Online estimation of PID controllers and plant dynamics via multi-recursive least squares estimation from closed-loop I/O data

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Abstract
This article proposes an online solution to address the problem of closed-loop system identification using multiple recursive least squares estimation protocols. Some control systems cannot be analysed in an open-loop form for stability reasons or the requirement for online control system operation. So, it is necessary to identify plant dynamics and controller parameters based on input–output data from the feedback structure. The presented method identifies real-time parameters of plant dynamics and controller parameters by utilising a series of recursive least square estimation algorithms that estimate open-loop data from noisy input–output data measured from the closed-loop feedback structure. The proposed method can effectively identify abrupt variations in both the controller parameters and plant dynamics. This capability makes it valuable for deployment as a supervisory component, enabling the detection of any faults that may arise in operating systems. Mathematical formulations and theorems are developed, and two numerical case studies are presented to examine the feasibility and performance of the presented closed-loop system identification protocol.

1 | INTRODUCTION

Feedback control systems (FCSs) play a crucial role in regulating real-world processes by utilising measurements of a system's output to adjust its control input, to achieve then maintain a desired output through appropriate controllers [1]. These systems, also known as closed-loop control systems, find widespread application across various domains. For instance, they are commonly used in industrial automation and consumer electronics to achieve consistent temperatures in heating and cooling systems [2]. In the manufacturing process, FCSs govern the movement of robotic arms and other elements of the process [3]. Also, traffic control systems rely on closed-loop control approaches to manage traffic flow and prevent congestion [4]. Additionally, medical devices such as insulin pumps and pacemakers utilise feedback control systems to control drug delivery and ensure the proper functioning of vital bodily processes [5].

In real-world scenarios, FCSs encounter uncertain and variable conditions, such as temperature fluctuations, humidity changes, and varying loads. Open-loop control systems are inadequate for handling such uncertainties as they fail to consider real-time system behaviour [6–11]. In contrast, closed-loop systems effectively utilise feedback to adjust inputs and accommodate changes in system behaviour caused by varying operating conditions [12–16]. Furthermore, external disturbances can affect control system performance, leading to deviations from the desired behaviour. Open-loop systems lack the ability to compensate for disturbances due to the absence of feedback. Additionally, certain closed-loop feedback structures cannot be simplified into open-loop forms due to stability requirements or inherent plant characteristics [17]. Consequently, many control systems must be identified and modelled as closed-loop structures to ensure stable, accurate, and reliable performance in the presence of uncertainty, disturbances, and the need for online control system operation. This approach guarantees more precise and dependable performance in complex real-world situations.

Accurate identification and modelling of control systems are essential for designing effective feedback controllers that stabilise the system, enhance performance, and meet design requirements [18, 19]. Without models that represent the plant or process with sufficient accuracy, designing a controller that
meets performance specifications becomes challenging, risking unstable or underperforming systems. Therefore, achieving accurate model identification is crucial in control system design, and the choice between offline and online methods depends on factors like the system and available resources [20, 21].

Offline and online model identification are two approaches used to determine mathematical models of systems. Offline identification involves gathering data from the system, temporarily halting it if necessary, and estimating a model based on the collected data. This method is suitable for systems with repeatable behaviour or when data can be acquired within a finite amount of time [20]. In contrast, online identification collects data while the system is actively operating, continuously updating the model. This method is preferred for systems that cannot be paused or have highly variable behaviour [20, 22, 23]. Understanding the individual contributions of the plant and controller to the overall closed-loop control system response can be challenging, especially when there is a limitation in data gathering from different parts of the closed-loop structure. Besides, environmental factors and ageing can cause changes in the plant parameters, while the controller may adapt to these changes. Therefore, introducing new offline/online methods for system identification is beneficial for estimating plant parameters. This estimation allows for updating controller design or making real-time adjustments to control actions. Generally, online methods are gaining popularity as they provide continuous monitoring and understanding of the plant and controller behaviour over time. Various techniques are employed for both offline and online identification, including least-squares estimation, frequency response analysis, time-domain analysis for offline identification [21, 24–27] and the recursive least-squares (RLS) parameter estimation technique for online identification goals as a robust method for real-time operating FCSs [28, 29].

In the context of offline and online model identification techniques, the landscape of system identification has witnessed a profound transformation propelled by the rapid advancements in machine learning over the past decade. These developments have ushered in a new era of sophistication in system identification strategies. They have introduced a plethora of statistical analyses designed to model plant dynamics, capitalising on diverse data-driven online monitoring strategies that utilise various input and output signals characterised by a wide array of statistical features. These cutting-edge methods offer an improved understanding of process behaviour, even in the face of intricate and complex dynamics. For instance, consider [30], where a novel approach, termed “learning identification,” has emerged, grounded in error-correcting training procedures within machine learning. This approach is noteworthy for being non-disruptive and applicable in scenarios involving random and nonstationary input signals, all while boasting swift execution times, making it an invaluable tool for identifying linear quasi-time-invariant systems where certain parameters evolve slowly in comparison to the time required for identification. Moreover, [31] presents an exhaustive survey, intended to demystify key mathematical tools, concepts, and computational aspects underpinning these learning techniques. The survey's focus on kernel-based regularization, its associations with reproducing kernel Hilbert spaces, and its connections to Bayesian estimation of Gaussian processes serve to highlight the second aim of this survey: to underscore that learning techniques tailored to the unique characteristics of dynamic systems may surpass conventional parametric approaches in identifying stable linear systems. Furthermore, within the context of the burgeoning concerns regarding climate change and environmental conservation, [32] offers a holistic perspective on the control techniques of smart power generation, spanning from the regulatory level to the planning level. This paper aptly demonstrates the advantages of machine learning and data-driven control techniques in terms of enhancing visibility, manoeuvrability, flexibility, profitability, and safety in the domain of smart power generation, positioning these techniques as indispensable tools in the face of evolving power generation paradigms.

The RLS method is a popular iterative algorithm for estimating plant dynamics. It continuously updates parameters based on new measurements, minimising the squared errors between predicted and actual outputs. By adjusting model parameters using a weighted average of new data and previous estimates, the RLS method iteratively refines parameter estimates, improving the accuracy of the plant model through recursive updating [20, 21, 28, 29]. The RLS method is advantageous for system identification and modelling in several ways. Firstly, it can handle time-varying systems by continuously updating parameters to accommodate evolving dynamics. Secondly, it is computationally efficient, enabling real-time processing of large data sets. Additionally, the RLS method effectively filters out noise in the system by using a weighted average of new measurements and previous parameter estimates. These strengths make the RLS method a powerful tool for system identification, particularly in scenarios with uncertainties [28].

After reviewing the existing approaches, the authors have identified several research gaps in the development of online system identification methods. First, there is a need for methods that specifically tackle the difficulties arising from a finite set of measured data in control processes, including I/O data from closed-loop feedback structures. Second, there is a lack of online closed-loop system identification approaches that can provide real-time parameters for the controller and process dynamics, particularly for commonly used PID controllers. Previous approaches [20, 28, 29] have limitations in proposing closed-loop system identification methods because they rely on considering the controller parameters over time. To address these gaps, it is of particular interest to reduce reliance on the controller's knowledge and instead utilise only the nominal parameters as initial guesses for system identification. This would enable the provision of real-time parameters of the controller's parameters online. In response to these challenges, the authors have developed a multi-stage RLS parameter estimation algorithm. This algorithm is capable of estimating real-time parameters for both time-invariant and time-variant feedback systems. Its primary objective is to minimise errors, and it focuses on employing the widely used modelling framework known as AutoRegressive
moving average model with exogenous inputs (ARMAX) as considered in \cite{21}.

In order to overcome the difficulties associated with online identification using I/O data from the feedback structure, we have developed a novel approach for closed-loop system identification. This approach is specifically tailored for processes governed by PID controllers, making it suitable for a wide range of industrial plants. To determine the maximum order of the process model that can be accurately estimated, we have conducted rigorous mathematical analyses and provided proofs. An important feature of our approach is its reliance on nominal pre-assigned parameters for the controller, facilitating a transparent and suitable identification of the process model and controller parameters. This approach also enables the detection of sudden changes in estimated parameters caused by uncertainties. In contrast, previous system identification methods either assumed knowledge of controller parameters throughout the identification process or did not account for the identification of abrupt changes in plant or controller parameters due to various uncertainties \cite{20, 21, 29, 33}. Our approach functions as a means of detecting real-time uncertainties or sudden changes. This ability allows us to identify faults or failures in the operating system caused by environmental uncertainties, disturbances, or emerging cyber threats from malicious intruders. To validate the performance of our proposed online closed-loop system identification approach, we conducted numerical evaluations. These evaluations serve as evidence of the effectiveness and reliability of our method.

In all domains where PID control is present, the method can be used to provide sufficiently accurate models for PID controller optimization. To mention them all would render a long list, but it can generally be said that the limitation in the application domains depends on which accuracy of the models is needed and if a PID controller is the appropriate choice.

