frac{\mu_a u_l}{h_T} \int_0^l \frac{1}{1 + k - \frac{x^2}{l}} \, dx = \]

\[ = f_f(0)_{\text{press}} + f_f(0)_{\text{shear}}, \]

where

\[ f_f(0)_{\text{press}} = \frac{3 \mu_a u_l}{h_T} \int_0^l \frac{1}{1 + k - \frac{x^2}{l}} \, dx - \frac{6 \mu_a U}{h_T} \frac{1 + k}{2 + k} \int_0^l \frac{1}{1 + k - \frac{x^2}{l}}^2 \, dx \]  

(5.96)

and

\[ f_f(0)_{\text{shear}} = \frac{\mu_a u_l}{h_T} \int_0^l \frac{1}{1 + k - \frac{x^2}{l}} \, dx. \]  

(5.97)

We note here that the first and the third term are of the same type, although the first originates from the pressure driven flow and the third from the shear driven flow. The pressure driven contribution is often omitted when estimating friction in lubricated contacts, from this expression it is not so clear that this will give a good estimate, although “experience” tells us so.

Proceeding with the derivation of the friction force from (5.96) and (5.97) we get

\[ f_f(0)_{\text{press}} = \frac{\mu_a u_l}{k h_T} \left( -3 \ln \left( 1 + k - \frac{x^2}{l} \right) \bigg|_0^l - \frac{6}{2 + k} \frac{(1 + k)}{1 + k - \frac{x^2}{l}} \bigg|_0^l \right) = \]

\[ = \frac{\mu_a u_l}{h_T} \left( \frac{3}{k} \ln (1 + k) - \frac{6}{2 + k} \right). \]  

(5.98)

and

\[ f_f(0)_{\text{shear}} = \frac{\mu_a u_l}{k h_T} \left( - \ln \left( 1 + k - \frac{x^2}{l} \right) \bigg|_0^l \right) = \]

\[ = \frac{\mu_a u_l}{h_T} \left( \frac{1}{k} \ln (1 + k) \right). \]  

(5.99)

Thus,

\[ f_f(0) = f_f(0)_{\text{press}} + f_f(0)_{\text{shear}} = \frac{\mu_a u_l}{h_T} \left( \frac{4}{k} \ln (1 + k) - \frac{6}{2 + k} \right). \]  

(5.100)

In the same manner as we obtained the friction force, we can obtain a closed form expression for the resulting force (per unit length) caused by the lubricant shear stresses acting at the upper surface by computing the expression (5.101) below. That is,

\[ f_f(h) = - \int_0^l h(x) \frac{dp}{dx} \frac{\mu_a u_l}{h(x)} \, dx. \]  

(5.101)

We again partition the contribution into two parts, one related to the pressure gradient and the other one related to the shear caused by the moving surface, i.e.,

\[ f_f(h) = f_f(h)_{\text{press}} + f_f(h)_{\text{shear}} \]
where
\[ f_f(h)_{\text{press}} = -3\mu a u_t \int_0^1 \frac{1}{1 + k - k\frac{x}{l}} dx + 6\mu a u_t \frac{1 + k}{2 + k} \int_0^1 \frac{1}{1 + k - k\frac{x}{l}}^2 dx = \]
\[ = \frac{\mu a u_t l}{h_T} \left( 3\frac{k}{k} \ln (1 + k) + \frac{6}{2 + k} \right) \]  \hspace{1cm} (5.102)

and
\[ f_f(h)_{\text{shear}} = \frac{\mu a u_t l}{h_T} \int_0^1 \frac{1}{1 + k - k\frac{x}{l}} dx. \]
\[ = \frac{\mu a u_t l}{h_T} \left( -\frac{1}{k} \ln (1 + k) \right) \]  \hspace{1cm} (5.103)

Thus
\[ f_f(h) = \frac{\mu a u_t l}{h_T} \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right). \]  \hspace{1cm} (5.104)

The inclination is very small and although its \( x \)- and \( z \)-components would be evaluated as
\[ (f_f(h))_x = \frac{\mu a u_t l}{h_T} \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right) \cos \left( \frac{kh_T}{l} \right) \]  \hspace{1cm} (5.105)
and
\[ (f_f(h))_z = \frac{\mu a u_t l}{h_T} \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right) \sin \left( \frac{kh_T}{l} \right), \]  \hspace{1cm} (5.106)
respectively, this is a nearly perfectly horizontal force. Notice that is is in good alignment with the thin film approximation. In Fig. 5.8 dimensionless friction force \( f_f / (\mu a u_t l / h_T) \) is depicted as a function of the inclination parameter \( k \). In the left side of the figure, the blue line corresponds to the friction force acting on the upper surface, according to (5.104) while the black line is for the lower surface, according to (5.100). In the figure’s right side, a partitioning of the friction force in its pressure driven contribution evaluated at the upper surface according to (5.102) is given by the blue line. The red line showing the pressure driven contribution evaluated at the lower surface according to (5.98). The shear contribution, according to (5.99), is given by the black line.

5.4 Dimensionless formulation

Let us now consider the dimensionless formulation of Reynolds equation. In this case we are not after the relative importance of each term to simplify the equation as in our scaling of the Navier-Stokes equations. Instead, we will use the scaling to learn some general concepts about Reynolds equation. Our goal is therefore to reduce the number of inputs that determine the solution. We will also try to ensure that, in the resulting equation, all terms are of similar size. This is beneficial whenever numerical solutions are considered.

In the following subsection we will apply a scaling to transform i) the system consisting of the time-dependent representation of the 1D Reynolds equation for incompressible and iso-viscous fluid flow coupled with the force balance equation and ii) apply a scaling to transform
5.4. DIMENSIONLESS FORMULATION

Figure 5.8: Left: Dimensionless friction force as a function of the inclination parameter $k$, evaluated at the upper surface according to (5.104) (blue) and lower surface according to (5.100) (black). Right: Dimensionless friction force partitioned into its pressure driven contribution evaluated at the upper surface according to (5.102) (blue), and at the lower surface according to (5.98) (red) and the shear contribution according to (5.99) (black).

the stationary 1D Reynolds equation that we have already found the analytical solution for in Section 5.3

5.4.1 Time dependent Reynolds’ equation and force balance

The system consisting of the time-dependent representation of the 1D Reynolds equation for incompressible and iso-viscous fluid flow coupled with the force balance equation can be formulated as

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{12\mu_a} \frac{\partial p}{\partial x} \right) = \frac{u_u + u_t \partial h}{2} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t},
\]

(5.107a)

\[
f_{\text{app}}(t) - \int_0^t p(x, t) \, dx = m \frac{d^2 h_0}{dt^2}.
\]

(5.107b)

Next we will transform this system into dimensionless form. This is accomplished by scaling the independent and dependent variables of the determining system, which in this case are $x$, $t$, $h$ and $p$. We remark that $\mu_a$, $u_t$, $u_u$, $m$ - being the system’s mass of inertia (unit kg/m), and $f_{\text{app}}$ are input parameters to this problem and thus need not be scaled. The length $L$ of the domain $\Omega$ is also an input parameter. The film thickness, $h$, is in this case

\[
h(x, t) = h_0(t) + h_g(x) + h_u(x - u_u t) - h_l(x - u_l t),
\]

(5.108)

where $h_0$ is the rigid body separation and thus a dependent parameter itself (see Section 5.2.4 for more details about this).

The most general form of scaling for the problem at hand reads

\[
X = x/x_r, \quad T = t/t_r, \quad H = h/h_r \quad \text{and} \quad P = p/p_r,
\]

(5.109)
where \( x_r, t_r, h_r \) and \( p_r \) are reference parameters, which will be determined in the following. First it is observed that the length of the domain \( l \) is given as an input parameter and we should therefore use this as the reference parameter for \( x_r \), i.e., \( x_r = l \).

By inserting (5.109) into (5.107) we have

\[
\frac{h_r^3 p_r}{l^2} \frac{\partial}{\partial X} \left( \frac{H^3}{12 \mu_a} \frac{\partial P}{\partial X} \right) = \frac{u_u + u_l}{2} \frac{h_r}{l} \frac{\partial H}{\partial X} + \frac{h_r}{t_r} \frac{\partial H}{\partial T},
\]

or

\[
\frac{\partial}{\partial X} \left( \frac{H^3}{12 \mu_a} \frac{\partial P}{\partial X} \right) = \frac{6 \left( u_u + u_l \right) \mu_a l}{p_r h_r^3} \left( \frac{\partial H}{\partial X} + \frac{2l}{u_u + u_l} \frac{1}{t_r} \frac{\partial H}{\partial T} \right),
\]

\[
\frac{f_{\text{app}}(T)}{l p_r} - \int_0^1 P(X, T) \frac{dX}{dX} = \frac{m h_r}{l p_r t_r^2} \frac{d^2 H_0}{dT^2}.
\]

Although the reference parameters can be chosen arbitrarily, the preferred choice should be made to i) minimise the number of input parameters in the resulting dimensionless determining system and ii) to scale the terms so that they will be of similar order of magnitude. The latter is particularly relevant whenever numerical computations are involved. Thus, the procedure is first to identify the dimensionless groups and then find the set of reference parameters that minimise the number of these groups. From (5.111), the following dimensionless groups are identified

\[
\frac{6 \left( u_u + u_l \right) \mu_a l}{p_r h_r^3},
\]

\[
\frac{2l}{u_u + u_l} \frac{1}{t_r},
\]

\[
\frac{f_{\text{app}}(T)}{l p_r},
\]

\[
\frac{m h_r}{l p_r t_r^2}.
\]

There are four groups and we have four reference parameters. Thus it appears like it is possible to choose the reference parameters so that the system explicit dependence on input parameters is totally removed. This is not possible, however, because we have already chosen \( x_r = l \) in order to remove the inputs from the specification of the domain. Let us now choose the other reference parameters. Since it is desirable for all terms in each equation to be of the same order of magnitude, it is best to choose \( t_r \) so that,

\[
t_r = \frac{2l}{u_u + u_l}.
\]

Expressed in other words, the characteristic time is the time it takes for a feature moving with the mean speed of the surfaces to travel the length \( l \). This particular choice equalise
the orders of the spatial- and the time derivatives and the system (5.111) becomes
\[
\frac{\partial}{\partial X} \left( H^3 \frac{\partial P}{\partial X} \right) = 6 \left( \frac{u_u + u_t}{p_r h_r^2} \right) \left( \frac{\partial H}{\partial X} + \frac{\partial H}{\partial T} \right),
\]
(5.114a)
\[
f_{\text{app}}(T) \frac{l_{pr}}{l_{p_r}} - \int_0^1 P(X, T) dX = \frac{m h_r (u_u + u_t)^2 d^2 H_0}{4 l^3 \mu_a^2}.
\]
(5.114b)
At this stage we note that there are three groups but only two parameters left and we realise
that it will not be possible to remove the influence of all input parameters in the dimensionless
system in the end. No matter if we choose \(p_r\) as a function of \(h_r\) or vice versa, as long as
we scale so that all terms in (5.114a) and the left-hand side in (5.114b) will be of the same
order of magnitude. We thus choose the following set of reference parameters,
\[
x_r = l, \quad t_r = \frac{2}{u_l}, \quad p_r = \frac{f_{\text{app}}(T)}{l}, \quad h_r = \sqrt{\frac{6 (u_u + u_t) \mu_a l^2}{f_{\text{app}}(T)}},
\]
(5.115)
which will reduce (5.110b) to
\[
\frac{\partial}{\partial X} \left( H^3 \frac{\partial P}{\partial X} \right) = \frac{\partial H}{\partial X} + \frac{\partial H}{\partial T},
\]
(5.116a)
\[
1 - \int_0^{1/x_r} P(X, T) dX = M \frac{d^2 H_0}{dT^2},
\]
(5.116b)
where \(M\) is a representation of dimensionless mass, defined by
\[
M = m \sqrt{\frac{4 f_{\text{app}}^3 (T)/l}{(u_u + u_t)^{3/2} (6 \mu_a)^{1/2}}}. \tag{5.117}
\]
We note that if \(M \ll 1\), i.e., if the mass (per meter) is small enough or the mass reference
parameter in the denominator of (5.117) is large enough, then (5.116) reduces to an input
parameter free system of equations, which after having been solved once through scaling gives
all possible solutions having the same dimensionless gap \(H\). Note that this is also the case
for systems operating in stationary conditions for which \(d H_0 / d T = 0\).

5.4.2 Stationary 1D Reynolds’ equation

Let us now have a look at the case we studied in Section 5.3. In this case the system operates
at stationary conditions which means that \(\partial H / \partial T = 0\) and \(\partial^2 H_0 / \partial t^2 = 0\). Therefore,
the system (5.116) becomes parameter free. The only parameters relevant for the problem
can thus only come from the definition of \(H\). We will indeed see that only the inclination
parameter \(k\) appears in the dimensionless solution. Note that the reference parameter \(p_r\) is
consistent with using the scaling factor (5.85) as a reference pressure to transform the fluid
pressure (5.84) into dimensionless form. More precisely, from (5.115) we obtain
\[
p_r = \frac{6 (u_u + u_t) \mu_a l}{h_r^2}.
\]
(5.118)
However, when we are considering the stationary linear slider problem without applying force balance (as in Section 5.3), then $h_r$ should be chosen in relation to the input parameter $h_T$, which is known and will ensure that $h_r$ is of order 1. Moreover, if $l$ is taken as the reference length for $x$, then the adapted version (5.118) reads

$$p_r = \frac{6u_l\mu_al}{l^2}. \quad (5.119)$$

Applying this as a scaling for the pressure, it is clear that the only input parameter that influence the dimensionless pressure distribution $P = P(X)$ is the inclination parameter $k$ and we have i.e.,

$$P(X) = \frac{1}{k} \left( \frac{1}{1 + k - kX} - \frac{1 + k}{2 + k (1 + k - kX)^2} - \frac{1}{2 + k} \right). \quad (5.120)$$

Moreover, the dimensionless expression for the film thickness, described in (5.76), becomes

$$H(X) = 1 + k - kX. \quad (5.121)$$

In fact, if the scaling

$$X = x/l, \quad H = h/h_T \quad \text{and} \quad P = p/\left(\frac{6\mu_al}{h_T^2} \right),$$

is used to transform (5.77) one obtain

$$\frac{d}{dX} \left( H^3 \frac{dP}{dX} \right) = \frac{dH}{dX}, \quad 0 \leq X \leq 1, \quad (5.123)$$

with boundary conditions

$$P(0) = 0 \quad \text{and} \quad P(1) = 0.$$  

Observe that (5.123) has no explicit dependence on the input parameters ($l, \mu_a, h_T$ and $k$). We will now continue to derive dimensionless expression for the velocity $u$ in the $x$-direction and $w$ in the $y$-direction. That is, by introducing the scaling $U = u/u_l$ as well as defining $Z = z/h_T$ we have

$$U(X, Z) = \frac{3Z(Z - H(X))}{H^2(X)} \left( 1 - 2 \frac{1 + k}{2 + k H(X)} \frac{1}{H(X)} - \frac{Z}{H(X)} \right) + 1. \quad (5.124)$$

In alignment with the scaling that we deduced for the continuity equation in Section 5.1.1, the dimensionless form of $W$ should be defined as $W = w/(u_l h_T / l)$ and thus we have

$$W(X, Z) = -k \frac{Z^2}{H^2(X)} \left( 2 - 2 \left( Z - 3 \frac{1 + k}{2 + k} \right) \frac{1}{H(X)} - 6 \frac{1 + k}{2 + k} \frac{Z}{H^2(X)} \right). \quad (5.125)$$

Next, we consider the dimensionless load carrying capacity $F_p = f_p / f_r$, with $f_p$ defined by (5.94). Note that $f_p$ is a priori unknown for this problem and by choosing

$$f_r = \frac{6\mu_al}{h_T^2},$$

(5.126)
we have that

\[ F_p = \int_0^1 P(X) \, dX = \frac{1}{k^2} \ln (1 + k) - \frac{1}{k} \frac{2}{2 + k}. \]  

(5.127)

The dimensionless expression for the friction force at the lower- and the upper surface can be obtained by introducing the reference parameter

\[ f_{fr} = \frac{\mu_a u_0 l}{h_T}. \]  

(5.128)

Then, from (5.100) we have

\[ F_f(0) = F_f(0)_{\text{press}} + F_f(0)_{\text{shear}} = 4 \frac{k}{k} \ln (1 + k) - \frac{6}{2 + k}. \]  

(5.129)

For the inclined upper surface the friction force has \( x \)- and \( z \)-components, although the inclination is very small. The dimensionless friction force acting on this surface and its components can be obtained from the expressions (5.104)-(5.106), i.e.

\[ F_f(h) = \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right), \]  

(5.130)

\[ (F_f(h))_x = \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right) \cos \left( \frac{k h_T}{l} \right) \]  

(5.131)

and

\[ (F_f(h))_z = \left( \frac{2}{k} \ln (1 + k) + \frac{6}{2 + k} \right) \sin \left( \frac{k h_T}{l} \right). \]  

(5.132)

### 5.5 On the applicability of the Reynolds equation

For incompressible and iso-viscous flow due to motion in the \( x \)-direction, the corresponding Reynolds equation can be applied to study the flows in narrow gaps operating under conditions such that

\[ R_x := \rho_0 u_0 l_0 \left( \frac{h_0}{l_0} \right)^2 \ll 1, \]  

(5.133)

where \( R_x \) is referred to as the modified Reynolds number. We note that, as seen in (5.16) to (5.19), most terms only require that \( \varepsilon^2 = (h_0/l_0)^2 \) to be small but that the terms regarding inertia require \( R_x \) to be small. We further note that since it is the square of \( \varepsilon \) that appears, it is often not required for \( \varepsilon = h_0/l_0 \) to be extremely small for Reynolds equation to be a valid approximation. Indeed, \( \varepsilon \approx 0.1 \) can, in some cases, be small enough to motivate the use of Reynolds equation. Note that the directionality, indicated through the subscript \( x \), comes from the problem’s motion and length. Similarly we have modified Reynolds’ numbers in the \( y \)- and the \( z \)-directions, i.e.

\[ R_y := \rho_0 v_0 b_0 \left( \frac{h_0}{b_0} \right)^2, \]  

(5.134)
and
\[ R_z := \frac{\rho_0 v_0 h_0}{\mu_0}. \] (5.135)

Obviously, there are other circumstances that may affect the applicability. For example, hydrodynamic cavitation, that will be thoroughly explained in Section 5.7, may occur in various lubricated interfaces and it require coupling the Reynolds equation with another physical model that describes it.

The applicability of Reynolds equation for the case of compressible fluids is not as clear as in the case of incompressible ones. In this case, it does not only depend on how thin the film is but also on the type of compressibility at hand. In the reminder of this section, we will focus on the case of ideal gases. Recall that we arrived at the reduced system (5.41) for this type of gases. To investigate the validity of Reynolds equation, numerical simulations predicting the velocity field and pressure build-up for an infinitely wide linear slider 2D model bearing, of the same type as the one depicted in Fig. 5.3, was performed. The inclination parameter \( k \) was chosen so that optimal load carry capacity is obtained. For a fluid modelled as an ideal gas \( k \approx 5.6 \), which means that the leading edge gap is \( \approx 6.6 \) times the height of the trailing edge gap. Recall that, in Section 5.3.2, it was shown that for this type of bearing lubricated with an incompressible and iso-viscous model fluid, the optimum occurs for \( k \approx 1.2 \), i.e., when the leading edge gap is \( \approx 2.2 \) times height of the trailing edge gap.

