Stefan Kempisty (1892–1940)

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Abstract

Stefan Kempisty was a Polish mathematician, working on the theory of real functions, set theory, integrals, interval functions and the theory of surface area. In 1919 he defended his Ph.D. thesis, *On semi-continuous functions*, at the Jagiellonian University in Cracow under the supervision of Kazimierz Żorawski. In December 1924 he did his habilitation at the University of Warsaw and continued his work at the Stefan Batory University in Vilnius. Kempisty published over forty scientific papers, three textbooks and one monograph. Kempisty's name in mathematics appears in connection with the definition of quasi-continuous functions, different kinds of continuity of functions of several variables, the classification of Baire, Young and Sierpiński functions, interval functions, and Denjoy or Burkill integrals.

This paper is prepared for a wide range of readers. It is an abridged version of the article written in Polish by the same authors (cf. Jóźwik et al., 2017), where can be found more detailed information.

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Streszczenie

Stefan Kempisty był polskim matematykiem zajmującym się funkcjami zmiennej rzeczywistej, teorią mnogości, całkami, funkcjami przedziału i teorią pola powierzchni. W 1919 roku obronił pracę doktorską *O funkcjach nawpółciągłych* na Uniwersytecie Jagiellońskim w Krakowie, a jego promotorem był Kazimierz Żorawski. W grudniu 1924 roku habilitował się na Uniwersytecie Warszawskim. W latach 1920–1939 pracował na Uniwersytecie Stefana Batorego w Wilnie. Opublikował ponad czterdzieści prac naukowych i trzy podręczniki z analizy rzeczywistej oraz jedną monografię. Reprezentował w swoich pracach i na seminariach szkołę warszawską. Nazwisko Kempistego w matematyce pojawia się w związku z definicją funkcji quasi-ciągłej, różnymi ciągłościami funkcji wielu zmiennych, klasyfikacją funkcji Baire'a, Younga i Sierpińskiego, funkcjami przedziału oraz całkami Denjoy i Burkilli.

Praca ta jest skróconą wersją artykułu napisanego po polsku przez tych samych autorów (cf. Jóźwik et al., 2017), w której można znaleźć bardziej szczegółowe informacje.

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1. Biography

Stefan Jan Kempisty was born on 23 July 1892 in Zamość. He attended a philological junior high school and Trade School in Lublin, from which he graduated in 1909. He completed his university studies in Paris obtaining a bachelor's degree of sciences (*licencié ès sciences*) in 1911. In 1912 the Kempisty family moved to Voronezh (Russia), where Stefan passed a state matriculation exam entitling him to university studies in Russia. In 1914 he received a certificate as a gymnasium teacher issued by the Examination Commission of The Imperial Novorossiysk University in Odessa and necessary for teaching mathematics in high schools. In the years 1915–1917 he continued his mathematical and philosophical studies at the Jagiellonian University (Uniwersytet Jagielloński in Polish=UJ) in Cracow (Kraków in Polish). His teachers included A.L. Birkenmajer, A. Hoborski, W. Natanson, A. Rosenblatt, J. Śleszyński, S. Zaremba, K. Żorawski and others.

In the years 1917–1919 he taught mathematics in private high schools in Warsaw and he conducted pedagogical courses for women.

In 1919 Kempisty received a Ph.D. based on the thesis, *On semi-continuous functions* (written in Polish and unpublished), which was presented at the Jagiellonian University in Cracow under the supervision of Kazimierz Żorawski¹ (cf. Archive of the Jagiellonian University, 1918/1919). (See Figure 2.)

From 1920 he became assistant professor of mathematics at the Faculty of Mathematics and Natural Sciences of the Stefan Batory University (Uniwersytet Stefana Batorego in Polish=USB) in Vilnius.

In December 1924 he received habilitation at the University of Warsaw. His thesis was titled, *About limited derivative functions*, and was published in 1925 as Kempisty (1925c).

On 18 February 1925, he married Eugenia Pogorzelska (1900–1982). Their daughter, Maria Kempisty (1925–1989), was an associate professor at the University of Warsaw and professionally worked on cybernetics and logic. She was the first and is still the best known female cyberneticist in Poland.

From 1 October 1925 Stefan Kempisty (see Figure 1) was appointed extraordinary professor at USB. He was one of the founders of the Vilnius Branch of the Polish Mathematical Society (Polskie Towarzystwo Matematyczne in Polish=PTM) and a member of the Polish Society of Naturalists named after Nicolaus Copernicus. Kempisty was one of the organizers of the Faculty of Mathematics and Natural Sciences at the Society of Friends of Sciences in Vilnius. From 1932 he was a member of the Société Mathématique de France, and from 1934 also a member of the London Mathematical Society.

In 1937 Kempisty became a full professor at the Stefan Batory University in Vilnius. In the following academic years, 1937/1938 and 1938/1939, he was the dean of the Faculty of Mathematics and Natural Sciences of USB. In April 1939 he became a vice president of the Polish Mathematical Society.

At USB Kempisty lectured (cf. Archives of Modern Records, 1919/1937; The Lithuanian Central State Archive, 1936) on the following subjects: Differential and integral calculus (1928–1931), Functions of interval (1928/29), Seminar on the theory of the function of a real variable (1928/29), Func-

¹ Kazimierz (Paulin Kazimierz Stefan) Żorawski (Żórawski) (1866–1953), Polish mathematician, professor at the Lvov Polytechnic (1893), the Jagiellonian University in Cracow (1895–1918), the Warsaw University of Technology (1919–1926), and the University of Warsaw (1926–1935).