The remaining sections of the article are structured as follows: In Section 2, we present the fundamental aspects of recursive parameter estimation for open-loop systems using ARMAX modelling. Additionally, we introduce the closed-loop feedback structure and its associated components. Furthermore, we establish parameter identifiability conditions that offer insights into considering the model order for processes being controlled by PID controllers. Moving on to Section 3, we propose a multi-step online closed-loop system identification approach. We employ the RLS parameter estimation strategy as the foundation for our developed approach. This section also includes the necessary formulations derived for parameter estimation of each component within the assumed closed-loop structure. We conclude Section 3 by presenting an algorithm that summarises our proposed approach. Furthermore, in Section 4, we validate the effectiveness of the online identification algorithm, particularly in the presence of uncertainties in the controller or process, using case studies. Numerical simulations using synthetic data and experimental data are used to demonstrate that the introduced closed-loop system identification approach successfully captures the actual values of the controller, process model, and measurement noise parameters accurately. This swift parameter identification capability significantly enhances the reliability and trustworthiness of the investigated approach. Finally, in Section 5, we draw conclusions and outline future research directions. Additionally, we provide mathematical derivations and proofs in the appendix.

2 | PRELIMINARIES AND PARAMETER IDENTIFIABILITY CONDITIONS

This section provides a concise overview of the RLS parameter estimation method based on ARMAX modelling. It also introduces the components of a closed-loop control system, which will be further analysed in subsequent identification discussions in this article.

2.1 | Fundamentals of recursive parameter estimation for ARMAX models

Assume the following linear, time-invariant (LTI) dynamical system, which is represented in the ARMAX model as

\[ y(k) = G(q)u(k) + L(q)e(k), \]

where \( G(q) = \frac{b(q)}{A(q)} \) and \( L(q) = \frac{Z(q)}{A(q)} \) are defined as plant and measurement noise filter dynamics, respectively. Furthermore,

\[
\begin{align*}
A(q) &= 1 - a_1 q^{-1} - a_2 q^{-2} - \cdots - a_{n_A} q^{-n_A}, \\
B(q) &= b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_B} q^{-n_B}, \\
Z(q) &= 1 + d_1 q^{-1} + d_2 q^{-2} + \cdots + d_{n_Z} q^{-n_Z},
\end{align*}
\]

with \( k \) as the time index, \( y(k) \) as the output signal and controlled variable, \( u(k) \) as the manipulated variable, and \( e(k) \) is the white Gaussian measurement noise being filtered by \( L(q) \) as a monic, Hurwitz, and invertible function. Moreover, the measurement noise \( e(k) \) is considered to involve a sequence of independent and identically distributed (i.i.d) normal random signals with the feature \( e(k) \sim \mathcal{N}(0, \sigma^2) \). Additionally, \( q^{-1} \) is the backward shift operator; that is, \( q^{-1} u(k) = u(k - 1) \). \( A(q) \), \( B(q) \) and \( Z(q) \) are polynomials in terms of \( q \) with orders \( n_A, n_B \) and \( n_Z \), respectively \cite{21}. The range of the coefficients is \( n_A - n_B \geq 0 \) and \( n_A - n_Z \geq 0 \). Besides, the general representation of the open-loop ARMAX model is illustrated in Figure 1. In this work, the coefficients of \( A(q) \) and \( B(q) \) are denoted as model parameters and those of \( Z(q) \) as the measurement noise model parameters.

Assumption 1. The ARMAX model (1), (2) satisfies the given conditions:

1. The introduced polynomials in the plant and the measurement noise models, denoted by \( A(q), B(q) \) and \( Z(q) \) are coprime.
2. The order of polynomials \( A(q), B(q) \) and \( Z(q) \), denoted by \( n_A, n_B \) and \( n_Z \), respectively, are known in advance. Also, we assume that \( n_Z < n_A \).
3. The coefficients of polynomials \( A(q), \) \( B(q) \) and \( Z(q) \) are unknown.
4. The polynomial \( Z(q) \) which is defined as the measurement noise model, is stable (strictly Schur), that is, all zeros of this polynomial are inside the unit circle.
5. The operational system under the control of conventional PID controllers is functioning in a stable manner in closed-loop.

Define the one-step-ahead prediction of the output signal as \( \hat{y}(k|\theta) \), where \( \theta \) is the vector of parameters to be estimated [21]. The introduced single-input, single-output ARMAX model \( \hat{y}(k|\theta) \) for one iteration data sampling is given as

\[
\hat{y}(k|\theta) = \frac{B(q)}{Z(q)} y(k) + \left(1 - \frac{A(q)}{Z(q)}\right) y(k).
\]  

(3)

Accordingly the one-step-ahead prediction of the output signal is obtained recursively each time new I/O data is available online as

\[
\hat{y}(k|\theta) = \varphi(k)\theta,
\]  

(4)

where the vector \( \varphi(k) \) is defined as

\[
\varphi(k) = [y(k-1), \ldots, y(k-n_Z), y(k-(n_Z+1)), \ldots, y(k-n_z), u(k), u(k-1), \ldots, u(k-n_u)],
\]  

(5)

and \( \theta \) is

\[
\theta = [(d_1 + a_1), \ldots, (a_{n_z} + a_{n_z}), a_{n_z+1}, \ldots, a_{n_z}, b_0, b_1, \ldots, b_{n_u}, -d_1, -d_2, \ldots, -d_{n_x}]^T.
\]  

(6)

where \( i \) denotes the subscript indicating the output prediction for the \( i \)-th iteration, yielding

\[
\hat{y}_i(k|\theta) = \varphi_i(k)\theta, \quad i = 1, 2, \ldots, N
\]  

(7)

for \( N \) samples performing output prediction and improving the parameter estimation.

\[
\sum_{k=1}^{N} y(k|\theta) = \hat{y}_1(k|\theta), \hat{y}_2(k|\theta), \ldots, \hat{y}_N(k|\theta)\]

and

\[
\mathcal{P}_N = [\varphi_1, \varphi_2, \ldots, \varphi_N]^T.
\]

(8)

So, we have

\[
\hat{\theta}_N = (\varphi_N^T \mathcal{P}_N)^{-1} \varphi_N^T \sum_{k=1}^{N} y(k|\theta).
\]

Adding the \((N + 1)\)-th sample to update the estimation process, we have

\[
\hat{\theta}_{N+1} = \hat{\theta}_N + K_{N+1}(y_{N+1} - \varphi_{N+1}^T \hat{\theta}_N),
\]

(9)

\[
P_{N+1} = \frac{1}{\lambda} \left(P_N - \frac{P_{N+1} \varphi_{N+1} \varphi_{N+1}^T P_N}{\lambda + \varphi_{N+1}^T P_N \varphi_{N+1}}\right),
\]

\[
K_{N+1} = \frac{P_{N+1} \varphi_{N+1}}{\lambda + \varphi_{N+1}^T P_{N+1} \varphi_{N+1}},
\]

where \( K_{N+1} \) denotes the estimation gain, and \( P_{N+1} \) is the inverse of the correlation matrix or commonly known as the parameter estimation error covariance matrix when the \((N + 1)\)-th sample is added, and \( \lambda \) is the forgetting factor in the RLS parameter estimation algorithm. It should be noted that in this article, the forgetting factor is set to \( \lambda = 1 \), meaning that all past observations are treated with equal importance in the parameter estimation process. Also, the initial guess for the inverse of the correlation matrix \( P_0 \) is set with a large positive definite value to ensure numerical stability during the early iterations of the algorithm. The choice of \( P_0 \) is a critical aspect of the algorithm’s performance, influencing its ability to adapt to different system dynamics and handle various types of input data. The investigations conducted in the studies [21, 29] thoroughly examine the RLS parameter estimation method.

### 2.2 Closed-loop control system structure

Figure 2 illustrates the block diagram of the closed-loop control system. Following this structure, the plant and measurement noise dynamics are defined as in (1) and (2) for ARMAX modelling. Besides, denote \( G_c(q) = \frac{C(q)}{M(q)} \) the controller’s transfer function with numerator and denominator's dynamics as

\[
\begin{align*}
C(q) &= a_0 + \epsilon_1 q^{-1} + \epsilon_2 q^{-2} + \cdots + \epsilon_{n_c} q^{-n_c}, \\
M(q) &= 1 + m_1 q^{-1} + m_2 q^{-2} + \cdots + m_{n_M} q^{-n_M},
\end{align*}
\]

(10)

with maximum orders \( n_c \) and \( n_M \). Moreover, \( \epsilon \) and \( u \) are the tracking error and control input signals, respectively. In this article, standard discrete-time PID controller is considered [21].
Therefore, \( G_e(q) \) takes the form
\[
G_e(q) = \frac{\bar{a}_0 + \bar{a}_1 q^{-1} + \bar{a}_2 q^{-2}}{1 + \bar{N}T_s q^{-1} (2 + \bar{N}T_s) + q^{-2}},
\]
where numerator coefficients of the controller are defined as \( \bar{a}_0 = K_p(1 + \bar{N}T_s) + K_i T_s + K_d \bar{N} \), \( \bar{a}_1 = K_p(2 + \bar{N}T_s) + K_i T_s + 2K_d \bar{N} \), and \( \bar{a}_2 = K_p + K_d \bar{N} \). The parameters \( K_p, K_i \) and \( K_d \) are the proportional, integral and derivative coefficients of the controller, respectively. The derivative term in the controller is realised by a low-pass filter (LPF) with an LPF coefficient of \( \bar{N} \). Also, \( T_s \) is the data sampling time. Define \( \lambda_1 = K_p T_s \), \( \lambda_2 = K_d \bar{N} \), and \( \alpha = 1 + \bar{N}T_s \), the controller's dynamics is
\[
G_e(q) = \frac{a_0 + a_1 q^{-1} + a_2 q^{-2}}{1 + m_1 q^{-1} + m_2 q^{-2}}.
\]
where the coefficients of the controller's transfer function are
\[
a_0 = \frac{1}{\alpha} (K_p \alpha + \lambda_1 + \lambda_2),
\]
\[
a_1 = -\frac{1}{\alpha} (K_p(1 + \alpha) + \lambda_1 + 2\lambda_2),
\]
\[
a_2 = \frac{1}{\alpha} (K_p + \lambda_2),
\]
\[
m_1 = -\frac{1 + \alpha}{\alpha},
\]
\[
m_2 = \frac{1}{\alpha}.
\]

In the upcoming subsection, we will discuss the identifiability conditions for plant and controller dynamics using the recursive least squares parameter estimation method, focusing specifically on PID controllers. These conditions are based on the feedback structure depicted in Figure 2.

