Weakly compact sets and weakly compact pointwise multipliers in Banach function lattices

Karol Leśnik1,2 | Lech Maligranda2,3 | Jakub Tomaszewski2

1 Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Uniwersytetu Poznańskiego 4, Poznań 61-614, Poland
2 Poznan University of Technology, Institute of Mathematics, Piotrowo 3A, Poznań 60-965, Poland
3 Luleå University of Technology, Department of Engineering Sciences and Mathematics, SE-971 87 Luleå, Sweden

Correspondence
Karol Leśnik, Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965, Poznań, Poland.
Email: klesnik@wp.pl

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Abstract
We prove that the class of Banach function lattices in which all relatively weakly compact sets are equi-integrable sets (i.e. spaces satisfying the Dunford–Pettis criterion) coincides with the class of 1-disjointly homogeneous Banach lattices. New examples of such spaces are provided. Furthermore, it is shown that Dunford–Pettis criterion is equivalent to de la Vallee Poussin criterion in all rearrangement invariant spaces on the interval. Finally, the results are applied to characterize weakly compact pointwise multipliers between Banach function lattices.

KEYWORDS
Banach function spaces, Orlicz spaces, pointwise multipliers, rearrangement invariant spaces, weakly compact sets

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1 INTRODUCTION

It is a classical question, whether one can describe all weakly compact sets in a given non-reflexive Banach space. The most well-known result in this direction is the Dunford–Pettis theorem, which says that relatively weakly compact sets in $L^1[0,1]$ are exactly the so-called equi-integrable sets (sometimes called uniformly-integrable) – see [19, pp. 376–378] (see also [1, Theorem 5.2.9], [15, Theorem 5.2.9] and [20, Corollary 11, p. 294]). This theorem has found many applications in analysis as well as in probability theory. Therefore, not surprisingly that much attention was paid to deciding whether such kind of result can be extended to other separable non-reflexive Banach function spaces.

In fact, it is independently interesting that Orlicz [38] was the first who proved a theorem in this direction and he did so four years before Dunford and Pettis. Namely, in 1936 he proved that in the Orlicz space $L^\varphi[0,1]$ relatively weakly compact sets are exactly $L^p$-equi-integrable sets, provided the function $\varphi$ is an $N$-function satisfying $\Delta_2$ and if the complementary function $\varphi^*$ to $\varphi$ satisfies the condition

$$\lim_{u \to \infty} \frac{\varphi^*(2u)}{\varphi^*(u)} = \infty. \tag{1.1}$$

These Orlicz spaces $L^\varphi[0,1]$ in a certain sense resemble the $L^1$-spaces but the assumption of being $N$-function excludes the space $L^1[0,1]$ from Orlicz considerations. In 1978, Luxemburg [32] recalled Orlicz’s result on relatively weakly compact sets in Orlicz spaces which at that time has not received the attention it deserves.
Later on, in 1994, Alexopoulos proved once again the Orlicz result, but he assumed a slightly weaker condition than (1.1). Namely, he showed (cf. [2, Corollary 2.9]) that if an $N$-function $\varphi \in \Delta_1^\infty$ and its complementary function $\varphi^*$ satisfies
\[
\lim_{u \to \infty} \frac{\varphi^*(\lambda u)}{\varphi^*(u)} = \infty \text{ for some } \lambda > 1,
\]
then a bounded set $K \subset L^\varphi$ is relatively weakly compact if and only if $K$ is $L^\varphi$-equi-integrable.

The property (1.2) appears to be crucial in the sequel, thus we say that an Orlicz function $\varphi$ satisfies the $\Delta_0$-condition and we write $\varphi \in \Delta_0$ if for some $\lambda > 1$
\[
\lim_{u \to \infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \infty.
\]

Since Orlicz spaces are members of the wide class of Banach function lattices, a natural question arises, in which Banach function lattices does the analogous characteristics of relatively weakly compact sets remains valid? Such spaces are said to satisfy the Dunford–Pettis criterion, according to [9]. Quite recently, in 2008, the question was completely answered for rearrangement invariant space on $[0, 1]$. Namely, Astashkin, Kalton and Sukochev in [9, Theorem 5.5] proved that a rearrangement invariant space on $[0, 1]$ satisfies the Dunford–Pettis criterion if and only if it has the property $(\mathcal{W}m)$, i.e. convergence in measure together with weak convergence implies convergence in the norm. Their method is based on techniques attributes to rearrangement invariant spaces and descriptions of weakly compact sets in such spaces from paper [17].

Very recently Astashkin in [6, Theorem 3.4] found yet another characterization of rearrangement invariant spaces on $[0, 1]$ satisfying the Dunford–Pettis criterion. Namely, he proved that this class coincides with the class of 1-disjointly homogeneous spaces (cf. also [6, Theorem 3.3] and [23, Proposition 4.9]). There is also result that 1-disjointly homogeneous lattices are the same as the lattices with the positive Schur property (see [43, Theorem 7] and [23, Proposition 4.9]).

In the paper we extend Astashkin’s characterization to the whole class of separable Banach function lattices (not only rearrangement invariant spaces) over arbitrary nonatomic measure space. Namely, our main result says that a separable Banach function lattice satisfies the Dunford–Pettis criterion if and only if it is 1-disjointly homogeneous space. Our method is quite elementary and different from the one in [6].

The paper is organized as follows. In the next section we provide necessary definitions and discuss the notion of $X$-equi-integrable sets. The third section contains the main results. Firstly, we show that in Banach function spaces $X$-equi-integrable sets are not only relatively weakly compact, but even Banach–Saks sets. Next, we compare de la Vallée Poussin condition with $X$-equi-integrability and show that those two notions are equivalent in the class of rearrangement invariant spaces on $[0, 1]$, while de la Vallée Poussin condition cannot be extended to spaces on $[0, \infty)$. This section is finished by the proof of the main result – Theorem 3.6. In the fourth section we give new examples of 1-disjointly homogeneous rearrangement invariant spaces on $[0, \infty)$. Moreover, we are able to characterize all 1-disjointly homogeneous Orlicz spaces on $[0, \infty)$ in quite a constructive way (compare with another characterization from [23, Theorem 5.1]). In the fifth section we comment the condition (1.2) on Orlicz function and give a number of examples. Finally, in the last section we apply previous results to discuss weak compactness of pointwise multipliers.

2  NOTATION AND PRELIMINARIES

2.1  Banach function lattices

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite, complete and nonatomic measure space. As usual, $L^0 = L^0(\Omega)$ is the space of all (equivalence classes of) real-valued $\Sigma$-measurable functions defined on $\Omega$. A Banach space $X \subset L^0$ is said to be a Banach function lattice if:

(i) $f \in L^0, g \in X$ and $|f| \leq |g|$ a.e. on $\Omega$ implies that $f \in X$ and $\|f\| \leq \|g\|$.

(ii) $X$ has a weak unit, i.e. an element $f \in X$ such that $f(t) > 0$ for a.e. $t \in \Omega$.

A Banach function lattice $X$ is said to satisfy the Fatou property if for each sequence $(f_n) \subset X$ satisfying $f_n \uparrow f$ $\mu$-a.e. on $\Omega$ and $\sup_{n \in \mathbb{N}} \|f_n\|_X < \infty$, there holds $f \in X$ and $\|f\|_X \leq \sup_{n} \|f_n\|_X$. 

NOTATION AND PRELIMINARIES
An element \( f \in X \) is said to be order continuous if for any sequence \( (f_n) \subset X \) with \( 0 \leq f_n \leq |f| \) and \( f_n \to 0 \) \( \mu \)-a.e. on \( \Omega \) there holds \( \|f_n\|_X \to 0 \). By \( X_a \) we denote the subspace of all order continuous elements of \( X \). A Banach function space \( X \) is called order continuous (we write \( X \in \text{OC} \)) if \( X_a = X \). It will be used few times in the sequel that \( f \in X_a \) if and only if \( \|f\|_X \to 0 \) for any sequence \( (A_n) \) satisfying \( A_n \downarrow \emptyset \), where \( A_n \downarrow \emptyset \) means that \( (A_n) \) is decreasing sequence of measurable sets with the intersection of measure zero (see [12, Proposition 3.5, p. 15]). The subspace \( X_a \) is always closed in \( X \) (see [12, Theorem 3.8, p. 16]).