Figure 1. Kempisty in 1927 and his signature.

tions of two variables (1929/30), Logical foundations of arithmetic (1929/30), Seminar on the theory of functions of two variables (1929/30), Seminar on the logical basis of mathematics (1930/31), Seminar on applications of set theory to functions of a real variable (1930/31), Set theory (1930/31). Moreover, at USB he ran Seminar III, where the following topics were discussed: functions of a real variable, analytically representable functions, Lebesgue integral, interval functions, set theory and foundations of mathematics (Seminar I was conducted by Wiktor Staniewicz,² Seminar II – by Juliusz Rudnicki³).

Kempisty was very active scientifically. He traveled abroad a lot (as is written in more detail in Section 2) and his scientific works were valued among mathematicians. However, he was not a good teacher. He was not pedagogically gifted and working with young inexperienced mathematicians was very difficult for him. According to his student Leon Jeśmanowicz (1914–1989), Kempisty lectured badly and was even a “pedagogical anti-talent”, but as an examiner he was forgiving (see Jeśmanowicz, 1971, cf. also Przeniosło, 2011).

His only Ph.D. student was M. Krzyżański,⁴ whom Kempisty promoted at USB in 1934. The dissertation was entitled *About the generalized absolute continuous functions of two variables* and included a generalization of Luzin’s theory regarding the Denjoy integral.

² Wiktor (Wiktor Emeryk Jan) Staniewicz (1866–1932), Polish mathematician, professor at the Peter the Great St. Petersburg Polytechnic University (1902–1919) and at the Stefan Batory University in Vilnius (1919–1932; rector 1921–1922).

³ Juliusz Rudnicki (1881–1948), Polish mathematician, professor at the Warsaw University of Technology (1919–1923), the Stefan Batory University in Vilnius (1923–1939), and the Nicolaus Copernicus University in Toruń (1945–1948).

⁴ Mirosław Godzimir Krzyżański (1907–1965), Polish mathematician, professor at the Academy of Mining and Metallurgy (later the Cracow University of Technology) (1949–1955) and the Jagiellonian University in Cracow (1955–1965).

Figure 2. PhD diploma of Stefan Kempisty from the Jagiellonian University in 1919. Scan comes from Archive of the Jagiellonian University (1918/1919).

From 19 September 1939 Vilnius was occupied by the Soviet Army and on 26 October 1939, the Soviets handed over the city to the Lithuanians. The Lithuanians deprived Kempisty of the right of permanent residence and the right to work. On 15 June 1940, the Soviets occupied Vilnius again. Kempisty was arrested by the Lithuanians on the night of 11/12 July 1940 and imprisoned in Vilnius Łukiszki prison. Kempisty was interrogated by the Lithuanians first, and after 20 July 1940 by the Soviets. The conditions in prison were too hard for Kempisty and on 3 August 1940 he committed suicide. A detailed description of the last days of his life, based on documents, can be found in Duda's work (Duda, 2017).

2. Kempisty's participation in conferences and his academic trips

Stefan Kempisty was an active participant in many conferences and very often stayed abroad. Below is the chronological list of conferences in which Kempisty took part (cf. Józwik et al., 2017; Archives of Modern Records, 1919/1937 and The Lithuanian Central State Archive, 1936):

1927 – **The First Polish Mathematical Congress** (Lvov, 7–10 September 1927), where he gave two lectures, *O całkach funkcji przedziału* [On the integrals of interval functions], and, *O pochodnych funkcji przedziału* [On the derivatives of interval functions]. Both talks were published in French (for references see Józwik et al., 2017).

1928 – **The International Congress of Mathematicians (ICM)** in Bologna (3–10 September 1928). Over 30 Polish mathematicians attended the congress (among others: L. Chwistek, A. Hoborski, F. Leja,

Figure 3. Participants of the 2nd Polish Congress of Mathematicians in Vilnius in 1931. In the first row from the right: **Stefan Kempisty**, probably his wife Eugenia, Samuel Dickstein, Arnaud Denjoy and Wacław Sierpiński.

L. Lichtenstein, Sz. Mandelbrojt, J. Sława-Neyman, A. Zygmund). Otton Marcin Nikodym and his wife Stanisława also attended the congress. She actively participated in it and presented the lecture, *Sur une propriété topologique du plan euclidien* [On a topological property of the Euclidean plane].

1929 – **The First Congress of Mathematicians of the Slavic Countries** (Warsaw, 23–27 September 1929), where he gave two lectures, *Całkowanie w przedziale nieskończonym* [Integration in an infinite interval], and *Sur les fonctions ponctuellement discontinues* [On punctually discontinuous functions]. Both talks were published in French (for references see Józwik et al., 2017).

1931 – **The Second Congress of Polish Mathematicians** (Vilnius, 23–26 September 1931), where Kempisty was the secretary general, and in Section 2 he gave the lecture: *O funkcjach kwaziciągłych* [On quasi-continuous functions]. We enclose a photo of the participants of the congress (Figure 3).

1932 – **The Second Congress of Romanian Mathematicians** (Turnu-Severin, 5–9 May 1932), where he delivered the lecture *Sur les fonctions partiellement continues* [On partially continuous functions], which was published as Kempisty (1935a). Apart from him, from Poland there were also A. Rosenblatt, A. Zygmund, O.M. Nikodym and his wife Stanisława, who was a regular participant at the congress and delivered the lecture *Sur une propriété topologique du plan* [On a topological property of the plane].

1932 – **The International Congress of Mathematicians (ICM)** in Zurich (4–12 September 1932). There were 30 Polish mathematicians, among them S. Saks and A. Zygmund.

1937 – **The Third Congress of Polish Mathematicians** (Warsaw, 28 September–3 October 1937), at which Kempisty was elected to the Presidium of the congress. The congress was connected with the jubilee

of the 65th anniversary of Samuel Dickstein's⁵ academic, pedagogical and social activity. The solemn academy dedicated to him took place on the last day of the congress. The Jubilee Book (Jubileusz, 1937) was also published, containing many interesting details about his activity.