Table 5.1 lists the properties of air necessary to analyse the hydrodynamic lubrication of air flow by means of (5.41). Indeed, at room temperature the value of \( M/(RT) \approx 10^{-5} \). This means that speeds of a few hundreds of m/s, are required for the left hand sides of (5.40b) and (5.40c) to be as influential as the terms in the right hand sides. To see why this is the case, notice in (5.41b) and (5.41c) that the terms on the left hand side are of order \( (Mu_0)/(RT) \) whereas those on the right hand side are of order 1. This is, of course, only occurring in a small number of applications. One example of a high speed application is a hard disk drive, where the read and write head may be flying over the disk spinning at 10000 rpm at a 5 cm radius. In this particular case, the speed is \( \approx 50 \) m/s and \( (Mu_0^2)/(RT) \approx 3\% \) meaning that \( (g_x, g_y, g_z) = (0, 0, g) \), where \( g \) is the gravitation acceleration. This means that \( (Mg_x)/(RT) = (Mg_y)/(RT) = 0 \). This implies that the compressibility plays a marginal role and that an incompressible solution could serve as a good approximation. There are, however, applications where the relative speed is even higher, e.g., high-speed dental drills. According to [31], there are high-speed dental drills with as large shaft diameter as 3/16” spinning at 500000 rpm, which implies a relative speed of \( \approx 125 \) m/s. Lubricated with air

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_a )</td>
<td>298.15</td>
<td>K</td>
<td>Ambient temperature</td>
</tr>
<tr>
<td>( p_a )</td>
<td>101325</td>
<td>Pa</td>
<td>Ambient pressure</td>
</tr>
<tr>
<td>( \rho_a = \rho(T_a, p_a) )</td>
<td>1.177</td>
<td>kg/m(^3)</td>
<td>Ambient density</td>
</tr>
<tr>
<td>( M )</td>
<td>29 \cdot 10^{-3}</td>
<td>kg/mol(^3)</td>
<td>Molar mass</td>
</tr>
<tr>
<td>( \mu_a = \mu(T_a, p_a) )</td>
<td>18.46 \cdot 10^{-6}</td>
<td>Pas</td>
<td>Ambient viscosity</td>
</tr>
<tr>
<td>( R )</td>
<td>8.314462175</td>
<td>kg/(mol K)</td>
<td>Universal gas constant</td>
</tr>
</tbody>
</table>
this means that \((Mu_0^2)/(RT) \approx 18\%\). There are also examples of foil bearing applications and of bearings found in turbo machinery \[32\] where the speed is high enough for the left hand sides of \((5.40b)\) and \((5.40c)\) to be as influential as the terms in the right hand sides.

In the following we will present the outcome of the numerical analysis of the infinitely wide 2D linear slider model bearing. Figure 5.9 depicts the velocity in the \(x\)-direction \(u = u(x, z)\), obtained with CFD, in this 2D case. Note that the colour range has been truncated at 125 m/s while the maximum velocity is \(\approx 346\) m/s. In Fig. 5.10 the CFD-pressure solution showing that it is not varying in the \(z\)-direction in this particular case, i.e., \(p = p(x)\). This is also consistent with our results \((5.41d)\). Figure 5.11 depicts the pressure solution to the corresponding Reynolds equation \((5.62)\), \(p = p_{\text{REY}}^{U}\), for air at room temperature, for \(U = 5, 50\) and 125 m/s. Finally, Fig. 5.12 present the relative difference between the CFD-pressure

![Figure 5.9: The CFD-velocity in the \(x\)-direction \(u = u(x, z)\). Sliding speed 125 m/s, \(h_T = 10\) \(\mu\)m, \(h_L = 6.6h_T\), \(l = 0.1\) m/s, and other data given in Table 5.1.](image)

![Figure 5.10: The CFD-pressure solution. Sliding speed 125 m/s, \(h_T = 10\) \(\mu\)m, \(h_L = 6.6h_T\), \(l = 0.1\) m/s, and other data given in Table 5.1.](image)

![Figure 5.11: The CFD-pressure solution. Sliding speed 125 m/s, \(h_T = 10\) \(\mu\)m, \(h_L = 6.6h_T\), \(l = 0.1\) m/s, and other data given in Table 5.1.](image)

![Figure 5.12: The relative difference between the CFD-pressure and the corresponding Reynolds equation.](image)
solution and the solution to (5.62), defined as

$$\frac{|p_{REY}^U - p_{CFD}^U|}{\max(p_{REY}^U)},$$

for each of the three sliding speeds $U = 5, 50$ and $125$ m/s. From this numerical example it is clear that the classical Reynolds equation predicts the pressure build-up and thus load carrying capacity quite accurately as long as the sliding speeds are moderate. The difference between the solutions for $U = 5$ m/s is so small that the black solid line, at the scale of the figure, coincides with the $x$-axis.
of the figure, coincides with the $x$-axis. But it also shows that, if high accuracy is desired
in a high velocity case, one is forced to resort to the more consistent (and complex) system
(5.41).

## 5.6 The finite difference method for Reynolds’ type of PDE:s

In general, solving Reynolds equation analytically is a complicated task. It is possible in some
simple 2D geometries such as the one presented in Section 5.3 but not generally. For 3D cases,
one can forget about obtaining an analytical solution altogether. Therefore, numerical tools
are used to solve Reynolds equation in general. In this section we will therefore present one
such tool. In particular, we will introduce the method of finite differences and apply it to
Reynolds equation.

The Reynolds equation is a special case of the Poisson second order order differential
equation, i.e.

$$
\frac{d_a}{dt} \frac{\partial u}{\partial t} + \nabla_x \cdot \left( -c \nabla_x u - \alpha u + \gamma \right) + \beta \cdot \nabla u + au = f, \tag{5.137}
$$

where

$$
d_a = d_a(t, x), \quad c = \begin{bmatrix} c_{11}(t, x) & c_{12}(t, x) \\ c_{21}(t, x) & c_{22}(t, x) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_{11}(t, x) \\ \alpha_{21}(t, x) \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_{11}(t, x) \\ \gamma_{21}(t, x) \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_{11}(t, x) \\ \beta_{21}(t, x) \end{bmatrix}, \quad a = a(t, x) \quad \text{and} \quad f = f(t, x)
$$

in 2D and

$$
\frac{d_a}{dt} \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left( -c \frac{\partial u}{\partial x} - \alpha u + \gamma \right) + \beta \frac{\partial u}{\partial x} + au = f, \tag{5.139}
$$

where

$$
d_a = d_a(t, x), \quad c = c(t, x), \quad \alpha = \alpha(t, x), \quad \gamma = \gamma(t, x), \quad \beta = \beta(t, x), \quad a = a(t, x) \quad \text{and} \quad f = f(t, x)
$$

in 1D. Note that the we will only consider the situation when $c_{12} = c_{21}$. We will start by a
derivation of a second order finite difference scheme, with uniform element size $\Delta x$ and time
step $\Delta t$, for the 1D case. The spatially discretised finite difference scheme will be realised
by linearisation of the coefficient functions (5.137) to (5.140) at a point $x_i$. In particular, we
need to discretise derivatives of the form

$$
\frac{\partial u}{\partial x}, \quad \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right) \quad \text{and} \quad \frac{\partial u}{\partial t}.
$$

The element size is defined by the length $l$ of the domain $[x_a, x_b]$ and the number of elements.
If we use $N + 2$ points to discretise this domain ($N$ interior points plus $x_a$ and $x_b$), the element
size becomes

$$
\Delta x := \frac{l}{N + 1}. \tag{5.141}
$$
Note that \( N \) elements are described through \( N + 1 \) points. To allow for a compact notation for the subsequent steps of the derivation we introduce the notation
\[
 u^k_i := u(t_k, x_i), \quad x_i = x_a + i \Delta x. \tag{5.142}
\]

Time is denoted by \( t_k \) here, but at this time it is not restricted to a particular set of discrete times. The foundation for the discretisation of the 1D Poisson equation (5.139) is depicted in Fig. 5.13. In the figure, also the midpoints \( x_{i \pm 0.5} \) are shown. These will be used to derive

![Schematics of a discretised domain in 1D.](image)

a second order finite difference scheme defined on the interval \( I_i = [x_{i-1}, x_{i+1}] \), for the second order derivative. Based on these definitions it is now possible to define the classical finite differences for the first derivative term \( \partial u/\partial x \). They are based on the Taylor series expansions at \( x + \Delta x \), \( x + \Delta x/2 \), \( x - \Delta x/2 \) and \( x - \Delta x \), i.e.,
\[
 f(x + \Delta x) = f(x) + \frac{\Delta x}{1!} f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + \frac{\Delta x^4}{4!} f^{(4)}(x) + \ldots, \tag{5.143a}
\]
\[
 f(x + \frac{\Delta x}{2}) = f(x) + \frac{\Delta x}{2!} f'(x) + \frac{\Delta x^2}{2^2 2!} f''(x) + \frac{\Delta x^3}{2^3 3!} f'''(x) + \frac{\Delta x^4}{2^4 4!} f^{(4)}(x) + \ldots, \tag{5.143b}
\]
\[
 f(x - \frac{\Delta x}{2}) = f(x) - \frac{\Delta x}{2!} f'(x) + \frac{\Delta x^2}{2^2 2!} f''(x) - \frac{\Delta x^3}{2^3 3!} f'''(x) + \frac{\Delta x^4}{2^4 4!} f^{(4)}(x) + \ldots, \tag{5.143c}
\]
\[
 f(x - \Delta x) = f(x) - \frac{\Delta x}{1!} f'(x) + \frac{\Delta x^2}{2!} f''(x) - \frac{\Delta x^3}{3!} f'''(x) + \frac{\Delta x^4}{4!} f^{(4)}(x) + \ldots \tag{5.143d}
\]

From (5.143d) get the so called upwind difference scheme directly, i.e.,
\[
 \Delta x f'(x) = f(x) - f(x - \Delta x) + O(\Delta x^2), \tag{5.144}
\]
or
\[
 f'(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + O(\Delta x). \tag{5.145}
\]

Thus, up to first order accuracy the \( x \)-derivative of the dependent variable \( u \), at the point \( x = x_i \), can be approximated as
\[
 \frac{\partial u}{\partial x} \bigg|_{x_i} \approx \frac{u_i - u_{i-1}}{\Delta x}. \tag{5.146}
\]
5.6. THE FINITE DIFFERENCE METHOD FOR REYNOLDS’ TYPE OF PDE:S

This is often referred to as an upwind scheme and is particularly suited for hyperbolic equations with initial value or a boundary condition upstream the point itself. By first order accuracy, it is meant that the correction terms are of order $\Delta x^1$. In practice, this means that one can expect the error to reduce by a factor $2^1$ upon halving $\Delta x$. In general, an $n$th order approximation will have corrections of order $\Delta x^n$ and the error will reduce by a factor $2^n$ upon halving $\Delta x$. The difference between (5.143a) and (5.143d) gives the central difference scheme for the first derivative of $f$, i.e.,

$$2\Delta x f'(x) = f(x + \Delta x) - f(x - \Delta x) + 2\frac{\Delta x^3}{3!} f'''(x) + 2\frac{\Delta x^5}{5!} f^{(5)}(x) + \ldots, \quad (5.147)$$

which for a function $u$ leads to a finite difference scheme of second order accuracy that reads

$$\frac{\partial u}{\partial x} \bigg|_{x_i} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}, \quad (5.148)$$

There are two ways two derive a finite difference scheme for the second order derivative of a function which is of second order accuracy, the most obvious being to add (5.143a) and (5.143d), i.e.,

$$\Delta x^2 f''(x) = f(x + \Delta x) + f(x - \Delta x) - 2f(x) + 2\frac{\Delta x^4}{4!} f^{(4)}(x) + \ldots, \quad (5.149)$$

which restated as a second order accurate finite difference scheme for the function $u$ reads

$$\frac{\partial^2 u}{\partial x^2} \bigg|_{x_i} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}, \quad (5.150)$$

For the second term in the 2D and the 1D Poisson equations, (5.137) and (5.139), an approach based on the points $x_{i\pm 1/2} := x \pm \Delta x/2$, must be used to obtain a finite difference scheme of second order accuracy. For simplicity we consider the 1D case. Indeed, (5.148) is used to obtain

$$\frac{\partial}{\partial x} \left( c(t, x) \frac{\partial u}{\partial x} \right) \bigg|_{x_i} \approx \frac{1}{\Delta x} \left( c_{i+1/2} \frac{\partial u}{\partial x} \bigg|_{x_{i+1/2}} - c_{i-1/2} \frac{\partial u}{\partial x} \bigg|_{x_{i-1/2}} \right) \approx \frac{1}{\Delta x} \left( c_{i+1/2} \frac{u_{i+1} - u_i}{\Delta x} - c_{i-1/2} \frac{u_i - u_{i-1}}{\Delta x} \right) = \frac{c_{i+1/2} u_{i+1} - c_{i-1/2} u_{i-1}}{(\Delta x)^2}, \quad (5.151)$$

which is a second order accurate finite difference scheme for the second term in the Poisson equation. What remains to be deduced, is the values of the function $c$ at the points $x_{i\pm 1/2} = x \pm \Delta x/2$. If an explicit mathematical expression defines $c$, $c_{i\pm 1/2}$ can be determined by evaluation directly at the points. Assuming it is not, it can, for example, be determined by
averaging based on its neighbouring values at $x_{i-1}$, $x_i$ and $x_{i+1}$. The most common method is, perhaps, to use the arithmetic average, i.e.
\[ c_{i \pm 1/2} = \frac{c_i + c_{i \pm 1}}{2}. \] (5.152)

Note that in the Reynolds equation, $c$ and $\gamma$ depend on the film thickness $h$, which is a function of $x$.

The two most typical finite differences of the time derivative are the forward- and backward Euler methods. Let us start by formulating (5.137) as
\[ \frac{\partial u}{\partial t} = F(t, x, u). \] (5.153)

The forward Euler is an explicit scheme which is obtained by approximating the time derivative with an upwind scheme similar to (5.146), viz.
\[ \frac{u^k - u^{k-1}}{\Delta t} = F(t_k, x, u^{k-1}). \] (5.154)

This is an initial value problem and starting from the initial value $u_0 = u(t_0, x)$ the solution at time $t_k$ is given by
\[ u^k = u^{k-1} + \Delta t F(t_k, x, u^{k-1}). \] (5.155)

With the backward Euler method the solution $u_k$ is obtained by solving
\[ \frac{u^{k+1} - u^k}{\Delta t} = F(t_{k+1}, x, u^{k+1}). \] (5.156)

This is an implicit method and it differs from the explicit forward Euler method in that it uses $F(t_{k+1}, x, u_{k+1})$ instead of $F(t_k, x, u_k)$. It is seen that the (new) approximation $u_{k+1}$ appears in both sides of the equation, and one have to solve an algebraic equation to obtain its value. Thus, the solution procedure must be chosen so that it aligns with the characteristics of the problem. For a non-stiff problem, fixed-point iteration can be used. More precisely,
\[ [u^{k+1}]^0 = u_k, \quad [u^{k+1}]^{i+1} = u^k + \Delta t F(t_{k+1}, x, [u^{k+1}]^i). \] (5.157)

where $i$ is iterated with hope that some specified error measure $\epsilon^i$ reduces to some specified tolerance $\tau_\epsilon$. The error measure is commonly chosen as
\[ \epsilon^i = \frac{|[u^{k+1}]^{i+1} - [u^k]^i|}{\max([u^k]^i)} \] (5.158)

If it converges, i.e., $\epsilon^i \leq \tau_\epsilon$, for lets say $i = N_\tau$ then $[u^{k+1}]^{N_\tau}$ is used as the approximation to find $u^{k+2}$ and so on. The NewtonRaphson (or some kind modification of it) can also be used to solve the algebraic equation. With this we have the tools required to develop finite difference schemes for all sorts of differential equations, specifically different variants of the Reynolds equation.
5.6. THE FINITE DIFFERENCE METHOD FOR REYNOLDS’ TYPE OF PDE:S

5.6.1 An FDM for the stationary 1D Reynolds equation

In this section we present a second order finite difference scheme for (5.77), i.e., the linear stationary 1D Reynolds equation, describing incompressible and iso-viscous flow, viz.

\[
\frac{d}{dx} \left( \frac{h^3}{12\mu_a} \frac{dp}{dx} \right) = \frac{u_l}{2} \frac{dh}{dx}, \quad 0 \leq x \leq l,
\]

with (pressure) boundary conditions

\[ p(0) = p_T \quad \text{and} \quad p(l) = p_L. \]

By comparing this form of Reynolds equation with (5.139) we can identify coefficients. Indeed

\[
\alpha = 0, \quad c = -\frac{h^3}{12\mu_a}, \quad \beta = 0, \quad \gamma = -\frac{u_l h}{2}, \quad f = 0.
\]

Thus, a second order accurate scheme for (5.77) can be formulated as

\[
\frac{c_{i+1/2}p_{i+1} - (c_{i+1/2} + c_{i-1/2})p_i + c_{i-1/2}p_{i-1}}{(\Delta x)^2} = \frac{\gamma_{i+1} - \gamma_{i-1}}{2\Delta x}, \quad (5.159)
\]

where \( c_{i \pm 1/2} \) are given by (5.152). The FDM-scheme in (5.159) may be written in a more compact form as

\[
e_i^w p_{i-1} + e_i^c \Delta x p_i + e_i^e p_{i+1} = F_i, \quad (5.160)
\]

where

\[
e_i^w = \frac{c_{i-1/2}}{\Delta x}, \quad e_i^c = -\frac{c_{i+1/2} + c_{i-1/2}}{\Delta x}, \quad e_i^e = \frac{c_{i+1/2}}{\Delta x} \quad \text{and} \quad F_i = \frac{\gamma_{i+1} - \gamma_{i-1}}{2}
\]

and \( p_0 = p_T \) and \( p_{N+1} = p_L \). Let us list a few of the equations of the algebraic system that often is referred to as the determining system which we will solve. Since \( i \) marks the internal nodes only, \( i = 1 \ldots N \) this system becomes,

\[
i = 1: \quad e_1^w p_0 + e_1^c p_1 + e_1^e p_2 = F_1,
\]

\[
i = 2: \quad e_2^w p_1 + e_2^c p_2 + e_2^e p_3 = F_2,
\]

\[
\vdots
\]

\[
i = N: \quad e_N^w p_{N-1} + e_N^c p_N + e_N^e p_{N+1} = F_N,
\]

(5.161)
or

\[ \begin{align*}
    i &= 1 : \quad e_i^cp_1 + e_i^wp_2 &= F_1 - e_i^wp_L, \\
    i &= 2 : \quad e_2^wp_1 + e_2^cp_2 + e_2^e_p &= F_2, \\
    & \quad \vdots \\
    i &= i_{i-1} : \quad e_i^wp_i + e_i^cp_{i+1} &= F_i, \\
    & \quad \vdots \\
    i &= N : \quad e_N^wp_{N-1} + e_N^cp_N &= F_N - e_N^e_pT,
\end{align*} \]

(5.162)

where the terms including the boundary conditions, which are known, have been moved to the right-hand side. The determining system is often presented in matrix-vector form, which in this case reads

\[
\begin{bmatrix}
    e_1^c & e_1^e \\
    e_2^w & e_2^c & e_2^e \\
    & \ddots & \ddots & \ddots \\
    e_i^w & e_i^c & e_i^e \\
    & \ddots & \ddots & \ddots \\
    e_N^w & e_N^c
\end{bmatrix}
\begin{bmatrix}
    p_1 \\
    p_2 \\
    \vdots \\
    p_i \\
    \vdots \\
    p_N
\end{bmatrix}
= \begin{bmatrix}
    F_1 - e_1^wp_L \\
    F_2 \\
    \vdots \\
    F_i \\
    \vdots \\
    F_N - e_N^e_pT
\end{bmatrix}
\]

(5.163)

An even more compact notation for this tri-diagonal system reads

\[ \mathbf{A}\mathbf{p} = \mathbf{b} \]

(5.164)

and its solution

\[ \mathbf{p} = \mathbf{A}^{-1}\mathbf{b}, \]

(5.165)

can be readily obtained with various methods.

### 5.6.2 An FDM for the stationary 2D Reynolds equation

In this section we address the 3D thin film flow situation and present a second order finite difference scheme for the stationary form of (5.58), i.e., the linear stationary 2D Reynolds equation, describing incompressible and iso-viscous flow in polar coordinates, viz.

\[
\nabla_x \cdot \left( \frac{h^3}{12\mu_a} \left[ \begin{array}{cc} x_2 & 0 \\ 0 & 1/x_2 \end{array} \right] \nabla_x p - \frac{\omega}{2} \left[ \begin{array}{c} x_2 \\ 0 \end{array} \right] \right) = 0, \quad \text{on} \quad \Omega
\]

(5.166)

where \( \Omega \) is the circle sector defined by \( \Omega = \{(x_1, x_2) : 0 < x_1 < \theta_p, \ R_i < x_2 < R_o \} \). We will also discuss the implications of two applicable variants of boundary conditions, i.e. Dirichlet in both directions and a combination of Dirichlet- and periodic boundary conditions.