For two Banach function lattices the symbol \( X \overset{C}{\hookrightarrow} Y \) means that the inclusion \( X \subset Y \) is continuous with a norm which is not bigger than \( C \), i.e., \( \|f\|_Y \leq C \|f\|_X \) for all \( f \in X \). In the case when the embedding \( X \overset{C}{\hookrightarrow} Y \) holds with some (unknown) constant \( C > 0 \) we simply write \( X \overset{C}{\hookrightarrow} Y \).

An important class of Banach function lattices is constituted by rearrangement invariant spaces. Consider \( I = [0, \infty) \), where \( 0 < \alpha \leq \infty \) with the Lebesgue measure \( m \). Recall that the distribution function of \( f \in L^0(I) \) is defined by

\[
d_f(\lambda) = m(\{ t \in I : |f(t)| > \lambda \}) \quad \text{for} \quad \lambda \geq 0.
\]

We say that two functions \( f, g \in L^0(I) \) are equimeasurable when they have the same distribution functions, i.e., \( d_f = d_g \). Then a Banach function lattice \( X \) on \( I \) is called rearrangement invariant (or symmetric) if for two given equimeasurable functions \( f, g \in L^0(I) \) with \( f \in X \) there holds \( g \in X \) and \( \|f\|_X = \|g\|_X \). In particular, \( \|f\|_X = \|f^*\|_X \), where \( f^* \) is the nonincreasing rearrangement of \( f \), i.e.,

\[
f^*(t) := \inf \{ \lambda > 0 : d_f(\lambda) \leq t \}
\]

for \( t \geq 0 \). For more information on rearrangement invariant spaces we refer to books [12,26] and [30].

If not specified otherwise, we will understand that all general Banach function lattices are defined on an arbitrary \( \sigma \)-finite, complete and nonatomic measure space \( (\Omega, \Sigma, \mu) \), while all rearrangement invariant spaces are on \( I = [0, 1] \) or \( I = [0, \infty) \) with the Lebesgue measure \( m \).

### 2.2 Orlicz functions and Orlicz spaces

In this paper \( \varphi \) will always denote an Orlicz function, i.e. a continuous increasing and convex function \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi(0) = 0 \). We will assume that Orlicz function \( \varphi \) is coercive, i.e. it has the additional property of \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty \), or is an \( N \)-function if both

\[
\lim_{u \to 0^+} \frac{\varphi(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty.
\]

Property of coerciveness of \( \varphi \) exclude the case of \( \varphi(u) = au \), but ensures that the conjugate function has finite values. For an Orlicz function \( \varphi \) the conjugate function \( \varphi^* \) is defined by

\[
\varphi^*(v) = \sup_{u \geq 0} [uv - \varphi(u)], \quad v \geq 0.
\]

Then \( \varphi^* \) is finite-valued if and only if \( \varphi \) is coercive. Moreover, if \( \varphi \) is an \( N \)-function then \( \varphi^* \) is also an \( N \)-function. To avoid pathologies through the paper, we will understand that all Orlicz functions are coercive.

We say that an Orlicz function \( \varphi \) satisfies the \( \Delta_2 \)-condition for large \( u \) (or at infinity) and we write \( \varphi \in \Delta_2^\infty \) if \( \limsup_{u \to \infty} \frac{\varphi(2u)}{\varphi(u)} < \infty \) or, equivalently, there exist constants \( C > 1 \) and \( u_0 \geq 0 \) such that

\[
\varphi(2u) \leq C \varphi(u) \quad \text{for all} \quad u \geq u_0.
\]

If condition (2.2) holds with \( u_0 = 0 \), then we say that an Orlicz function \( \varphi \) satisfies the \( \Delta_2 \)-condition for all \( u \) and we write \( \varphi \in \Delta_2^\infty \).
Further definitions, properties and results about Orlicz’s functions or N-functions are taken from the books [25,34] and [40].

The Orlicz space $L^\varphi = L^\varphi(\Omega)$ on a $\sigma$-finite complete nonatomic measure space $(\Omega, \mu)$ is the space of all $f \in L^0(\Omega)$ satisfying $I_\varphi(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$, where the modular $I_\varphi$ is given by

$$I_\varphi(f) := \int_\Omega \varphi(|f(x)|) \, d\mu(x).$$

This space is a Banach function lattice with the Luxemburg–Nakano norm defined as

$$\|f\|_\varphi := \inf \{ \lambda > 0 : I_\varphi(f/\lambda) \leq 1 \}.$$

For $\Omega = I$ with the Lebesgue measure $m$, the Orlicz space $L^\varphi(I)$ is a rearrangement-invariant space with the Fatou property (see [12] and [26]).

### 2.3 Equi-integrable sets

Let us recall that, classically, a bounded subset $K$ of $L^1[0,1]$ is called equi-integrable (or uniformly integrable) when for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every set $A \subset [0,1]$ with $m(A) < \delta$ we have

$$\sup_{f \in K} \int_A |f| \, dm = \sup_{f \in K} \|f\chi_A\|_1 < \varepsilon,$$

i.e., $\lim_{m(A) \to 0} \sup_{f \in K} \int_A |f| \, dm = 0$. This notion generalizes easily to an arbitrary Banach function lattice on a finite measure space – it is enough to replace the norm $\| \cdot \|_1$ by an abstract norm $\| \cdot \|_X$. However, it is not the right definition when we wish to deal with an infinite measure spaces as well. Therefore, we define $X$-equi-integrable sets in a slightly different way.

Let $X$ be a Banach function lattice and $K \subset X$ be a bounded set. We say that a set $K$ is $X$-equi-integrable (or $K$ has equi-absolutely continuous norms in $X$) if for each sequence of measurable sets $(A_n), A_n \subset \Omega$ such that $A_n \downarrow \emptyset$, there holds

$$\sup_{f \in K} \|f\chi_{A_n}\|_X \to 0 \text{ when } n \to \infty.$$

The following trivial lemma ensures that in the case of finite measure our definition of $X$-equi-integrability coincides with another ones.

**Lemma 2.1.** Let $X$ be a Banach function lattice on a finite measure space $(\Omega, \mu)$ and let $K \subset X$. The following statements are equivalent:

1. For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $A \subset \Omega$ with $\mu(A) < \delta$ we have

   $$\sup_{f \in K} \|f\chi_A\|_X < \varepsilon.$$

2. For each sequence of measurable sets $(A_n), A_n \subset \Omega$ such that $\mu(A_n) \to 0$ there holds

   $$\sup_{f \in K} \|f\chi_{A_n}\|_X \to 0 \text{ when } n \to \infty.$$

3. For each sequence of measurable sets $(A_n), A_n \subset \Omega$ such that $A_n \downarrow \emptyset$ there holds

   $$\sup_{f \in K} \|f\chi_{A_n}\|_X \to 0 \text{ when } n \to \infty.$$
Proof. Evidently, \((i) \Rightarrow (ii) \Rightarrow (iii)\), thus we will explain only \((iii) \Rightarrow (i)\). Suppose there is \(\varepsilon > 0\) such that for each \(n \in \mathbb{N}\) there is \(B_n \subset \Omega\) with \(\mu(B_n) < 1/2^n\) satisfying
\[
\sup_{f \in K} \|f \chi_{B_n}\|_X > \varepsilon.
\]
It is enough to take \(A_n = \bigcup_{k=n}^\infty B_k\), to see that \((iii)\) does not hold. \(\square\)

In the sequel we will need the following technical lemma generalizing Lemma 2.6 from \([2]\).