Kempisty's additional stays abroad are listed chronologically:

1927 Sorbonne – a three-month scientific stay paid by the fund of the “Ministry of Religious Beliefs and Public Enlightenment” (Ministerstwo Wyznań Religijnych i Oświecenia Publicznego in Polish=MWRiOP) (cf. Przeniosło, 2011).

1934 autumn and 1935 autumn (half a year), 1936 Paris. He collaborated with J. Hadamard, G. Julia, A. Denjoy and also delivered lectures (cf. Przeniosło, 2011).

1 September 1935–20 April 1936: two trimesters in Paris and three months in Cambridge, England.

In the April 1936 report, we read that in November and December 1935, Stefan Kempisty took part in the Hadamard Seminar at Collège de France with a lecture *On the Denjoy integral of functions of several variables*, a summary of which was published in Kempisty (1936b), in the lectures of Borel and Darmon on probability theory.

He took part in the “Topology Seminar” at the Poincaré Institute, led by Professor Gaston Julia. He also participated in special lectures of A. Denjoy, P. Montel, M. Fréchet and G. Julia.

On March 4 and 5, 1936, in the presence of A. Denjoy, P. Montel and G. Darmon, he delivered lectures on nonadditive functions of the interval. Denjoy suggested that Kempisty develop these lectures in the form of a book and publish them as a monograph (Kempisty, 1939a).

3. Scientific achievements of Stefan Kempisty

Stefan Kempisty published one monograph (Kempisty, 1939a), one lithographic script, *Rachunek różniczkowy i całkowy* [Differential and integral calculus] in three volumes published in 1931, and 42 scientific papers in the following languages: Polish (8 papers), English (1 paper) and French (33 papers).

The monograph (Kempisty, 1939a) concerns nonadditive functions of interval. As we mentioned earlier, it was published after his lectures in Paris in March 1936. In this book, Kempisty developed Burkill's ideas of the integral of nonadditive interval functions and used them in the discussion about area and length. Moreover, he examined the Denjoy integral of functions of many variables. (See Figure 4.)

Kempisty published his articles mainly in *Bulletin de la Société Mathématique de France* (3 papers), *Fundamenta Mathematicae* (9 papers), *Comptes Rendus Hebdomadaires des Seances de l'Académie des Sciences* (10 papers), and *Annales de la Société Polonaise de Mathématique* (6 papers), but also in Polish journals *Wektor* (3 papers) and *Wiadomości Matematyczne* (1 paper).

He worked primarily on functions of a real variable, theory of integrals, set theory and foundations of mathematics. He examined different kinds of continuity, analytically representable functions, classification of Baire, Young and Sierpiński,⁶ integrals of Denjoy and Burkill.

Kempisty's name in mathematics appears especially in relation to the definition of quasi-continuity, and the classification of Baire, Young and Sierpiński, and Denjoy integrals. While topology was not among Kempisty's interests, his achievements do include one paper connected with this field (Kempisty, 1935b). It concerns the operations on topological spaces and was probably inspired by Kuratowski's theorem (1922)

⁵ Samuel Dickstein (12 May 1851 Warsaw–28 September 1939 Warsaw), Polish mathematician, pedagogue and historian of science. Professor at the University of Warsaw and founder and publisher of the journals *Wiadomości Matematyczne* [Mathematical News] and *Prace Matematyczno-Fizyczne* [Mathematical-Physical Works].

⁶ Waław (Waław Franciszek) Sierpiński (1882–1969), Polish mathematician, professor at the University of Warsaw (1919–1960), leading figure in point-set topology and one of the founding fathers of the Polish school of mathematics, which flourished between World Wars I and II.

Figure 4. The title pages of Kempisty's book script and monograph from 1931 and 1939.

which asserts that by using alternately the closure and complement operations on a set in a topological space E we may obtain 14 different sets (even if a topological space is $\mathbb{R}$).

The first papers of Kempisty were written in Polish and presented the author's wide knowledge derived from the articles of foreign mathematicians such as: B. Russell, H. Lebesgue, R. Baire, O. Veblen, and D. Hilbert.

In Kempisty (1935b), he began his research concerning properties of different kinds of operations defined on subsets A of a fundamental set E . By an operation he meant a function Φ which uniquely assigns a set $\Phi(A) \subset E$ to a set $A \subset E$. Kempisty defined so-called conjugate operations: We say that two operations Φ, Ψ are *conjugate* if

$$\Phi(E - A) + \Psi(A) = E \quad \text{and} \quad \Phi(E - A) \cdot \Psi(A) = 0.$$

With the present denotation the above formulas have the following form:

$$\Phi(E \setminus A) \cup \Psi(A) = E \quad \text{and} \quad \Phi(E \setminus A) \cap \Psi(A) = \emptyset$$

(it is easy to see that in any topological space E interior and closure are such operations: $\Phi(A) = \text{Int}(A)$ and $\Psi(A) = \text{Cl}(A)$). We say that a set A is *open of type* Φ if $A \subset \Phi(A)$, it is *closed of type* Ψ if $\Psi(A) \subset A$. Having the above definitions, Kempisty considered compositions of these operations, their monotonicity, additivity and multiplicativity.

3.1. The Baire, Young and Sierpiński classifications

In the monograph of Baire one can find a well-known classification of functions, according to which

1. a continuous function belongs to the class 0,
2. a function belongs to the class α , if it is the limit of a sequence of functions belonging to classes with numbers less than α (on condition that this function does not belong to any of the classes of smaller numbers).