2.3 Key conditions for parameter identifiability

If the parameter estimation algorithm consistently shows a trend in measuring I/O data within the feedback structure shown in Figure 2, it is deemed that the feedback structure is parameter identifiable [20]. The next part of this section outlines the conditions required to achieve parameter identifiability.

**Condition 1.** The mathematical representation of the closed-loop form introduced in Figure 2 is given as
\[
(A + B \frac{C}{M}) y(k) = Z e(k).
\]
To provide a modified closed-loop representation of Figure 2, a polynomial \( \Lambda(q) \) is introduced to be added to and then subtracted from (18). Moreover, for the sake of simplicity the operator \( (q) \) is subsequently dropped. So, (18) is altered to
\[
(A + A + \left( B - \frac{M}{C} \frac{\Lambda}{M} \right) \frac{C}{M}) y(k) = Z e(k).
\]

Consequently, (19) is simplified to
\[
\left( C(A + A) + (CB - \Lambda M) \frac{C}{M} \right) y(k) = C Z e(k).
\]

Denoting \( \bar{A} = CA + \Lambda C \), \( \bar{B} = CB - \Lambda M \) and \( \bar{Z} = CZ \), we have
\[
\left( \bar{A} + \frac{B C}{M} \right) y(k) = \bar{Z} e(k).
\]

Accordingly, we can formulate
\[
\bar{B} \frac{\bar{A}}{A} = \frac{CB - \Lambda M}{CA + \Lambda C} \quad \text{and} \quad \bar{Z} \frac{\bar{A}}{A} = \frac{CZ}{CA + \Lambda C},
\]
which relate the modified feedback structure with the added polynomial \( \Lambda \) to the originally given feedback structure according to Figure 2. Moreover, the controller dynamics \( G_e(q) = \frac{C}{M} \) will have the same I/O signal as the originally given feedback structure. Since the polynomial \( \Lambda \) can be selected arbitrarily, the plant cannot be identified unconditionally from the closed-loop I/O data even if we consider that the controller dynamics are available with accurate parameters. Consequently, the order of process model, which are polynomials \( A \) and \( B \), should be known in advance for the closed-loop system identification purposes, the same condition as introduced in [20]. Thus, the first parameter identifiability condition is stated next.

**Lemma 1.** The model order must be known in advance as the initial information for the identifier [20].

**Condition 2.** According to Figure 2, the transfer function between the output signal \( y \) and the noise signal \( e \) is calculated as:
\[
\frac{y}{e} = \frac{L}{1 + \bar{G} G} = \frac{Z M}{A M + B C}
\]
\[
= \frac{1 + \delta_1 q^{-1} + \delta_2 q^{-2} + \cdots + \delta_r q^{-r}}{1 + \zeta_1 q^{-1} + \zeta_2 q^{-2} + \cdots + \zeta_p q^{-p}}
\]
\[
= \frac{H}{F}.
\]
According to (23), the number of \( n_A + n_B \) coefficients considered to describe the plant model needs to be identified from the number of \( \rho \) coefficients related to the parameters \( \xi_i, i = 1, \ldots, \rho \). Assume polynomials \( Z \) and \( P \) are coprime and have no common roots. As a result, for the straightforward estimation of the process model parameters, we can conclude that \( \sigma = n_A + n_B \). In other words, the following identification requirements are introduced:

\[
\max (n_A + n_M, n_B + n_C) \geq n_A + n_B, \tag{24}
\]

\[
\max (n_M - n_B, n_C + n_A) \geq 0, \tag{25}
\]

As a result, the second parameter identifiability condition is stated as in the following lemma, with a complete proof in [20].

**Lemma 2.** The controller order has to be sufficiently large and should satisfy

\[
\text{if } n_C > n_M + n_A - n_B \Rightarrow n_C \geq n_A, \tag{26}
\]

\[
\text{if } n_C < n_M + n_A - n_B \Rightarrow n_M \geq n_B. \tag{27}
\]

Thus, the controller order must be \( n_C \geq n_A \) or \( n_M \geq n_B \) [20]. It should be noted that if the order of the controller is not large enough to satisfy the given conditions in (26) and (27), some studies state that closed-loop identification strategies can be developed under two different selections of controller parameters [20].

**Remark 1.** Based on (26) and (27) and the assumption of including the PID controller, the order of \( G_c(q) \) is \( n_C = n_M = 2 \) and accordingly, the maximum order for the process parameters is \( n_A = n_B = 2 \). Hence, to comply with Condition 2, the highest order of a process model is two. As a result, \( G(q) \) is modelled as a linear second order process with the following representation:

\[
G(q) = \frac{B(q)}{A(q)} = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}}, \tag{28}
\]

Previous studies have emphasised the importance of incorporating controller information in indirect closed-loop system identification methods [20, 21, 28].

**Condition 3.** To reduce the reliance on knowledge of controller parameters during the identification process, subsequent RLS formulations assume only nominal parameters of the controller are accessible.

It will be shown that even in the presence of measurement noise, parameters of both controller and process model can be estimated. Therefore, to achieve an accurate estimation of both plant and controller parameters, the third parameter identifiability condition is proposed in the given lemma.

**Lemma 3.** The nominal parameters of the controller are known and provided as initial values for the identification procedure [20, 21].

The next section will examine the closed-loop system identification protocol using the corresponding RLS formulations.

### 3 | ONLINE RECURSIVE PARAMETER ESTIMATION: CLOSED-LOOP SYSTEM IDENTIFICATION

To estimate system parameters using the ARMAX modelling approach with closed-loop I/O data, three RLS estimators that perform hierarchically are introduced. These estimators provide one-step-ahead predictions for the output and control input signals, aiming to estimate controller and process model dynamics. As a result, the online closed-loop system identification problem is addressed. This approach utilises closed-loop I/O data and nominal controller parameters to provide real-time modelling of controller and plant.

One-step ahead prediction of the output signal in Figure 2, denoted by \( \hat{y}(k|\Theta) \) using ARMAX modelling is calculated as follows:

\[
\hat{y}(k|\Theta) = \Phi(k)\Theta, \tag{29}
\]

where the vector \( \Phi(k) \) is defined as

\[
\Phi(k) = [y(k-1), \ldots, y(k-4)],
\]

\[
r(k-1), \ldots, r(k-4),
\]

\[
\hat{y}(k-1|\Theta), \hat{y}(k-2|\Theta), \hat{y}(k-3|\Theta), \tag{30}
\]

and \( \hat{y}(k-i|\Theta) \) is defined as the estimated value of the output signal at \( i \)th prior step. Moreover, \( \Theta = [\theta_1, \ldots, \theta_{11}]^T \) is defined with the given parameters:

\[
\theta_1 = \frac{1}{\alpha} (-b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) + \alpha (d + a_1)), \tag{31}
\]

\[
\theta_2 = \frac{b_1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2 \lambda_2) - \frac{b_2}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) \tag{32}
\]

\[-\frac{1}{\alpha} (1 + \alpha) (d + a_1) + a_2, \]

\[
\theta_3 = -\frac{b_1}{\alpha} (K_p + \lambda_2) + b_2 \left( \frac{K_p (1 + \alpha) + \lambda_1 + 2 \lambda_2}{\alpha}\right) \tag{33}
\]

\[
+ \frac{d + a_1}{\alpha} - \frac{a_2 (1 + \alpha)}{\alpha}, \]

\[
\theta_4 = \frac{1}{\alpha} (a_2 - b_2 (K_p + \lambda_2)), \tag{34}
\]

\[
\theta_5 = \frac{1}{\alpha} (b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2)), \tag{35}
\]

\[
\theta_6 = \frac{b_1}{\alpha} (-K_p (1 + \alpha) - \lambda_1 - 2 \lambda_2) \tag{36}
\]

\[
+ \frac{b_2}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2), \]

\[
\theta_7 = \frac{1}{\alpha} (b_1 (K_p + \lambda_2) + b_2 (-K_p (1 + \alpha) - \lambda_1 - 2 \lambda_2)). \tag{37}
\]
\[ \begin{align*}
\theta_8 &= \frac{1}{\alpha} \left( k_2 (K_p + \lambda_2) \right), \\
\theta_9 &= \frac{1}{\alpha} \left( \alpha d - (1 + \alpha) \right), \\
\theta_{10} &= \frac{1}{\alpha} \left( 1 - d(1 + \alpha) \right), \\
\theta_{11} &= -\frac{d}{\alpha}
\end{align*} \]

Accordingly, with the provided RLS estimation formula in (29), the one-step-ahead estimation of the output signal is calculated at each time step. The derivation of (29) is given in the Appendix.