**Lemma 2.2.** Let \(X\) be an order continuous Banach function lattice on a measure space \((\Omega, \mu)\). If a bounded set \(K \subset X\) is not \(X\)-equi-integrable, then there exist a sequence \((f_n) \subset K\), a number \(\varepsilon > 0\) and a sequence of disjoint measurable sets \((A_n)\), \(A_n \subset \Omega\) such that for every \(n \in \mathbb{N}\)
\[
\|f_n \chi_{A_n}\|_X > \varepsilon.
\]

**Proof.** When \(K \subset X\) is not \(X\)-equi-integrable then for some \(\varepsilon > 0\) there exist a sequence \((B_n)\), \(B_n \downarrow \emptyset\), and a sequence \((f_n) \subset K\) such that
\[
\|f_n \chi_{B_n}\|_X \geq \varepsilon.
\]
Choose \(n_1 = 1\). Since \(f_{n_1} \in X_a = X\) we can find \(n_2\) such big that
\[
\|f_{n_1} \chi_{B_{n_1} \setminus B_{n_2}}\|_X \geq \varepsilon/2.
\]
We proceed in the same way with \(n_2\) in place of \(n_1\). In consequence, we get a sequence \((n_k)\) such that sets \(A_k = B_{n_k} \setminus B_{n_{k+1}}\) and functions \(f_{n_k}\) satisfy the thesis with \(\varepsilon := \varepsilon/2\). \(\square\)

It is known \([36, \text{Proposition 3.6.5.}]\) that in any Banach lattice \(X\) every \(X\)-equi-integrable set is relatively weakly compact (see \([36]\) for notion of \(X\)-equi-integrable sets in abstract lattices). The question is whether the opposite implication also holds, or rather, in which spaces the opposite implication also holds. Following \([9]\) we will say that a Banach function lattice \(X\) satisfies the Dunford–Pettis criterion when each relatively weakly compact set in \(X\) is \(X\)-equi-integrable.

Trivially, none of \(L^p\) with \(1 < p < \infty\) satisfies the Dunford–Pettis criterion. Moreover, it cannot hold also in any Banach function lattice which is not order continuous (take just \(K = \{f\}\) for any \(f \in X \setminus X_a\)). In particular, if \(K \subset X\) is \(X\)-equi-integrable then \(K \subset X_a\).

### 2.4 1-disjointly homogeneous spaces and the positive Schur property

Let \(X\) be a Banach function lattice. We say that \(X\) is 1-disjointly homogeneous space (1-DH for short) if every normalized sequence \((f_n) \subset X\) of disjointly supported functions has a subsequence \((f_{n_k})\) equivalent to the \(\ell^1\)-basis, i.e. there exists a constant \(c > 0\) such that for every \(a = (a_k) \in \ell^1\)
\[
c \|a\|_1 \leq \left\| \sum_{k=0}^{\infty} a_k f_{n_k} \right\|_X \leq \|a\|_1.
\]
Evidently, if we required only that the sequence \((\|f_n\|_X)\) is just bounded from above and from below, we would get the equivalent definition.

1-disjointly homogeneous spaces and, more generally, \(p\)-disjointly homogeneous spaces \((1 \leq p < \infty)\), have been intensively investigated during the last decade – see \([22–24,5,7]\).

Examples of 1-DH Banach lattices are: the Orlicz spaces \(L^\varphi[0,1]\) when \(\varphi \in \Delta_2^\infty\) and \(\varphi^* \in \Delta_0\) (cf. \([29]\), see also \([22]\)) and the Lorentz spaces \(\Lambda_w[0,1]\) with the norms \(\|f\|_w = \int_0^1 f^*(t)w(t)\,dt\), where \(w\) is a positive, nonincreasing
function on \((0, 1]\), such that \(\lim_{t \to 0^+} w(t) = \infty, w(1) > 0\) and \(\int_0^1 w(t) \, dt = 1\) (cf. [21, Theorem 5.1]). In particular, for \(w_p(t) = \frac{1}{p} t^{1/p-1}, 1 \leq p < \infty\), we have \(\Lambda_{w_p}[0, 1] = L^{p,1}[0, 1]\), which are 1-DH spaces.

Let us also mention that Banach lattices being 1-DH have previously been considered under a different approach. Wnuk [42] started to investigate the positive Schur property: a Banach lattice \(X\) has the positive Schur property if every weakly null sequence with positive terms is norm convergent, see also [42, Theorem 5.1]. It follows from, e.g. [36, Corollary 2.3.5] that it is sufficient to verify this condition for disjoint sequences. Using Rosenthal’s \(l^1\)-theorem, Wnuk proved in [43, Theorem 7] that a Banach lattice \(X\) is 1-DH if and only if \(X\) has the positive Schur property (see also [23, Proposition 4.9] and [39, Proposition 1.2]). Note that the positive Schur property (as well as 1-DH) is not preserved by isomorphisms (see [43, p. 18]).

Surprisingly, in [8], it was proved that in the case of rearrangement invariant spaces on \([0,1]\) the situation is completely different (see [8, Theorem 5 and Corollary 2]).

3 WEAKLY COMPACT SETS IN BANACH FUNCTION LATTICES

As we mentioned in the previous section, each \(X\)-equi-integrable set is relatively weakly compact even in abstract lattices. For subsets of Banach function lattices we can prove more, i.e. that each equi-integrable set is also a Banach–Saks set. In the case of Orlicz spaces on \([0,1]\) such theorem was proved in [2], while for order continuous rearrangement invariant spaces on \([0,\infty)\) it is the statement of [18, Theorem 4.10].

Recall, that given a Banach space \(X\) and a set \(K \subset X\) we call \(K\) the Banach–Saks set if for each sequence \((f_n) \subset K\) there exist \(f \in X\) and a subsequence \((f_{n_k})\) such that every further subsequence \((f_{n_{k_j}})\) has means norm converging to \(f\), i.e. \(\left(\frac{1}{n} \sum_{i=1}^{n} f_{n_k}\right)\) converges to \(f\) in norm. Note that each Banach–Saks set is relatively weakly compact (see, for example, [31, Proposition 2.3]).

**Theorem 3.1.** Let \(X\) be a Banach function lattice with the Fatou property such that \(L^1 \cap L^\infty \hookrightarrow X\). If \(K \subset X\) is a bounded and \(X\)-equi-integrable set, then \(K\) is a Banach–Saks set in \(X\).

**Proof.** Let \(K \subset X\) be an \(X\)-equi-integrable set and let \((f_n) \subset K\). By the Day–Lennard theorem (cf. [14, Theorem 3.1]) there exists a function \(f \in X\) and a subsequence \((f_{n_k})\) such that for each further subsequence \((f_{n_{k_j}})\)

\[
\frac{1}{n} \sum_{j=1}^{n} f_{n_{k_j}}(t) \to f(t) \text{ for a.e. } t \in \Omega,
\]
as \(n \to \infty\). Firstly, we will show that \(f \in X_a\). Let \(h \in X'\) with \(\|h\|_{X'} \leq 1\) and \(A \subset \Omega\) be a measurable set. We have

\[
\left| \int_{\Omega} h \chi_A \, f \, d\mu \right| \leq \int_{\Omega} |h \chi_A f| \, d\mu \leq \liminf_{n \to \infty} \int_{\Omega} h \chi_A \left(\frac{1}{n} \sum_{k=1}^{n} f_{n_k}\right) \, d\mu
\]

\[
\leq \sup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \sum_{k=1}^{n} \int_{\Omega} |h \chi_A f_{n_k}| \, d\mu \right\} \leq \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \|f_{n_k} \chi_A\|_X \|h\|_{X'}
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \sup_{g \in K} \|g \chi_A\|_X = \sup_{g \in K} \|g \chi_A\|_X.
\]

Thus, by the Fatou property of \(X\), \(\|f \chi_A\|_X \leq \sup_{g \in K} \|g \chi_A\|_X\) for arbitrary \(A \subset \Omega\). Since \(K\) is \(X\)-equi-integrable set, we conclude that \(f \in X_a\).