The sum of finite number of functions of class α is also a function of class α . If instead of the limit we consider the absolutely convergent series, then we obtain the Sierpiński classification. The classes of the Sierpiński classification were called by Kempisty *the absolute classes*.

Another classification of functions was presented by W.H. Young who used monotone sequences. The limits of nonincreasing sequences of functions were called *the functions of type u* (as it was an upper semi-continuous function), and the limits of nondecreasing sequences of functions were called *the functions of type l* (lower semi-continuous). Then the limits of nonincreasing (nondecreasing) sequences of functions of type l were denoted by ul (ll), etc. (in the first place Young put the letter corresponding to the last sequence).

These denotations were sufficient for Young because he considered the functions obtained by a finite number of limits. The Young classification may be simplified if we do not demand taking the elements of the sequences of the same type (if there are infinitely many of them we may take a subsequence of the same type). In the paper (Hahn, 1921), functions of Young class α were called *functions of α range*. As was written above, the continuous functions are of zero class, limits of monotone sequences of range less than α are of the range α (unless they are of a lower range). Functions of α range are of type l or u depending on whether or not they are limits of nonincreasing or nondecreasing sequences.

These classifications were the basis of two papers: Kempisty (1921a,b), published in the second volume of *Fundamenta Mathematicae*.

In the first one Kempisty considered the Baire classification and proved that any function of Baire class one is the limit of a nonincreasing (nondecreasing) sequence of lower (upper) semi-continuous functions. Moreover, he showed that any function which is a limit of a monotone sequence of semi-continuous functions (of the same type) is expandable into an absolutely convergent series of continuous functions. It follows from here that any function of Baire class one may be presented as an absolutely convergent series of continuous functions, so it is of the Sierpiński second class. Simultaneously, since any semi-continuous function is a limit of a monotone sequence of continuous functions, it can also be classified as of Young class one. It turns out that those three classifications: Baire, Young and Sierpiński include all the functions analytically representable (that is which may be obtained by addition, multiplication or by taking limits), which results from the fact that:

1. Every function of Baire class α is at most Young range $\alpha + 1$ (of both possible types).
2. Every function of Young range α is at most α absolute class (of Sierpiński classification).
3. Every function of Baire class α is at most Sierpiński class $\alpha + 1$.

Using modern mathematical language we may describe α classes of Baire (B_α) and Sierpiński (S_α) classifications in the following way: If $f: X \rightarrow \mathbb{R}$, where X is a metric space, then

$B_0(X) = S_0(X) = C(X)$ – the class of continuous functions,

$B_\alpha(X) = \left\{ \lim_{n \rightarrow \infty} f_n : (f_n) \text{ – a sequence convergent for any } x \in X, f_n \in B_{\alpha_n}(X) \text{ for } \alpha_n < \alpha \right\},$

$S_\alpha(X) = \left\{ \sum_{n=1}^{\infty} f_n : \sum_{n=1}^{\infty} |f_n| < \infty \text{ for any } x \in X, f_n \in S_{\alpha_n}(X) \text{ for } \alpha_n < \alpha \right\}.$

Then $S_1(X)$ is the class of differences of upper (or lower) semi-continuous functions. Evidently, $S_\alpha(X) \subset B_\alpha(X)$ for any $\alpha < \omega_1$.

In the paper (Kempisty, 1921a) Kempisty posed a problem, whether $S_\alpha([0, 1]) \neq B_\alpha([0, 1])$ for any α such that $1 \leq \alpha < \omega_1$. W. Sierpiński (1921) and S. Mazurkiewicz (1921) showed (independently) that $S_1([0, 1]) \neq B_1([0, 1])$. Kempisty's problem for $\alpha > 1$ was solved by the Polish mathematician M. Morayne much later in the paper (Morayne, 1992). He presented the following result: In the α class of the Baire classification of Borel measurable real functions defined on some uncountable Polish space (a *Polish space* is a separable completely metrizable topological space) there is a function which cannot be expressed as a countable union of functions from the α class in Sierpiński's classification. Moreover, Morayne showed that for every $\alpha < \omega_1$, in Sierpiński's α class there is a function which cannot be expressed as a countable union of functions each of which is in one of the two Young's α classes. Morayne did not assume that classes with different numbers were disjoint.

F. Medvedev mentioned the article (Kempisty, 1921a) in the book (Medvedev, 1991, p. 167) in his history of real functions.

In Kempisty (1921b) Kempisty considered some results obtained independently by Mazurkiewicz and de la Vallée Poussin. They proved that for any bounded function of Baire class one, defined on $\mathbb{R}$, and arbitrary $\varepsilon > 0$ there exists a function φ which is the difference of two upper semi-continuous functions and

$$|f(x) - \varphi(x)| < \varepsilon \quad \text{for } x \in \mathbb{R}.$$

Stefan Kempisty generalized this result to unbounded functions.

Theorem 1 (Kempisty, 1921). *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function of Baire class one. Then for any $\varepsilon > 0$ there exists a function $\chi(x)$ such that it is the difference of two upper semi-continuous functions and $|f(x) - \chi(x)| < \varepsilon$ for all $x \in \mathbb{R}$.*

To prove this theorem, first Kempisty showed that if $f_1 < f_2$ and f_1, f_2 are functions of Baire class one, then there exists a function f which is the difference of two upper semi-continuous functions such that

$$f_1(x) \leq f(x) \leq f_2(x).$$

Then for $f_1(x) = f(x) - \varepsilon$ and $f_2(x) = f(x)$ there exists a function $\chi(x) = \varphi(x) - \psi(x)$ (φ, ψ are upper semi-continuous) which fulfills, the inequalities $f_1(x) \leq \chi(x) \leq f_2(x)$.

