DOCTORAL THESIS

Characterizing Prospective Elementary Teachers' Mathematical Knowledge for Teaching Fractions

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Mathematics Education

LULEÅ UNIVERSITY OF TECHNOLOGY

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by

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To my family

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Abstract

Although research has delved into the challenges faced by mathematics teachers and students across various educational levels when working with fractions, these challenges persist. Fractions are an important but difficult content area in elementary school mathematics, and many future elementary teachers enter teacher education with limited knowledge of fractions. This lack of prior knowledge, combined with diverse experiences and attitudes toward the teaching and learning of mathematics, presents a challenge for teacher education programs aiming at enhancing prospective teachers' knowledge to enable them to teach fractions effectively. Teacher education is a crucial time to attain a comprehensive pedagogical knowledge of mathematics, and this thesis characterizes prospective elementary teachers' (n = 61) knowledge for teaching fractions as an outcome of Swedish teacher education. Through four empirical studies, the thesis also addresses the concept of mathematical identity and investigates the topic in light of the Mathematical Knowledge for Teaching framework.

The findings highlight substantial differences between the participants in the subject matter and pedagogical content knowledge domain of fractions, revealing difficulties in both their procedural and conceptual knowledge. Challenges also arise in analyzing different fraction solutions and combining knowledge of fraction content with knowledge about elementary students, teaching, and the national curriculum document. This difficulty can be a hindrance for the quality teaching of fractions. Moreover, incoherent fraction knowledge occurs across the different knowledge categories included in the Mathematical Knowledge for Teaching framework. The findings also reveal uncertainty in relation to work with fractions and prospective teachers' mathematical identities. A new framework for the analysis of different aspects of fractions is also presented in the thesis, and the findings are discussed considering the context of Swedish teacher education and the Mathematical Knowledge for Teaching framework.

Keywords: elementary teacher education, fractions, mathematical identity, mathematical knowledge for teaching, prospective elementary teacher

List of papers

The following papers are included in this thesis and referred to in the text by their Roman numerals:

I. Tossavainen, A. (2022). Student teachers' common content knowledge for solving routine fraction tasks. *LUMAT: International Journal on Math, Science and Technology Education, 10*(2), 256–280. https://doi.org/10.31129/LUMAT.10.2.1656

II. Tossavainen, A., & Helenius, O. (2024). Student teachers' conceptions of fractions: A framework for the analysis of different aspects of fractions. *Mathematics Teacher Education and Development*, *26*(1), Article 4. https://mted.merga.net.au/index.php/mted/article/view/889

Tossavainen was the corresponding author of Paper II, being responsible for the conceptualization of the study, data collection, development of the analytical framework, and the analysis process. Helenius contributed with ideas of how to visualize the results and in the final preparation of the paper.

III. Tossavainen, A. (in press). Pedagogical content knowledge in prospective elementary teachers' descriptions of teaching and learning of fractions. *Scandinavian Journal of Educational Research.*

IV. Tossavainen, A., & Johansson, M. (2023). An insight into prospective elementary teachers' mathematical knowledge for teaching: An example of fraction division. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 3303–3310). Alfréd Rényi Institute of Mathematics and ERME. https://hal.science/CERME13/hal-04421423v1

Tossavainen was the corresponding author of Paper IV, being responsible for the analysis process. Data were collected in collaboration with Johansson, who was responsible for the transcription of the interview data.

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List of abbreviations

The following abbreviations are used in this thesis:

CCK	Common Content Knowledge
CK	Content Knowledge
ECTS	European Credit Transfer and Accumulation System; a standard for comparing the performance of students of higher education across the European Union
EPA	*Enskilt-Par-Alla;* a Swedish version of the cooperative learning activity Think-Pair-Share
FMK	Fundamental Mathematical Knowledge
HCK	Horizon Content Knowledge
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KQ	Knowledge Quartet
MKT	Mathematical Knowledge for Teaching
PCK	Pedagogical Content Knowledge
PT	Prospective Elementary Teacher
SCK	Specialized Content Knowledge
SMK	Subject Matter Knowledge

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At the end of my PhD studies, it is now finally time for me to attempt to express my gratitude to my supervisors as well. I am especially thankful to Dr. Monica Johansson, who followed me through the whole journey from the beginning to the end of my PhD studies—thank you for your enormous support. During my last year as a PhD student, I also received important guidance from Professor Ola Helenius, who deserves my deepest thanks for giving me his expertise. I also want to thank Professor Iiris Attorps and Associate Professor Maria Johansson for sharing their time with my project. I am also very thankful to Professor Hanna Palmér, whose constructive comments at my 90% seminar showed me valuable insights, helping me find a direction for the thesis frame.