Let \((f_{n_k})\) be a subsequence of \((f_{n_k})\). We will show that means of \((f_{n_k})\) are norm convergent to \(f\). Denote

\[g_n = \frac{1}{n} \sum_{i=1}^{n} f_{n_k}.
\]
Let $\varepsilon > 0$ be arbitrary. Consider the sequence $B_n = \bigcup_{k>n} \Omega_k \downarrow \emptyset$ with $n \to \infty$, where the sequence $(\Omega_n)$ comes from the definition of $\sigma$-finite measure space. By $X$-equi-integrability of $K$ and order continuity of $f$ there is $i \in \mathbb{N}$ such that

$$\|f \chi_{A}\|_X \leq \varepsilon \text{ and } \sup_{g \in K} \|g \chi_{B_i}\|_X \leq \varepsilon.$$ 

Denote $B := B_i$ and notice that for $B' = \Omega \setminus B$ there holds $\mu(B') < \infty$, by definition of $B_n$ sets. Moreover, by Lemma 2.1, there is $\delta > 0$ such that

$$\|f \chi_A\|_X \leq \varepsilon \text{ and } \sup_{g \in K} \|g \chi_A\|_X \leq \varepsilon,$$

whenever $\mu(A) < \delta$ and $A \subset B'$. In consequence, also

$$\|g_n \chi_A\|_X \leq \varepsilon \text{ and } \|g_n \chi_B\|_X \leq \varepsilon,$$

for each $n \in \mathbb{N}$ and each $A$ like above. On the other hand, by the Egorov theorem there exists $C \subset B'$ such that $\mu(B' \setminus C) \leq \delta$ and $g_n$ converges uniformly to $f$ on $C$. Let $N \in \mathbb{N}$ be such

$$\|(f - g_n) \chi_C\|_\infty \leq \frac{\varepsilon}{\|\chi_C\|_X}$$

for $n \geq N$. Therefore,

$$\|f - g_n\|_X \leq \|(f - g_n) \chi_C\|_X + \|(f - g_n) \chi_{B' \setminus C}\|_X + \|(f - g_n) \chi_B\|_X \leq \frac{\varepsilon}{\|\chi_C\|_X} \|\chi_C\|_X + \|f \chi_{B' \setminus C}\|_X + \|g_n \chi_{B' \setminus C}\|_X + \|f \chi_B\|_X + \|g_n \chi_B\|_X \leq 5\varepsilon$$

for $n \geq N$, which finishes the proof. \hfill $\square$

While the Dunford–Pettis description of relatively weakly compact sets in terms of equi-integrability attracted considerable attention and the notion of equi-integrable sets found its counterparts in general function lattices, the alternative description by de la Vallée Poussin seems to be much less popular. On the other hand, some counterparts of de la Vallée-Poussin theorems appear in the subject of strong embeddings of rearrangement invariant spaces (see, for example, \[4, Lemma 4\] and references therein). The following theorem shows that de la Vallée-Poussin characterization works in all rearrangement invariant spaces on $[0,1]$.

**Theorem 3.2.** Let $X$ be a rearrangement invariant space on $I = [0,1]$ such that $X_a \neq \{0\}$ and let a set $K \subset X$ be bounded. The following statements are equivalent:

(i) $K$ is $X$-equi-integrable.

(ii) There holds

$$\lim_{\gamma \to \infty} \sup_{f \in K} \|f \chi_{\{|f| > \gamma\}}\|_X = 0.$$

(iii) There exists an increasing convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ satisfying $\lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty$ such that $\sup_{f \in K} \|\varphi(f)\|_X < \infty$.

**Proof.** (i) $\Rightarrow$ (ii). Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\sup_{f \in K} \|f \chi_A\|_X \leq \varepsilon$$
when \( m(A) \leq \delta \). Since \( X \) is rearrangement invariant space, thus \( X \hookrightarrow CL^1 \) for some constant \( C \geq 1 \) (cf. [12] or [26]). By the Chebyshev inequality

\[
m(|f| > \gamma) \leq \frac{\|f\|_1}{\gamma} \leq C \frac{\|f\|_X}{\gamma} \leq \delta
\]

for each \( f \in K \) and \( \gamma \) large enough. Consequently

\[
\lim_{\gamma \to \infty} \sup_{f \in K} \left\| f \mathbf{1}_{|f| > \gamma} \right\|_X = 0.
\]

\( (ii) \Rightarrow (i) \). Let \( \varepsilon > 0 \). By the assumption there exists \( \gamma_0 \) such that

\[
\sup_{f \in K} \left\| f \mathbf{1}_{|f| > \gamma_0} \right\|_X \leq \varepsilon.
\]

Let \( (A_n) \) be a sequence of measurable sets such that \( A_n \downarrow \emptyset \). Since \( X_a \neq \{0\} \) (and thus characteristic functions are order continuous, cf. [12, Theorem 5.5 (b)]), we can choose \( N \in \mathbb{N} \) such that

\[
\|X_{A_n}\|_X < \frac{\varepsilon}{\gamma_0}
\]

for \( n \geq N \). In consequence,

\[
\left\| f \mathbf{1}_{A_n} \right\|_X \leq \left\| f \mathbf{1}_{|f| > \gamma_0 \cap A_n} \right\|_X + \left\| f \mathbf{1}_{|f| > \gamma_0 \cap A_n} \right\|_X
\]

\[
\leq \gamma_0 \left\| \mathbf{1}_{A_n} \right\|_E + \varepsilon = 2\varepsilon,
\]

for \( n \geq N \). It means that \( K \) is \( X \)-equi-integrable set.

\( (iii) \Rightarrow (ii) \). Let \( \varepsilon > 0 \). By the assumption there exists \( u_\varepsilon > 0 \) such that

\[
u \leq \varepsilon \varphi(u)
\]

for all \( u \geq u_\varepsilon \). Denote \( M := \sup_{f \in K} \| \varphi(f) \|_X \). We have for \( f \in K \), by the lattice property of \( X \),

\[
\left\| f \mathbf{1}_{|f| > u_\varepsilon} \right\|_X \leq \varepsilon \left\| \varphi(|f|) \mathbf{1}_{|f| > u_\varepsilon} \right\|_X \leq \varepsilon \sup_{f \in K} \| \varphi(f) \|_X \leq M \varepsilon,
\]

where the first inequality follows from 3.1 and lattice property of \( X \).

\( (ii) \Rightarrow (iii) \). Let \( u_1 = 0 \). For each \( n \geq 2 \) we can choose \( u_n > 0 \) in such a way that

\[
\sup_{f \in K} \left\| f \mathbf{1}_{|f| > u_n} \right\|_X \leq \frac{1}{n^2}.
\]

and \( u_{n+1} \geq 2u_n \) for each \( n \geq 2 \). For \( u \geq 0 \) define a function

\[
\varphi(u) = \sum_{n=1}^{\infty} (u - u_n)_+.
\]

Clearly, \( \varphi \) is an increasing convex function with \( \varphi(0) = 0 \). Let us explain that also \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty \). In fact, for \( u \in [u_n, u_{n+1}) \) there holds

\[
\varphi(u) = \sum_{k=1}^{n} (u - u_k)_+ = nu - \sum_{k=1}^{n} u_k \geq nu - 2u_n,
\]
which means that \( \frac{\varphi(u)}{u} \geq n - 2 \frac{u_n}{u} \geq n - 2 \) and proves the claim. It remains to see that \( \sup_{f \in K} \| \varphi(f) \|_X < \infty \). For \( f \in K \) we have \( \varphi(|f|) \leq \sum_{n=1}^{\infty} |f| \chi_{|f| > u_n}, \) thus

\[
\| \varphi(|f|) \|_X \leq \sum_{n=1}^{\infty} \| f \chi_{|f| > u_n} \|_X \leq \frac{\pi^2}{6},
\]

and the proof is finished. \( \square \)

**Remark 3.3.** Observe that one cannot get similar characterization of \( X \)-equi-integrable sets in rearrangement invariant spaces on \( I = [0, \infty) \). Indeed, if \( \varphi > 0 \) is an arbitrary positive function defined on \( [0, \infty) \), then the set

\[
K = \{ \chi_{[n,n+1)} : n \in \mathbb{N} \} \subset X
\]

is not \( X \)-equi-integrable, while we have

\[
\sup_{f \in K} \| \varphi(f) \|_X = \varphi(1)\| \chi_{[0,1)} \|_X < \infty.
\]

It means that de la Vallée Poussin condition cannot imply equi-integrability in the case of infinite measure space.