In the 2D case, the grid is enumerated in two directions and when posing the determining system, as we did in the 1D case obtaining (5.162), we first need to decide how we order
the elements. In Fig. 5.14 we find a discretised solution grid, where blue squares marks the boundary nodes, where the solution is known and the red circles marks the internal nodes, where the solution is unknown. We also have a so called five-point formula, centred at the node \((i, j)\). We will here follow use alphabetic order, starting with increasing \(i\) and then each \(N + 1\) time increase \(j\). If we assume that the value of the pressure is prescribed on the boundaries \(i = 0 \wedge N + 1\) and \(j = 0 \wedge M + 1\), i.e. Dirichlet boundary conditions, then the solution \(p_{ij}\) can be formulate as a column vector (here denoted) \(p\), structured as in (5.173), viz.

\[
p = \left[ \{p_{11}, p_{21}, \ldots, p_{N1}\} \quad \{p_{12}, p_{22}, \ldots, p_{N2}\} \quad \cdots \quad \{p_{1M}, p_{2M}, \ldots, p_{NM}\} \right]^T.
\]

We remark here that \(N\) and \(M\) will determine the positions of the diagonals in the determining matrix \(A\) and that it in some cases can be wise to change the order. For example, in the case when \(N\) is significantly larger than \(M\) the bandwidth can be reduced by ordering the other way around. Let us proceed. We will use the formulae in (5.148) and (5.151) with both \(i\) and \(j\) to obtain an FDM scheme for the 2D stationary Reynolds equation (5.166) with two different variants of boundary conditions. By comparing (5.166) with (5.137) we can identify coefficients. Indeed

\[
\alpha = 0, \quad c = -\frac{h^3}{12\mu_a} \begin{bmatrix} x_2 & 0 \\ 0 & 1/x_2 \end{bmatrix}, \quad \beta = 0, \quad \gamma = -\frac{\omega}{2} \begin{bmatrix} x_2 & 0 \\ 0 & 0 \end{bmatrix} h, \quad f = 0.
\]
Therefore, the discrete equation becomes

\[
\begin{align*}
\frac{c_{i+1/2,j}^{11}p_{i+1,j} - (c_{i+1/2,j}^{11} + c_{i-1/2,j}^{11})p_{i,j} + c_{i-1/2,j}^{11}p_{i-1,j}}{(\Delta x)^2} + \\
\frac{c_{i,j+1/2}^{22}p_{i,j+1} - (c_{i,j+1/2}^{22} + c_{i,j-1/2}^{22})p_{i,j} + c_{i,j-1/2}^{22}p_{i,j-1}}{(\Delta y)^2} &= \gamma_{i+1,j}^{11} - \gamma_{i-1,j}^{11} \\
\frac{2\Delta x}{2\Delta x}
\end{align*}
\]

(5.167)

where \(c_{i\pm1/2,j}\) and \(c_{i,j\pm1/2}\) can be obtained from (5.152). In a compacted notation, the five-point FDM scheme (5.167) can be formulated as

\[
e^{s}_{ij}p_{i,j-1} + e^{w}_{ij}p_{i-1,j} + e^{e}_{ij}p_{i,j} + e^{c}_{ij}p_{i+1,j} + e^{n}_{ij}p_{i,j+1} = F_{ij},
\]

(5.168)

where

\[
\begin{align*}
e^{w}_{ij} &= \frac{c_{i-1/2,j}^{11}}{(\Delta x)^2}, & e^{c}_{ij} &= -\frac{c_{i+1/2,j}^{11} + c_{i-1/2,j}^{11}}{(\Delta x)^2} - \frac{c_{i,j+1/2}^{22} + c_{i,j-1/2}^{22}}{(\Delta y)^2}, & e^{e}_{ij} &= \frac{c_{i+1/2,j}^{11}}{(\Delta x)^2} \\
e^{s}_{ij} &= \frac{c_{i,j-1/2}^{22}}{(\Delta y)^2}, & e^{n}_{ij} &= \frac{c_{i,j+1/2}^{22}}{(\Delta y)^2}, & F_{ij} &= \frac{\gamma_{i+1,j}^{11} - \gamma_{i-1,j}^{11}}{2\Delta x}
\end{align*}
\]

We note that if the matrix \(c\) would have had non-zero off-diagonal terms, this would have rendered 4 more diagonals, i.e. sub- and super-diagonals to \(e^{s}\) and \(e^{n}\), being multipliers for \(p_{i\pm1,j}\) and \(p_{i,j\pm1}\). Like we did in the 1D case before, we will now just list the most essential equations of the algebraic system for the solution \(p_{i,j}\), commonly referred to as the determining system.

In this case, due to size requirements we will separate it into a left- and a right-hand side.
part. The left-hand side then reads
\[
\begin{bmatrix}
e_{11}^p p_{1,0} + e_{11}^w p_{0,1} + e_{11}^c p_{1,1} + e_{11}^n p_{1,2} \\
e_{21}^p p_{2,0} + e_{21}^w p_{1,1} + e_{21}^c p_{2,1} + e_{21}^n p_{2,2} \\
\vdots \\
e_{N1}^p p_{N,0} + e_{N1}^w p_{N-1,1} + e_{N1}^c p_{N,1} + e_{N1}^n p_{N+1,1} + e_{N1}^n p_{N,2} \\
e_{12}^p p_{1,1} + e_{12}^w p_{0,2} + e_{12}^c p_{1,2} + e_{12}^n p_{1,3} \\
e_{22}^p p_{2,1} + e_{22}^w p_{1,2} + e_{22}^c p_{2,2} + e_{22}^n p_{2,3} \\
\vdots \\
e_{N2}^p p_{N,1} + e_{N2}^w p_{N-1,2} + e_{N2}^c p_{N,2} + e_{N2}^n p_{N+1,2} + e_{N2}^n p_{N,3} \\
\vdots \\
e_{1M}^p p_{1,M-1} + e_{1M}^w p_{0,M} + e_{1M}^c p_{1,M} + e_{1M}^n p_{1,M+1} \\
e_{2M}^p p_{2,M-1} + e_{2M}^w p_{1,M} + e_{2M}^c p_{2,M} + e_{2M}^n p_{2,M+1} \\
\vdots \\
e_{NM}^p p_{N,M-1} + e_{NM}^w p_{N-1,M} + e_{NM}^c p_{N,M} + e_{NM}^n p_{N+1,M} + e_{NM}^n p_{N,M+1}
\end{bmatrix}
\]

and the right-hand side becomes
\[
\begin{bmatrix}
\{F_{11}, F_{21}, \ldots, F_{N1}\} \\
\{F_{12}, F_{22}, \ldots, F_{N2}\} \\
\vdots \\
\{F_{1M}, F_{2M}, \ldots, F_{NM}\}
\end{bmatrix}^T. \tag{5.170}
\]

It is now time to address the boundary conditions and we will devote one subsection for each of the cases.

**Dirichlet boundary conditions**

For a typical tilting pad bearing, such as the one illustrated in [2.1] each segment is surrounded by lubricant supplied at a given pressure \(p_a\). If it is assumed that the segments behave identically, the numerical analysis may be focussed on a single element reducing the computational burden by a factor equalling the number of pads. This means that the pressure distribution on each individual segment assuming Dirichlet boundary conditions is pressure solution depicted in [5.2]

\[
p(0, x_2) = p(\theta, x_2) = p(x_1, R_i) = p(x_1, R_\alpha) = p_a. \tag{5.171}
\]

Looking at the system above, this means that \(p_{0,j} = p_{N+1,j} = p_{i,0} = p_{i,M+1} = p_a\) for \(i = 1 \ldots N\) and \(j = 1 \ldots M\) are known and should be moved to the right-hand side. This makes it possible to formulate the determining system in matrix form - in which the subscript \(D\) indicate that Dirichlet boundary conditions are considered, as

\[
A_D p_D = b_D. \tag{5.172}
\]

where \(A_D\) is an \((NM) \times (NM)\) five-diagonal matrix and where \(p_D\) and \(b_D\) are \((NM) \times 1\) vectors structured in the same way, i.e.

\[
\begin{bmatrix}
\{f_{11}, f_{21}, \ldots, f_{N1}\} \\
\{f_{12}, f_{22}, \ldots, f_{N2}\} \\
\vdots \\
\{f_{1M}, f_{2M}, \ldots, f_{NM}\}
\end{bmatrix}^T, \tag{5.173}
\]
where the $M$ sets, one for each $j$ listing the equations $i = 1 \ldots N$, have been grouped together with curly braces $\{ \}$. The matrix $A_D$ consist of the coefficients $c_{ij}^*$ for the solution $p_{ij}$.

When carrying out the implementation of the FDM solution it is convenient to represent the coefficients $c_{ij}^*$ and $F_{ij}$ on matrix form. To this end, one can use the following template

$$
g = \begin{bmatrix}
g_{0M} & g_{10} & \cdots & g_{N,0} & g_{N+1,0} 
g_{10} & g_{11} & \cdots & g_{N,1} & g_{N+1,1} 
\vdots & \vdots & \ddots & \vdots & \vdots 
g_{0M} & g_{1M} & \cdots & g_{N,M} & g_{N+1,M} 
g_{0,M+1} & g_{1,M+1} & \cdots & g_{N,M+1} & g_{N+1,M+1}
\end{bmatrix} \quad (5.174)
$$

where we have also coloured the components corresponding to the Dirichlet boundary conditions in blue. Note that the corner coefficients will not be used for the five-point scheme, but if the matrix $c$ would have had non-zero off-diagonal terms $c_{12} = c_{21}$, they would have.

For the readers convenience, we will give the explicit description of the determining system so as to easily obtain $A_D$ and $b_D$. We start with the left-hand side of the determining system,
which if colouring the components corresponding to the boundary conditions in blue reads

\[
\begin{align*}
\begin{bmatrix}
    e_{11}^s P_a & e_{11}^u t x p_a & e_{11}^P p_{1,1} & e_{11}^e p_{2,1} & e_{11}^n P_{1,2} \\
    e_{21}^s t x p_a & e_{21}^u p_{1,1} & e_{21}^P p_{2,1} & e_{21}^e p_{3,1} & e_{21}^n p_{2,2} \\
    \vdots & & & & \\
    e_{N1}^s t x p_a & e_{N1}^u P_{N-1,1} & e_{N1}^P p_{N,1} & e_{N1}^e t x p_a & e_{N1}^n P_{N,2} \\
\end{bmatrix} & = \\
\begin{bmatrix}
    e_{12}^s P_a & e_{12}^u p_{1,1} & e_{12}^P p_{1,2} & e_{12}^e p_{2,2} & e_{12}^n P_{1,3} \\
    e_{22}^s p_{1,2} & e_{22}^u p_{2,2} & e_{22}^P p_{3,2} & e_{22}^e p_{3,3} & e_{22}^n p_{2,3} \\
    \vdots & & & & \\
    e_{N2}^s P_{N,1} & e_{N2}^u P_{N,2} & e_{N2}^P p_{N,2} & e_{N2}^e P_a & e_{N2}^n P_{3,3} \\
\end{bmatrix} \\
\vdots & & & & \\
\begin{bmatrix}
    e_{1M}^s P_{1,M-1} & e_{1M}^u P_a & e_{1M}^P P_{1,M} & e_{1M}^e P_{N+1,M} & e_{1M}^n P_{1,M} \\
    e_{2M}^s P_{2,M-1} & e_{2M}^u P_{2,M} & e_{2M}^P P_{2,M} & e_{2M}^e P_{N+1,M} & e_{2M}^n P_{2,M} \\
    \vdots & & & & \\
    e_{NM}^s P_{N,M-1} & e_{NM}^u P_{N,M-1} & e_{NM}^P P_{N,M} & e_{NM}^e P_a & e_{NM}^n P_{3,3} \\
\end{bmatrix}
\end{align*}
\]  
\[
(5.175)
\]

This means that

\[
A_D =
\begin{bmatrix}
    e_{11}^e & e_{11}^n \\
    e_{21}^e & e_{21}^n \\
    \ddots & \ddots \\
    e_{12}^s & e_{N1}^e & e_{N1}^n \\
    e_{22}^s & e_{22}^e & e_{22}^n \\
    \ddots & \ddots & \ddots \\
    e_{1M}^s & e_{1M}^e & e_{1M}^n \\
    e_{2M}^s & e_{2M}^e & e_{2M}^n \\
    \ddots & \ddots & \ddots \\
    e_{NM}^s & e_{NM}^e & e_{NM}^n
\end{bmatrix}
\]  
\[
(5.176)
\]

This five-diagonal matrix can be seen as a block matrix consisting of sub-, main- and super-diagonal blocks

\[
\begin{bmatrix}
    e_{1j}^s & e_{2j}^s \\
    \ddots & \ddots \\
    e_{Nj}^s
\end{bmatrix}, \quad
\begin{bmatrix}
    e_{1j}^e & e_{2j}^e \\
    \ddots & \ddots \\
    e_{Nj}^e
\end{bmatrix}, \quad
\begin{bmatrix}
    e_{1j}^n & e_{2j}^n \\
    \ddots & \ddots \\
    e_{Nj}^n
\end{bmatrix}
\]  
\[
(5.177)
\]
After moving over the known components from (5.175) to the right-hand side we get

\[
\mathbf{b}_D = \begin{bmatrix}
\begin{Bmatrix}
F_{11} & e_{11}^s & e_{11}^w \\
F_{21} & e_{21}^s & 0 \\
\vdots & \vdots & \vdots \\
F_{N1} & e_{N1}^s & e_{N1}^w
\end{Bmatrix} - \begin{bmatrix}
p_a \\
p_a \\
p_a
\end{bmatrix}
\end{Bmatrix}
\]

(5.178)

Mixed periodic and Dirichlet boundary conditions

Let us now consider the situation when the cylindrically shaped bearing exhibits symmetry in the circumferential direction so that the solution domain can be divided into a number of identical sections. The spiral groove thrust bearing, with film thickness map as schematically illustrated in [5.16] is a typical example this type of configuration. For this type of application, it is appropriate to pose Dirichlet boundary conditions in the radial direction and periodic boundary conditions in the circumferential direction, i.e.

\[
p(0, x_2) = p(\theta_{\text{per}}, x_2) \quad \text{and} \quad p(x_1, R_i) = p(x_1, R_o) = p_a.
\]

(5.179)

The periodic boundary conditions becomes \( p_{0,j} = p_{N+1,j} \) for \( j = 0 \ldots M + 1 \). The numerical solution is defined in the ‘periodic window’ \( i = 0 \ldots N \), and we denote the vector representation as

\[
\mathbf{p}_{\text{per}} = \begin{bmatrix}
\{p_{01}, p_{11}, \ldots, p_{N,1}\} & \{p_{02}, p_{12}, \ldots, p_{N,2}\} & \cdots & \{p_{0M}, p_{1M}, \ldots, p_{N,M}\}
\end{bmatrix}^T.
\]

(5.180)

Note that the periodicity means that \( p_{ij} = p_{i+N+1,j} \). Specifically, this means that \( p_{-1,j} = p_{N,j} \), which will be utilised when generating the determining matrix for this problem.

The Dirichlet conditions can be expressed as \( p_{i,0} = p_{i,M+1} = p_a \) for \( i = 1 \ldots N \), and as described above these are known and should be moved to the right-hand side.
5.6. THE FINITE DIFFERENCE METHOD FOR REYNOLDS’ TYPE OF PDE:S

Figure 5.16: A model of the film thickness map of a spiral groove bearing with a \( \theta_{\text{per}} = 72^\circ \) period, repeating the same section five times. The colours indicating the film thickness for the plateaux, grooves and central and external recess.

We recall that

\[
e^{w}_{ij} = \frac{c^{11}_{i-1/2,j}}{(\Delta x_1)^2}, \quad e^{c}_{ij} = -\frac{c^{11}_{i+1/2,j} + c^{11}_{i-1/2,j}}{(\Delta x_1)^2} - \frac{c^{22}_{i,j+1/2} + c^{22}_{i,j-1/2}}{(\Delta x_2)^2}, \quad e^{e}_{ij} = \frac{c^{11}_{i+1/2,j}}{(\Delta x_1)^2}
\]

\[
e^{e}_{ij} = \frac{c^{22}_{i,j-1/2}}{(\Delta x_2)^2}, \quad e^{n}_{ij} = \frac{c^{22}_{i,j+1/2}}{(\Delta x_2)^2},
\]

\[
F_{ij} = \frac{\gamma^{11}_{i+1,j} - \gamma^{11}_{i-1,j}}{2\Delta x_1}
\]

where

\[
c^{11} = -x_2 \frac{h^3}{12\mu_a}, \quad c^{22} = -\frac{1}{x_2} \frac{h^3}{12\mu_a}, \quad \gamma^{11} = -x_2 \frac{\omega h}{2}.
\]

For the periodic problem it is required that \( e^*_{ij} \) and \( F_{ij} \) are extended to enable a formulation of the five-point FDM scheme for the periodic window. We can generate extended forms \( e^*_{\text{per}} \) and \( F_{\text{per}} \) by inserting the penultimate column before the first one in the previously defined matrix \((5.174)\), i.e.

\[
\mathbf{g}_{\text{per}} = \begin{bmatrix}
  g_{N,0} & g_{0M} & g_{10} & \cdots & g_{N,0} & g_{N+1,0} \\
  g_{N,1} & g_{01} & g_{11} & \cdots & g_{N,1} & g_{N+1,1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  g_{N,M} & g_{0M} & g_{1M} & \cdots & g_{N,M} & g_{N+1,M} \\
  g_{N,M+1} & g_{0,M+1} & g_{1,M+1} & \cdots & g_{N,M+1} & g_{N+1,M+1}
\end{bmatrix}.
\]
Here we have also coloured the components of $p_{\text{per}}$ corresponding to the periodic and Dirichlet boundary conditions in red and blue, respectively.

By using the same colouring scheme, the left-hand side in the determining system becomes

$$
\begin{bmatrix}
  e^s_{01}p_a + e^w_{01}P_{N,1} + e^e_{01}p_{0,1} + e^e_{01}p_{1,1} + e^e_{01}p_{0,2} \\
  e^s_{11}p_a + e^w_{11}P_{0,1} + e^e_{11}p_{1,1} + e^e_{11}p_{2,1} + e^e_{11}p_{1,2} \\
  \vdots \\
  e^s_{N,1}p_a + e^w_{N,1}P_{N-1,1} + e^e_{N,1}p_{N,1} + e^e_{N,1}p_{0,1} + e^e_{N,1}p_{N-1,2} \\
  e^s_{02}p_{0,1} + e^w_{02}P_{N,2} + e^e_{02}p_{0,2} + e^e_{02}p_{1,2} + e^e_{02}p_{0,3} \\
  e^s_{12}p_{1,1} + e^w_{12}P_{0,2} + e^e_{12}p_{1,2} + e^e_{12}p_{2,2} + e^e_{12}p_{1,3} \\
  \vdots \\
  e^s_{N2}p_{N,1} + e^w_{N2}P_{N-1,2} + e^e_{N2}p_{N,2} + e^e_{N2}p_{0,2} + e^e_{N2}p_{N,3} \\
  \vdots \\
  e^s_{0M}p_{0,M-1} + e^w_{0M}P_{N,M} + e^e_{0M}p_{0,M} + e^e_{0M}p_{1,M} + e^e_{0M}p_{0,M} \\
  e^s_{1M}p_{1,M-1} + e^w_{1M}P_{0,M} + e^e_{1M}p_{1,M} + e^e_{1M}p_{2,M} + e^e_{1M}p_{1,M} \\
  \vdots \\
  e^s_{NM}p_{N,M-1} + e^w_{NM}P_{N-1,M} + e^e_{NM}p_{N,M} + e^e_{NM}p_{0,M} + e^e_{NM}p_{N,M} 
\end{bmatrix}
$$

Thus, moving the known components to the right-hand side gives

$$
\begin{bmatrix}
  F_{01} \\
  F_{11} \\
  \vdots \\
  F_{N1}
\end{bmatrix}
- 
\begin{bmatrix}
  e^e_{01} \\
  e^e_{11} \\
  \vdots \\
  e^e_{N1}
\end{bmatrix}
\begin{bmatrix}
  p_a
\end{bmatrix}
$$

$$
b_{\text{per}} = 
\begin{bmatrix}
  F_{02} \\
  F_{12} \\
  \vdots \\
  F_{N2}
\end{bmatrix}
$$

This makes it possible to write the determining system on the form

$$
A_{\text{per}}p_{\text{per}} = b_{\text{per}}.
$$
We note that the pressure solution consists of $(N + 1) \times M$ unknowns and that, for each $j = 1 \ldots M$, the column in $A_{per}$ corresponding to $N \times j + 1$ has the element $e_{N,j}^e$ at row $N \times j$ and that the column $N \times j$ has the element $e_{0,j}^e$ at row $N \times j + 1$. If we express the sub-, main- and super-diagonal blocks, for the determining matrix for this mixed boundary condition situation, they read

$$\begin{bmatrix}
    e_{0,j}^s & e_{1,j}^s & \ldots & e_{N,j}^s \\
    e_{0,j}^c & e_{1,j}^c & \ldots & e_{N,j}^c \\
    \vdots & \vdots & \ddots & \vdots \\
    e_{0,j}^w & e_{1,j}^w & \ldots & e_{N,j}^w
\end{bmatrix}, \begin{bmatrix}
    e_{0,j}^e & e_{0,j}^c & \ldots & e_{0,j}^w \\
    e_{1,j}^e & e_{1,j}^c & \ldots & e_{1,j}^w \\
    \vdots & \vdots & \ddots & \vdots \\
    e_{N,j}^e & e_{N,j}^c & \ldots & e_{N,j}^w
\end{bmatrix}, \begin{bmatrix}
    e_{0,j}^n & e_{1,j}^n & \ldots & e_{N,j}^n
\end{bmatrix}. \quad (5.185)$$

## 5.7 Modelling mass-conserving hydrodynamic cavitation

Hydrodynamic cavitation is found in various lubrication situations. For example, at the divergent section between the shaft and the bushing in a plain journal bearing, where the fluid film is subjected to a tensile stress situation. The fluid cannot withstand these stresses and thus the fluid film ruptures. In this situation, Reynolds equation will fail to capture this phenomenon. Instead, it will wrongly predict negative (tensile) pressures. An early approach to avoid this issue was simply to ignore the negative pressures. This, however, does not preserve mass continuity and thus also leads to inaccurate results. The first attempts to model mass-conserving hydrodynamic cavitation was presented by three authors Jacobson, Floberg and Olsson [33–36]. They described the so called rupture and reformation boundary conditions and showed how they could be incorporated in the Reynolds equation leading to a mathematical model of hydrodynamic cavitation.