**Remark 2.** Based on (29), when examining the obtained parameters \( \theta_1, \ldots, \theta_{11}, \) it becomes evident that estimations of both the controller and the plant parameters cannot occur simultaneously due to the concurrent presence of unknown parameters in both the controller and the process model dynamics. However, if \( \hat{y}(k|\bar{\theta}) \) is used to formulate another RLS estimation procedure for the one-step-ahead prediction of the control input \( \hat{u}(k|\bar{\theta}) \), this issue can be addressed.

In the next step, one-step ahead prediction of the control input signal in Figure 2, denoted by \( \hat{u}(k|\bar{\theta}) \) using ARMAX modelling is calculated as follows (the derivation is provided in the Appendix):

\[ \hat{u}(k|\bar{\theta}) = \overline{\Phi}(k)\overline{\theta}, \]

where the vector \( \overline{\Phi}(k) \) is defined as

\[ \Phi(k) = [\hat{y}(k|\bar{\theta}), y(k - 1), y(k - 2), r(k), r(k - 1), r(k - 2), \hat{u}(k - 1|\bar{\theta}), \hat{u}(k - 2|\bar{\theta})], \]

and \( \hat{u}(k - i|\bar{\theta}) \) is the estimate of the control input signal in \( i \)th step prior, and \( \overline{\theta} = [\overline{\theta}_1 \ldots \overline{\theta}_8]^T \) is defined with the parameters:

\[ \begin{align*}
\overline{\theta}_1 &= -\frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2), \\
\overline{\theta}_2 &= \frac{1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2\lambda_2), \\
\overline{\theta}_3 &= -\frac{1}{\alpha} (K_p + \lambda_2), \\
\overline{\theta}_4 &= \frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2), \\
\overline{\theta}_5 &= -\frac{1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2\lambda_2)
\end{align*} \]

According to (42), we have

\[ \hat{\theta} = -\frac{1}{\overline{\theta}_8} \]

\[ \hat{\lambda}_2 = \frac{\overline{\theta}_2 + \frac{\hat{\theta} \overline{\theta}_3}{\overline{\theta}_4} \overline{\theta}_2 + \overline{\theta}_4 \overline{\theta}_4 + \hat{\theta} \overline{\theta}_3}{\overline{\theta}_8 \overline{\theta}_8 + 3}, \]

\[ \hat{\theta}_1 = \frac{\hat{\theta} \overline{\theta}_3 + \hat{\theta} \overline{\theta}_2}{\overline{\theta}_8}, \]

\[ \hat{\theta}_4 = \frac{\hat{\theta} \overline{\theta}_3}{\overline{\theta}_8}. \]

Furthermore, using Equations (29) and (42)–(58), we have

\[ \hat{\gamma} = -\frac{\hat{\theta} \overline{\theta}_3 + (1 + \hat{\theta})}{\overline{\theta}_8}, \]

\[ \hat{\gamma}_1 = \frac{\hat{\theta} \overline{\theta}_5 + \hat{\theta} \overline{\theta}_4 + \hat{\theta} \overline{\theta}_3}{\overline{\theta}_8 \overline{\theta}_8 + 3}, \]

\[ \hat{\gamma}_2 = \frac{\hat{\theta} \overline{\theta}_5 + \hat{\theta} \overline{\theta}_4 + \hat{\theta} \overline{\theta}_3}{\overline{\theta}_8 \overline{\theta}_8 + 3}. \]

Now, the only parameters left to estimate are the denominator variables of the process model, denoted by \( a_1 \) and \( a_2 \). In the following, the third RLS estimator is proposed to provide an estimation of these parameters.

When utilising the outcomes of the two previously examined RLS estimators described, it becomes experimentally clear that precise values for \( a_1 \) and \( a_2 \) cannot be attained with few errors. Consequently, an additional RLS estimator is proposed in the subsequent discussion as a third-level RLS estimation approach, aiming to address this concern. The third-level RLS estimator suggests generating a forecast of the output signal similar to that in the first RLS parameter estimation procedure, but utilising the obtained estimation of the control input signal, \( \hat{u}(k|\bar{\theta}) \).
One-step ahead prediction of the output signal in Figure 2 utilising $\hat{n}(k|\theta)$ into the RLS estimation procedure is calculated as

$$\hat{y}(k|\theta) = \overline{\varphi}(k) \overline{\theta}, \quad (62)$$

where the vector $\overline{\varphi}(k)$ is defined as

$$\overline{\varphi}(k) = [y(k-1), y(k-2), \ldots, y(k-N)], \quad (63)$$

and $\overline{\theta} = [\overline{\theta}_1, \ldots, \overline{\theta}_N]$ with the given parameters as:

$$\overline{\theta}_1 = \theta_1 + a_1, \quad \overline{\theta}_2 = \theta_2 + a_2, \quad \overline{\theta}_3 = \theta_3, \quad \overline{\theta}_4 = \theta_4, \quad \overline{\theta}_5 = -d. \quad (64)$$

The derivation of (62) is provided in the Appendix.

Accordingly, the denominator parameters of the plant model are calculated as

$$\hat{a}_2 = \overline{\theta}_2, \quad \hat{a}_1 = \overline{\theta}_1 + \overline{\theta}_3. \quad (65)$$

By utilising the developed hierarchical RLS parameter estimators, it is possible to simultaneously estimate the controller and process model parameters in real-time using the I/O data, while also considering the nominal controller parameters from synthesis. The outlined procedure is presented in Algorithm 1 at the end of Section 3.

In order to ensure the reliability and accuracy of the identification process, Theorem 1 is presented, providing the mean and covariance values of the parameter estimation errors in the developed RLS estimators.

**Theorem 1.** Consider the feedback structure represented in Figure 2 by the ARMAX modelling in (1), (2) and (10) governed by the PID controller. The derived hierarchical RLS estimation protocols in (29), (42) and (62) for $N$ number of experiment iterations and with $k$ number of data sampling remain unbiased with constant covariance matrices of estimation errors. Consequently, the estimators remain consistent.

More specifically, by defining $\tilde{\theta}(k), \tilde{\theta}(k) \text{ and } \tilde{\theta}(k)$ as the estimation errors in the first, second and third RLS estimators, the expectation of these parameters are obtained as

$$E\{\tilde{\theta}(k)\} = E\{\tilde{\theta}(k)\} = E\{\tilde{\theta}(k)\} = 0, \quad (66)$$

with the calculated corresponding covariance matrices for $N$ iterations as

$$P^\theta(k) = E\{\tilde{\theta}(k)\tilde{\theta}^T(k)\} = \mu \sigma^2 \tilde{P}(k),$$

$$P^\theta(k) = E\{\tilde{\theta}(k)\tilde{\theta}^T(k)\} = \mu \sigma^2 \tilde{P}(k),$$

$$P^\theta(k) = E\{\tilde{\theta}(k)\tilde{\theta}^T(k)\} = \mu \sigma^2 \tilde{P}(k),$$

where $\mu$, $\mu$ and $\mu$ are constants, $\tilde{P}(k)$, $\tilde{P}(k)$ and $\tilde{P}(k)$ are the estimation error covariance matrices for each RLS estimator with $N$ iterations and $\sigma^2$ is the variance of the considered noise vector.

**Proof of Theorem 1.** See the Appendix.

The block diagram in Figure 3 summarises the proposed closed-loop system identification protocol presented in Algorithm 1.

**Remark 3.** Resetting the estimation error covariance matrix $\tilde{P}(k)$, $\tilde{P}(k)$ and $\tilde{P}(k)$ periodically in the RLS algorithm offers advantages in adapting to dynamic system conditions and preventing the accumulation of errors over time. This strategy allows the algorithm to better handle changes in system parameters and enhances numerical stability by preventing ill-conditioning or singularity issues associated with long-term operation. The decision to reset instead of applying forgetting factor $\lambda$ depends on the specific system dynamics and the trade-off between continuous adaptation and stability that best aligns with the goals of the parameter estimation process.

**Remark 4.** Algorithm 1 is proposed with the initial thought of applying it to real-world applications. A common case in real applications is that the process is represented as a second-order
dynamical plant and is controlled using a standard PID controller. However, our proposed algorithm can be extended and modified to more general cases where the process is modelled as a high order dynamics.

4 NUMERICAL ASSESSMENTS

In this section, we evaluate our closed-loop parameter estimation algorithm using two case studies, the first using synthetic data and the second using real data obtained from a Quanser QUBE's Servo Motor Laboratory experiment [34]. We will show that while our proposed estimation procedure is much simpler than for instance [28], the estimation performance is plausible.

4.1 Case study 1: Test using synthetic data

This subsection validates the effectiveness of the developed closed-loop parameter estimation algorithm under different scenarios. In the first scenario, the algorithm’s performance is demonstrated in the absence of uncertainties, showcasing the estimated parameters of the plant, controller and noise amplitude. Second scenario focuses on altered controller parameters after a specific time index, while the plant model parameters remain unchanged. This scenario presents the estimated parameters of the system, controller, and noise amplitude before and after this event. The third scenario involves time-varying parameters of the plant model, highlighting the continuous estimation of plant model parameters over time.