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Anne
Luleå, Easter 2024

1 Introduction

In Sweden, prospective elementary teachers enter teacher education from different starting points. The youngest prospective teachers enter their teacher education programs straight after finishing upper secondary education, while others already have prior university studies or several years of experience in working life. In addition to differences in prospective teachers' ages and educational backgrounds, they may also vary in their expectations about the studies in teacher education and conceptions of the teaching profession. Prospective teachers may also have varying attitudes toward mathematics as a school subject, including negative prior experiences of themselves as mathematics learners. As prospective teachers bring their different backgrounds with them to teacher education, challenges arise for teacher education programs when developing mathematical identity and knowledge for teaching the subject. Even though mathematics is one of the core elementary school subjects studied in Swedish teacher education, many prospective teachers express their concern about mathematics studies. Internationally, fractions have been shown to be among the most difficult mathematical topics to learn and teach (e.g., Behr et al., 1983; Getenet & Callingham, 2021; Lamon, 2007; Ma, 2010; Van Steenbrugge et al., 2014), and many prospective teachers attending teacher education also have limited prior knowledge of fractions (e.g., Chinnappan & Forrester, 2014).

Fractions are one of the most important content areas in school mathematics, since knowledge of fractions affects the understanding of other mathematical content areas, for example, aiding proficiency with ratios, proportions, and percentages and laying the foundation for algebraic reasoning and more advanced mathematical concepts (e.g., Ball et al., 2005; Booth & Newton, 2012; Siegler et al., 2012). Understanding fractions requires, among other things, knowledge of several different notions, such as the numerator and denominator, and knowledge of different mathematical notations and representation forms for fractions. Several interpretations can also be related to fractions, and fractions also have many connections to other mathematical concepts and constructions. However, Siegler and Braithwaite (2017, p. 195) state, "[a]lthough written fraction notation is usually introduced in early elementary school, connecting written fractions with the magnitudes that they represent remains challenging even for many adults." Moreover, the prior knowledge of procedures for natural numbers is often incorrectly overgeneralized to concern fractions as well (e.g., Siegler et al., 2011).

Mathematics textbooks have a significant role in Swedish teachers' teaching practices (Jablonka & Johansson, 2010). According to TIMSS data from 2011, 89% of Swedish Grade 4 teachers use textbooks as a basis for instruction in their mathematics teaching (Mullis et al., 2012). However, textbooks used in Swedish elementary schools have been shown not to provide a proper presentation of fractions, and they may even create different opportunities to learn (Ahl & Helenius, 2021; Berggren, 2023). As discussed later in this thesis, fraction content is also represented on a general level in the Swedish national curriculum document for the compulsory school and in the commentary material connected to it. The sparse representation of fraction content in the supporting materials for mathematics teaching requires mathematics teachers themselves to have the knowledge needed for the effective teaching of fractions. For prospective elementary teachers, teacher education is the crucial time to attain this comprehensive knowledge of fractions.

To clarify the challenges that Swedish teacher education has when aiming to enhance prospective elementary teachers' mathematical knowledge, it is of great importance to examinate the character of the mathematical knowledge for teaching fractions that prospective teachers possess before starting in their professions as elementary mathematics teachers.

1.1 Aim and research questions

The overall aim of this thesis is to characterize prospective elementary teachers' mathematical knowledge for teaching fractions as an outcome of Swedish teacher education. With this aim, their knowledge of fractions is investigated before their graduation from teacher education programs, and the consequences of the results to be addressed in teacher education are discussed. In this sense, the thesis is also an implicit study of teacher education, discussing it in terms of the participating prospective elementary teachers' displayed knowledge of fractions and mathematical identity.

The topic of this thesis is framed by the Mathematical Knowledge for Teaching framework developed by Ball et al. (2008). Briefly expressed, the framework describes different knowledge categories needed for the work of teaching mathematics effectively. In this thesis, the work of mathematics teachers is examined from the point of view of teaching fractions. To reach the stated aim, the thesis addresses the following research question:

- How can prospective elementary teachers' knowledge for teaching fractions be characterized in light of the Mathematical Knowledge for Teaching framework?