**Remark 3.4.** An analogue of de la Vallée Poussin condition for a class of quasi-Banach function spaces on a finite measure space may be found in the recent paper \([13, \text{Theorem 5.1}]\).

To prove our main result we will need the following Rosenthal’s type lemma (cf. \([15, \text{p. 82}]\)) proved by Alexopoulos in \([2, \text{Lemma 2.7}]\).

**Lemma 3.5.** Let \( X \) be a Banach space, \((x_n) \subset X \) be a weakly null sequence and \((x^*_n) \subset X^* \) be a weakly\(^*\) null sequence. For every \( \varepsilon > 0 \) there exists an increasing sequence \((k_i) \subset \mathbb{N} \) such that

\[
\sum_{j \neq i} \left| \left( x_{n_i}, x^*_{n_j} \right) \right| < \varepsilon \text{ for each } i \in \mathbb{N}.
\]

**Theorem 3.6.** Let \( X \) be a Banach function lattice. Then \( X \) satisfies the Dunford–Pettis criterion if and only if \( X \) is 1-disjointly homogeneous.

**Proof.** Sufficiency. Assume that \( X \) is not 1-disjointly homogeneous, i.e. there exists a normalized sequence \((f_n) \subset X \) of disjointly supported functions without subsequences equivalent to the \( \ell^1 \) basis. By the Rosenthal’s \( \ell^1 \)-theorem, the sequence \((f_n) \) contains some weakly Cauchy subsequence \((f_{n_k}) \). We will show that \((f_{n_k}) \) is weakly null. Suppose for a contradiction, that there exist \( g \in X' \) and \( a \neq 0 \) such that

\[
\lim_{k \to \infty} \int_{\Omega} gf_{n_k} \, d\mu = a.
\]

Denote \( A = \bigcup_{k=0}^{\infty} \text{supp} \ (f_{n_{2k}}) \). Observe that evidently \( h = g\chi_A \in X' \). We have

\[
\lim_{k \to \infty} \int_{\Omega} hf_{n_{2k}} \, d\mu = a \quad \text{while} \quad \int_{\Omega} hf_{n_{2k+1}} \, d\mu = 0,
\]

which means that the sequence \( (\int_{\Omega} hf_{n_k} \, d\mu) \) is not convergent. Thus, we arrived to the contradiction with \((f_{n_k}) \) being weakly Cauchy. Therefore, \((f_{n_k}) \) is weakly null and, in particular, the set \( K = \{ f_{n_k} \} \) is weakly compact. The proof of sufficiency will be finished when we prove that \( K \) is not \( X \)-equi-integrable set. Let

\[
A_i = \bigcup_{k=i}^{\infty} \text{supp} \ (f_{n_k}).
\]
Clearly \( A_i \downarrow \emptyset \). However,
\[
\sup_{f \in K} \| f \chi_{A_i} \|_X \geq \| f_{n_i} \chi_{A_i} \|_X = \| f_{n_i} \|_X = 1.
\]
Thus \( X \) does not satisfy the Dunford–Pettis criterion.

**Necessity.** Let \( K \subset X \) be an arbitrary relatively weakly compact set and assume it is not \( X \)-equi-integrable. One can choose \( (g_m) \subset K \) which is not \( X \)-equi-integrable. Since \( K \) is weakly compact there exists a subsequence \( (g_{m_n}) \) converging weakly to some \( g \in X \). Define
\[
f_n = g_{m_n} - g.
\]
Then \( (f_n) \) is weakly null. Moreover, it is also not \( X \)-equi-integrable set, since \( g \in X_n = X \). By Lemma 2.2 there exist \( \varepsilon > 0 \) and a sequence of disjoint measurable sets \( (A_k) \) such that for some subsequence \( (f_{n_k}) \) of \( (f_n) \)
\[
\| f_{n_k} \chi_{A_k} \|_X > \varepsilon
\]
for each \( n \in \mathbb{N} \). Since \( X \) is 1-disjointly homogeneous there exists a further subsequence \( (f_{n_{k_j}} \chi_{A_{k_j}}) \) of \( (f_{n_k} \chi_{A_k}) \) equivalent to the \( \ell^1 \) basis. Denote \( w_1 := f_{n_{k_j}} \chi_{A_{k_j}} \). We claim that there is \( h \in X' \) such that
\[
\left\langle \sum_{l=0}^\infty a_l w_l, h \right\rangle = \sum_{l=0}^\infty a_l
\]
for each \( a \in \ell^1 \). Indeed, let \( U = [w_1] \) be a closed span of \( (w_l) \) and \( T : U \to \ell^1 \) be an isomorphism transforming \( (w_l) \) into \( \ell^1 \) basis, i.e. \( T(w_l) = e_l \), for \( l \in \mathbb{N} \). Let \( y^* \in \ell^\infty \) be such that \( \langle a, y^* \rangle = \sum_{l=0}^\infty a_l \) for each \( a \in \ell^1 \). Denote \( h^* = T^* y^* \in U^* \). By the Hahn–Banach theorem there exists \( h \in X^* = X' \) (the equality of the dual space \( X^* \) with the associated space \( X' \) follows from \( X \) being order continuous, which follows in turn from \( X \) being 1-DH) such that \( h|_U = h^* \). Clearly \( \sum_{l=0}^\infty a_l w_l, h \rangle = \sum_{l=0}^\infty a_l \) for each \( a \in \ell^1 \) and the claim follows.

Denote \( h_l = h \chi_{A_{k_j}} \) and observe that for each \( l \in \mathbb{N} \)
\[
\left\langle f_{n_{k_j}}, h_l \right\rangle = \int f_{n_{k_j}} h_l \, d\mu = \int f_{n_{k_j}} h \chi_{A_{k_j}} \, d\mu = \langle w_1, h \rangle = 1.
\]
Moreover, \( (h_l) \) is weak*-null sequence. Indeed, for each \( f \in X \) we have
\[
\left| \int \Omega f h_l \, d\mu \right| = \left| \int_{A_{k_j}} f h_l \, d\mu \right| \leq \| f \chi_{A_{k_j}} \|_X \| h_l \|_{X'} \leq \| f \chi_{A_{k_j}} \|_X \| h \|_{X'} \to 0,
\]
since \( X \) is order continuous space, as is each 1-DH space.

By Lemma 3.5 there exists an increasing sequence \( (l_j) \subset \mathbb{N} \) such that for \( j \in \mathbb{N} \)
\[
\sum_{j \neq j} \left| \left\langle f_{n_{k_j}}, h_{l_j} \right\rangle \right| < \frac{1}{2}.
\]
We have for each \( i \in \mathbb{N} \)
\[
\left| \left\langle f_{n_{k_i}}, h \right\rangle \right| = \left| \left\langle f_{n_{k_i}}, \sum_{j=1}^\infty h_{l_j} \right\rangle \right| \geq \left| \left\langle f_{n_{k_i}}, h_{l_i} \right\rangle \right| - \sum_{j \neq i} \left| \left\langle f_{n_{k_j}}, h_{l_j} \right\rangle \right| \geq 1 - \frac{1}{2} = \frac{1}{2}.
\]
However, \( (f_n) \) was weakly null, so we get a contradiction, which finishes the proof. \( \square \)
4  |  1-DISJOINTLY HOMOGENEOUS SPACES ON \([0, \infty)\)

The main known examples of 1-disjointly homogeneous spaces are Lorentz \(L^{p,1}\) spaces on \(I = [0,1]\) (generated by the norm 
\[
\|f\|_X = \frac{1}{p} \int_I x^{1/p-1} f^*(x) \, dx \quad \text{for} \quad 1 < p < \infty
\]
and Orlicz spaces \(L^\varphi\) spaces on \([0,1]\), when \(\varphi \in \Delta_2^\infty\) and its complementary function \(\varphi^*\) satisfies condition (1.2).

It seems however, that much less is known about such spaces on semiaxis. The following theorems provide such examples.

**Theorem 4.1.** For each \(1 < p < \infty\) the space \((L^{p,1} \cap L^1)(0, \infty)\) is 1-disjointly homogeneous.

**Proof.** To simplify the notation let us denote 
\(E = (L^{p,1} \cap L^1)(0, \infty)\). Let \((f_n) \subset (L^{p,1} \cap L^1)(0, \infty)\) be a normalized sequence of positive functions with disjoint supports. We need to consider two cases.