Elrod and Adams [37] developed a cavitation algorithm using a single equation throughout the lubrication region without the need for explicit equations to locate the cavitation boundaries, and used a switch function to terminate the pressure gradient in the region of cavitation. In [38] Elrod prented a variant of the cavitation algorithm proposed in [37], in which he used a different constitutive relationship between pressure and density. Vijayaraghavan and Keith introduced a more rigorous derivation and presented their contribution to cavitation modelling in [39,40]. As in [37] their algorithm was derived based on the constant bulk modulus type of compressibility. For real lubricants the bulk modulus varies with pressure. The importance of using more realistic models for the compressibility was investigated further in the paper [41] by Sahlin et al. However, treating the bulk modulus as a constant could produce good results in a narrow pressure range. Examples of other work addressing the difficulties associated with cavitation modelling are [42,44].

### 5.7.1 JFO theory

Jakobsson and Floberg [33] developed a mass preserving cavitation theory. They assumed a constant pressure in the cavitation region, i.e. the pressure gradient is zero. They also derived a set of conditions to locate the cavitation boundaries. Later, Floberg and Olsson [34,36]
extended the theory of cavitation and implemented it in the numerical solution procedures for numerous bearing types. The JFO-theory is based on the complementary assumption that the fluid is either fully saturated, i.e. \( \theta = 1 \) and the fluid pressure is larger than the cavitation pressure \( p > p_c \), or cavitated, i.e. \( \theta < 1 \) and the pressure equals the cavitation pressure \( p = p_c \). They formulated this mathematically as a boundary condition for the location \( x_c \) of the cavitation inception or rupture, i.e.

\[
p(x_c) = p_c \text{ and } \left. \frac{\partial p}{\partial n} \right|_{x=x_c}
\]

and as a condition for preservation of mass flow at the point \( x_r \) of where the fluid film reforms

\[
\frac{h^3}{12 \mu_a} \left. \frac{\partial p}{\partial n} \right|_{x=x_r} = \frac{V_n}{2} \left( 1 - \theta \right) \left. \right|_{x=x_r}.
\]

Though the JFO-rupture and reformation conditions can be applied to various lubrication problems, such as the ones for journal bearings, piston ring - cylinder liner conjunctions and rolling element bearings, it is difficult to handle situations where rupture and cavitation occurs many times inside the interface, as would be the case with e.g. a textured bearing surface. This lead successors to develop “universal” cavitation algorithms, as will be presented below.

### 5.7.2 Elrod’s and Adams’ universal cavitation algorithm

It is not so easy to describe the universal cavitation algorithm developed by Elrod and Adams. One reason is that their work starts in [37], with a paper in which they employ a constant bulk modulus type of constitutive relationship between density and pressure, i.e.,

\[
p = p_c + \beta g(\theta) \ln \theta,
\]

where \( \theta \) is the saturation or the dimensionless density given by

\[
\theta(x) = \frac{\rho(p(x))}{\rho_c},
\]

and where \( \rho_c = \rho(p_c) \) and \( g(\theta) \) is a so called switch function

\[
g(\theta) = \begin{cases} 
1, & p > p_c, \\
0, & p = p_c. 
\end{cases}
\]

The expression (5.188) comes from

\[
\rho = \rho_a e^{-(p-p_a)/\beta}
\]

with \( p_a = p_c \). By means of precisely this constitutive relationship between \( p \) and \( \theta \), Elrod and Adams presented a “universal differential equation” originating from (5.62), which reads

\[
\frac{\partial (\theta h)}{\partial t} = \nabla \cdot \left( \frac{\beta h^3}{12 \mu_a} g(\theta) \nabla \theta - \frac{u_s}{2} \theta h \right).
\]
Elrod then continues the development in [38] but there another constitutive relationship between density and pressure is adopted, i.e. (5.191). Elrod describes his model like this:

“Within the cavitated zone, the liquid everywhere possesses the density, $\rho_c$, but the actual mass content is $\rho_c \theta h$ per unit area. Here $1 - \theta$, then, is the same as the void fraction. Within the complete film, due to variation in pressure, the fluid density also varies. By reason of slight compression, the film mass content exceeds the content that would exist if the pressure were $p_c$. In other words, $\theta = \rho/\rho_c$ and the corresponding film pressure is:

$$p = p_c + \beta g(\theta) (\theta - 1),$$

(5.191)

where $\beta = \mathcal{O}(10^9)$ in SI units for a typical lubricating oil. It is realised, of course, that $\theta$ will be very nearly unity in the full-film zone, but when equation (5.191) constitutes an analytical convenience, it will be used.” In a more condensed form this means that the fluid behaves more or less as incompressible in the full-film zones and that it expands as a homogeneous blend in the cavitated zones.

Based on (5.191), the corresponding Reynolds equation (5.62) would become

$$\frac{\partial (\theta h)}{\partial t} = \nabla_x \cdot \left( \frac{\beta h^3}{12 \mu_a} \nabla_x (g(\theta)(\theta - 1)) - \frac{u_s}{2} \theta h \right).$$

(5.192)

The finite difference scheme presented in [38] implies that it would originate from

$$\frac{\partial (\theta h)}{\partial t} = \nabla_x \cdot \left( \frac{\beta h^3}{12 \mu_a} \nabla_x (g(\theta)(\theta - 1)) - \frac{u_s}{2} \theta h \right),$$

(5.193)

which differs from (5.192) as it reflects a flow situation where the fluid behaves like an incompressible liquid in the full-film zones and as a homogeneous compressible gas-liquid mixture in the cavitated zones. This is, also, to some extent consistent with Elrod’s description of his model, reprinted above. However, they found, and so did successors, that obtaining a converged numerical solution to this equation can be quite challenging if not even impossible sometimes.

In the following sections, the inconsistency of the introduction of the constitutive expression in (5.191) and the controversy that Reynolds equation, (5.62), actually reduces to (5.192) when adopting the constitutive relation given by (5.188) will be elaborated upon.

### 5.7.3 Vijayaraghavan’s and Keith Jr’s cavitation model

As Elrod and Adams did in [37], Vijayaraghavan and Keith Jr. [39,40] also used the constant bulk modulus type of compressibility as a starting point in their derivation of a cavitation model. They, however, presented a more rigorous derivation, which finally lead to a cavitation algorithm similar to the one presented by Elrod in [38]. More precisely, they use the density-pressure relation defined in (5.27) and reach again

$$\frac{\partial (\theta h)}{\partial t} = \nabla_x \cdot \left( \frac{\beta h^3}{12 \mu_a} g(\theta) \nabla_x \theta - \frac{u_s}{2} \theta h \right).$$

Studying this equation they conclude

$$g(\theta) \nabla_x \cdot (\theta) = g(\theta) \nabla_x \cdot (\theta - 1) \nabla_x \cdot (g(\theta)(\theta - 1)) - (\theta - 1) \nabla_x \cdot (g(\theta)).$$

(5.194)
However, in the full-film zones $\theta = 1$ and in the cavitated ones $g(\theta) = 0$ thus the last term in (5.194) vanishes everywhere except at the point of rupture and we have

$$g(\theta)\nabla_x \cdot (\theta) = \nabla_x \cdot (g(\theta)(\theta - 1))$$  \hspace{1cm} (5.195)

so that (5.190) remarkably reduces to (5.193), which is the continuous interpretation of the cavitation algorithm that Elrod proposed in [37]. Moreover, they introduce the concept of type differencing from transonic flow computations to obtain a finite difference stencil for the shear flow term that effectively considers that the governing equation goes from elliptic to hyperbolic when flow goes from fully flooded to cavitated. That is,

$$\frac{\partial(\theta h)}{\partial x} = \frac{\partial}{\partial x} \left( \theta h - (1 - g(\theta)) \frac{\partial(\theta h)}{\partial x} \right)$$  \hspace{1cm} (5.196)

where $\Delta x$ is the element size in the $x$-direction. When discretised this leads to the finite difference scheme

$$\left. \frac{\partial(\theta h)}{\partial x} \right|_{x_i} = \frac{(\theta h)|_{x_i} - (\theta h)|_{x_{i-1}}}{\Delta x} + \mathcal{O}(\Delta x),$$  \hspace{1cm} (5.197)

for the shear flow term within the cavitated zone, while it becomes

$$\left. \frac{\partial(\theta h)}{\partial x} \right|_{x_i} = \frac{(\theta h)|_{x_{i+1}} - (\theta h)|_{x_{i-1}}}{2\Delta x} + \mathcal{O}((\Delta x)^2),$$  \hspace{1cm} (5.198)

in the full-film zone. In comparison, the scheme presented in [38], yields

$$\left. \frac{\partial(\theta h)}{\partial x} \right|_{x_i} = \frac{h|_{x_i} - h|_{x_{i-1}}}{\Delta x} + \mathcal{O}(\Delta x),$$  \hspace{1cm} (5.199)

which clearly represents shear flow for an incompressible fluid. The scheme (5.197) is of first order, while (5.198) is of second order. In [40], Vijayaraghavan and Keith present the remedy for this by extending the type differencing scheme to

$$\left. \frac{\partial E}{\partial x} \right|_{x_i} = \frac{1}{2\Delta x} \left( (g_{i+1/2}E_{i+1} - (2 - g_{i+1/2} - g_{i-1/2})E_i - (2 - g_{i-1/2})E_{i-1} + \frac{\partial^2 E}{\partial x^2} \frac{(\Delta x)^2}{2} - \frac{\partial^3 E}{\partial x^3} \frac{(\Delta x)^3}{8} \right),$$  \hspace{1cm} (5.200)

where $E = \theta h$. This leads to the second order finite difference scheme

$$\left. \frac{\partial E}{\partial x} \right|_{x_i} = \frac{1}{2\Delta x} \left( (g_{i+1/2}E_{i+1} - (2 - g_{i+1/2} - g_{i-1/2})E_i - (2 - g_{i-1/2})E_{i-1}) + \frac{\partial^2 E}{\partial x^2} \frac{(\Delta x)^2}{2} - \frac{\partial^3 E}{\partial x^3} \frac{(\Delta x)^3}{8} \right) + \mathcal{O}((\Delta x)^2).$$  \hspace{1cm} (5.201)

The pressure driven flow can then be discretised with a standard second order accurate finite difference scheme. This method leads to a much more robust solution than previous approaches.
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5.7.4 Arbitrary compressibility switch function based cavitation algorithm

In the paper by Sahlin et al. [41], they adopt the expression (5.24), viz.
\[ \rho = \rho_a f(p), \]
as the constitutive relationship for the fluid compressibility. We recall, from above, that \( f(p_a) = 1 \) and \( \rho_a \) is the density at the ambient pressure \( p_a \). This means that
\[ \nabla \rho = \rho_a f'(p) \nabla p, \quad (5.202) \]
can be inverted to express the pressure gradient as,
\[ \nabla p = \frac{1}{f'(p)} \nabla \theta, \quad (5.203) \]
where \( \theta(x) = \rho(p(x))/\rho_a \). This means that (5.62) including the switch function \( g(\theta) \) (5.189) can be posed as
\[ \frac{\partial (\theta h)}{\partial t} = \nabla_x \cdot \left( \frac{h^3}{12f'(p)\mu_a} g(\theta) \nabla_x \theta - \frac{u_s}{2} \theta h \right). \quad (5.204) \]
In [41] they analyse the resulting predictions based on both the constant bulk modulus type of compressibility (5.27) and another kind of density-pressure relationship, the well-known relation,
\[ \rho = \rho_a + C_1 (p - p_a) - \frac{C_1 + p - p_a}{C_1 + p - p_a}, \quad (5.205) \]
deduced by Dowson and Higgins and presented in [45], together with the constants \( C_1 = 0.59 \) GPa and \( C_2 = 1.34 \) that the found to be best fit to mineral oil density-pressure data they had access to. Sahlin et al. [41] fitted the constants to another set of mineral oil density-pressure data for pressures up to 1 GPa, and found a close fit for \( C_1 = 2.22 \) GPa and \( C_2 = 1.66 \).

5.7.5 The linear complementarity problem formulation

In 2005, Bayada presented a [46] a continuous complementarity formulation of Elrod’s and Adams’ cavitation algorithm [37]. Later [47] presented the same formulation together with the discretised linear complementarity problem formulation of the same cavitation algorithm on the basis that the fluid in the fully flooded regions behaves if as incompressible. Independent of each other Bertocchi et al. [48] and Almqvist et al. [49, 50] developed this further. The work [49] is built upon that the pressure solution given by the Reynolds equation (expressed in its most fundamental form):
\[ \nabla \cdot q = 0, \quad (5.206) \]
may well be below \( p_c \). It proceeds by stating that in the full film zones, the density can be expressed as \( \rho = \rho_c e^{(p-p_c)/\beta} \). In the cavitation zones the density, or the saturation, is an unknown, here denoted by \( \delta \), hence
\[ \rho(p) = \rho_c \left\{ \begin{array}{ll} e^{(p-p_c)/\beta}, & p > p_c \\ \delta, & p = p_c \end{array} \right. \quad (5.207) \]
The unknown saturation function \( \delta \) satisfies \( 0 \leq \delta \leq 1 \). Since \( \nabla p = 0 \) in the region where \( p = p_c \) we have that

\[
q = \rho_c \begin{cases} 
\frac{e^{(p-p_c)/\beta}}{2} - \frac{e^{(p-p_c)/\beta}}{12\mu} \nabla p, & p > p_c \\
\frac{\delta h}{2} - \frac{\beta h^3}{12\mu}, & p = p_c
\end{cases}
\]  

(5.208)

which is a nonlinear expression in \( p \). By introducing the following change of variables

\[
u = e^{(p-p_c)/\beta} - 1, \quad u \geq 0
\]

(5.209)

we get that

\[
q = \rho_c \begin{cases} 
\frac{uh}{2} + \frac{h}{2} - \frac{\beta h^3}{12\mu} \nabla u, & u > 0 \\
\frac{h}{2} - \frac{\eta h}{2}, & u = 0
\end{cases}
\]  

(5.210)

A key point in the derivation of the cavitation model they present in [49] is that they rewrite (5.210) by introducing a new unknown variable \( \eta \), which is complementary to \( u \) in the whole domain, i.e., \( u\eta = 0 \). They define the variable \( \eta \) as

\[
\eta = 1 - \delta = \begin{cases} 
0, & u > 0 \\
1 - \delta, & u = 0
\end{cases}
\]

(5.211)

This means that, if \( u > 0 \) then \( \eta = 0 \) and if \( u = 0 \) then \( 0 \leq \eta \leq 1 \). The expression for the mass flow (5.210) can now be rewritten as:

\[
q = \rho_c \begin{cases} 
\frac{hu}{2} + \frac{h}{2} - \frac{\beta h^3}{12\mu} \nabla u, & u > 0 \\
\frac{h}{2} - \frac{\eta h}{2}, & u = 0
\end{cases}
\]

or alternatively

\[
q = \rho_c \left( \frac{hu}{2} + \frac{h}{2} - \frac{\beta h^3}{12\mu} \nabla u - \frac{\eta h}{2} \right), \quad u \geq 0.
\]

(5.212)

With this expression for the mass flow the continuity equation, becomes

\[
\nabla \cdot q = \rho_c \nabla \cdot \left( \frac{hu}{2} + \frac{h}{2} - \frac{\beta h^3}{12\mu} \nabla u - \frac{\eta h}{2} \right) = 0,
\]

and summing up, they expresses their mass preserving cavitation model as

\[
\nabla \cdot \left( \frac{\beta h^3}{12\mu} \nabla u - \frac{hu}{2} \right) = \nabla \cdot \left( \frac{h}{2} \right) - \nabla \cdot \left( \frac{\eta h}{2} \right),
\]

(5.213)

\[
\begin{array}{c}
u \geq 0, \quad 0 \leq \eta \leq 1, \quad u\eta = 0.
\end{array}
\]

The beauty of this formulation is that it permits a subsequent numerical LCP analysis by means of readily available methods, such as Lemke’s algorithm, presented in Section [4.5].
Having the solution \((u, \eta)\) the fluid pressure \(p\) and the saturation \(\delta\) can be obtained from (5.209) and (5.211). It should also be noted that MATLAB code for the numerical solution of this cavitation algorithm has been made available at MATLAB file central [51]. This model was later generalised in to include elastic deformation as well as the situation where the distance between the surfaces varies with time in [50].

By varying the bulk modulus, the compressibility of the lubricant is varied. A low value of the bulk modulus corresponds to a highly compressible lubricant, while a high value corresponds to a nearly incompressible lubricant. In fact, in the limit \(\beta \to \infty\), (5.213) becomes

\[
\nabla \cdot \left( \frac{h^3}{12\mu} \nabla p \right) = \nabla \cdot \left( \frac{h}{2} U \right) - \nabla \cdot \left( \frac{\eta h}{2} U \right),
\]

\[
p \geq 0, \quad 0 \leq \eta \leq 1, \quad p\eta = 0.
\]

which is the same cavitation algorithm as in [47]. Remark that, the system (5.214) can also be obtained by starting from the assumption \(\rho = \rho_c\) and thereafter following the procedure presented above.