The theorem of Kempisty was generalized by Sierpiński (1924). It was done in the following way: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is of Baire class $\leq \alpha$, then for any number $\varepsilon > 0$ there is a function $g: \mathbb{R} \rightarrow \mathbb{R}$ of Baire class $\leq \alpha$ such that $g(x)$ is a multiple of ε for any $x \in \mathbb{R}$ and $|f(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$.

Kempisty examined functions of Baire class one, and in Kempisty (1921b) he also showed how between two functions of Baire class one $f_1 < f_2$ one can insert a function f ($f_1 \leq f \leq f_2$) that is the difference of two upper semi-continuous functions.

Considerations concerning functions which are the difference of two upper semi-continuous functions may be found in the work (Sierpiński, 1921). In that paper the author proved that any function $f: \mathbb{R} \rightarrow \mathbb{R}$ is expandable into absolutely convergent series if and only if it is a difference of two upper semi-continuous functions (so the first class of Sierpiński's classification consists of the differences of upper semi-continuous functions).

In 1908, Young proved so-called *Rome theorem*, named after the city where it was presented by Young for the first time. It was among the first theorems concerning arbitrary functions without any additional properties.

For any real function of a real variable Young denoted by $C^+(f, x)$ the set of all numbers which are limits of sequences $f(x_n)$ for $x_n \rightarrow x^+$. Analogously, $C^-(f, x)$ stands for the set of all numbers which are

limits of sequences $f(x_n)$ for $x_n \rightarrow x^-$. Young proved that for any function f there exists a denumerable set E , such that the equality $C^-(f, x) = C^+(f, x)$ holds for all $x \in \mathbb{R} \setminus E$.

In the work (Kempisty, 1924) instead of ordinary limits Kempisty considered approximate limits and proved that for any function $f: \mathbb{R} \rightarrow \mathbb{R}$ the set of points such that left- and right-hand approximate limits exist and are different is denumerable. Recall that y is the approximate limit of a function f at a point x_0 if there is a measurable set A such that $\lim_{h \rightarrow 0+} \frac{m(A \cap [x_0-h, x_0+h])}{2h} = 1$ and $y = \lim_{x \in A, x \rightarrow x_0} f(x)$.

In 1925, Sierpiński (1925) presented an example of a function of two variables which was upper semi-continuous with respect to each variable and nonmeasurable. Kempisty (1929b) showed that the measurability of a function is preserved if it is upper semi-continuous with respect to one variable and lower semi-continuous with respect to the other one. More precisely, Kempisty proved the following

Theorem 2 (Kempisty, 1929). *If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is upper semi-continuous with respect to one variable and lower semi-continuous with respect to the other one, then f is at least of Baire class one.*

The proof took place in several steps. Firstly Kempisty showed that if f is lower semi-continuous with respect to x and $M_k^{(y)}(x_0, y_0)$ stands for the upper limit of the function $\varphi(y) = f(x_0, y)$ on the interval $y \in (y_0 - k, y_0 + k)$, where $k \in \mathbb{R}$, then $M_k^{(y)}$ (as a function of two variables) is lower semi-continuous. Simultaneously, if $m_k^{(y)}(x_0, y_0)$ denotes the lower limit of φ on the same interval and f is upper semi-continuous with respect to x , then $m_k^{(y)}$ is upper semi-continuous. In the proof of the theorem Kempisty assumed, without loss of generality, that f is lower semi-continuous with respect to x and is upper semi-continuous with respect to y . Kempisty noticed that for any $n \in \mathbb{N}$ and $(x, y) \in \mathbb{R}^2$

$$m_{\frac{1}{n}}^{(x)}(x, y) \leq f(x, y) \leq M_{\frac{1}{n}}^{(y)}(x, y).$$

Moreover, the sequence $\{M_{\frac{1}{n}}^{(y)}\}_{n \in \mathbb{N}}$ is nonincreasing, so $M^{(y)} = \lim_{n \rightarrow \infty} M_{\frac{1}{n}}^{(y)}$ is the limit of a nonincreasing sequence of lower semi-continuous functions. Analogously, $m^{(x)} = \lim_{n \rightarrow \infty} m_{\frac{1}{n}}^{(x)}$ is the limit of a nonincreasing sequence of upper semi-continuous functions. Therefore $M^{(y)}$ is of type *ul* and $m^{(x)}$ of type *lu* in Young's classification. Since $m_{\frac{1}{n}}^{(x)} \leq M_{\frac{1}{n}}^{(y)}$ and $m_{\frac{1}{n}}^{(x)}$ is upper semi-continuous, $M_{\frac{1}{n}}^{(y)}$ is lower semi-continuous, so for any $n \in \mathbb{N}$ there exists a continuous function f_n such that

$$m_{\frac{1}{n}}^{(x)}(x, y) \leq f_n(x, y) \leq M_{\frac{1}{n}}^{(y)}(x, y)$$

for $(x, y) \in \mathbb{R}^2$. Hence, we obtain a sequence of continuous functions $\{f_n\}_{n \in \mathbb{N}}$ converging to the function f , so f is at least of Baire class one.

Baire and Young classifications also appear in the paper (Kempisty, 1934b), in which he examined the classification of partial derivatives of a function of two variables and the measurability of Dini derivatives. In 1922, S. Banach proved that the Dini derivatives of measurable functions are measurable. A characteristic function of a nonmeasurable set is an example of a function with nonmeasurable Dini derivatives. J. C. Burkill and U. S. Haslam-Jones in 1932 showed the measurability of partial derivatives of functions of two variables, while Kempisty proved (Kempisty, 1934b, Theorem 5) that if $f(x, y)$ is defined and measurable on a set E , then its extreme partial derivatives $\underline{D}_x f$ and $\overline{D}_x f$ are also measurable on E .