Consider the given block diagram in Figure 2, with the plant’s transfer function

\[ G(s) = \frac{8}{1296s^2 + 72s + 1}, \quad (67) \]

The discrete-time form of (67) with sampling time \( T = 1 \) sec is

\[ G(q^{-1}) = \frac{0.0030q^{-1} + 0.0030q^{-2}}{1 - 1.9450q^{-1} + 0.9460q^{-2}}, \quad (68) \]

with the selected nominal parameters of the standard discrete-time PID controller chosen as

\[ K_p,n = 0.4333, K_i,n = 0.0084, K_d,n = 5.2452, \hat{N}_n = 5.1231, \]

with subscript \( n \) denotes the nominal values description. Hence, \( \alpha_n = 6.1231 \). The noise dynamics are \( \nu(k) = Z(q) \epsilon(k) \), where \( \epsilon(k) \) is a white-Gaussian noise and it is arbitrarily selected as \( \epsilon(k) \sim \mathcal{N}(0, 0.05) \). Also, \( Z(q) = 1 + dq^{-1} \), with \( d_w = 0.2 \).

4.1.1 Case 1 Scenario 1 (C1-S1), ideal case

In this scenario, no uncertainties in the plant or controller parameters are considered. By applying Algorithm 1 with \( T_{\text{final}} = 1000 \) s, the estimated values for the actual operating closed-loop system are obtained as in Tables 1 and 2.

To evaluate the accuracy of the proposed method, two common error metrics, the root mean square error (RMSE) and the integral of squared error (ISE) of the estimated parameters are assessed, with the results provided in Tables 3 and 4, demonstrate that the proposed method estimates the parameters with minor errors.

Moreover, Figure 4 shows the reference, real and estimated output signals’ changes, Figure 5 plots the estimated values of

\[ \begin{array}{c|c|c|c|c|c}
\text{Metric} & K_p & K_i & K_d & \hat{N} \\
\hline
\text{RMSE} & 0.0001 & 0.0004 & 0.0003 & 0.0001 \\
\text{ISE} & 0 & 0.0001 & 0 & 0 \\
\end{array} \]

TABLE 1 Estimated actual values of the plant parameters in C1-S1.

\[ \begin{array}{c|c|c|c|c}
\hat{b}_1 & \hat{b}_2 & \hat{a}_1 & \hat{a}_2 & \hat{d} \\
\hline
0.0027 & 0.0049 & 1.9314 & -0.9324 & 0.3424 \\
\end{array} \]

TABLE 2 Estimated actual values of the controller parameters in C1-S1.

\[ \begin{array}{c|c|c|c|c}
\hat{K}_p & \hat{K}_i & \hat{K}_d & \hat{N} & \hat{\alpha} \\
\hline
0.4333 & 0.0080 & 5.2455 & 5.1232 & 6.1232 \\
\end{array} \]

TABLE 3 Calculated estimation error measures for the plant in C1-S1.

\[ \begin{array}{c|c|c|c|c|c}
\text{Metric} & K_p & K_i & K_d & \hat{N} \\
\hline
\text{RMSE} & 0.0002 & 0.0016 & 0.0842 & 0.0833 & 0.1458 \\
\text{ISE} & 0 & 0.0016 & 0.0842 & 0.0833 & 0.9603 \\
\end{array} \]

TABLE 4 Calculated estimation error measures for the controller in C1-S1.

FIGURE 3 Block diagram of the developed closed-loop system identification protocol under the developed hierarchical recursive least-square (RLS) estimation algorithm.
the plant and noise parameters over time, while Figure 6 shows the estimated values of the controller parameters over time. It is important to note that the plots in Figures 5 and 6 exhibit step-formed changes in the estimated values at $T = 500$ s, indicating the occurrence of a reference signal change for the first time, starting from that time step.

The results in this example demonstrate that the introduced identification protocol is capable to estimate feedback parameters online and exhibits a relatively high identification rate even when encountering different changes in the reference signal.

In our proposed identification algorithm, we set the forgetting factor $\lambda = 1$, which is the maximum value, as it can be chosen as $0 < \lambda \leq 1$. This choice is due to several factors, such as the presence of noise, uncertainties, and changes in reference signals over time, which can lead to transient responses in the closed-loop system. In such situations, the practical use of a forgetting factor becomes challenging. A very low $\lambda$ could cause the identifier to become unstable, as illustrated in Figure 7. Increasing the value of $\lambda$ to be close to 1 improves the stability, but inaccuracies still persist in the estimated parameters and signal predictions, as illustrated in Figure 8.

While we only show the effect of $\lambda$ in this S1-C1, the conclusion is consistent in other scenarios and case studies. In the subsequent scenario, we assess the performance of the estimator under various parameter variations and uncertainties, encompassing the controller and the plant.

4.1.2 Case 1 Scenario 2 (C1-S2), deviation to controller parameters

In this scenario, we examine the impact of controller parameters deviation over time while keeping the plant parameters consistent with the previous scenario. The plant and controller parameters are initially the same as in Scenario 1 (C1-S1). However, starting from $T = 500$ s, the controller gain values undergo changes as given in Table 5.

We conduct simulations to determine whether these changes can be detected in the estimation of controller values. The estimation results are provided Tables 6 and 7.

The responses and changes in the parameters during the recursions are also plotted. In Figure 9, the changes in the reference, real and estimated output signals are showcased.

Figure 10 shows the estimated values of the plant and noise parameters over time and Figure 11 shows the estimated values of the controller parameters over time. The effect of the presence of uncertainties or changes in the controller parameters at $T = 500$ s can be seen, particularly in these two figures.

To analyse the presence of uncertainties in the parameters, whether it is the controller or the plant, it is advisable to periodically reset the estimation error covariance matrices in the RLS parameter estimation procedures. This ensures that the latest observed data are more effectively incorporated into the estimations after a certain number of iterations. In Scenario 2, we also arbitrarily reset the covariance matrices for the RLS estimation procedures at $T = 600$ s to enhance the identifier’s sensitivity to incorporate new data more than previous data. This explains the spikes observed in the estimated values at $T = 600$ s in Figures 10 and 11, which subsequently gradually normalise over time. Without resetting the parameter estimation covariance matrices, new parameters will still be estimated but with a slower convergence rate [21].

Moreover, the same as in Scenario 1, to evaluate the accuracy of the proposed method, the root mean squared error (RMSE) and the integral of squared error (ISE) of the estimated parameters are analysed and the results are provided in Tables 8 and 9.

### Table 5

<table>
<thead>
<tr>
<th>$K_p$</th>
<th>$K_i$</th>
<th>$K_d$</th>
<th>$\hat{N}$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2520</td>
<td>0.0050</td>
<td>2.7880</td>
<td>3.3850</td>
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</tr>
</tbody>
</table>

### Table 6

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<thead>
<tr>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\vartheta$</th>
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</thead>
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<td>0.0031</td>
<td>0.0052</td>
<td>1.7240</td>
<td>-0.7281</td>
<td>0.3919</td>
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</table>

### Table 7

<table>
<thead>
<tr>
<th>$K_p$</th>
<th>$K_i$</th>
<th>$K_d$</th>
<th>$\hat{N}$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2810</td>
<td>0.0094</td>
<td>2.7510</td>
<td>3.3859</td>
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</table>

### Table 8

<table>
<thead>
<tr>
<th>Metric</th>
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<th>$b_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>0.0001</td>
<td>0.0023</td>
<td>0.2015</td>
<td>0.1989</td>
<td>0.1872</td>
</tr>
<tr>
<td>ISE</td>
<td>0.0011</td>
<td>8.2051</td>
<td>7.9886</td>
<td>7.0790</td>
<td></td>
</tr>
</tbody>
</table>

Abbreviations: ISE, integral of squared error; RMSE, root mean squared error.
As evident from the estimation performance in Scenario 2, the identification protocol is capable of online computation of parameter parameters, even when confronted with various changes in the reference signal. This holds true particularly for cases involving parameter variations in the controller dynamics. In application, this method can be utilised to assess the controller values over time and reset the controller parameters by
4.1.3 Case 1 Scenario 3 (C1-S3), deviations to plant dynamics

In this scenario, we examine the impact of deviations to plant dynamics over time while keeping the controller parameters consistent as in Scenario 1 (C1-S1). The objective is to observe how the plant or process uncertainties affect the overall performance of the closed-loop system and whether these types of uncertainties are identifiable by the proposed approach or not. To illustrate this scenario, from $T = 500$ s onward, it is set that uncertainties arise in the plant model parameters, by changing the coefficients of the transfer function (68) with the values provided in Table 10, accordingly.

Running the simulations applying Algorithm 1, we obtain the estimation results as tabulated in Tables 11 and 12.

Moreover, Figure 12 demonstrates the variations in reference, real, and estimated output signals. The estimated plant and noise parameters over times are shown in Figure 13, while the estimated controller parameters over time are plotted in Figure 14.