### Analytical solution

The analytical solution to the cavitation algorithm (5.213) can be obtained for some simple cases. One of these is the two-dimensional pocket slider bearing, defined as

\[
h(x) = \begin{cases} 
h_0 & 0 \leq x \leq a \\
h_1 & a < x < b \\
h_0 & b \leq x \leq l \end{cases}, \quad \text{with } h_1 > h_0,
\]

with graphical representation is presented in Fig. 5.17. The analytical solution to this pocket bearing, for an the incompressible case (5.214) can be found in [52], see also [53]. For 2D
pocket bearing geometry, the continuity equation becomes one-dimensional and reads:

\[
\frac{dq}{dx} = \rho c \frac{d}{dx} \left( \frac{hu}{2} U + \frac{h}{2} U - \frac{\beta h^3}{12\mu} \frac{du}{dx} - \frac{\eta h}{2} U \right) = 0.
\]

Together with the boundary conditions \( p(0) = p_{in} \) and \( p(l) = p_{out} \), for the inlet and outlet respectively, the cavitation model can be formulated as

\[
\frac{d}{dx} \left( \frac{\beta h^3}{12\mu} \frac{du}{dx} - \frac{U^2}{2} hu \right) = \frac{U}{2} \frac{dh}{dx} - \frac{U}{2} \frac{d}{dx} (\eta h),
\]

(5.216)

\[
u \geq 0, \quad 0 \leq \eta \leq 1, \quad \eta = 0.
\]

It is assumed that the fluid cavitates inside the pocket between the point of rupture at \( x = a \) and the point of reformation at \( x = z \), where \( a \leq z \leq b \). This means that the bearing can be subdivided into three liquid phase zones; \( 0 \leq x \leq a \) and \( z \leq x \leq b \) and \( b \leq x \leq l \) and one gas phase zone \( a \leq x \leq z \), where \( u = 0 \) and \( 0 \leq \eta \leq 1 \). In each of the liquid phase zones, \( \eta = 0 \) and \( u \) is given by

\[
\frac{d}{dx} \left( \frac{\beta h^3}{12\mu} \frac{du}{dx} - \frac{U}{2} hu \right) = \frac{U}{2} \frac{dh}{dx}.
\]

Summing up, for each of the zones the solution explicitly reads

\[
u = C_1 + C_2 \exp \left( \frac{6\mu U}{\beta h_0^2} x \right), \quad \eta = 0, \quad 0 \leq x \leq a,
\]

(5.217)

\[
u = 0, \quad \eta = C, \quad a \leq x \leq z,
\]

(5.218)

\[
u = C_3 + C_4 \exp \left( \frac{6\mu U}{\beta h_1^2} x \right), \quad \eta = 0, \quad z \leq x \leq b,
\]

(5.219)

\[
u = C_5 + C_6 \exp \left( \frac{6\mu U}{\beta h_0^2} x \right), \quad \eta = 0, \quad b \leq x \leq l.
\]

(5.220)

The boundary conditions \( u(0) = e^{(p_{in} - p_c) / \beta} - 1 \) and \( u(a) = 0 \), can be used to determine the constants \( C_1 \) and \( C_2 \), i.e.

\[
C_1 = -\frac{1 - \exp \left( (p_{in} - p_c) / \beta \right)}{1 - \exp \left( -\frac{6\mu U a}{\beta h_0^2} \right)}, \quad (5.221)
\]

\[
C_2 = -C_1 \exp \left( -\frac{6\mu U a}{\beta h_0^2} \right).
\]

(5.222)

Knowing the solution \( u \) it can be used to equate the mass flow \( q \) according to

\[
q = \frac{\rho c U h_0}{2} \left( 1 - \frac{1 - \exp \left( (p_{in} - p_c) / \beta \right)}{1 - \exp \left( -\frac{6\mu U a}{\beta h_0^2} \right)} \right).
\]

(5.223)
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The mass flow is preserved throughout the whole domain. According to equation (5.212), the mass flow in $a < x < z$ is given by

$$q = \frac{\rho_c U h_0}{2} (1 - C),$$

(5.224)

and the constant $C$ can be determined. More precisely,

$$C = \frac{1 - \exp\left(\frac{(p_{in} - p_c)/\beta}{1 - \exp\left(-\frac{6\mu U a}{\beta h_0^2}\right)}\right)}{1 - \exp\left(-\frac{6\mu U a}{\beta h_0^2}\right)} \quad (= -C_1).$$

The remaining constants $C_i, i = 3, \ldots, 6$ and $z$, can be found by using the conditions that both $u$ and the mass flow are continuous at $x = z$ and $x = b$ and the boundary condition $u(l) = e^{(p_{out} - p_c)/\beta} - 1$. Summing up:

$$C_3 = \frac{h_0}{h_1} (1 - C) - 1,$$

(5.225)

$$C_5 = -C,$$

(5.226)

$$C_6 = \frac{\exp\left(\frac{(p_{out} - p_c)/\beta - 1 - C_5}{\exp\left(\frac{6\mu U l}{\beta h_0^2}\right)}\right)}{\exp\left(\frac{6\mu U l}{\beta h_0^2}\right)} \quad (= \frac{6\mu U b}{\beta h_0^2}),$$

(5.227)

$$C_4 = \frac{-C_3 + C_5 + C_6 \exp\left(\frac{6\mu U b}{\beta h_0^2}\right)}{\exp\left(\frac{6\mu U b}{\beta h_0^2}\right)} \quad (= \frac{6\mu U b}{\beta h_0^2}),$$

(5.228)

and (if it exists) the point of reformation reads

$$z = \frac{\beta h_1^2}{6\mu U} \ln \left(\frac{-C_3}{C_4}\right) - a.$$  

(5.229)

Note that $C_3$ in (5.225) is the correct value. In [49] it was not correct but an errata with the expression in (5.225) was published shortly after.

**Numerical solution procedures**

In the following subsections, we will discuss how to discretise the cavitation model (5.213) such that the discretised problem is on the form of a standard linear complementary problem (LCP). Notice that once written as a LCP, the problem can be solved using various methods. For example, Lemke’s algorithm, presented in Section 4.5, can be used. Although a two-dimensional formulation is straightforward, a one-dimensional version is presented here in order not to complicate the notation unnecessarily. Two different types of differencing schemes are discussed. In the first type, central and upwind differencing is combined, while the second type is based on central differences only.
Combined central and upwind differencing

Let us recall the one-dimensional form of (5.213):
\[ \frac{d}{dx} \left( a \frac{du}{dx} + bu \right) = \frac{dF}{dx} - \frac{d}{dx} (\eta F), \]
(5.230)

\[ u \geq 0, \quad 0 \leq \eta \leq 1, \quad u\eta = 0. \]

where
\[ a = \frac{\beta h^3}{12\mu}, \quad b = -\frac{U}{2} h, \quad F = \frac{U}{2} h. \]
(5.231)

The problem is discretised again using finite differences by dividing the domain \(0 < x < l\) into a uniform grid with \(N + 2\) points, thus having elements of size \(\Delta x = l/(N + 1)\). Note that, by interpreting \(u\) as pressure and choosing coefficients as
\[ a = \frac{h^3}{12\mu}, \quad b = 0, \quad F = \frac{U}{2}, \]
(5.232)

the model obtained corresponds to the one presented in [47], where the lubricant is assumed to be incompressible in the full film regions. The following notation is adopted
\[ x_i = il/N, \quad u_i := u(x_i). \]

The problem at hand is elliptic in the full film domain, where \(\eta = 0\). A central difference scheme is therefore used to approximate the derivatives in (5.230). Following the scheme presented in Section 5.6 and using the notation
\[ a_{i+1/2} = \frac{a_{i+1} + a_{i}}{2}, \]

and the approximation
\[ \frac{du}{dx}\bigg|_{i+1/2} \approx \frac{u_{i+1} - u_i}{\Delta x} \quad \text{and} \quad \frac{du}{dx}\bigg|_{i-1/2} \approx \frac{u_i - u_{i-1}}{\Delta x}, \]

we obtain
\[
\frac{d}{dx} \left( a \frac{du}{dx} + bu \right) \approx \frac{a_{i+1/2} du}{dx}\bigg|_{i+1/2} - \frac{a_{i-1/2} du}{dx}\bigg|_{i-1/2} + \frac{(bu)_{i+1} - (bu)_{i-1}}{2\Delta x} \\
\approx \frac{(a_{i+1} + a_i)}{2} \left( \frac{u_{i+1} - u_i}{\Delta x} \right) - \frac{(a_{i-1} + a_i)}{2} \left( \frac{u_i - u_{i-1}}{\Delta x} \right) + \frac{b_{i+1} u_{i+1} - b_{i-1} u_{i-1}}{2\Delta x} \\
= \frac{1}{2\Delta x^2} [(a_i + a_{i-1}) u_{i-1} - (a_{i-1} + 2a_i + a_{i+1}) u_i + (a_i + a_{i+1}) u_{i+1}] + \\
+ \frac{b_{i+1} u_{i+1} - b_{i-1} u_{i-1}}{2\Delta x},
\]
for the left hand side. The first term in the right hand side becomes

\[
\frac{dF}{dx} \approx \frac{F_{i+1} - F_{i-1}}{2\Delta x}.
\]

In the cavitated regions, where \( u = 0 \), we observe that the equation is hyperbolic in \( \eta \) and an upwind difference scheme is employed accordingly, i.e.,

\[
\frac{d}{dx} (\eta F) \approx \frac{\eta_i F_i - \eta_{i-1} F_{i-1}}{\Delta x}.
\]

Let us introduce the following notation

\[
\begin{align*}
    e_w^i &= a_{i-1} + a_i - b_{i-1} \frac{b_{i-1}}{2\Delta x^2} - \frac{b_{i-1}}{2\Delta x}, \\
    e_c^c &= -a_{i-1} + 2a_i + a_{i+1}, \\
    e_c^c &= a_i + a_{i+1} - \frac{b_{i+1}}{2\Delta x^2} \frac{b_{i+1}}{2\Delta x}, \\
    z_i &= \frac{F_{i+1} - F_{i-1}}{2\Delta x}, \\
    g_c^c &= -\frac{F_i}{\Delta x}, \\
    g_w^w &= \frac{F_{i-1}}{\Delta x}.
\end{align*}
\]

Using this notation, we define the following matrices and vectors:

\[
A = \begin{bmatrix}
e_c^c & e_c^c & 0 & 0 & 0 & \cdots \\
e_w^w & e_c^c & 0 & 0 & 0 & \cdots \\
0 & e_w^w & e_c^c & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
& & & & \cdots & \cdots & \cdots \\
& & & & & \cdots & \cdots \\
e_w^w & e_c^c & 0 & 0 & 0 & \cdots \\
\end{bmatrix}, \quad (5.233)
\]

\[
f = \begin{bmatrix}
z_1 - e_w^w u_0 - g_w^w \eta_0 \\
z_2 \\
z_3 \\
\vdots \\
z_{N-2} \\
z_{N-1} - e_w^w u_{N-1} - g_w^w \eta_{N-1} \\
z_N - e_w^w u_N - g_w^w \eta_N
\end{bmatrix}, \quad (5.234)
\]
where the values of \( \eta \) on the boundaries are computed from the complementarity conditions \( u_0 \eta_0 = 0 \) and \( u_N \eta_N = 0 \), and

\[
B = \begin{bmatrix}
  g_1^c & 0 & 0 & 0 & 0 & \cdots \\
  g_2^w & g_2^c & 0 & 0 & 0 & \cdots \\
  0 & g_3^w & g_3^c & 0 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  g_N^w & g_N^c & 0 & \cdots & \cdots & \cdots \\
  0 & g_{N-1}^w & 0 & \cdots & \cdots & \cdots \\
\end{bmatrix}.
\]

The discretised form of (5.216) can now be written as

\[
Au = f + B \eta, \quad u_i, \eta_i \geq 0, \quad u_i \eta_i = 0.
\]

Solving this system for \( u \) gives

\[
u = q + M \eta, \quad u_i, \eta_i \geq 0, \quad u_i \eta_i = 0,
\]

where \( q = A^{-1} f \) and \( M = A^{-1} B \).

The linear complementarity problem (5.237) can readily be solved by employing standard numerical methods. One which is frequently used is Lemke’s pivoting algorithm, see [17] and Section 4.5. One advantage, is that Lemke’s pivoting algorithm finds the solution in a finite number of steps. Hence, the solution obtained is numerically exact. The method chosen for the present work is a vectorized Matlab version of a pivoting algorithm solving linear complementarity problems [51].

Note that in the Lemke algorithm it not explicitly stated that \( \eta_i \leq 1 \). However, by using the same ideas as in [46], it can be proved that any solution to (5.213), with Dirichlet boundary conditions, without the condition \( \eta \leq 1 \) still satisfies \( 0 \leq \eta \leq 1 \). This implies that the numerical solution found with the Lemke algorithm automatically satisfies \( \eta \leq 1 \). This agrees with the physical interpretation that the saturation \((1 - \eta)\) must be positive and cannot be larger 1.

**Elliptic formulation and central differencing**

In the previous subsection, central differences were used above in the full film region and upwind differences in the cavitated regions. However, it is possible to use central differences throughout the whole domain by introducing a small perturbation, which makes the problem (5.216) elliptic also in \( \eta \) and not only in \( u \). Indeed,

\[
\frac{d}{dx} \left( a \frac{du^\varepsilon}{dx} + bu^\varepsilon \right) = \frac{dF}{dx} - \frac{d}{dx} \left( \varepsilon \frac{d\eta^\varepsilon}{dx} + \eta^\varepsilon F \right),
\]

\( u^\varepsilon \geq 0, \quad 0 \leq \eta^\varepsilon \leq 1, \quad u^\varepsilon \eta^\varepsilon = 0. \)
Table 5.2: Input parameters for the pocket bearing problem.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>l</th>
<th>h₀</th>
<th>h₁</th>
<th>p₀</th>
<th>U</th>
<th>µ</th>
<th>pᵢᵣ = pₒᵤᵣ</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 mm</td>
<td>5 mm</td>
<td>20 mm</td>
<td>1 µm</td>
<td>10 µm</td>
<td>0 Pa</td>
<td>1 m/s</td>
<td>0.01 Pa/s</td>
<td>100 kPa</td>
</tr>
</tbody>
</table>

where \( \varepsilon > 0 \) is a small parameter. Discretised by central differences, in the same manner as described above, this can be written as

\[
A u^{\varepsilon} = f + (\varepsilon D + B) \eta^{\varepsilon}, \quad u^{\varepsilon}, \eta^{\varepsilon} \geq 0, \quad u^{\varepsilon} \eta^{\varepsilon} = 0.
\] (5.239)

In order to get the standard form for linear complementarity problems we can rewrite this as

\[
u^{\varepsilon} = q^{\varepsilon} + M^{\varepsilon} \eta^{\varepsilon}, \quad u^{\varepsilon}, \eta^{\varepsilon} \geq 0, \quad u^{\varepsilon} \eta^{\varepsilon} = 0,
\]

where

\[q^{\varepsilon} = A^{-1} f \quad \text{and} \quad M^{\varepsilon} = A^{-1} (\varepsilon D + B).
\]

For small values of \( \varepsilon \), \( u^{\varepsilon} \) and \( \eta^{\varepsilon} \) in (5.238) are good approximations of \( u \) and \( \eta \) in (5.230).

Numerical vs. analytical solution

In this example, the numerical simulation procedure described above is verified against the analytical solution for the pocket bearing problem earlier in this section. The selected set of input parameters are presented in Table 5.2.

Figure 5.18 depicts the analytical and the numerical pressure solutions for the pocket bearing with input parameters given Table 5.2. The solutions are obtained for \( \beta = 5 \times 10^8 \) and \( N = 512 \). The relative error, defined as

\[
\frac{\sum_{i=0}^{N} |p_i^{\text{analytical}} - p_i^{\text{numerical}}|}{\sum_{i=0}^{N} p_i^{\text{analytical}}},
\]

was 6%, 2.5% and 1% (and the computational times were 3 ms, 22 ms and 106 ms on a standard laptop) for \( N = 128 \), \( N = 256 \) and \( N = 512 \) respectively. The numerical difficulties are concentrated to the reformation occurring close to the discontinuity in \( h \).

5.7.6 The FEM and cavitation modelling

It turns out that neither the switch-function nor the LCP-based cavitation models are well suited for implementation in an FE-based framework. It is, however, possible to establish mass-conserving cavitation models in other ways. In the following, a method that assumes the fluid to be incompressible in its liquid phase, comparable to the model presented in [47], will be described. Söderfjäll demonstrated that it is a mass-conserving method in [55], albeit not conveyed typographically entirely correct. The method was then utilised in numerical
investigations of texture’s effect in the piston ring - cylinder liner contact operating under mixed lubrication conditions [56]. Even more recently it was used to model inter-asperity cavitation in a two-scale EHL model, based on the Heterogenous Multiscale Method [57]. It should be mentioned that these examples were implemented in COMSOL Multiphysics®. This model assumes that both the lubricant viscosity and density depend on the pressure in the following way

\[
\rho = \rho_0 \frac{f(p) + \alpha}{1 + \alpha},
\]

and

\[
\mu = \mu_0 \frac{f(p) + \alpha}{1 + \alpha},
\]

where the dimensionless auxiliary parameter \(\alpha\), should be made as small as possible and where the function \(f\) is specified as

\[
f(p) = \begin{cases} 
0, & p - p_c < -\beta \\
1 - 2 \left(\frac{p - p_c}{\beta}\right)^3 - 3 \left(\frac{p - p_c}{\beta}\right)^2, & -\beta \leq p - p_c \leq 0 \\
1, & p - p_c > 0
\end{cases}
\]

The parameter \(\beta\) is also an auxiliary parameter, with dimensions of pressure, that needs to be specified for the problem at hand. For example, in [57], \(p_c\) was set to zero and they chose \(\alpha = 0.01\) and \(\beta = 0.2\) MPa.
5.8 Homogenisation of the Reynolds equation

Up until this point, we have considered cases in which the surfaces are smooth. This is, however, not generally the case. A more realistic situation is depicted in Fig. 5.19. On it, the geometry of a step bearing is presented. The geometry is, however, not only composed of the step itself but roughness is also present. This roughness poses a problem when solving the problem numerically. Indeed, a very fine discretisation would be needed to resolve the roughness with sufficient accuracy, which would result to very long computations. An alternative is to use a two-scale formulation. Roughly speaking, a two-scale formulation works by solving the problems at two distinct scales. The smooth geometry is considered in the global scale. Since the geometry is smooth, a coarse discretisation can be used in this scale avoiding lengthy computation times. To incorporate the effect of roughness, the equation is modified by introducing so-called ‘flow factors’. These are computed by solving a similar flow problem at the local scale. In this scale, the domains are very small and thus can be resolved in the detail required by the roughness. The translation of these loose concepts to mathematical terms is known as homogenization, which is the topic of this section.

Homogenisation is a type of averaging which has been found applicable for two-scale problems with highly varying coefficients. There are many papers reporting the successful application of Homogenisation in the field of lubrication, where the fluid flow may be governed by the Reynolds equation \[24, 58–65\]. This has lead to highly effective numerical tools where the effects caused by the surface roughness are embedded in the derived homogenised equations. Moreover, the equations are unambiguously determined and their nature allow for straightforward parallelisation. These tools enables studies of rough surface hydrodynamically lubricated problems such as that arising in the bearing configuration visualized in Fig. 2.1. This means that the theoretical model concerns different types of the non-stationary Reynolds equation in two dimensions.

The key ingredient when homogenising the effect of surface roughness in the Reynolds equation is the two-scale description of the gap between the surfaces. This is done via a mathematical description of the film thickness that include the surface roughness and appears in the Reynolds equation. More precisely, it is assumed that the film thickness can be modelled by means of the following auxiliary function

\[
h(x, t, y, \tau) = h_g(x, t) + h_u(y - u_u \tau) - h_l(y - u_l \tau),
\]

where \(h_g\), \(h_u\) and \(h_l\) are (mathematically) smooth functions, expressed in the global-scale variables \(x\) and \(t\), the local-scale variables \(y\) and \(\tau\). The expression \(u_u = [u_u, v_u]^T\) defines the velocity of the upper surface and \(u_l = [u_l, v_l]^T\) defines the velocity of the lower surface. Note that the velocities may depend on position, \(x\). The global scale is modelled by the function \(h_g\), which describes the geometry of the problem, and the local scale is modelled via the \(y\) and \(\tau\)-periodic functions \(h_u\) and \(h_l\) that describes the surface roughness of the upper and the lower surfaces, respectively. Indeed, the functions \(h_u\) and \(h_l\) are assumed to be periodic in their second argument and the cell of periodicity is denoted by \(Y\) for both \(h_u\) and \(h_l\). It is also assumed that \(u_t, u_u, v_l\) and \(v_u\) are such that \(h\) is periodic in \(\tau\) and the cell of periodicity, in \(\tau\), is denoted by \(Z\). By means of this auxiliary function, a simplified physical description of the gap \(h_\varepsilon\) between the surfaces can be achieved. That is

\[
h_\varepsilon(x, t) = h(x, t, x/\varepsilon, t/\varepsilon), \quad \varepsilon > 0.
\]
In this expression for the film thickness, $\varepsilon$ is a (small) parameter which moderates the

wavelength of the surface roughness. As mentioned above (5.244) admits a velocity field that varies with position $x$ (encountered in e.g. a rotating application). Moreover, since $h_l$ and $h_u$ are functions of $x$, this also allows for different roughness descriptions on different positions. An illustration of a bearing with step-shaped geometry (global-scale) and idealised periodic surface roughness (local-scale) at a given time, is presented in Fig. 5.19. The lower surface is moving in the $x_1$-direction with speed $u_l$, i.e. $u_t = [u_l, 0]^T$, and the upper surface is stationary. In terms of the small wavelength parameter $\varepsilon$, the Reynolds equation for iso-viscous and compressible flow (5.62) may be stated as

$$\frac{\partial}{\partial t} c_\varepsilon(x, t) = \nabla_x \cdot (A_\varepsilon(x, t) \nabla_x p_\varepsilon - B_\varepsilon(x, t)), \quad x \in \Omega$$

(5.245)

where

$$c_\varepsilon(x, t) = \rho(p_\varepsilon) h_\varepsilon, \quad A_\varepsilon(x, t) = \frac{\rho(p_\varepsilon) h_\varepsilon^3}{12 \mu_a}, \quad B_\varepsilon(x, t) = \frac{u_s}{2} \rho(p_\varepsilon) h_\varepsilon.$$

(5.246)

and $\Omega = \{(x_1, x_2)|0 < x_1 < l_1 \land 0 < x_2 < l_2\}$. Due to roughness, the coefficients $c_\varepsilon(x, t)$, $A_\varepsilon(x, t)$ and $B_\varepsilon(x, t)$ are rapidly oscillating functions that require high spatial and time resolution for a mesh independent subsequent numerical analysis.