3.2. Quasi-continuous functions in the sense of Kempisty

The idea of quasi-continuous functions in the sense of Kempisty, presented in 1932 in Kempisty (1932b), became a permanent notion. The paper in which it was widely presented for the first time is Kempisty's most-cited article. It had a huge impact on the work of many mathematicians. This article is cited in many books: Bruckner (1978), Thomson (1994), Hart and Nagata (2004) and in a lot of papers, among others in Piotrowski (1987/1988), Lee and Piotrowski (1985/1986, 1988) and Grande (2006) (for more information see Jóźwik et al., 2017).

The definition of quasi-continuous functions was already introduced in the talk of Kempisty during The Second Congress of Polish Mathematicians in Vilnius, September 1931. Let us note that according to Kempisty (Kempisty, 1932b), the notion of quasi-continuity has its beginning in the paper (Hahn, 1919). However, it was used by Baire in 1899 (Baire, 1899) in connection with the study of continuity points of functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ which are continuous with respect to each variable separately (see also Piotrowski, 1987/1988, p. 91).

Definition 1 (Kempisty, 1932). A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is quasi-continuous (in the sense of Kempisty), if for any $\varepsilon > 0$, any real x and any open neighborhood U of x there exists a nonempty open set $V \subset U$ (V need not contain x), such that $|f(x) - f(y)| < \varepsilon$ for all $y \in V$.

A similar definition applies to functions defined on $\mathbb{R}^n$.

It is evident that all continuous functions as well as one-sided continuous functions are quasi-continuous. The following function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is an example of function quasi-continuous at the point $x = 0$ but discontinuous at this point.

Kempisty proved a classical theorem about quasi-continuity:

Theorem 3 (Kempisty, 1932). Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. If the functions $f_x(\cdot) = f(x, \cdot)$ and $f^y(\cdot) = f(\cdot, y)$ are quasi-continuous on $\mathbb{R}$, then f is quasi-continuous on $\mathbb{R}^2$.

In 1961, Kempisty's definition was generalized by N. Martin (1961): Assume that X and Y are topological spaces; we say that $f: X \rightarrow Y$ is *quasi-continuous* if the preimage of any open set is quasi-open, which means it has a dense interior. In the same year S. Marcus showed (Marcus, 1961) that there exists a quasi-continuous function $f: [0, 1] \rightarrow \mathbb{R}$ which is not Lebesgue measurable. In the same paper he proved that for any ordinal number $\alpha < \omega$ there is a quasi-continuous function $f: [0, 1] \rightarrow \mathbb{R}$, which belongs to the Baire class α and does not belong to any Baire class with a smaller number.

A wide survey of quasi-continuity with numerous generalizations (for example for multivalued functions) was presented by Tibor Neubrunn in Neubrunn (1988–1989).

3.3. Approximate continuity

The definition of approximate continuous function was introduced and examined by Denjoy in his paper about derivatives (Denjoy, 1915). A function f defined in a neighborhood of x_0 is *approximate continuous* at x_0 , if the set $\{x \in E: |f(x) - f(x_0)| < \varepsilon\}$ has density 1 at this point for any $\varepsilon > 0$.

Denjoy proved that a function f measurable and bounded on the interval $[a, b]$ is equal to the derivative of its integral at each point of approximate continuity of f (one can find the proof of this fact in Kempisty's paper (Kempisty, 1925c, pp. 18–19) and the book (Bruckner et al., 1997, Theorem 7.36)).

In Kempisty (1925c) Kempisty defined quasi-approximate continuity: a function f is called *quasi-approximate continuous* at a point x_0 , if for any $\varepsilon > 0$ the set $\{x \in E : |f(x) - f(x_0)| < \varepsilon\}$ has upper density 1 at this point. He also proved the analogue of the Denjoy theorem (see (Kempisty, 1925c, pp. 25–26)).

Theorem 4 (Kempisty, 1925). *A function bounded on the interval $[a, b]$ is equal to a derived number (Dini derivative) of its integral at any point of quasi-approximate continuity.*

Both papers: Kempisty (1925c) and its abridged version (Kempisty, 1925a) were sent to journals in December 1922 and published in 1925. The first one was Kempisty's habilitation thesis.

In Kempisty (1925b) Kempisty defined extreme approximate limits based on the notion of infimum and supremum neglecting sets of density λ for $\lambda \in (0, 1)$. He proved that *extreme approximate limits belong to the third class of Young classification*. Moreover, he simplified the proofs of theorems: of Denjoy (Denjoy, 1915) – *An approximate continuous function is of Baire first class*, and of Carathéodory (Carathéodory, 1918) – *A measurable function is almost everywhere equal to a function of the second class of Young*.

In Kempisty (1926) Kempisty proved generalizations of the following theorems: The Newman theorem about derivatives of functions with bounded variation (1923) – *If the regular derivative of a set function is integrable then it is approximate continuous*, and the well-known Denjoy theorem (Denjoy, 1915) – *A bounded approximate continuous function is a regular derivative of its integral*.

In another work (Kempisty, 1929a) Kempisty formulated a new property of regular derivative. From this property he obtained continuity (in the sense of Darboux) of a derivative and of an approximate continuous function.

In 1922, W. Sierpiński showed that the set of points x at which $f_+(x) \neq f_-(x)$ is denumerable. Two years later, Kempisty (1924) obtained the analogous result for the set of points, at which the upper approximate right-hand limit is less than the lower approximate left-hand limit.