These simulations are done in the presence of plant uncertainties. It is particularly visible in Figure 14 that the estimated parameters change to the new values at $T = 500$ s. The same as in Scenario 2, an arbitrarily reset of the covariance matrices is done at $T = 600$ s to enhance the identifier’s sensitivity to the operator if any deviations are detected. By doing so, the closed-loop performance and efficiency can be enhanced.
incorporate new data more than previous data, as also seen in the figure.

Furthermore, the RMSE and ISE error metrics of the estimated parameters are provided in Tables 13 and 14.

From the presented estimation values and the plots it can be observed that the identification protocol exhibits the ability to perform online computation of parameter values. This capability is maintained even when encountering diverse changes...
in the reference signal. Therefore, the proposed method has proven to perform effectively in detecting uncertainties and deviations in the process parameters with minimal errors, especially when the closed-loop system is stable and is performing as a slow process with higher settling times. Notably, the protocol proves particularly effective in handling parameter variations in plant dynamics.

### 4.2 Case study 2: Test using real experimental Quanser QUBE data

In this experiment, a Quanser QUBE servo device is used, with the NI MyRIO to implement the PID controller for the introduced plant in [34]. The experimental setup is depicted in Figure 15. The Quanser QUBE is a versatile generic servo based platform for control engineering experiments. This equipment is ideal for real-life case studies in various servo control applications due to its adaptability and support for a wide range of control topics, like PID control, adaptive control, and system identification. With a precise servo motor, it faithfully replicates real-life scenarios, offering an interactive environment for hands-on experimentation, benefiting both educational and research purposes.

Consider the block diagram in Figure 2. Setting the QUBE as a position control system, the nominal transfer function of the servo motor describes the motor voltage to its angular position as given by

$$G(s) = \frac{23}{0.13s^2 + s}. \quad (69)$$

The discrete-time representation of (69) with a sampling time $T_s = 0.02$ sec is

$$G(q^{-1}) = \frac{0.0336q^{-1} + 0.0320q^{-2}}{1 - 1.8512q^{-1} + 0.8512q^{-2}}. \quad (70)$$

A standard discrete-time PID controller is designed to control the system, with the following set of selected gains:

$$K_{p, n} = 0.70, K_{i, n} = 0.025, K_{d, n} = 0.07, N_n = 49.97.$$  

In addition, $e(k)$ is selected as $e(k) \sim \mathcal{N}(0, 0.05)$ and $Z(q) = 1 + d_0q^{-1}$, with $d_0 = 0.04$.

The time stamped input and output data from the experiment with sampling time $T_s = 0.02$ s are recorded. Then Algorithm 1 is applied to these data for a duration of $T_{\text{final}} = 5.5$ s. The estimated values for the actual operating closed-loop system are obtained, as presented in Tables 15 and 16. Moreover, Figure 16 illustrates the reference, real, and estimated output signals’ changes.

Comparing the estimated values presented in Tables 15 and 16 with the real values, and also observing Figure 16, it is evident that the proposed method estimates parameters accurately, and that the algorithm can effectively track the closed-loop system’s I/O changes.

Furthermore, using the RMSE and ISE error analysis to evaluate the accuracy of the introduced method, consistently small estimation error values are obtained, as provided in Tables 17 and 18. From these results, it can be concluded that the introduced identification protocol demonstrates the capability to estimate feedback parameters online and exhibits a rela-
Converges of plant and noise parameters’ estimated values in time in C1-S3 (From top to bottom: \( \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2 \)).

Converges of controller parameters’ estimated values in time in C1-S3 (From top to bottom: \( \hat{K}_p, \hat{K}_i, \hat{K}_d, \hat{\nu} \)).

Effectively high identification rate even when encountering different changes in the reference signal.

Consequently, from the two case studies with various scenarios, the proposed closed-loop system identification approach, validated through numerical simulations, effectively estimates real-time parameters of the controller, plant model, and measurement noise dynamics. Notably, when combined with PID controllers and ARMAX models, the approach demonstrates superior accuracy. It promptly detects inconsistencies and uncertainties in the actual values of the controller and plant...
parameters, bolstering the reliability of the identification process.

Furthermore, the developed approach can serve as a valuable tool for detecting false data trends in interconnected cyber-physical systems (CPSs) [35]. The approach offers remarkable flexibility, accommodating various PID controllers and ARMAX models, making it applicable to diverse system configurations. Continuous monitoring enables timely identification of real-time changes or uncertainties. By swiftly addressing deviations and uncertainties, the approach enhances overall performance. Its versatility extends to a wide range of closed-loop systems, making it suitable for various domains and applications.

5 | CONCLUSIONS AND FUTURE WORK

The article introduces an innovative solution for online identification of closed-loop control systems. The proposed approach utilizes recursive least squares parameter estimation methods and focuses on estimating the plant and controller dynamics, as well as the output measurement noise, using closed-loop input–output information and nominal controller parameters based on ARMAX modelling. The article also presents new parameter identifiability conditions, supported by mathematical proofs, for each stage of the system identification protocol. The effectiveness of the approach is demonstrated through numerical results, using both synthetic and real experimental data, which show its ability to estimate real-time plant and controller dynamics even under uncertainties. The simulations also highlight the approach’s capability to detect sudden changes in plant and controller parameters. Overall, the presented approach enhances important aspects of the closed-loop identification process, including flexibility, real-time monitoring and applicability. Future work will involve applying the developed approach to design online resilient secure-based controllers through adaptive control protocols, using various real-life systems or process control case studies. Considering PID controller with anti-windup could also be a future extension of this work.

AUTHOR CONTRIBUTIONS

Amirreza Zaman: Conceptualization; data curation; formal analysis; investigation; methodology; project administration; resources; software; validation; visualization; writing—original draft; writing—review and editing. Wolfgang Birk: Conceptualization; formal analysis; project administration; supervision; writing—review and editing.

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CONFLICT OF INTEREST STATEMENT
The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT
Data availability is not applicable to this paper.

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33. Shu, Y.: Neural dynamic surface control for stochastic nonlinear systems with unknown control directions and unmodelled dynamics. IET Control Theory Appl. 17(6), 649–661 (2023)
A.1 Deriving the initial RLS estimator from Section 3

According to Equations (10),(11),(12) and (28) one-step-ahead closed-loop output prediction is written as

\[
\hat{y}(k|\theta) = \frac{B(q^{-1})}{Z(q^{-1})}y(k) + \left(1 - \frac{A(q^{-1})}{Z(q^{-1})}\right)y(k) \\
= \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + dq^{-1}} C(q^{-1})(r(k) - y(k)) + \left(1 - \frac{1 - a_1 q^{-1} - a_2 q^{-2}}{1 + dq^{-1}}\right)y(k),
\]

therefore, Equation (A1) is rewritten as

\[
(1 + dq^{-1})\hat{y}(k|\theta) = (b_1 q^{-1} + b_2 q^{-2}) C(q^{-1}) \\
\times (r(k) - y(k)) + \left((d + a_1) q^{-1} + a_2 q^{-2}\right)y(k) \\
= (b_1 q^{-1} + b_2 q^{-2}) \left( K_p + \frac{\lambda_2}{\alpha - q^{-1}} (1 - q^{-1}) \right) \cdot (r(k) - y(k)) \\
+ \left(\frac{\lambda_1}{\alpha - q^{-1}} (b_1 q^{-1} + b_2 q^{-2}) \right) (r(k) - y(k)) \\
+ \left( (d + a_1) q^{-1} + a_2 q^{-2}\right)y(k) \\
= \left( K_p \tau + \frac{\lambda_1}{1 - q^{-1}} \tau + \frac{\lambda_2}{\alpha - q^{-1}} (1 - q^{-1}) \right) r(k) \\
- \left( K_p \tau + \frac{\lambda_1}{1 - q^{-1}} \tau + \frac{\lambda_2}{\alpha - q^{-1}} \right) y(k) \\
- \left( (d + a_1) q^{-1} + a_2 q^{-2}\right)y(k),
\]

where \( \tau = b_1 q^{-1} + b_2 q^{-2} \). Consequently, we have

\[
(1 + dq^{-1})(1 - q^{-1})\hat{y}(k|\theta) \\
= \left( K_p (1 - q^{-1}) \gamma \tau + \lambda_1 \gamma \tau + \lambda_2 (1 - q^{-1})^2 \right) r(k) \\
+ \left( (1 - q^{-1})\gamma (d + a_1) q^{-1} + a_2 q^{-2}\right)y(k) \\
+ \left(- K_p (1 - q^{-1}) \gamma \tau - \lambda_1 \gamma \tau - \lambda_2 (1 - q^{-1})^2 \right)y(k),
\]

where \( \gamma = \alpha - q^{-1} \). To simplify (A3), following calculations are used:

\[
(1 - q^{-1})(\alpha - q^{-1}) = \alpha - (1 + \alpha) q^{-1} + q^{-2},
\]

\[
(1 + dq^{-1})(\alpha - q^{-1}) = \alpha + (d + (1 + \alpha)) q^{-1} + (1 - d(1 + \alpha)) q^{-2} + dq^{-3},
\]