The main idea in homogenization is to prove that there exist a solution $p_0$, solving a so called homogenized equation that does not involve rapidly oscillating coefficient functions, such that

$$p_\varepsilon \to p_0, \quad \text{as} \quad \varepsilon \to 0.$$ 

This means that for small values of epsilon - which is the case for realistic surfaces, $p_0$ is a good approximation of $p_\varepsilon$.

In the subsections below, the multiple scale expansion homogenisation procedure will be applied to the Reynolds equation in order to obtain the equation that governs $p_0$. This will be done for the flow of incompressible fluids, as well as compressible fluids governed by constant
bulk modulus and ideal gas type of density-pressure relationships, will be presented. The fluid is in all three of these cases assumed to be iso-viscous.

5.8.1 Incompressible fluid

Notice that, the stationary form of the Reynolds equation for iso-viscous and compressible flow of ideal gases, in both cartesian and polar coordinates, admits the following generalisation

\[ \nabla \cdot (A(x) \nabla u(x)) - \nabla \cdot B(x) = 0, \]

where \( u \) is the dependent variable, \( A \) and \( B \) are known functions and \( x = (x_1, x_2)^T \). In a cartesian coordinate formulation, \( A = I h^3/(12\mu a) \) and \( B = u_s h/2 \) and in polar coordinates, for the fluid film formation in a rotating device - as discussed [64], they are the matrix and the vector given by

\[ A(x) = \frac{h^3}{12\mu a} \begin{bmatrix} x_2 & 0 \\ 0 & 1/x_2 \end{bmatrix}, \]

\[ B(x) = \frac{\omega h}{2} \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \]

where \( x_1 \) is the angular coordinate and \( x_2 \) is the radial coordinate. For the subsequent homogenisation process we consider the following auxiliary equation

\[ \nabla \cdot (A(x,y) \nabla u(x,y)) - \nabla \cdot (B(x,y)) = 0, \]

where, again, \( u \) is the dependent variable and \( A \) and \( B \) are known functions. The variables \( x = (x_1, x_2)^T \) and \( y = (y_1, y_2)^T \) refer to the global and local domains, respectively. It holds that \( y = x/\varepsilon \).

Assume that \( A(x,y) \) and \( B(x,y) \) are periodic in \( y \) and also assume that the following expansion holds:

\[ u(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + ..., \]

where \( u_i \) are periodic in \( y \). Note that (5.251) reflects that \( u(x,y) \) can be expanded with respect with the perturbation \( \varepsilon \). Indeed, roughness is treated as a perturbation of the smooth geometry. What (5.251) claims is therefore that, when \( \varepsilon = 0 \) (i.e., when the surface is smooth), \( u = u_0 \). It further claims that, when there is some roughness (\( \varepsilon > 0 \), small), the solution can be approximated by adding to \( u_0 \) correction terms of the form \( \varepsilon^i u_i \), which become very small as the index \( i \) increases. This approach is commonly used in mathematics under the name of perturbation theory. In this terminology (5.250) becomes

\[
\left( \nabla_x + \frac{1}{\varepsilon} \nabla_y \right) \cdot \left( A(x,y) \left( \nabla_x + \frac{1}{\varepsilon} \nabla_y \right) \left( u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... \right) \right) - \left( \nabla_x + \frac{1}{\varepsilon} \nabla_y \right) \cdot B(x,y) = 0,
\]

where \( \nabla_x \) and \( \nabla_y \) indicate that the \( \nabla \) operator is applied to the set of variables \( x \) and \( y \), respectively. In the following we will consider the asymptotic behaviour of (5.252) as \( \varepsilon \) goes to zero and we realise that the reasonable result when expanding (5.252) require that the
coefficients for $\varepsilon^{-2}$, $\varepsilon^{-1}$ and $\varepsilon^0$ are equivalent to zero independently of each other. Indeed, as $\varepsilon \to 0$ the coefficients of higher orders of $\varepsilon$, i.e., $\varepsilon^k$ with $k \geq 1$, will not influence the result and we obtain following set of determining equations

\[ \varepsilon^{-2} : 0 = \nabla_y \cdot (A \nabla_y u_0), \]  
\[ \varepsilon^{-1} : 0 = \nabla_y \cdot (A \nabla_y u_1) + \nabla_y \cdot (A \nabla_x u_0) + \nabla_x \cdot (A \nabla_y u_0) - \nabla_y \cdot B, \]  
\[ \varepsilon^0 : 0 = \nabla_y \cdot (A \nabla_y u_2) + \nabla_y \cdot (A \nabla_x u_1) + \nabla_x \cdot (A \nabla_y u_1) + \nabla_x \cdot (A \nabla_x u_0) - \nabla_x \cdot B \]  

where the dependency to $x$ and $y$ has dropped for the readers convenience.

From (5.253a), we obtain that $u_0 = u_0(x)$. By means of this fact, we can simplify (5.253b) to

\[ \nabla_y \cdot (A \nabla_y u_1) + \nabla_y (A \nabla_x u_0) - \nabla_y \cdot B = 0. \]  

Moreover, if we let $\nabla_y A$ denote the gradient of each of the columns of $A$ independently, then (5.254) can be written as

\[ \nabla_y \cdot (A \nabla_y u_1) + \nabla_y A \cdot \nabla_x u_0 - \nabla_y \cdot B = 0, \]  

This means that the solution $u_1$ to this equation must be on the form

\[ u_1 = \chi_0 + \chi_1 \frac{\partial u_0}{\partial x_1} + \chi_2 \frac{\partial u_0}{\partial x_2}, \]  

where $\chi_i$ are the solution of the following periodic local problems

\[ 0 = \nabla_y \cdot (A \nabla_y \chi_0) - \nabla_y \cdot B \text{ in } Y, \]  
\[ 0 = \nabla_y \cdot (A \nabla_y \chi_1) + \nabla_y \cdot (A e_1) \text{ in } Y, \]  
\[ 0 = \nabla_y \cdot (A \nabla_y \chi_2) + \nabla_y \cdot (A e_2) \text{ in } Y, \]

where $Y$ is the domain defining one period in $y$, with the periodic boundary conditions $\chi_i(x, 0) = \chi_i(x, 1)$.

Now, the fact that $u_0 = u_0(x)$ together with (5.256) can be inserted into (5.253c) and we can complete the homogenisation process of (5.247). We start by rewriting (5.253c) as

\[ \nabla_x \cdot (A \nabla_x u_0 + A \nabla_y u_1 - \nabla_x \cdot B) = \nabla_y \cdot (A \nabla_x u_1 + A \nabla_y u_2) \]  

and we realise that, due to periodicity

\[ \nabla_x \left( \int_Y A(x, y) \nabla_x u_0 + \int_Y A(x, y) \nabla_y u_1 \, dy - \nabla_x \cdot \int_Y B(x, y) \, dy \right) = 0. \]  

Making use of (5.256) we can proceed and we finally get the homogenised equation

\[ \nabla_x \cdot (A_0(x) \nabla_x u_0) - \nabla_x \cdot B_0(x) = 0, \]
where $A_0$ is a $2 \times 2$ matrix and $B_0$ is a $2 \times 1$ vector, i.e.,

$$A_0(x) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ (5.261) 

and

$$B_0(x) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$ (5.262)

The coefficient functions of $A_0$ and $B_0$ reads

$$a_{11}^0(x) = \int_Y A(x, y) \left( 1 + \frac{\partial \chi_1}{\partial y_1} \right) dy,$$ (5.263)

$$a_{12}^0(x) = \int_Y A(x, y) \frac{\partial \chi_2}{\partial y_1} dy,$$ (5.264)

$$a_{21}^0(x) = \int_Y A(x, y) \frac{\partial \chi_1}{\partial y_2} dy = \int_Y A(x, y) \frac{\partial \chi_2}{\partial y_1} dy = a_{12}^0(x),$$ (5.265)

$$a_{22}^0(x) = \int_Y A(x, y) \left( 1 + \frac{\partial \chi_2}{\partial y_2} \right) dy,$$ (5.266)

$$b_1^0(x) = \int_Y \left( B(x, y) - A(x, y) \frac{\partial \chi_0}{\partial y_1} \right),$$ (5.267)

and

$$b_2^0(x) = \int_Y \left( B(x, y) - A(x, y) \frac{\partial \chi_0}{\partial y_2} \right).$$ (5.268)

Notice that the homogenised equation, (5.260) is on the same form as the original one (5.250), where the coefficients matrices $A(x, y)$ and $B(x, y)$ have been replaced by the homogenised counterparts $A_0(x)$ and $B_0(x)$.

### 5.8.2 Constant bulk-modulus fluid

Recall that an iso-viscous fluid with constant bulk modulus is characterized by a constant viscosity and a density of the form

$$\rho = \rho_a e^{p-p_a}/\beta.$$ (5.269)

By replacing the pressure by the density as the main dependent variable, the Reynolds equation can, in this case, be written as

$$\nabla \cdot \left( \frac{h^3 \beta}{12 \mu_a} \nabla \rho \right) = \frac{u_s}{2} \nabla (\rho h).$$ (5.270)

More generally, this can be written, both in Cartesian and polar coordinates as

$$\nabla \cdot (A(x, y) \nabla u(x, y)) - \nabla \cdot (B(x, y) u(x, y)) = 0,$$ (5.271)
where, again, $u$ is the dependent variable and $A$ and $B$ are known functions. Notice that $u$ in this case represent fluid density and not pressure. We again assume that $A(x,y)$ and $B(x,y)$ are periodic in $y$ and also assume that the expansion (5.251). In this terminology (5.271) becomes

$$
\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \cdot \left(A(x,y) \left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\right)\right) - \left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \cdot \left(B(x,y) \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\right)\right) = 0.
$$

(5.272)

Following the procedure introduced in the previous section for incompressible and iso-viscous flow, this leads to the following set of determining equations

$$
\varepsilon^{-2} : \ 0 = \nabla_y \cdot \left(A \nabla_y u_0\right), \tag{5.273a}
$$

$$
\varepsilon^{-1} : \ 0 = \nabla_y \cdot \left(A \nabla_y u_1\right) + \nabla_y \cdot \left(\nabla_x \cdot \left(A \nabla_y u_0\right)\right) - \nabla_y \cdot \left(B u_0\right), \tag{5.273b}
$$

$$
\varepsilon^0 : \ 0 = \nabla_y \cdot \left(A \nabla_y u_2\right) + \nabla_y \cdot \left(\nabla_x \cdot \left(A \nabla_y u_1\right)\right) + \nabla_y \cdot \left(\nabla_x \nabla_x u_0\right) + \nabla_x \cdot \left(B u_0\right) - \nabla_y \cdot \left(B u_1\right) \tag{5.273c}
$$

Similarly to what we had before, (5.273a) implies that $u_0 = u_0(x)$ and we can, therefore, simplify (5.273b) to

$$
\nabla_y \cdot \left(A \nabla_y u_1\right) + \nabla_y \nabla_x u_0 - u_0 \nabla_y \cdot B = 0. \tag{5.274}
$$

This implies that $u_1$ is on the form

$$
u_1 = \chi_0 u_0 + \chi_1 \frac{\partial u_0}{\partial x_1} + \chi_2 \frac{\partial u_0}{\partial x_2}, \tag{5.275}
$$

which means that $\chi_i$ is the solution to the same set of local problems as in the previous case (5.257). Following the same procedure as before leads to the homogenised equation

$$
\nabla_x \cdot \left(A_0(x) \nabla_x u_0\right) - \nabla_x \cdot \left(B_0(x) u_0\right) = 0 \tag{5.276}
$$

and remarkably, the coefficient functions of $A_0$ and $B_0$ are the same as for the incompressible fluid, thus given (5.263)-(5.268).

### 5.8.3 Ideal gas

An ideal gas has a constant viscosity and a density of the form $\rho = \kappa p$. The Reynolds equation for these gases, in both cartesian and polar coordinates, admits the following generalisation

$$
\nabla \cdot \left(A(x) u(x) \nabla u(x)\right) - \nabla \cdot \left(B(x) u(x)\right) = 0, \tag{5.277}
$$

where $u$ is the dependent variable, representing fluid pressure. For the subsequent homogenisation process we consider the following auxiliary equation

$$
\nabla \cdot \left(A(x,y) u(x,y) \nabla u(x,y)\right) - \nabla \cdot \left(B(x,y) u(x,y)\right) = 0 \tag{5.278}
$$
and that the expansion (5.251) holds. Following the procedure introduced in Section 5.8.1 for incompressible and iso-viscous flow, this leads to the following set of determining equations

\[ \varepsilon^{-2} : 0 = \nabla_y \cdot (A u_0 \nabla_y u_0), \]  

\[ \varepsilon^{-1} : 0 = \nabla_y \cdot (A u_1 \nabla_y u_0) + \nabla_y \cdot (A u_0 \nabla_y u_1) + \nabla_y \cdot (A u_2 \nabla_x u_0) + \nabla_x \cdot (A u_0 \nabla_y u_0) - \nabla_y \cdot (B u_0), \]  

\[ \varepsilon^0 : 0 = \nabla_y \cdot (A u_0 \nabla_x u_1 + A u_1 \nabla_x u_0 + A u_0 \nabla_y u_2 + A u_2 \nabla_y u_1 + A u_1 \nabla_y u_0) + \nabla_x \cdot (A u_0 \nabla_y u_0 + A u_0 \nabla_y u_1 + A u_1 \nabla_y u_0) - \nabla_x \cdot (B u_0) - \nabla_y \cdot (B u_1). \]  

Similarly to what we had before, (5.279a) implies that \( u_0 = u_0(x) \) and we can, therefore, simplify (5.279b) to

\[ u_0 \nabla_y \cdot (A \nabla_y u_1) + u_0 \nabla_y A \cdot \nabla_x u_0 - u_0 \nabla_y \cdot B = 0, \]

which since we have \( u_0 > 0 \) (note that \( u_0 = 0 \) would imply a zero density) becomes

\[ \nabla_y \cdot (A \nabla_y u_1) + \nabla_y A \cdot \nabla_x u_0 - \nabla_y \cdot B = 0, \]

i.e., the same as (5.255) with solution (5.256). Thus we have the same (periodic) local problems (5.257) as for the two previous cases. Finally we obtain the homogenised equation for this case

\[ \nabla_x \cdot (A_0(x) u_0 \nabla_x u_0) - \nabla_x \cdot (B_0(x) u_0) = 0, \]

We remark that coefficient functions of \( A_0 \) and \( B_0 \) are the same as the ones for iso-viscous and incompressible case, as well as the one for the iso-viscous and constant bulk modulus compressible case, thus given (5.263)-(5.268).

5.8.4 Homogenised coefficients and Patir and Cheng flow-factors

The method proposed by Patir and Cheng in [66, 67] considers a representative part of the surface roughness and model the complete surface as its periodic extension. This means that the film thickness can be expressed in the form of (5.243), which in its stationary form will be utilized in the following presentation of the Patir and Cheng averaging technique. Note that the \( x_1 \)-direction must be chosen so that it is aligned with the direction of motion. Retaining as much as possible of the already introduced notation, the averaged Reynolds equation presented in [66, 67] is restated as

\[ \nabla \cdot \left( \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \frac{h^3}{12 \mu_a} \nabla p^{pe} \right) = \nabla \cdot \left( \begin{pmatrix} \frac{u_s}{2} \\ 0 \end{pmatrix} \right) \left( \frac{1}{l_1 l_2 h_g} \int_Y h \ dy + \phi_0 \right) h_g \right) \quad \text{in} \ \Omega, \]  

where the flow factors \( \phi_i \) are given by

\[ \phi_0 = -\frac{1}{l_1 l_2 h_g} \int_Y h \frac{\partial v_0}{\partial y_1} \ dy, \]  

\[ \phi_1 = \frac{1}{l_1 l_2 h_g} \int_Y h \frac{\partial v_0}{\partial y_1} \ dy. \]
\[ \phi_1 = \frac{1}{l_2 h_y} \int_Y h^3 \frac{\partial v_1}{\partial y_1} \, dy, \]  
(5.284)

and

\[ \phi_2 = \frac{1}{l_1 h_y} \int_Y h^3 \frac{\partial v_2}{\partial y_2} \, dy. \]  
(5.285)

The Patir and Cheng local problems, i.e. the so-called micro bearing problems, determining \( v_i \) read

\[ \nabla_y \cdot (h^3 \nabla_y v_0) = \frac{\partial h}{\partial y_1} \text{ in } Y, \]  
(5.286a)

\[ v_0 (x, 0, y_2) = v_0 (x, l_1, y_2) = 0, \quad \frac{\partial v_0}{\partial y_2} \bigg|_{(x,y_1,0)} = \frac{\partial v_0}{\partial y_2} \bigg|_{(x,y_1,l_2)} = 0, \]  
(5.286b)

\[ v_1 (x, 0, y_2) = 0, \quad v_1 (x, l_1, y_2) = 1, \quad \frac{\partial v_1}{\partial y_2} \bigg|_{(x,y_1,0)} = \frac{\partial v_1}{\partial y_2} \bigg|_{(x,y_1,l_2)} = 0, \]  
(5.286c)

\[ \nabla_y \cdot (h^3 \nabla_y v_2) = 0 \text{ in } Y, \]  
(5.286d)

\[ \frac{\partial v_2}{\partial y_1} \bigg|_{(x_0,y_2)} = \frac{\partial v_2}{\partial y_1} \bigg|_{(x_1,y_2)} = 0, \quad v_2 (x, y_1, 0) = 0, \quad v_2 (x, y_1, l_2) = 1. \]

In [66] and [67] these local solutions \( v_i \) are interpreted as local pressures.

To facilitate a comparison between Patir and Cheng flow factors and homogenized coefficients the results presented above are reformulated in the following, see also [68]. Indeed, another way of formulating (5.282) is

\[ \nabla \cdot (A^{pc} \nabla p^{pc}) = \nabla \cdot B^{pc}, \]  
(5.287)

where

\[ A^{pc} = \frac{1}{12 \mu_a} \begin{pmatrix} a_{11}^{pc} & a_{12}^{pc} \\ a_{12}^{pc} & a_{22}^{pc} \end{pmatrix}, \]

\[ a_{11}^{pc} (x) = \frac{1}{l_2} \int_Y h^3 \frac{\partial v_1}{\partial y_1} \, dy, \]  
(5.288a)

\[ a_{12}^{pc} (x) = a_{21}^{pc} (x) = 0, \]  
(5.288b)

\[ a_{22}^{pc} (x) = \frac{1}{l_1} \int_Y h^3 \frac{\partial v_2}{\partial y_2} \, dy, \]  
(5.288c)

and

\[ B^{pc} = \frac{u_s}{2} \begin{pmatrix} b_{11}^{pc} (x) \\ b_{22}^{pc} (x) \end{pmatrix} = \frac{u_s}{2} \begin{pmatrix} 1 & \frac{1}{l_1} \int_Y \left( h - h^3 \frac{\partial v_0}{\partial y_1} \right) \, dy \end{pmatrix}, \]  
(5.289)

and where the functions \( v_i \) solve the local problems defined in (5.286).