3.4. Kempisty integral

Let a function $F : [a, b] \rightarrow \mathbb{R}$ be differentiable. It is known that if its derivative F' is Riemann integrable, then

$$\int_a^x F'(t) dt = F(x) - F(a) \quad \text{for any } x \in [a, b], \quad (3.1)$$

but *a priori* F' need not be Riemann integrable. If we consider the Lebesgue integral instead of the Riemann integral it is still necessary to have Lebesgue integrability of F' , which may be not fulfilled by a function, as we know that there exist functions F' which are not Lebesgue integrable.

Mathematicians kept searching for such a definition of integral on $[a, b]$, so that any derivative is integrable (in the sense of this definition) on $[a, b]$ and the condition (3.1) is fulfilled. It led to the definition of the Denjoy integral in 1912 (so-called *Denjoy integral in the narrow sense*), Perron integral in 1914, and independently to the Henstock and Kurzweil integrals (1955 and 1957). It turned out that all these definitions are equivalent.

In the 1930s in Kempisty (1934a) Kempisty introduced two-dimensional Denjoy integral, sometimes called the *Denjoy–Kempisty integral* or simply the *Kempisty integral*. This theory was developed in Kempisty (1936c) and the case of n variables was introduced in the following way. Take any set $R \subset \mathbb{R}^n$ and the family $\mathcal{S}$ of all intervals $I = [a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$, which have disjoint interiors. By a partition of the set R we mean a finite subfamily $\mathcal{S}_0$ of $\mathcal{S}$, such that $R = \bigcup_{\mathcal{S}_0} I$. Let $|I|$ be the measure of interval I . Put

$p(I) = |I|/|K|$ where K is the smallest cube containing I . The number $p(I)$ is called the regularity factor of the interval I and $p(\mathcal{S}) = \max p(I)$ for $I \in \mathcal{S}$ is called the regularity factor of the family $\mathcal{S}$. Consider an interval function $F(I)$ defined on $\mathcal{S}$, additive and continuous in the sense of Saks (continuity in the sense of Saks means that $F(I) \rightarrow 0$ for $|I| \rightarrow 0$). If for a fixed regularity factor $p(\mathcal{S})$ and all partitions $\mathcal{S}_0$ of the set R there exists the limit of the sum $\sum_{I \in \mathcal{S}_0} F(I)$ as $|I| \rightarrow 0$ and it is finite, then this limit is called the Kempisty integral of F .

Functions F integrable with $p(\mathcal{S}) = \frac{1}{2}$ were called by Kempisty regular integrable. In Kempisty (1934a) he examined the properties of such integrals and he proved that any continuous function is regular integrable. He showed that his integral is additive and homogeneous. Moreover, he proved that any regular integrable function on the set R is also regular integrable on any $R_0 \subset R$.

The results obtained by Kempisty were used by Ralph Henstock in his Ph.D. thesis (Henstock, 1948). One of its chapters (Chapter 3 titled *Special functions*) was constructed analogously as the paper (Kempisty, 1934a). Henstock considered some types of integrals of interval functions and examined their properties for the functions with bounded variation, absolutely continuous functions, absolutely semi-continuous functions and functions integrable with respect to some set (following Kempisty it was called *around a set*).

The whole Chapter 4 of Ostaszewski's book (Ostaszewski, 1986) was devoted to the Kempisty integral (it was titled *The integral of Kempisty*). Ostaszewski considered the integral for the regularity factor $p(\mathcal{S}) \geq \rho$, where $\rho \in (0, 1)$ was a fixed number. He showed that any function integrable in the Kempisty sense is Lebesgue integrable on a certain nondegenerate closed interval. Moreover, he presented the definition of the Kempisty integral with a help of the suitable differentiation (derivation) basis.

The Kempisty integral (or the Kempisty–Denjoy integral) is mentioned in the books: Benedetto (1975), Ostaszewski (1986), Čelidze and Džvaršeišvili (1989), Benedetto and Czaja (2010), and in some articles (cf. Józwik et al., 2017).

3.5. Theory of surface area

It would seem that we may calculate the area $A(S)$ of a bounded surface S by using the same method as for calculation the length of a curve. The length of the continuous curve $f : [a, b] \rightarrow \mathbb{R}^n$ can be defined as the supremum or the limit of the polygons, that is, the sum of line segment lengths for a normal partition of $[a, b]$ given by $\sum_{k=1}^n |f(t_k) - f(t_{k-1})|$. If this number is finite, the curve is called rectifiable (has finite length). If f is continuously differentiable on $[a, b]$, then the length of f is always finite.

It turns out that we can not directly transfer both definitions of the length of a curve to the case of surface area. One definition would state that the area $A(S)$ is the upper limit of the areas $A(S_k)$ for all inscribed polyhedra (but then, even in simple cases we may obtain $+\infty$). The second definition would specify the area $A(S)$ as a limit of the sequence of numbers $A(S_k)$, but then such a limit may not exist even with the assumption that the diameters (we mean maximum of diameters) of the polyhedra tend to zero.

One of the subtleties of surface area is that it cannot be defined simply as the limit of areas of polyhedral shapes approximating a given smooth surface. This definition was shown to be untenable by H.A. Schwarz (1880) and independently by G. Peano (1882). It was demonstrated that already for the cylinder, different choices of approximating flat surfaces can lead to different limiting values of the area (this example is known as Schwarz boot). Moreover, while for piecewise smooth surfaces there is a unique natural notion of surface area, if a surface is very irregular, or rough, then it may not be possible to assign an area to it at all.

Since the publication of Schwarz's counterexample became known, a great number of new interpretations for a theory of surface areas have been proposed. In 1902, H. Lebesgue defined surface area in the following way. He considered all sequences of polyhedra $(S_k)_{k \in \mathbb{N}}$, which converge uniformly to S and by the area of S he means the number $A(S) = \inf[\liminf_{k \rightarrow \infty} A(S_k)]$, where the infimum is taken over all such sequences of polyhedra. This definition of surface area circumvented the contradictions posed by the

cylinder area paradox and also maintained the surface area concept as basically geometrical. According to the method of approximating the area $A(S)$ by areas of inscribed polyhedra, we obtain two definitions: by Lebesgue and Geöcze. Both are presented in Radó (1932).