By applying (A4) into (A3), we have

\[
\alpha\hat{y}(k|\theta) + (\alpha d - (1 + \alpha)) q^{-1}\gamma \hat{y}(k|\theta) + q^{-2}\gamma \hat{y}(k|\theta) \\
- d(1 + \alpha) q^{-3}\gamma \hat{y}(k|\theta) + dq^{-3}\gamma \hat{y}(k|\theta) \\
= q^{-1} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k) + q^{-2} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k) \\
+ q^{-3} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k) \\
+ q^{-3} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k) \\
+ q^{-4} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k) \\
+ q^{-1} \alpha (d + a_1) y(k) \\
+ q^{-2} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k) \\
+ q^{-2} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k) \\
+ q^{-3} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k) \\
+ q^{-3} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k) \\
+ q^{-4} b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) + a_2 y(k)
\]

Thus, (A5) finally leads to \( \hat{y}(k|\theta) = \varphi(k) \theta \), where

\[
\varphi(k) = [y(k - 1), \ldots, y(k - 4), \\
r(k - 1), \ldots, r(k - 4)],
\]

\[
\hat{y}(k) - \hat{y}(k-1|\theta), \hat{y}(k - 2|\theta), \hat{y}(k - 3|\theta)],
\]

and \( \hat{y}(k - n|\theta) \) is defined as the estimated value of the output signal at \( n \)th prior step. Moreover, \( \theta = [\theta_1, \cdots, \theta_{11}]^T \) is

\[
\theta_1 = \frac{1}{\alpha} (-b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2) + \alpha (d + a_1)),
\]

\[
\theta_2 = \frac{b_1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2 \lambda_2) - \frac{b_2}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2)
\]
hence, (A7) is simplified to

\[
y(k) = \left( \frac{-b_1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) \right) y(k-1) \\
+ \frac{1}{\alpha} (1 + a_1 + a_2 \alpha) y(k-1) \\
+ \frac{b_1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2 \lambda_2) y(k-2) \\
- \frac{b_2}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k-2) \\
+ \frac{1}{\alpha} (1 + a_1 + a_2 (1 + \alpha)) y(k-2) + a_2 y(k-2) \\
+ \frac{b_1}{\alpha} (K_p + \lambda_2) y(k-3) \\
+ \frac{1}{\alpha} (a_2 - b_2 (K_p + \lambda_2)) y(k-4) \\
+ \frac{1}{\alpha} (b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2)) y(k-1) \\
+ \frac{1}{\alpha} (b_2 (K_p + \lambda_2)) r(k-1) \\
+ \frac{b_1}{\alpha} (-K_p (1 + \alpha) - \lambda_1 - 2 \lambda_2) r(k-2) \\
+ \frac{b_2}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k-2) \\
+ \frac{b_1}{\alpha} (K_p + \lambda_2) r(k-3) \\
+ \frac{b_2}{\alpha} (-K_p (1 + \alpha) - \lambda_1 - 2 \lambda_2) r(k-3) \\
+ \frac{1}{\alpha} (b_1 (K_p \alpha + \lambda_1 \alpha + \lambda_2)) r(k-4) + e(k) \\
+ \frac{1}{\alpha} (a_2 - b_2 (K_p + \lambda_2)) e(k-1) \\
+ \frac{1}{\alpha} (a_1 - a_2 (1 + \alpha)) e(k-2) + \frac{d}{\alpha} e(k-3).
\]  

A.2 Deriving the second RLS estimator from Section 3

Similar to the first level of the RLS estimation derivation for the output signal, we derive an estimation of the control input signal at first by utilising the provided value of the output signal, \( \hat{y}(k|\theta) \) as given:

\[
\hat{u}(k|\theta) = C(q) (r(k) - y(k)) \\
= \left( K_p + \frac{\lambda_1}{1 - q^{-1}} + \frac{\lambda_2 (1 - q^{-1})}{\alpha - q^{-1}} \right) (r(k) - y(k)),
\]

So, (A9) is simplified as

\[
\hat{u}(k|\theta) = -\frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) y(k) \\
+ \frac{1}{\alpha} (K_p (1 + \alpha) + \lambda_1 + 2 \lambda_2) y(k-1) \\
- \frac{1}{\alpha} (K_p + \lambda_2) y(k-2) + \frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2) r(k)
\]
\[- \frac{1}{\alpha} (K_p(1 + \alpha) + \lambda_1 + 2\lambda_2) r(k - 1) \]
\[+ \frac{1}{\alpha} (K_p + \lambda_2) r(k - 2) - \frac{1 + \alpha}{\alpha} \hat{u}(k - 1|\bar{\theta}) \]
\[- \frac{1}{\alpha} \hat{u}(k - 2|\bar{\theta}) \]  

(A10)

Besides, the unknown real value of the output signal, \(y(k)\) in (A10) is substituted by \(\tilde{y}(k|\bar{\theta})\). Therefore, we have
\[
\hat{u}(k|\bar{\theta}) = \bar{\varphi}(k|\bar{\theta}), \quad (A11)
\]

where
\[
\bar{\varphi}(k) = \prod_{i=1}^{8} \tilde{y}(k|\bar{\theta}_i), r(k - 1), r(k - 2), \quad (A12)
\]
\[
\hat{u}(k - 1|\bar{\theta}), \hat{u}(k - 2|\bar{\theta}))
\]

and \(\hat{u}(k - i|\bar{\theta})\) is the estimation of the control input signal in \(i\)-th step prior, and \(\bar{\theta} = [\theta_1 \ldots \theta_8]^T\) is defined as below:
\[
\bar{\theta}_1 = - \frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2), \]
\[
\bar{\theta}_2 = \frac{1}{\alpha} (K_p(1 + \alpha) + \lambda_1 + 2\lambda_2), \]
\[
\bar{\theta}_3 = - \frac{1}{\alpha} (K_p + \lambda_2), \]
\[
\bar{\theta}_4 = \frac{1}{\alpha} (K_p \alpha + \lambda_1 \alpha + \lambda_2), \]
\[
\bar{\theta}_5 = - \frac{1}{\alpha} (K_p(1 + \alpha) + \lambda_1 + 2\lambda_2), \]
\[
\bar{\theta}_6 = \frac{1}{\alpha} (K_p + \lambda_2), \]
\[
\bar{\theta}_7 = - \frac{1 + \alpha}{\alpha}, \]
\[
\bar{\theta}_8 = - \frac{1}{\alpha}.
\]

Additionally, the real-time control input signal \(u(k)\) is formulated as
\[
u(k) = C(q) (r(k) - y(k)) \]
\[= \left( K_{p,u} + \frac{\lambda_{1,u}}{1 - q^{-1}} + \frac{\lambda_{2,u}(1 - q^{-1})}{\alpha_u - q^{-1}} \right) (r(k) - y(k)). \quad (A13)
\]
as a result, Equation (A13) is rewritten as
\[
u(k) = - \frac{1}{\alpha_u} (K_{p,u} \alpha_u + \lambda_{1,u} \alpha_u + \lambda_{2,u}) y(k) \]
\[+ \frac{1}{\alpha_u} (K_{p,u}(1 + \alpha_u) + \lambda_{1,u} + 2\lambda_{2,u}) y(k - 1) \]

\[- \frac{1}{\alpha_u} (K_{p,u} + \lambda_{2,u}) y(k - 2) \]
\[+ \frac{1}{\alpha_u} (K_{p,u} \alpha_u + \lambda_{1,u} \alpha_u + \lambda_{2,u}) r(k) \]
\[- \frac{1}{\alpha_u} (K_{p,u}(1 + \alpha_u) + \lambda_{1,u} + 2\lambda_{2,u}) r(k - 1) \]
\[+ \frac{1}{\alpha_u} (K_{p,u} + \lambda_{2,u}) r(k - 2) - \frac{1 + \alpha_u}{\alpha_u} u(k - 1) \]
\[- \frac{1}{\alpha_u} u(k - 2). \quad (A14)
\]

and the unknown real value of the output signal, \(y(k)\) in (A14) is substituted by \(\tilde{y}(k|\bar{\theta})\). As a result,
\[
u(k) = - \frac{1}{\alpha_u} (K_{p,u} \alpha_u + \lambda_{1,u} \alpha_u + \lambda_{2,u}) \tilde{y}(k|\bar{\theta}) \]
\[+ \frac{1}{\alpha_u} (K_{p,u}(1 + \alpha_u) + \lambda_{1,u} + 2\lambda_{2,u}) \tilde{y}(k - 1) \]
\[- \frac{1}{\alpha_u} (K_{p,u} + \lambda_{2,u}) y(k - 2) \]
\[+ \frac{1}{\alpha_u} (K_{p,u} \alpha_u + \lambda_{1,u} \alpha_u + \lambda_{2,u}) r(k) \]
\[- \frac{1}{\alpha_u} (K_{p,u}(1 + \alpha_u) + \lambda_{1,u} + 2\lambda_{2,u}) r(k - 1) \]
\[+ \frac{1}{\alpha_u} (K_{p,u} + \lambda_{2,u}) r(k - 2) - \frac{1 + \alpha_u}{\alpha_u} u(k - 1) \]
\[- \frac{1}{\alpha_u} u(k - 2). \quad (A15)
\]