An alternative way of presenting the homogenization results in Section [5.8.1] is obtained by introducing the new dependent variable \( \psi_i \);

\[ \psi_i (y) = \frac{y_i + \chi_i (y)}{l_i}, \quad i = 1, 2, \]  
(5.290)
for which (5.257b) and (5.257c) become
\[ \nabla_y \cdot (h^3 \nabla_y \psi_1) = 0 \text{ in } Y, \]
\[ \psi_1(x, 0, y_2) + 1 = \psi_1(x, l_1, y_2), \quad \psi_1(x, y_1, 0) = \psi_1(x, y_1, l_2), \]
\[ \psi_1(x, 0, y_2) + 1 = \psi_1(x, l_1, y_2), \quad \psi_1(x, y_1, 0) = \psi_1(x, y_1, l_2). \]

Hence (5.263)-(5.266) become
\[ a_{11}(x) = \frac{1}{l_2} \int_Y h^3 \frac{\partial \psi_1}{\partial y_1} dy, \]  
\[ a_{12}(x) = \frac{1}{l_1} \int_Y h^3 \frac{\partial \psi_2}{\partial y_1} dy, \]  
\[ a_{21}(x) = \frac{1}{l_2} \int_Y h^3 \frac{\partial \psi_1}{\partial y_2} dy, \]  
\[ a_{22}(x) = \frac{1}{l_1} \int_Y h^3 \frac{\partial \psi_2}{\partial y_2} dy. \]

With this, we have equalised the Patir and Cheng results with the homogenised and they can now be easily compared. It is clear that the methods share quite a few features, and that the main differences are the boundary conditions for the local problems and that the off-diagonal terms of $A^{pc}$ and the $y_2$-direction coefficient of $B^{pc}$ are identical to zero while they are not for the homogenised model. The homogenisation method gives the correct flow-factors for any kind of topography, which fulfils the Reynolds roughness assumption. The Patir and Cheng method will give the same result as the homogenised method for surface topographies that are symmetric in both the $x$- and $y$-directions. To elucidate on this, the three figures 5.20, 5.21 and 5.22 are presented. In Fig. 5.20 the solution of the local problem (5.286b), in the Patir and Cheng method, for a bi-sinusoidal surface is depicted. The figure illustrates the effect of the Neumann boundary conditions, which means that there is no flow over the upper- and lower boundaries. This result should be compared to the solution of the local problem in (5.291a) in the homogenization method, for the same bi-sinusoidal surface patch. This is depicted in Fig. 5.21 illustrating how the periodic boundary conditions allows for flow around the protrusions of the bi-sinusoidal texture as well as to take place over the upper and lower boundaries. Figure 5.22 show what the result becomes when replacing the bi-sinusoidal surface with a symmetric bi-cosinusoidal one. In this case, due to the horizontal symmetry, solving (5.291a) with periodic boundary conditions or (5.286b) with Neumann boundary conditions gives identical results.

5.8.5 Homogenised flow factors for mixed lubrication conditions

In the previous sections the expressions for coefficient functions $a_{ij}$ and $b_i$ in the homogenised matrix $A_0$ and vector $B_0$ were derived. Indeed, the explicit expressions (being the same for each of the three types of fluids considered) are given by (5.263)-(5.268). It is also clear that they are functions of the global coordinates $(x_1, x_2)$ and this is also what couples the local- and the global scale. Obviously, computing the coefficients for each node $(x_{1i}, x_{2j})$, 

Figure 5.20: Streamlines computed from the solution of the local problem in (5.286b) in the Patir and Cheng method. The contour map depicts the film thickness for a bi-sinusoidal surface roughness patch. Red for thinner and blue for thicker film. Arrows visualize fluid velocity.

Figure 5.21: Streamlines computed from the solution of the local problem in (5.291a) in the homogenization method. The contour map depicts the film thickness for a bi-sinusoidal surface roughness patch. Red for thinner and blue for thicker film. Arrows visualize fluid velocity.

in the discretised global domain require solving the set (5.257) of the local problems for all \((i,j)\) belonging to the grid. Although this procedure could be used it is impractical for (at least) the following reasons, i) it renders an unnecessarily large set of data, ii) the values of the coefficient functions are computed only for the points \((x_{1i}, x_{2j})\). Thus if they are not ‘tabulated’ together with the values of the discretised film thickness function \(h\) equated at exactly the same points \((x_{1i}, x_{2j})\), it would not be possible to used them together with another representation of the global scale geometry. However, precisely as Patir and Cheng [66] did,
it is possible to render a more versatile set of coefficient functions by means of a simplistic parametrisation. In [69], Patir presented a routine for generating randomly rough surfaces with Gaussian height distribution and with the possibility of specifying the aspect ratio (or lay) of the topography. This routine was also used to generate the set of surfaces they used to generate the flow factors they presented. In connection to this, Pérez-Ráfols and Almqvist [70], developed a routine that can be used to generate self-affine fractal surfaces with given height distribution and power spectrum. Let us now describe a procedure that can be applied in order to accomplish this.

We start by noting that on the local scale $h_g$ is to be treated as a parameter representing the global scale average interfacial separation. We denote this parameter $\alpha$ and let $h_\alpha$ replace $h$ in (5.243). More precisely, for full film conditions we define $h_\alpha$ as

$$h_\alpha(y, \tau) = \alpha + h_r(y, \tau), \quad \alpha > \bar{h}_r$$

(5.293)

where,

$$h_r := (h_u - h_l) - \min_{v(y, \tau)} (h_u - h_l).$$

(5.294)

to ensure that $h_r \geq 0$ and $\bar{h}_r$ is the arithmetic mean of $h_r$.

For mixed lubrication conditions $\alpha \leq h_r$ and the shape of the gap depend on the contact mechanics between the two rough surfaces being pressed together and before we proceed, we will very briefly explain how $h$ (in (5.243)) is connected to the contact mechanics model. Indeed, let

$$h_d = h_r + u - g_{00}$$

(5.295)

describe the (local scale) gap that between the deformed surfaces $h_u$ and $h_l$, that results due to the application of normal force $F$ pressing them together. Note that $u = u(p_d)$, where $p_d$ is the contact pressure, is the (local scale) displacement of $h_r$ and $g_{00}$ is the rigid
body displacement. Then we can formulate the corresponding (local scale contact mechanics) complementarity problem

\[ h_d = 0, \quad p_d > 0, \quad (5.296a) \]
\[ h_d > 0, \quad p_d = 0, \quad (5.296b) \]

subject to the force balance constraint

\[ F - \int_Y p_d dy = 0. \quad (5.297) \]

Solving (5.296)-(5.297) for a given load \( F \) returns the dependent variables \( p_d, u \) and \( g_{00} \). Recall that (5.295) describes the relation between \( h_d, u \), and \( g_{00} \). Moreover, by solving (5.296)-(5.297) for a whole range of loads gives us the input required to solve the local problems (5.257) for a range of \( \alpha \)-values. To this end, we supplement the \( \alpha \)-parametrised film thickness equation defined for full film conditions with an expression valid for mixed lubrication conditions. This is accomplished by forcing \( h \) to take exactly the values of \( h_d \) that was computed while solving the complementarity problem (5.296) for the specified range of loads \( F_k \). This means that \( h \) for mixed lubrication conditions reads

\[ h_\alpha(y, \tau) = \alpha + h_r(y, \tau) + u(y, \tau) + \epsilon, \quad \alpha = -g_{00} \quad (5.298) \]

where \( \epsilon \) is a (small) auxiliary parameter that makes sure that \( h_\alpha > 0 \). Note now that there are two different specifications of the local scale film thickness, i.e. i) the expression (5.293) for full film conditions and ii) the expression (5.298) for mixed lubrication conditions, which require solving the contact mechanics problem for a range of loads.

We will now summarise the procedure described above, to obtain a widely applicable set of coefficient functions \( a_{ij} \) and \( b_i \). That is,

1. Specify a range of separations \( \alpha > h_r \) and a range of loads \( F \).
2. Solve the complimentarity contact mechanics problem (5.296) for the specified range of loads \( F \).
3. Solve the local problems (5.257) for the auxiliary film thickness descriptions (5.293) and (5.298) for the specified \( \alpha \)-values (including the values of \( g_{00} \) obtained from the contact mechanics model).
4. Compute the coefficient functions \( a_{ij} \) and \( b_i \) for all the local problems obtained in the previous steps.

### 5.9 Modelling mixed lubrication

A mixed lubrication model can be established by by a combination of a model governing the hydrodynamic contribution and a model that accounts for contact mechanics. One of the first examples of such a model was presented by Patir and Cheng in [66,67]. They derived an averaged form of the Reynolds equation, in which the surface roughness was accounted for by
means of what we know today as “flow factors”. Then, in order to simulate partial contact they include the effect of surface roughness by comparing the average interfacial separation $\bar{h}$ with the combined variance, $\sigma = \sqrt{R_{q1}^2 + R_{q2}^2}$, of any two digitised surface roughness height descriptions, which full fill the assumptions for Reynolds equation to be valid on the local scale. Since the effect of roughness diminishes for large values of $\bar{h}/\sigma$ they focus their analysis to situations where $\bar{h}/\sigma < 3$ and their work has been formed the starting point for a large number of similar contributions.

Obviously, Patir and Cheng did not include a contact mechanics model to account for local deformation of the contacting surfaces. This can e.g. be achieved by incorporating an asperity based contact model, which was also done by Rohde et al. in [74]. More precisely, they combined the averaged Reynolds equation that Patir and Cheng derived, with the model by Greenwood and Tripp [72] and considered cavitation by means of the half-Sommerfeld boundary condition. Bolander et al. [73] took this concept further by replacing the half-Sommerfeld condition with cavitation algorithm proposed by Elrod and Adams [37].

A mixed lubrication model based on the homogenised Reynolds equation was later presented by Sahlin et al. [3, 74]. Indeed, this model combines half-space theory based contact mechanics of rough surfaces with the homogenised Reynolds equation in which the flow factors has been obtained in the way described in Section 5.8.5. In addition, to make the methodology even more versatile one can always generalise by transforming $h_\alpha$ into dimensionless form, i.e.

$$H_\alpha = h_\alpha/h_{ref},$$

(5.299)

where $h_{ref}$ is an appropriate reference parameter. The local problems (5.257b) and (5.257c) are invariant under this transformation, meaning that the solutions $\chi_1$ and $\chi_2$ will be the same for any choice of $h_{ref}$. The local problem (5.257a) is, however, not invariant and becomes

$$0 = \nabla_y \cdot \left( \tilde{A}_\alpha \nabla_y \chi_0 \right) - \nabla_y \cdot \tilde{B}_\alpha \text{ in } Y,$$

(5.300)

where $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ indicates that they are transformed and where $\chi_0 = h_{ref}^2 \chi_0$. Moreover the subscript also indicates that they are parametrised in $\alpha$. This means that the homogenised matrix $A_0$ and vector $B_0$ transforms and their transformed correspondences should be computed from

$$\tilde{A}_0^\alpha = \iint_Y \begin{bmatrix} \tilde{a}_{11}^\alpha & \tilde{a}_{12}^\alpha \\ \tilde{a}_{21}^\alpha & \tilde{a}_{22}^\alpha \end{bmatrix} (e_i + \nabla_y \chi_i) dy$$

(5.301a)

$$\tilde{B}_0^\alpha = \iint_Y \tilde{B}_\alpha e_1 + \tilde{A}_\alpha \nabla_y \chi_0 dy$$

(5.301b)

$$\tilde{B}_0^\alpha = \iint_Y \tilde{B}_\alpha e_1 + \tilde{A}_\alpha \nabla_y \chi_0 dy$$

(5.301c)

Before the homogenised solution $u_0$ can be obtained the $\alpha$-parametrised homogenised matrix $\tilde{A}_0^\alpha$ and vector $\tilde{B}_0^\alpha$ must be mapped onto $\Omega$. This is achieved by interpolating $\tilde{A}_0^{h_{\beta}(x_1, x_2)}$ and $\tilde{B}_0^{h_{\beta}(x_1, x_2)}$ for each point $(x_1, x_2)$ in the global scale grid point. Thus

$$\nabla_x \cdot \left( \tilde{A}_0(x) \nabla_x \tilde{u}_0 \right) - \nabla_x \cdot \tilde{B}_0(x) = 0,$$

(5.302)
where \( \tilde{u}_0 = h_{ref}^2 u_0 \).

This closes the last section of this chapter and the authors thanks the readers for the attention.
Chapter 6

Modelling Wear in Lubrication

In mixed and boundary lubrication, there is contact and relative sliding between the surfaces. Because of this interaction between the surfaces, wear will occur. Wear is often defined as the loss of material due to sliding, although there is no agreed clear definition and in some even plastic deformation is included. What is clear is that wear will result on a change of the geometry of a body and of its surface roughness. In turn, this will change the way in which a given machine element operates. It is thus important to understand it and, when possible, reduce it. Understanding and predicting wear, however, is a very complex problem. This is because, besides the contact between surfaces, the analysis of wear also involves complex physics and chemistry. For example, temperature increase, fluid-solid interaction, growth and break down of oxide layers, strain hardening and adhesion are some of the aspects that might be relevant in a general wear problem [75], which makes it very difficult to isolate the effect of any individual parameter. This complexity has often led to a descriptive and qualitative analysis of wear [75], in which the behaviour of a given experiment will be classified according to a certain mechanism. Many such mechanisms have thus arisen in the literature, being abrasive, adhesive and corrosion wear among the most often encountered ones. This variety of behaviours have further lead to a large number of different empirical laws, which depend on an even larger list of parameters [76].

It is not the intent of this compendium to analyse all existing types of wear and the proposed ways to model them. In this chapter we will thus only focus on the modelling of a particular type of wear, which can be easily modelled by Archard’s wear equation. For this, we will first derive this equation for two different types of wear, i.e., adhesive and abrasive wear. Finally, a model for the wear of rough surfaces based on this equation will be presented and its applicability will be discussed.

6.1 Archard’s model for adhesive wear

In adhesive wear, the attractive forces between the surfaces are large enough so that particles can be detached from one or both surfaces. One of the first theories to explain the phenomena of adhesive wear was proposed by Archard [77] and has since been commonly used to model this type of wear. There are several ways to reach the expression Archard arrived to and thus the presentation here might not follow exactly that proposed by Archard. It is, however,
Figure 6.1: Schematics of the adhesive wear situation modelled by Archard.

Start considering the contact between two asperities, as seen in Fig. 6.1. This contact has a size of $2a$ and thus an area $\delta A \propto a^2$, where $\propto$ means proportional to. Note that we use $\delta$ to refer to quantities of an asperity as opposed to quantities related to the whole surface. In this situation, adhesive wear will occur if the two surfaces bond together stronger than the substrate does with itself. In this case, a particle will break lose. In this theory, it is assumed that the depth of the loose part is of a size similar to the contact size. This leads to a volume removed $\delta V \propto a^3$. Equivalently, $\delta V \propto a \delta A$. It is more interesting to write the volume in terms of the contact area as it can be related to the load. In particular, Archard assumed that wear occur in plastically deformed asperities. In this case, it can be assumed that the pressure has saturated to its maximum value possible, i.e., the hardness, $H$, of the softer material. The load carried by the asperity is thus

\[ \delta W = H \delta A. \]  

(6.1)

Therefore, we can write the wear volume of an asperity as

\[ \delta V \propto a \frac{\delta W}{H}. \]  

(6.2)

Now, consider the case of a rough surface. We shall assume that at a given time, a given pair of asperities is in full contact, i.e., all with a contact size $2a$. After sliding a certain distance, the contact becomes smaller until eventually, when the sliding distance is $\delta s = 2a$, the contact is fully lost. It can be assumed, however, that during this time a new contact has been formed. The total wear per unit sliding length will then be the sum of all the asperities in contact at a given time, with a caveat. It is further assumed that wear only occurs at a small proportion $K$ of the asperities in contact. The total wear volume per sliding distance is thus

\[ \frac{dV}{ds} \propto \sum K \frac{\delta W}{H} = K \frac{w}{H}, \]  

(6.3)

where $w$ is the total load applied on the surface. Finally, if we assume that the conditions do not change, i.e., that wear is not altering the surface in a way that the wear properties are changed, then the wear volume should be independent on the sliding distance. Therefore, the total wear volume is

\[ V \propto K s \frac{w}{H}. \]  

(6.4)
6.2. ARCHARD’S MODEL FOR ABRASIVE WEAR

To find out the constant of proportionality, one needs to assume a certain shape of the wear particles created as well as of the contact area. In particular, if assumes that the contacts are perfectly circular and that the particles are semi-spherical, one reaches

$$V = K s \frac{W}{3H}$$  \hspace{1cm} (6.5)

This, however, does not usually match the experimental observations, as the shapes assumed are far from reality. Moreover, computing the probability that an asperity contact will lead to wear, $K$, is by no means a simple task. Therefore, one usually writes

$$V = k_{adhesion} s \frac{W}{H}$$  \hspace{1cm} (6.6)

where $k_{adhesion}$ is known as the wear rate and is usually determined experimentally. Notice that $k_{adhesion}$ is now not only related to the probability that a collision will result in a wear particle but also includes other system-based parameters such as shape of the particles. Moreover, this wear rate can also include some deviation in the assumptions. For instance, it can be shown that, at low loads, the contact area is proportional to the load even for elastic contacts. The different constant of proportionality will then be reflected in a change of $k_{adhesion}$. It is therefore clear that the value of $k_{adhesion}$ is case dependent. Moreover, it depends on the whole system, including the materials in contact, the type of surface roughness, the lubrication in-between the surfaces, the ambient conditions, etc.

### 6.2 Archard’s model for abrasive wear

Another common type of wear is abrasive wear, in which a harder surface scratches a softer one. In this case, an equation similar to (6.6) can be derived from different considerations. As before, we start by considering the wear caused by a single asperity. As shown in Fig. 6.2, the situation now is quite different. A hard, pointy, asperity has indented the softer counter surface. When sliding, this has created a scratch that can be identified as wear. Let us now compute the wear volume caused by this scratch. If we assume, again, that this indentation is caused primarily by plastic deformation, the load $\delta W$ carried by the asperity is

$$\delta W = H \delta A = H \pi a^2,$$  \hspace{1cm} (6.7)
where $H$ is the hardness of the softer surfaces, $\delta A$ is the contact area and $a$ the contact size. Assuming the conical shape of the asperity represented in Fig. [6.2] the volume of the scratch after sliding a length $s$ is

$$\delta V = s\pi a^2 \tan \theta = \frac{\delta W}{H}s \tan \theta.$$  \hspace{1cm} (6.8)

When considering many asperities, the total wear thus becomes

$$V = \sum \delta V = \tan \theta s \frac{W}{H}.$$  \hspace{1cm} (6.9)

Again, the geometrical assumptions made to reach this equation are too restrictive in reality. Therefore, we write

$$V = k_{\text{abrasive}}s \frac{W}{H}.$$  \hspace{1cm} (6.10)

Similarly to the previous case, the interpretation of $k_{\text{abrasive}}$ is far from trivial as it includes the combined effect of different effects such as roughness, material properties, etc. Therefore, it is also obtained via experimental analysis.

### 6.3 Application of Archard’s wear model to rough surfaces

As discussed previously, there are many ways in which wear occurring when two rough surfaces slide can be simulated. Indeed, different types of wear can and should be modelled by different means. Conversely, a particular wear model will only lead to accurate prediction for a limited type of wear conditions. This also concerns the model we introduce here, which is not the only one available. Therefore, although it has shown that it can accurately predict wear in some situations [78], one must ensure that the type of wear modelled fits the capabilities of the model. This said, let us introduce the model.

The model presented is based on Archard’s equation, as presented previously, but applies it locally. The strategy to model wear is depicted in Fig. [6.3]. The wear volume is given by Archard’s equation,

$$V = ksW,$$  \hspace{1cm} (6.11)

where we have included $1/H$ into $k$ for simplicity. Now, this equation is applied in rectangular elements, following the discretisation we presented for the contact mechanics model, see Section 4.4. This element is under a total load

$$W = pdx_1 dx_2,$$  \hspace{1cm} (6.12)

where $dx_1$, $dx_2$ are the sizes of the element and $p$ is the contact pressure, computed through a contact mechanics model such as the one presented in Chapter 4. Now, the wear volume can be assumed to also occur under this rectangular element, thus it can be computed as

$$V = h_W dx_1 dx_2,$$  \hspace{1cm} (6.13)
where $h_W$ is the wear depth. Combining (6.12) and (6.13) with Archard’s equation, (6.11), we obtain an equation for the depth loss due to wear,

$$h = ksp, \quad (6.14)$$

If the time step is sufficiently small to consider $p$ constant through the sliding distance $s$, this equation can be used to numerically compute wear. To do so, a simple algorithm can be used, as presented in Fig. 6.3. At a given time, the relative position between the two surfaces is given. In this position, the contact mechanics model presented in Chapter 4 can be used to compute the contact pressure distribution. Then, (6.14) can be used to compute the amount of depth lost at each point due to wear. Knowing this, the surface can be updated by removing the lost depth. Then, the relative position can be shifted slightly and the process is repeated again.