(i) Firstly, if there is \(\delta > 0\) such that \(\|f_n\|_1 > \delta\) for infinitely many \(n\)’s, then composing these \(n\)’s into a sequence \((n_k)\) we have for each \(a = (a_k) \in l^1\)
\[
\sum_{k=1}^{\infty} |a_k| \geq \sum_{k=1}^{\infty} a_k f_{n_k} \geq \sum_{k=1}^{\infty} a_k f_{n_k} \geq \delta \sum_{k=1}^{\infty} |a_k|.
\]

(ii) Otherwise \(\|f_n\|_1 \to 0\) and thus \(\|f_n\|_{p,1} = 1\) for almost all \(n\)’s. We may assume that actually \(\|f_n\|_{p,1} = 1\) for all \(n\)’s. We claim that for each \(k \in \mathbb{N}\) there is \(n \in \mathbb{N}\) such that
\[
\|f_n \chi_{\{f_n > k\}}\|_{p,1} \geq 1/2.
\]

Suppose for the moment that we have proved the claim. Then the claim together with \(\|f_n\|_{p,1} = 1\) imply that there is an increasing sequence \((n_k)\) such that
\[
m(\{f_{n_k} > k\}) \to 0 \quad \text{when} \quad k \to \infty \quad \text{and} \quad \|f_{n_k} \chi_{\{f_{n_k} > k\}}\|_{p,1} \geq 1/2 \quad \text{for each} \quad k.
\]

Thus we can select another subsequence \((n_{k_i})\) such that
\[
m(\{f_{n_{k_i}} > k_i\}) < \frac{1}{2i} \quad \text{for each} \quad i.
\]

Define a new sequence \((g_i)\) by
\[
g_i(t) = \left[f_{n_{k_i}} \chi_{\{f_{n_{k_i}} > k_i\}}\right]^*(t - 1/2^i) \quad \text{for} \quad t \geq 1/2^i
\]
and \(g_i(t) = 0\) otherwise. Notice that \(g_i\) is equimeasurable with \(f_{n_{k_i}} \chi_{\{f_{n_{k_i}} > k_i\}}\) for each \(i\). Moreover, \((g_i)\) is disjoint sequence and all supports of \((g_i)\) fit into \([0,1]\), thus we may regard \((g_i)\) as a subset of \(L^{p,1}[0,1]\). We have \(1/2 \leq \|g_i\|_{p,1} \leq 1\) for each \(i\). However, the space \(L^{p,1}[0,1]\) is 1-disjointly homogeneous (cf. [21] and Subsection 2.4) and so there is subsequence of \((g_i)\) which is equivalent to the \(l^1\) basis. Without loss of generality we may assume that \((g_i)\) itself is equivalent to the \(l^1\) basis. Then there is \(\eta > 0\) such that for each \(a = (a_i) \in l^1\) we have
\[
\eta \sum_{i=1}^{\infty} |a_i| \leq \left\| \sum_{k=1}^{\infty} a_{i}g_{k} \right\|_{L^{p,1}[0,1]} \leq \left\| \sum_{k=1}^{\infty} a_{i}f_{n_{k}}\chi_{\{f_{n_{k}} > k\}} \right\|_{E} \leq \left\| \sum_{k=1}^{\infty} a_{i}f_{n_{k}} \right\|_{E} \leq \sum_{i=1}^{\infty} |a_i|.
\]

Thus it remains to prove the claim. Suppose (4.1) does not hold. This means there is \(k_0\) such that for each \(n\)
\[
\|f_n \chi_{\{f_n > k_0\}}\|_{p,1} < 1/2.
\]
which implies
\[ \left\| f_n \chi_{\{ f_n \leq k_0 \}} \right\|_{p,1} \geq 1/2, \quad (4.3) \]
for each \( n \). Then
\[ 1/2 \leq \left\| f_n \chi_{\{ f_n \leq k_0 \}} \right\|_{p,1} \leq \int_0^{k_0} \frac{1}{p} f_n(t) \, dt \leq \left( \int_0^{k_0} d_{f_n}(t) \, dt \right)^{1/p} k_0^{1/p'}.
\]
This is
\[ \| f_n \|_1 \geq \frac{1}{2^p k_0^{p/p'}}, \]
which contradicts \( \| f_n \|_1 \to 0 \) and the claim is proved.

Remark 4.2. Notice that the point (ii) above could be concluded also in the following way. Since \( \| f_n \|_1 \to 0 \), it follows that \( f_n^*(t) \to 0 \) for every \( t \in (0, \infty) \). In consequence, applying [16, Proposition 1], we conclude that \( (f_n) \) contains a subsequence which is equivalent in \( L^{p,1} \) to the unit vector basis of \( L^1 \).

Astashkin, Kalton and Sukochev [9] proved that order continuous Orlicz space \( L^\varphi[0,1] \) satisfies the Dunford–Pettis criterion if and only if complementary function \( \varphi^* \) satisfies \( \Delta_0 \) condition, that is,
\[ \lim_{u \to \infty} \frac{\varphi^*(\lambda u)}{\varphi^*(u)} = \infty \text{ for some } \lambda > 1. \quad (4.4) \]
In case of Orlicz spaces on \( I = [0, \infty) \) it is in order to ask if the analogous theorem may be proved with \( (1.2) \) condition on \( \varphi^* \) extended to small arguments. This is, if, analogously as for the finite measure space, the condition
\[ \lim_{t \to 0} \frac{\varphi^*(\lambda t)}{\varphi^*(t)} = \infty \text{ for some } \lambda > 1 \]
together with \( (4.4) \) is sufficient or necessary for \( L^\varphi[0, \infty) \) to satisfy the Dunford–Pettis criterion. However, it appears, that this is not the case and we have the following characterization of Orlicz spaces on \( [0, \infty) \) satisfying the Dunford–Pettis criterion. Notice that already another characterization of 1-DH Orlicz spaces on \( I = [0, \infty) \) was given in [23, Theorem 5.1]. It was, however, formulated in terms of \( C_\varphi \) set, which is much more difficult to understand than simply saying that a respective function satisfies the condition \( \Delta_0 \). Moreover, the proof of [23, Theorem 5.1] relies on the general construction of [37], while ours is quite elementary.

Theorem 4.3. The Orlicz space \( L^\varphi[0, \infty) \) is 1-disjointly homogeneous if and only if \( \varphi \in \Delta_2, \varphi^* \in \Delta_0 \) and \( \varphi \) is equivalent to the linear function for small arguments, i.e. there are constants \( C, c, d, u_0 > 0 \) such that for each \( 0 < u < u_0 \) there holds
\[ cu \leq \varphi (du) \leq C u. \]

Proof. Sufficiency follows by an argument analogous to the one from previous theorem, since under our assumptions
\[ L^\varphi[0, \infty) = (L^\varphi \cap L^1)[0, \infty). \]
The only difference appears when proving the claim (4.1) from the previous proof, thus we will provide a detailed argument only for the claim: if \( (f_n) \subset L^\varphi[0, \infty) \) is a sequence of disjoint positive functions such that \( \| f_n \|_1 \to 0 \) and \( \| f_n \|_\varphi = 1 \), then for each \( k \in \mathbb{N} \) there is \( n \in \mathbb{N} \) such that
\[ \left\| f_n \chi_{\{ f_n > k \}} \right\|_\varphi > 1/2. \quad (4.5) \]
Assume it is not the case, this means there is $k_0$ such that for each $n$

$$\left\|f_n \chi_{\{f_n > k_0\}} \right\|_\varphi \leq 1/2,$$

(4.6)

which implies

$$\left\|f_n \chi_{\{f_n \leq k_0\}} \right\|_\varphi > 1/2.$$

(4.7)

The latter means that

$$\int_{\{f_n \leq k_0\}} \varphi(2f_n(t)) \, dt > 1.$$

Notice that convexity of $\varphi$ implies

$$\varphi(2t) \leq \frac{\varphi(2k_0)}{k_0} t \quad \text{for each } 0 < t \leq k_0.$$

Consequently,

$$1 < \int_{\{f_n \leq k_0\}} \varphi(2f_n(t)) \, dt \leq \frac{\varphi(2k_0)}{k_0} \int_{\{f_n \leq k_0\}} f_n(t) \, dt,$$

this is

$$\|f_n\|_1 \geq \frac{k_0}{\varphi(2k_0)},$$

for each $n$ and we arrived to contradiction with $\|f_n\|_1 \to 0$ and the assumption on $\varphi$.