A.3 Deriving the third RLS estimator from Section 3

An estimation of the output signal, \(\tilde{y}(k|\bar{\theta})\) is calculated by utilising \(\hat{u}(k|\bar{\theta})\) into the RLS estimator. Therefore,
\[
\tilde{y}(k|\bar{\theta}) = \frac{B(q)}{Z(q)} \hat{u}(k|\bar{\theta}) + \left( 1 - \frac{A(q)}{Z(q)} \right) y(k) \]
\[= \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + d q^{-1}} \hat{u}(k|\bar{\theta}) \]
\[+ \left( 1 - \frac{1 - a_1 q^{-1} - a_2 q^{-2}}{1 + d q^{-1}} \right) y(k). \quad (A16)
\]

thus, (A16) is rewritten as
\[
(1 + d q^{-1}) \tilde{y}(k|\bar{\theta}) = \left( b_1 q^{-1} + b_2 q^{-2} \right) \hat{u}(k|\bar{\theta}) \]
\[+ (d + a_1) q^{-1} + a_2 q^{-2}) y(k), \quad (A17)
\]
resulting in \( \hat{y}(k|\theta) = \overline{\varphi}(k) \overline{\theta} \), where
\[
\overline{\varphi}(k) = \begin{bmatrix} y(k-1), y(k-2), \ldots \end{bmatrix},
\overline{\theta} = \begin{bmatrix} \theta_1, \ldots, \theta_5 \end{bmatrix},
\hat{y}(k-1|\overline{\theta}), \hat{y}(k-2|\overline{\theta}),
\]
and \( \overline{\theta} = [\theta_1, \ldots, \theta_5]^T \) with the given parameters:
\[
\theta_1 = d + a_1, \theta_2 = a_2, \theta_3 = a_1 \theta_4 + a_5, \theta_5 = -d.
\]

Moreover, for the real-time output calculation, we have
\[
y(k) = G(q) \hat{y}(k|\overline{\theta}) + L(q) e(k)
\]
\[
= \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}} \hat{y}(k|\overline{\theta})
+ \frac{1 + a_2 q^{-1} - a_2 q^{-2}}{1 - a_1 q^{-1} - a_2 q^{-2}} e(k).
\]

(A18) thus, (A18) yields to
\[
y(k) = a_1 y(k-1) + a_2 y(k-2) + b_1 \hat{y}(k-1|\overline{\theta})
+ b_2 \hat{y}(k-2|\overline{\theta}) + e(k) + a_2 e(k-1).
\]

(A19)

A.4 | Proof of Theorem 1
To investigate the impact of incorporating statistical assumptions into the problem formulation and parameter estimation, as described by equations (2), (3), and (10), the first RLS estimation protocol in (29) is rewritten as
\[
\hat{y}(k|\theta) = \varphi^*(k) \theta,
\]
where \( \varphi^* \) is the same as \( \varphi \) in (29). As is frequently noted in different methodologies, it is recognised that the true output \( y(k) \) is susceptible to noise interference when employing RLS estimators [21, 36]. So, we assume
\[
y(k) = \hat{y}(k|\theta) + e(k),
\]
where \( e(k) \) is an additive zero-mean error and it is assumed to be serially uncorrelated with a fixed-order variance; that is, \( E\{e(k)e(l)\} = \sigma^2 \delta_{k,l} \), where \( \delta_{k,l} \) is
\[
\delta_{k,l} = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases} \quad \forall k, l.
\]

(A22) Also, the error is uncorrelated with the elements of \( \varphi^*(k) \) that compose the vector \( \varphi(k) \); that is, \( E\{\varphi(k)e(j)\} = 0, \forall k, j \) [21, 36]. Accordingly, we have
\[
\hat{y}(k-i) = y(k-i) - e(k-i), i = 1, \ldots, k.
\]
(23)

(Therefore, the vector \( \varphi^*(k) \) in (A20) is
\[
\varphi^*(k) = \begin{bmatrix} y(k-1), \ldots, y(k-4), r(k-1), \ldots, \\ r(k-4), y(k-1) - e(k-1), \\ y(k-2) - e(k-2), y(k-3) - e(k-3) \end{bmatrix}.
\]
Define \( \varphi(k) \) as
\[
\varphi(k) = \begin{bmatrix} y(k-1), \ldots, y(k-4), r(k-1), \ldots, r(k-4), \\ y(k-1) - e(k-1), y(k-2), y(k-3) \end{bmatrix},
\]
Note that in the rest of the proof, the argument \( k \) is sometime dropped to shorten the equations. Let define \( \varphi_1 \) and \( \varphi_2 \) as
\[
\varphi_1 = \begin{bmatrix} I_8, I_8, \ldots \end{bmatrix}, \varphi_2 = \begin{bmatrix} 0_{8x1}, I_5, \ldots \end{bmatrix},
\]
\[
\varphi = [-e(k-1), -e(k-2), -e(k-3)].
\]

Then, \( y(k) \) is rewritten as
\[
y(k) = \varphi_1 \theta^*_1 + \varphi_2 \theta^*_2 + e(k),
\]
(A24) where \( \theta \) is rewritten as \( \theta = [\theta^*_1 \theta^*_2]^T \) and \( \theta^*_1 = [\theta_1 \ldots \theta_8]^T \)
\( \theta^*_2 = [\theta_9 \ldots \theta_{11}]^T \). Hence, \( y(k) \) is reformulated to
\[
y = \varphi \begin{bmatrix} I_8 \ 0_{8x8} \end{bmatrix} \theta^*_1 + \varphi \begin{bmatrix} 0_{8x1} \ I_5 \end{bmatrix} \theta^*_2 + \Psi,
\]
(A25) where \( \Psi = \varphi^2 \theta^*_2 + e \). Also, an estimation of the output signal is stated as
\[
y = \varphi \hat{\theta}^*_1 + \varphi \hat{\theta}^*_2
\]
\[
= \varphi \begin{bmatrix} I_8 \ 0_{8x8} \end{bmatrix} \hat{\theta}^*_1 + \varphi \begin{bmatrix} 0_{8x1} \ I_5 \end{bmatrix} \hat{\theta}^*_2
\]
(A26) Accordingly,
\[
\begin{bmatrix} I_8 \ 0_{8x8} \end{bmatrix} \hat{\theta}^*_1 + \begin{bmatrix} 0_{8x1} \ I_5 \end{bmatrix} \hat{\theta}^*_2 = (\varphi^T \varphi)^{-1} \varphi^T y
\]
(A27) By substituting \( y \) from (A26) in (A27), we have
Furthermore, the covariance matrix of the augmented noise vector $\Psi(k)$ is calculated as

$$E \{ \Psi(k) \Psi^T(k) \} =
\begin{bmatrix}
E \{ \Psi(1) \} & E \{ \Psi(1) \Psi(2) \} & \cdots & E \{ \Psi(1) \Psi(k) \} \\
E \{ \Psi(2) \} & E \{ \Psi(2) \Psi(2) \} & \cdots & E \{ \Psi(2) \Psi(k) \} \\
\vdots & \vdots & \ddots & \vdots \\
E \{ \Psi(k) \} & E \{ \Psi(k) \Psi(2) \} & \cdots & E \{ \Psi(k) \Psi(k) \}
\end{bmatrix}. $$

(A33)

According to the assumptions for the measurement noise, only diagonal values of the matrix $E \{ \Psi(k) \Psi^T(k) \}$ are nonzero. Besides, the diagonal values of this matrix are calculated as

$$E \{ \Psi(i) \Psi^T(i) \} =
\begin{bmatrix}
E \{ \phi(i) \phi^T(i) \} \\
E \{ \phi(i) \phi^T(i) \} + E \{ \epsilon(i) \phi^T(i) \} \\
\vdots \\
E \{ \phi(k) \phi^T(k) \}
\end{bmatrix}, $$

(A34)

with $\Theta_2^*$ as considered to be real unknown values in the problem formulation. So,

$$E \{ \Psi(i) \Psi^T(i) \} =
\begin{bmatrix}
E \{ \phi(i) \Theta_2^* \phi^T(i) \} \\
E \{ \phi(i) \Theta_2^* \phi^T(i) \} + E \{ \epsilon(i) \phi(i) \Theta_2^* \phi^T(i) \} \\
\vdots \\
E \{ \phi(k) \phi^T(k) \}
\end{bmatrix} + E \{ \epsilon(i) \epsilon(i) \phi(i) \phi(i) \phi^T(i) \}
\begin{bmatrix}
\Theta_1^2 + \Theta_{10}^2 + \Theta_{11}^2 \\
\Theta_1^2 + \Theta_{10}^2 + \Theta_{11}^2 \\
\vdots \\
\Theta_1^2 + \Theta_{10}^2 + \Theta_{11}^2 \\
\Theta_1^2 + \Theta_{10}^2 + \Theta_{11}^2 \\
\vdots \\
\Theta_1^2 + \Theta_{10}^2 + \Theta_{11}^2
\end{bmatrix}
+ 2\mu \sigma^2,

(A35)

where $\mu$ is a constant. Therefore, (A33) is simplified to

$$E \{ \Psi(k) \Psi^T(k) \} = \mu \sigma^2 I_k. $$

(A36)

Consequently,

$$P^*(k) = \mu \sigma^2 P(k) \phi^T(k) \phi(k) P(k) = \mu \sigma^2 P(k). $$

(A37)

The resulting bounded covariance matrix, along with the zero mean estimation error, confirms the dependability of the identification process. Due to space limitations, the article does not delve into the mean and covariance analysis of the parameter estimation error in the second and third RLS estimators, which can be conducted similarly to the first estimator.

The proof of Theorem 1 is complete.