To answer the main research question, it is complemented with the following two sub-questions, which are addressed in the empirical studies connected to the thesis:

- How is subject matter knowledge of fractions reflected in prospective elementary teachers' work with fraction tasks and in their conceptions of fractions?
- How is pedagogical content knowledge in the fractional number domain displayed among prospective elementary teachers and connected to their mathematical identities at the end of their teacher education?

The Mathematical Knowledge for Teaching framework and the different knowledge categories that the research questions are built on, as well as the concept of mathematical identity, will be specified later in Chapter 3, "Theoretical framework".

1.2 Research context

The overarching context for this thesis is Swedish teacher education. The empirical studies connected to the thesis were conducted in connection to one Swedish university, and the participants represent one cohort of prospective elementary teachers in the university. The cohort, which consisted of 61 prospective teachers, attended teacher education programs that aim to prepare teachers for the preschool class and Grades 1–6 of the nine-year compulsory school—that is, concerning children between the ages of 6 and 12. A more detailed description of Swedish elementary teacher education and the participants will be given in Chapters 2 and 4.

The research context also represents the teacher education context where I was working for a while before starting the project for this thesis, teaching some of the participating prospective elementary teachers (PTs) along with many other teacher educators. However, during the research project, I was not involved in the teaching of the participants. The section "Ethical considerations" in Chapter 4 will elaborate on the researcher's connection to his/her own research context.

1.3 Overview of the thesis

The thesis comprises a thesis frame and four appended studies, presented in Papers I–IV. The thesis is grounded in the studies, which are synthesized and discussed in relation to previous research and their contribution to the topic of the thesis. The thesis frame is structured in six chapters. The present chapter serves as an introduction to the thesis, presenting the aim and the research questions as well as a brief overview of the research context and the participants.

Chapter 2 presents the background for the research in the thesis, describing Swedish elementary teacher education and fractions in Swedish elementary school mathematics teaching. An overview of different aspects of fractional numbers and previous research on PTs' knowledge of fractions is also given in the second chapter. Chapter 3 focuses on the theoretical framework of the thesis by presenting different aspects of the teacher knowledge base and the main framework, Mathematical Knowledge for Teaching. The chapter also discusses mathematical identity. Chapter 4 presents the procedures and methodological choices made for data collection and the analysis of the empirical data, and the chapter closes with some ethical considerations. The results of Papers I–IV and the main findings of the thesis are presented in Chapter 5. The last chapter discusses the findings and methodological choices as well as implications for teacher education and provides suggestions for further research.

2 Background

This chapter first gives an overview of the Swedish elementary teacher education system where the research project for the thesis was carried out and presents research on Swedish teacher education. Then, the chapter describes fractions in Swedish elementary school mathematics teaching and presents background on fractions in general. The chapter is closed by reviewing previous research on PTs' knowledge of fractions.

2.1 Swedish elementary teacher education

Swedish compulsory school consists of Grades 1–9, where most children are between the ages of 7 and 15. Before starting Grade 1, six-year-old children attend a mandatory one-year preschool class, which is often located in school. In this thesis, Grades 1–6 of the compulsory school are called elementary school. In Sweden, like in many other countries, elementary teachers are educated as generalists, teaching several subjects in Grades 1–6. Consequently, they are usually non-specialists in mathematics and other school subjects, and their educational background is largely linked to the teaching profession as a whole.

Entering teacher education in Sweden does not require prior university studies, thus making teacher education accessible for applicants with diverse backgrounds. In 2011, a new teacher education model was implemented in Sweden. The reform aimed, among other things, to improve the aspects of scientific basis and proven experience in teacher education, as well as to strengthen elementary teachers' subject knowledge base (Utbildningsdepartementet, 2010). Within the implemented model, elementary school teachers are educated in two separate teacher education programs: one focusing on the teaching of the preschool class and Grades 1–3 and the other focusing on teaching in Grades 4–6. The four-year academic elementary teacher education comprises 240 ECTS (European Credit Transfer and Accumulation System points) in both programs. The length of the education and the core content for the programs are nationally determined, which implies the same outcomes for the programs across all universities (Christiansen et al., 2021). However, the implementation of the content and the order of the courses is decided at the university level, as well as the formulation of specific course outcomes and decisions on assessment (Christiansen et al., 2021).