Necessity of $\varphi \in \Delta_0$ is evident by the Astashkin, Kalton and Sukochev [9, Proposition 5.8]. Also, each 1-DH space is separable, so $\varphi \in \Delta_2$. To see that $\varphi$ has to be equivalent to the linear function for small arguments it is enough to consider the following sequence

$$f_n = \chi_{[n-1,n]}, \quad \text{where } n \in \mathbb{N}.$$

It is evident that each subsequence of $(f_n)$ is isometrically equivalent to the unit basis of Orlicz sequence space $l^\varphi$. Thus $l^\varphi$ has to be equal to $l^1$, which happen if and only if $\varphi$ is equivalent to the identity function for small arguments. \qed

## 5

**THE $\Delta_0$-CONDITION AND EXAMPLES**

Let us recall that an Orlicz function $\varphi$ satisfies the $\Delta_0$-condition if for some $\lambda > 1$

$$\lim_{u \to \infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \infty.$$

(5.1)

Since it played the crucial role in the previous section, let us discuss this condition. First of all notice, that contrary to the $\Delta_2^\infty$-condition, in the definition of $\Delta_0$-condition one cannot equivalently assume that (5.1) holds for all $\lambda > 1$, as can be seen in [28, Example 7]. Furthermore, the $\Delta_0$-condition may be seen as a strong negation of the $\Delta_2^\infty$-condition. Going a slightly deeper into this analogy, if $\varphi \notin \Delta_2^\infty$ then $L^\varphi$ contains a copy of $l^\infty$ based on a sequence of disjointly supported functions, while if $\varphi \in \Delta_0$ then each normalized sequence of disjointly supported functions contains a subsequence that spans $l^\infty$ (and this is how $\Delta_0$-condition connects with 1-DH spaces by duality).

In addition to the previously mentioned papers, condition $\varphi^* \in \Delta_0$ appears in the papers [10, 27, 3] and [41].
The so-called Matuszewska–Orlicz indices of Orlicz functions are helpful in presenting specific examples of Orlicz's functions with the $\Delta_0$-condition. The lower and upper Matuszewska–Orlicz indices (for large arguments or at infinity) $\alpha_\varphi^\infty, \beta_\varphi^\infty$ of an Orlicz function $\varphi$ are defined by

\[
\alpha_\varphi^\infty = \lim_{t \to 0^+} \frac{\ln M(t, \varphi)}{\ln t}, \quad \beta_\varphi^\infty = \lim_{t \to \infty} \frac{\ln M(t, \varphi)}{\ln t},
\]

where $M(t, \varphi) = \limsup_{u \to \infty} \frac{\varphi(tu)}{\varphi(u)}$. The basic properties of these indices are: $1 \leq \alpha_\varphi^\infty \leq \beta_\varphi^\infty \leq \infty$, if and only if $\varphi \in \Delta_2^\infty$ and

\[
\frac{1}{\alpha_\varphi^\infty} + \frac{1}{\beta_\varphi^\infty} = \frac{1}{\alpha_{\varphi^*}^\infty} + \frac{1}{\beta_{\varphi^*}^\infty} = 1.
\]

(5.2)

More information on indices can be found in [26,33] and [34].

**Proposition 5.1.** If $\varphi \in \Delta_0$, then $\alpha_\varphi^\infty = \beta_\varphi^\infty = \infty$ and $\alpha_{\varphi^*}^\infty = \beta_{\varphi^*}^\infty = 1$.

**Proof.** If $\varphi \in \Delta_0$, then $\varphi \not\in \Delta_2^\infty$ and $\beta_\varphi^\infty = \infty$. Hence, by (5.2), $\alpha_\varphi^\infty = 1$. On the other hand, if $\varphi \in \Delta_0$, then for any $\eta > \lambda$ we have

\[
\liminf_{u \to \infty} \frac{\varphi(\eta u)}{\varphi(u)} \geq \liminf_{u \to \infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \lim_{u \to \infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \infty,
\]

and so $\alpha_\varphi^\infty = \infty$, from which by (5.2) we get $\beta_{\varphi^*}^\infty = 1$. In particular, $\varphi^* \in \Delta_2^\infty$.

We do not know any example of an Orlicz function $\varphi$ with $\alpha_\varphi^\infty = \beta_\varphi^\infty = \infty$ and $\varphi \not\in \Delta_0$. Let us therefore state the question:

**Question 1.** Does $\alpha_\varphi^\infty = \beta_\varphi^\infty = \infty$ imply $\varphi \in \Delta_0$?

Nevertheless, using Simonenko indices instead of Matuszewska–Orlicz we can formulate sufficient condition for $\varphi \in \Delta_0$.

**Theorem 5.2.** If an Orlicz function $\varphi$ satisfies

\[
\lim_{u \to \infty} \frac{u \varphi'(u)}{\varphi(u)} = 1,
\]

then $\lim_{u \to \infty} \frac{\varphi^*(\lambda u)}{\varphi^*(u)} = \infty$ for any $\lambda > 1$. In particular, $\varphi^* \in \Delta_0$.

**Proof.** First of all, let's see that if $\lim_{u \to \infty} \frac{u \varphi'(u)}{\varphi(u)} = \infty$, then

\[
\lim_{u \to \infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \infty \text{ for any } \lambda > 1.
\]

(5.4)

Really, since $\psi(t) := \frac{\varphi'(t)}{\varphi(t)} > c$ for any large $c$ and large enough $t$, it follows for arbitrary $\lambda > 1$ that

\[
\frac{\varphi(\lambda u)}{\varphi(u)} = \exp\left(\int_u^{\lambda u} \frac{\psi(t)}{t} \, dt\right) \geq \exp\left(\int_u^{\lambda u} \frac{c}{t} \, dt\right) = \lambda^c
\]

which gives (5.4).

Now, we are ready to prove Theorem 5.2. Define for $t, u > 0$

\[
p(t) = \frac{\varphi(t)}{t} \quad \text{and} \quad \varphi_1(u) = \int_0^u p(t) \, dt.
\]
Then, since \( p \) is non-decreasing, for each \( \lambda > 1 \) there holds

\[
1 \geq \lim_{u \to \infty} \frac{\varphi_1(u)}{u p(u)} \geq \lim_{u \to \infty} \frac{\int_{u/\lambda}^u p(t) \, dt}{u p(u)} \geq \lim_{u \to \infty} \frac{p(u/\lambda)(1 - 1/\lambda)}{p(u)} = 1 - 1/\lambda,
\]

which means that

\[
\lim_{u \to \infty} \frac{\varphi_1(u)}{u p(u)} = 1. \quad (5.5)
\]

Let \( \nu > 0 \) and \( u = p^{-1}(\nu) \) (where \( p^{-1} \) denotes the right continuous inverse of \( p \)), then

\[
\nu u = u p(u) = \varphi_1(u) + \varphi_1^*(\nu),
\]

i.e.

\[
\frac{\varphi_1(u)}{u p(u)} + \frac{\varphi_1^*(\nu)}{\nu p^{-1}(\nu)} = 1.
\]

Thus (5.5) implies

\[
\lim_{\nu \to \infty} \frac{\nu p^{-1}(\nu)}{\varphi_1^*(\nu)} = \infty.
\]