Let us finish by discussing some of the limitations of this model. The most clear one is that (6.14) contains the wear rate $k$, which can only be obtained experimentally. Now, it is clear from (6.11) that the total wear volume is fixed for a given $k$, $s$ and $w$. As indicated, $k$ is obtained via experiments and $s$ and $w$ are often specified by the operating conditions. The procedure to study wear for a given case would then start by performing an experimental measurement to obtain $k$. This means that the total wear can actually not be predicted. What the model will predict, however, is the distribution of this wear. This can be, in itself, extremely important. For example if the model is applied to a machine component, knowing where the wear will occur can be very helpful in the design phase to understand whether a given amount of wear will critically affect the performance of the component. It can also
inspire a change in design to mitigate the impact of wear on the performance. Also, it can serve to optimize the tolerance in production so that a good performance is retained after a given amount of wear. The model can also be useful in a different scale, i.e., when we study roughness. Again, the total amount of wear can only be found experimentally. Despite that, by providing the distribution of wear, the model provides for the shape of the roughness after wear. Knowing this can be important in a large range of application including seals, bearings, gears, etc.

A second limitation concerns the type of wear that can be modelled. We have, indeed, started deriving Archard’s equation for adhesive wear. In the derivation, however, the final equation is found for the average wear of a rough surface. This clearly can work if the wear of a component is studied, in which the roughness of the surface can be treated in an averaged sense. If one wants to focus on rough surfaces, however, some limitations arise. Indeed, the condition to use (6.14) is that the time step is sufficiently small for the pressure not to vary notably. This, however, cannot be reconciled with the assumption of a large particle being removed at a time. Therefore, the model will have a hard time predicting wear in adhesive conditions. In fact, it can easily be seen that in the case of a rough surfaces sliding against a harder, smoother one, (6.14) will lead, in general, to the smoothing of the softer surface \(^{78}\). Thus, it is clear that the model works at its best when modelling a polishing type wear, which sometimes falls into the category of abrasive wear.
Bibliography


Appendix A

Fourier Techniques

In Section 4.6 we saw that the Fourier transform can be used to accelerate the computations of the elastic deformation caused by a certain pressure distribution. Let us therefore give here a brief reminder of several related concepts such as the Fourier series, the Fourier transform and the Discrete Fourier transform.

A.1 The Fourier series

A Fourier series is a way to represent periodic functions as a summation of infinitely many sinusoidal waves. In particular, if $f(x)$ is periodic with period $(0,L]$, then we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x), \quad \omega_n = \frac{2\pi n}{L}. \quad \text{(A.1)}$$

which is valid as long as $f(x)$ is bounded and piecewise differentiable. Let us now see how to compute the coefficients. For this, we shall use the following results

$$\int_{0}^{L} \cos \left( \frac{2\pi n}{L} \right) \cos \left( \frac{2\pi m}{L} \right) = \frac{L}{2} \delta_{m,n}, \quad \text{(A.2a)}$$

$$\int_{0}^{L} \sin \left( \frac{2\pi n}{L} \right) \sin \left( \frac{2\pi m}{L} \right) = \frac{L}{2} \delta_{m,n}, \quad \text{(A.2b)}$$

$$\int_{0}^{L} \cos \left( \frac{2\pi n}{L} \right) \sin \left( \frac{2\pi m}{L} \right) = 0, \quad \text{(A.2c)}$$

where $\delta_{m,n}$ is one if $m = n$ and zero otherwise. These results can be shown by realizing that the following relations hold

$$2 \cos \left( \frac{2\pi n}{L} \right) \cos \left( \frac{2\pi m}{L} \right) = \cos \left( \frac{2\pi (m+n)}{L} \right) + \cos \left( \frac{2\pi (m-n)}{L} \right), \quad \text{(A.3a)}$$

$$2 \sin \left( \frac{2\pi n}{L} \right) \sin \left( \frac{2\pi m}{L} \right) = \cos \left( \frac{2\pi (n-m)}{L} \right) - \cos \left( \frac{2\pi (m+n)}{L} \right), \quad \text{(A.3b)}$$

$$2 \sin \left( \frac{2\pi n}{L} \right) \cos \left( \frac{2\pi m}{L} \right) = \sin \left( \frac{2\pi (n-m)}{L} \right) - \sin \left( \frac{2\pi (m+n)}{L} \right), \quad \text{(A.3c)}$$
and that the integral of a sine or cosine along one period is zero. It thus becomes clear that the integrals become zero unless \( m = n \). We can now use these results to extract the coefficients from (A.1). To do this, we multiply \( f(x) \) by a cosine and integrate, i.e.,

\[
\int_0^L f(x) \cos \left( \frac{2\pi n}{L} \right) = \int_0^L a_0 \cos \left( \frac{2\pi n}{L} \right) + \sum_{m=1}^{\infty} \int_0^L a_n \cos \left( \frac{2\pi m}{L} \right) \cos \left( \frac{2\pi n}{L} \right) + \sum_{m=1}^{\infty} \int_0^L b_n \cos \left( \frac{2\pi n}{L} \right) \sin \left( \frac{2\pi m}{L} \right). \tag{A.4}
\]

From (A.2), we can see that most of these equations are actually zero, leaving

\[
\int_0^L f(x) \cos \left( \frac{2\pi n}{L} \right) = \frac{L}{2} a_n. \tag{A.5}
\]

Therefore, the coefficients can be computed as

\[
a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{2\pi n}{L} \right), \tag{A.6a}
\]

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{2\pi n}{L} \right), \tag{A.6b}
\]

where the \( b_n \) are obtained in a similar manner, changing only the cosine by the sine function. Note that this equation holds even for \( a_0 \), which is simply twice the mean of the function over a period.

An alternative, yet equivalent representation makes use of the complex numbers. In particular, we can write

\[
f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\omega_n x}, \tag{A.7}
\]

where \( A_n \) is a complex number computed as

\[
A_n = \frac{1}{L} \int_0^L f(x) e^{-i\omega_n x} dx. \tag{A.8}
\]

By exploiting the relation

\[
e^{i\theta} = \cos \theta + i \sin \theta, \tag{A.9}
\]

we can see that

\[
A_n = \frac{1}{2} \begin{cases} a_n + ib_n & n > 0 \\ a_0 & n = 0 \\ a_n - ib_n & n < 0 \end{cases}. \tag{A.10}
\]

Notice that the 1/2 is related to the introduction of negative values of \( n \) in the latter formulation. With this representation it is a bit easier to interpret the coefficients. The modulus of \( A_n, |A_n| = \sqrt{a_n^2 + b_n^2} \) is the amplitude of the sinusoidal function with frequency \( \omega_n \). It therefore tells you how much of that frequency is there in the function \( f \). The phase in \( A_n, \angle A_n = \tan^{-1}(b_n/a_n) \) indicates the relative position of the peaks of each frequency. Note that it is only the relative phase that matters.
A.2 The Fourier transform

Up until now, we have considered periodic functions, in an interval \((0, L]\). Let us now consider how can a similar analysis be applied to non-periodic functions. The result will be the Fourier transform, which is similar but not equal to the Fourier series representation of a periodic function. Notice that considering a non-periodic function can be seen as enlarging the periodicity interval to \((-\infty, \infty)\). We notice that, by doing so, frequencies \(\omega_n\) are no longer discrete but become continuous. To see why, notice that

\[
\omega_{n+1} - \omega_n = \frac{2\pi}{L}, \tag{A.11}
\]

which tends to zero as \(L\) goes to infinity. Clearly, this continuous representation of \(\omega\) means that the summation in \(\text{(A.1)}\), or \(\text{(A.7)}\) will become an integral. It should thus be no surprise that the Fourier transform is

\[
\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \tag{A.12}
\]

and its inverse is

\[
\mathcal{F}^{-1}\{f\}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x}d\omega, \tag{A.13}
\]

Notice that the latter is the equivalent to \(\text{(A.1)}\) and \(\text{(A.7)}\) whereas the former is equivalent to \(\text{(A.6)}\) and \(\text{(A.8)}\).

The relevance of Fourier transform lays in some of its properties, which make it useful in many context, including equations including partial differentials and integrals. Without going into detail, let us comment few of these properties. The first one is linearity, i.e.,

\[
\mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}, \tag{A.14}
\]

which can be deduced from the linearity properties of integrals. It is also interesting to see what does Fourier transform to operations such as derivation and integration. If we define \(f'\) and \(F\) as the derivative and the primitive of \(f\), respectively, we have

\[
\mathcal{F}\{f'\} = i\omega\mathcal{F}\{f\} \tag{A.15}
\]

and

\[
\mathcal{F}\{F\} = \frac{1}{i\omega}\mathcal{F}\{f\}. \tag{A.16}
\]

To see why \(\text{(A.15)}\) holds, one just needs to integrate by parts in the definition of Fourier transform \(\text{(A.12)}\), i.e.,

\[
\mathcal{F}\{f'\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx = -\int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x}dx = i\omega\mathcal{F}\{f\}. \tag{A.17}
\]

The correctness of \(\text{(A.16)}\) then follows from the fact that \(F' = f\).
A.3 The discrete Fourier transform

Up until now, we have presented both the Fourier series representation of a function and the Fourier transform. Both of this, however, operate in a continuous space. In most cases, however, we will be dealing with discrete functions. Let us thus define a discrete function, periodic in the domain \((0, L]\), as

\[ f_n := f(x_n), \quad x_n = n \frac{L}{N}, \quad n = 0, 1, \ldots, N - 1, \tag{A.18} \]

where \(N\) is the number of points used to discretise the domain. In the context of Fourier analysis, the relevant transformation for this discrete function is the Discrete Fourier Transform (DFT). To define it, let us start, again from the Fourier series representation of a periodic function. Assume that the discrete function we consider is also periodic, with period \((0, L] \). Then, we can construct a representation of \(f\) as the sum of sinusoids as

\[ f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i \frac{2\pi}{N} k n} = \mathcal{D}F^{-1} \{ F_k \}. \tag{A.19} \]

Now, let us compare this equation with (A.7) to identify the differences. The first clear one is that the exponent of the exponential part has changed. Notice, however, that the new exponent is simply the discrete version of the previous one, in the sense of (A.18). The interpretation of \(F_k\) is then equal (up to a constant) to the interpretation of \(A_n\), i.e., it is closely related to the amplitude of the sinusoidal waves used to describe \(f\). Another relevant difference is that the summation is now done only from 0 to \(N - 1\) instead of from \(-\infty\) to \(\infty\). This is because with \(N\) points, we cannot resolve frequencies higher than \(\omega_{N/2}\). Note that (A.19) is known as the Inverse Discrete Fourier Transform (IDFT), the operation used to compute the coefficients \(F_k\) is then the Discrete Fourier Transform (DFT), which has the form

\[ F_k = \mathcal{D}F \{ f_n \} = \sum_{n=0}^{N-1} f_n e^{-i \frac{2\pi}{N} kn}. \tag{A.20} \]

The derivation of this equation can be done following the same steps as for the coefficients of the Fourier series.

The Discrete Fourier Transform also has some of the properties we saw for the Continuous Fourier Transform. For example, the DFT is also a linear operation, i.e.,

\[ \mathcal{D}F \{ af_n + bg_n \} = a\mathcal{D}F \{ f_n \} + b\mathcal{D}F \{ g_n \}, \tag{A.21} \]

which derives from the linearity of the sum operator. In some sense, we can also define the derivative and integration of this discrete functions as

\[ \mathcal{D}F \{ f'_n \} = i\omega_n \mathcal{D}F \{ f_n \}, \tag{A.22a} \]

\[ \mathcal{D}F \{ F_n \} = \frac{1}{i\omega_n} \mathcal{D}F \{ f_n \}, \tag{A.22b} \]

where \(\omega_n = \frac{2\pi}{L} n\). Now, of course, the derivative and integration only makes sense in the context of continuous functions, so what do we mean by \(f'_n\) and \(F_n\)? Consider first the
continuous function \( f \) to be an interpolation passing through the points in \( f_n \). To interpolate, the DFT is used, i.e.,
\[
f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{ik2\pi x},
\]
which is the continuous form of (A.18). It is this function which we integrate and derivative and then sample in order to obtain \( f'_n \) and \( F_n \). Note that this interpolation is the best fit one can make in the sense of least square error.

We have started the discussion about the Discrete Fourier Transform considering periodic discrete functions. Let us now show that this assumption is, in fact, a requirement. For this, consider the discrete function
\[
F_k = \mathcal{Df}^{-1} \{ f_n \}
\]
at points \( k' = k + aN \), where \( a \) is an integer.

From (A.19) one then has
\[
F_{k+aN} = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N} (k+aN)n} = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N} (kn+2\pi an)} = F_k,
\]
where we have used that for any integer \( m \)
\[
e^{i2\pi m} = 1.
\]

In a similar manner, we can see that
\[
f_{n+aN} = \sum_{k=0}^{N-1} F_k e^{i\frac{2\pi}{N} (n+aN)k} = \sum_{k=0}^{N-1} F_k e^{i\frac{2\pi}{N} kn} e^{i2\pi ak} = f_n.
\]

It is thus clear that the DFT only makes sense in the context of periodic functions.

Let us now discuss in more detail the effect of a specific difference between the Discrete Fourier Transform and the Fourier series, i.e., the truncation of the summation. As we have seen, the DFT of a function is periodic, with period \( N \) and thus expanding the summation will not provide new information. It is clear, however, that this limits the frequencies that can be considered. This, of course, comes from the discretisation itself. It is impossible to represent a sinusoid with less than two points. Therefore, frequencies above \( N/2L \) are not captured in the discretised function \( f_n \) and cannot either be represented in its DFT. Note that, in reality, capturing them with accuracy will put an even more stringent limit. If the signal contains frequencies larger than this limit, aliasing will occur. This means that the contribution of larger frequencies will be wrongly attributed to smaller ones. This will, of course, distort the representation of the function. From this considerations, and noting the symmetry in the coefficients, we can see that aliasing can be avoided if the signal has a limited frequency range. If \( f_s \) is the maximum frequency present, then we will avoid aliasing if \( \Delta x < 1/2f_s \) or \( N > 2f_sL \). Equivalently, one can say that aliasing is avoided if the signal has no frequencies larger than \( f_s > N/2L \). In this case the amplitudes that would be wrongly attributed to lower frequencies will all be zero and therefore cause no issue. In an equivalent manner, the inverse DFT will be free of aliasing if the real signal is zero for \( x > L/2 \).
A.4 The continuous convolution theorem

A property of Fourier Transforms, particularly relevant to the contact mechanics problem presented in Chapter 4, is the convolution theorem. Two versions of this exist, concerning continuous and discrete convolutions. Let us discuss the former in this section and leave the latter for the next one. Let us then start by defining a continuous convolution. This can be written as

\[ g(x) = h \ast f = \int_{-\infty}^{\infty} h(x-x') f(x') dx'. \]  \hspace{1cm} (A.27)

There are different interpretations of a convolution, depending on the physical phenomena it represents. A common way to interpret it is that it measures the overlap or correlation between \( h \) and \( f \) when they have been shifted a distance \( x \) between each other. Indeed, if \( h(x-x') \) and \( f(x') \) overlap significantly, both have the same sign most of the time and \( g(x) \) will be large. Another way to interpret it, closer to the contact mechanics problem considered here, is that it is adding all the contributions of \( f \), scaled by a weight dependent on the distance from the points at which \( f \) is applied and the one at which we measure. More simply, the convolution can be seen as a mathematical tool with certain properties that one encounters in some physical problem.

Whatever the interpretation, computing the convolution of two functions is not an easy task. A simpler path to the solution, however, can be found through Fourier Transform. Indeed, the Continuous convolution theorem states that

\[ g = h \ast f \leftrightarrow \mathcal{F}\{g\} = \mathcal{F}\{h\} \mathcal{F}\{f\}. \]  \hspace{1cm} (A.28)

To motivate the validity of this relation, one can simply substitute the definition of Fourier transform, (A.12), into the right hand side of the second equality in (A.28),

\[ \mathcal{F}\{h\} \mathcal{F}\{f\} = \int_{-\infty}^{\infty} h(y)e^{-i\omega y} dy \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega(x+y)} h(y)f(x) dx dy, \]  \hspace{1cm} (A.29)

which, by introducing the change of variables \( u = x + y \), becomes

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u} h(u-x)f(x) dx du = \int_{-\infty}^{\infty} e^{-i\omega u} \left[ \int_{-\infty}^{\infty} h(u-x)f(x) dx \right] du = \mathcal{F}\{g\}. \]  \hspace{1cm} (A.30)

A.5 The discrete convolution theorem

Let us in this section consider the convolution of discrete samples. In this case, two types of convolution can be defined, depending on the functions at hand. Considering first periodic functions, the circular convolution is defined as

\[ g_n = h \ast f = \sum_{i=0}^{N-1} h_{n-i} f_i \hspace{0.5cm} n = 0, 1, \ldots, N - 1 \]  \hspace{1cm} (A.31)

Notice that the index \( n - i \) can be negative. It should then be interpreted by means of the periodicity of \( h \). As before, this convolution turns into a multiplication in the Fourier space.
In particular, the discrete convolution theorem states that

\[ g = h \ast f \iff \mathcal{D}F\{g\} = \mathcal{D}F\{h\} \mathcal{D}F\{f\}. \]  
(A.32)

Again, we can motivate the validity of this expression by studying the right-hand side of the second equality,

\[ \mathcal{D}F\{h\} \mathcal{D}F\{f\} = \sum_{n=0}^{N-1} h_n e^{-i \frac{2\pi}{N} kn} \sum_{m=0}^{N-1} f_m e^{-i \frac{2\pi}{N} km} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} h_n f_m e^{-i \frac{2\pi}{N} k(m+n)}. \]  
(A.33)

We can, again, introduce the index change \( l = m + n \) to see that this equals to

\[ \sum_{m=0}^{N-1} \sum_{l=m}^{N-1+n} h_{l-m} f_m e^{-i \frac{2\pi}{N} kl} = \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} h_{l-m} f_m e^{-i \frac{2\pi}{N} kl} = \sum_{l=0}^{N-1} \left[ \sum_{m=0}^{N-1} h_{l-m} f_m \right] e^{-i \frac{2\pi}{N} kl} = \mathcal{D}F\{g\}. \]  
(A.34)

Notice that we have used the periodicity of \( h \) to change the integration limits in the second step.

The linear convolution is another type of discrete convolution in which non-periodic functions are involved. This linear convolution can be computed even with functions of different size, as it assumes that the functions are zero outside the sampled range. For simplicity, however, let us assume that both \( f \) and \( h \) have a size \( N \). Then, the linear convolution can be written as

\[ g_n = \sum_{i=0}^{N-1} h_{i-n} f_i, \quad n = 0, 1, ..., 2N - 2 \]  
(A.35)

Notice that now the resulting function is actually longer than the original. As this convolution applies to non-periodic functions and DFT to periodic functions, the discrete convolution theorem cannot be applied directly. A common approach that allows using the discrete convolution theorem is to replace the original functions for equivalent but periodic ones, in such a manner that the linear convolution becomes a circular convolution. Notice that, as discussed in Section A.3, there will be no aliasing problems as long as the all values are zero after \( L/2 \). Therefore, we can construct the new functions as

\[ f'_n = \begin{cases} f_n, & n = 0, 1, ..., N - 1 \\ 0, & n = N, \ldots, 2N - 1 \end{cases} \]  
(A.36a)

\[ h'_n = \begin{cases} h_n, & n = 0, 1, ..., N - 1 \\ 0, & n = N, \ldots, 2N - 1 \end{cases} \]  
(A.36b)

With this new functions, we can now read the convolution in (A.35) as a circular convolution and thus

\[ \mathcal{D}F\{g\} = \mathcal{D}F\{h'\} \mathcal{D}F\{f'\}. \]  
(A.37)