Using the methods mentioned above we obtain the following formulas describing the area of S . For any surface S given by a continuous function $z = f(x, y)$ for $(x, y) \in D$, where a domain $D = [a, b] \times [c, d]$, with the assumption that the partial derivatives f_x and f_y are also continuous on D , so that surface S is smooth and has a nonvertical tangent plane at $(x, y, f(x, y))$ for any $(x, y) \in D$, then the area of S is equal to

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2} dx dy. \quad (3.2)$$

In the case of parametric equations describing S : $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $(u, v) \in P$, with the rectangular $P = [a, b] \times [c, d]$ lying in the uv -plane: If the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives in P , then the area of S is given by formula

$$A(S) = \iint_P \sqrt{X^2 + Y^2 + Z^2} dudv \quad (3.3)$$

(X, Y, Z stand for the Jacobians: $X = \frac{\partial(y, z)}{\partial(u, v)}$, $Y = \frac{\partial(z, x)}{\partial(u, v)}$, $Z = \frac{\partial(x, y)}{\partial(u, v)}$).

The formulas (3.2) and (3.3) raise questions connected with the wider class of surfaces. Generalizations were carried out in two directions. Primarily, if S was given by the formula

$$S: z = f(x, y), \quad \text{where } (x, y) \in D, D \subset \mathbb{R}^2, \quad (3.4)$$

one tried to weaken the assumptions about the existence and continuity of partial derivatives of f and searched for more general conditions so that the surface area was finite and given by (3.2). In 1926, L. Tonelli proved that the area given by (3.4) is finite if and only if the function f is of bounded variation in D (variation in the sense of Tonelli). Let us note that if the surface (3.4) has finite area, then the partial derivatives f_x and f_y exist almost everywhere and the integral (3.2) exists in the sense of Lebesgue. In the same year Tonelli also proved that if S has finite Lebesgue measure, f is a function of bounded variation in D , partial derivatives f_x, f_y exist almost everywhere and are Lebesgue integrable in D , then

$$A(S) \geq \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2} dx dy$$

and the equality holds if and only if f is absolutely continuous in D (see Saks, 1933, Chapter VI).

The second direction of generalization was connected with surfaces described by parametric formulas. It turned out that the formula (3.3) holds if the functions describing S fulfill the Lipschitz condition. However, the necessary and sufficient conditions for the area to be finite and possibly described by (3.3) were not known.

In (Kempisty, 1936d), he gave an application for a rectangular function (analogous to an interval function) to the designation of an area of a surface (3.4). For fixed $h, k > 0$ we may consider the rectangle $R = [a, a + h] \times [b, b + k]$. We say that it is semi-regular if $1/2 \leq h/k \leq 2$. Assume that the surface S may be covered by the family of semi-regular rectangles $R_1, \dots, R_n$ such that they have disjoint interiors and F is an additive rectangular function. In analogy with Kempisty's examination of integrals of interval functions (Kempisty, 1927, 1929a, 1931a,b, 1932a, 1936b,c, 1938a), in that paper he examined semi-regular

integrals of rectangular functions F on the surface S and also considered absolutely semi-regular functions. Among other things he proved the following.

Theorem 5 (Kempisty, 1936). *If F is absolutely semi-regular continuous on R , then*

$$\int_R F = \iint_R \sqrt{X^2 + Y^2 + Z^2} dx dy.$$

In the same article he expanded his method on families of triangles: A rectangular R is divided into two rectangular triangles T_1 and T_2 in such a way that the triangle T_1 has vertices with coordinates (a, b) , $(a + h, b)$ and $(a + h, b + k)$, the vertices of T_2 are the points (a, b) , $(a, b + h)$ and $(a + h, b + k)$. The inscribed polyhedra had triangular faces and their projections on the plane xy were just the triangles T_1 and T_2 . Kempisty showed that the introduction of triangle functions (in the same way as interval functions) allowed him to extend the surface classes to which his method (called *the triangulation method*) could be used.

Kempisty continued his considerations in Kempisty (1945). This paper concerned the Lebesgue area $A(S)$ of a surface S given by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, where the functions x, y, z are continuous on the square $Q: 0 \leq u, v \leq 1$ and have partial derivatives almost everywhere on Q .

Kempisty's results presented in Kempisty (1936a,d, 1938b, 1939b, 1945) were discussed in the books (chronologically) (Saks, 1937; Radó, 1948; Cesari, 1956), and in some papers (cf. Józwik et al., 2017).

3.6. Kempisty spaces

The notion of quasi-continuity introduced in Kempisty (1932b) was used for defining *Kempisty space*. This definition was introduced in 1999 by V.K. Maslyuchenko in his habilitation thesis (Maslyuchenko, 1999, in Ukrainian). It may be found also in the more readily available article (Mykhaylyuk, 2006) written by V.V. Mykhaylyuk and published in 2006.

A topological space Y is called a *Kempisty space*, if for any Baire space X a function $f: X \times Y \rightarrow \mathbb{R}$ quasi-continuous with respect to the first variable and continuous with respect to the other one has the Namioka property. We say that $f: X \times Y \rightarrow \mathbb{R}$ has the *Namioka property* if there exists a set A of type G_δ and dense in X such that f is continuous at any point of the set $A \times Y$. In Mykhaylyuk (2006) Mykhaylyuk showed that the product of any family of compact Kempisty spaces is also a Kempisty space. This subject was continued in Mirmostafae (2011).

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