According to (5.4) it means that

\[
\lim_{u \to \infty} \frac{\varphi_1^*(\lambda u)}{\varphi_1^*(u)} = \infty, \quad (5.6)
\]

for each \( \lambda > 1 \). But, once again, monotonicity of \( p \) gives \( \varphi_1(u) \leq \varphi(u) \) for each \( u > 0 \), which, in turn, implies \( \varphi_1^*(u) \geq \varphi^*(u) \) for each \( u > 0 \). On the other hand, for each \( \eta > 1 \) and \( u > 0 \) we have

\[
\varphi_1(\eta u) \geq \int_u^{\eta u} p(t) \, dt \geq u p(u)(\eta - 1) = \varphi(u)(\eta - 1).
\]

Thus, making respective substitution in the definition of conjugate functions, we conclude that also \( \frac{1}{\eta - 1} \varphi_1^*(\eta^{-1} u) \leq \varphi^*(u) \) for each \( u > 0 \). Finally, let \( \lambda > 1 \) and choose \( \eta > 1 \) in such a way, that \( \lambda \eta^{-1} > 1 \). We have finally by (5.6)

\[
\lim_{u \to \infty} \frac{\varphi^*(\lambda u)}{\varphi^*(u)} \geq \lim_{u \to \infty} \frac{1}{\eta - 1} \frac{\varphi_1^*(\lambda \eta^{-1} u)}{\varphi_1^*(u)} = \infty. \quad \square
\]

The lower and upper Simonenko indices (at infinity) \( a_\varphi^\infty, b_\varphi^\infty \) of an Orlicz function \( \varphi \) are defined by

\[
a_\varphi^\infty = \liminf_{u \to \infty} \frac{u \varphi'(u)}{\varphi(u)}, \quad b_\varphi^\infty = \limsup_{u \to \infty} \frac{u \varphi'(u)}{\varphi(u)}.
\]

The basic properties are (see [25, 33, 34, 40]): \( 1 \leq a_\varphi^\infty \leq \alpha_\varphi^\infty \leq \beta_\varphi^\infty \leq b_\varphi^\infty \leq b_\varphi^\infty \leq \infty, b_\varphi^\infty < \infty \) if and only if \( \varphi \in \Delta_2^\infty \), and for an \( N \)-function \( \varphi \) we have

\[
\frac{1}{a_\varphi^\infty} + \frac{1}{b_\varphi^\infty} = \frac{1}{a_\varphi^\infty} + \frac{1}{b_\varphi^\infty} = 1. \quad (5.7)
\]
Corollary 5.3. If for an N-function \( \varphi \) we have \( a_\varphi^\infty = b_\varphi^\infty = \infty \), then \( \varphi \in \Delta_0 \).

Proof. Using relations (5.7) we obtain condition (5.3) for \( \varphi^* \) and by the fact that conjugation \( \varphi \mapsto \varphi^* \) is an involution, we conclude that \( \varphi \in \Delta_0 \).

Notice that \( a_\varphi^\infty = b_\varphi^\infty = \infty \) cannot be necessary for \( \varphi \in \Delta_0 \) because of Theorem 5.2 and Example 7 in [28].

Example 5.4. The N-function

\[
\varphi(u) = \begin{cases} 
\frac{u^2}{2} & \text{if } 0 \leq u \leq 1, \\
u \ln u + \frac{1}{2} & \text{if } u \geq 1,
\end{cases}
\]
satisfies the \( \Delta_2 \)-condition (even for all \( u > 0 \)) and its complementary function

\[
\varphi^*(u) = \begin{cases} 
\frac{u^2}{2} & \text{if } 0 \leq u \leq 1, \\
e^{u-1} - \frac{1}{2} & \text{if } u \geq 1,
\end{cases}
\]
satisfies the \( \Delta_0 \)-condition.

In many cases it is impossible to find explicit formula for the complementary N-function, but using Theorem 5.2 we can easily see why \( \varphi^* \in \Delta_0 \).

Example 5.5. For Orlicz functions \( \varphi_r(u) = u \ln^r(1 + u), r > 0 \), \( \varphi_a(u) = u \sqrt{1 + a \ln(1 + u)}, a > 0 \), and \( \varphi_b(u) = u \exp \left( \sqrt{1 + a \ln^+ u} \right), b > 0 \) we have

\[
\frac{u \varphi'_r(u)}{\varphi_r(u)} = 1 + \frac{ru}{(1 + u) \ln(1 + u)} \to 1 \text{ as } u \to \infty,
\]

\[
\frac{u \varphi'_a(u)}{\varphi_a(u)} = 1 + \frac{au}{2(1 + u)[1 + a \ln(1 + u)]} \to 1 \text{ as } u \to \infty,
\]

and

\[
\frac{u \varphi'_b(u)}{\varphi_b(u)} = 1 + \frac{a}{2 \sqrt{1 + a \ln^+ u}} \to 1 \text{ as } u \to \infty.
\]

From Theorem 5.2 we obtain that \( \varphi_r^* , \varphi_a^* , \varphi_b^* \in \Delta_0 \).

6 | WEAK COMPACTNESS OF POINTWISE MULTIPLIERS

Finally we are in a position to characterize weakly compact pointwise multipliers between Banach function lattices.

Given two Banach function lattices \( X \) and \( Y \) on \( \Omega \) we define the space of pointwise multipliers from \( X \) to \( Y \) as

\[
M(X, Y) = \{ f \in L^0(\Omega) : fg \in Y \text{ for all } g \in X \},
\]
with the operator norm

$$\|f\|_M = \sup_{\|g\|_X \leq 1} \|fg\|_Y.$$ 

If $Y$ has the Fatou property, then the space $M(X, Y)$ has the Fatou property (see [35]). Of course, each $f \in M(X, Y)$ defines a pointwise multiplication operator $M_f : X \to Y$ by

$$M_f(g) = fg, \ g \in X$$

and $\|M_f\| = \|f\|_{M(X,Y)}$.

**Lemma 6.1.** Let $X$ and $Y$ be Banach function lattices on $\Omega$. If $f \in (M(X,Y))_\alpha$, then $M_f : X \to Y$ is a weakly compact operator.

**Proof.** We will show that $M_f B(X)$ is $Y$-equi-integrable set. Let $(A_n)$ be a sequence of measurable sets such that $A_n \downarrow \emptyset$. We have

$$\sup_{h \in M_f B(X)} \|h \chi_{A_n}\|_X = \sup_{g \in B(X)} \|fg \chi_{A_n}\|_X = \|f \chi_{A_n}\|_M \to 0,$$

as $n \to \infty$. □

In the case $Y$ satisfies the Dunford–Pettis criterion, we see that order continuity of $f \in M(X,Y)$ is also necessary for weak compactness of $M_f$.

**Theorem 6.2.** Let $X$ and $Y$ be Banach function lattices on $\Omega$ such that $Y$ is 1-DH and let $f \in M(X,Y)$. Then $M_f : X \to Y$ is weakly compact if and only if $f \in (M(X,Y))_\alpha$.

**Proof.** Since $M_f$ is weakly compact then $M_f B(X)$ is relatively weakly compact subset of $Y$. From Theorem 3.6 it follows that $M_f B(X)$ is $Y$-equi-integrable set. Thus for arbitrary sequence of measurable sets $(A_n)$ such that $A_n \downarrow \emptyset$ we have

$$\|f \chi_{A_n}\|_M = \sup_{g \in B(X)} \|fg \chi_{A_n}\|_Y \to 0 \text{ when } n \to \infty,$$

which means that $f \in (M(X,Y))_\alpha$. □

Finally we present one immediate consequence of the previous theorem. Notice that there are in general no results describing properties of the space $M(X,Y)$ in terms of properties of spaces $X,Y$. It is just because properties of $M(X,Y)$ depend rather on coincidence of $X$ and $Y$, than on their properties separately. From this point of view the following conclusion seems surprising.

**Corollary 6.3.** Let $X$ and $Y$ be Banach function lattices over $I$. If $X$ is a reflexive space and $Y$ is 1-DH, then $M(X,Y) \in (OC)$.

**Remark added in proofs.** After the submission of this paper, Astashkin and Strakov gave a negative answer to Question 1 (see [11, pp. 638–641]).

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REFERENCES

[34] S. V. Astashkin, Orlicz spaces and interpolation, Seminars in Math. 5, University of Campinas, Campinas SP, Brazil, 1989.
Reference list:


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