This thesis is conducted in connection to one Swedish university where the elementary teacher education programs are based on distance studies, including only a few lectures given at the university campus during each teacher education term. Elementary teacher education programs in the university consist of studies of educational sciences (60 ECTS), elementary school subject disciplines and subject-specific didactics (150 ECTS), as well as teaching practicum (30 ECTS). In both programs, each core elementary school subject—that is, Swedish, English and mathematics—are studied for 30 ECTS. In addition to the focus on teaching different elementary school grade levels, there are no practical differences between the studies in the two programs, but PTs can choose, for example, whether to focus in their final bachelor thesis on topics related to mathematics education, pedagogics in English or Swedish, or to some other school subject studied during teacher education. Since the mathematics courses and the last teacher practicum course (15 ECTS) that has a focus on mathematics teaching are mainly the same for both programs, the PTs attending the two teacher education programs in the university are considered as one cohort in this thesis.

Teacher education mathematics studies in the programs attended by the PTs in this thesis consist of two 15 ECTS courses called *Mathematics for compulsory school teachers*, *part 1* and *part 2*. The first course is given during the autumn term in the second year of teacher education studies while the other is scheduled in the spring term of the third academic year. For the participating cohort in this thesis, the focus of the first mathematics course was on deepening the PTs' knowledge of different mathematical content areas and strengthening their computational skills, while the latter course was more focused on different theoretical aspects in mathematics teaching. It was stated as a goal of the courses that after completing the courses, PTs must be able to use the concepts, symbols, forms of representation, rules, and algorithms covered in the course. The requirements also expected the PTs to be able to solve problems with different mathematical content, present and discuss different solutions, and express and argue for their mathematical ideas, as well as carry out didactical analyses of specific mathematical content and plan teaching sequences based on their analyses. The goals for the courses also required knowledge of how to stimulate and develop elementary students' mathematical abilities and how to choose learning materials and working methods for mathematics teaching. Fractions as a core elementary school mathematics content were handled in both teacher education mathematics courses, where the main course literature, *Mathematics for elementary teachers with activities* by Beckmann (2018) also provided a comprehensive presentation of fractions.

2.1.1 Research on Swedish teacher education

Studies published after the establishment of the new Swedish teacher education model in 2011 have examined different aspects of Swedish teacher education. The national law for higher education (SFS, 1992:1434) requires the content and organization of teacher education to be based on scientific research. Alvunger and Wahlström (2018) argued in their study that Swedish teacher education can be regarded as "research-based in the sense that the content of the education are [*sic*] based on scientific research and that the education stimulates students to demonstrate openness toward different perspectives on school and education when reading and discussing course literature" (p. 344). However, the course literature used in Swedish teacher education is mainly based on secondary sources of research—that is, research descriptions that are adapted to PTs as a target group by its language and format (Alvunger & Wahlström, 2018). Thus, PTs may not develop the ability to read original research literature before writing their final thesis in teacher education and for their future needs as knowledgeable professionals. Moreover, the represented research mainly focuses on different classroom or school contexts based on sociocultural learning theory, providing few possibilities for argumentation from different perspectives (Alvunger & Wahlström, 2018).

Edling and Liljestrand (2020) analyzed how four major Swedish newspapers discussed teacher education between 2016 and 2017. The findings of the study showed that the media mainly emphasized the negative aspects of teacher education, especially highlighting problems related to the scientific knowledge base, which is regarded as "insignificant and woolly" (Edling & Liljestrand, 2020, p. 258) due to the insufficient knowledge of teachers and PTs. The media discourses, framed by so-called *outside-in-professionalism*, also perceived that examples of good teacher professionalism from places such as Singapore could be transferred as teaching models for the Swedish context. Despite the views raised by non-educational researchers and teacher educators, Edling and Liljestrand (2020) considered that media debates about teacher education are important to take into account, as teacher education is also influenced by public opinion.

In the field of mathematics education, Hemmi and Ryve (2015) examined Swedish teacher education discourses from the point of view of effective mathematics teaching. They found that teacher educators in Sweden conceptualize effective teaching as interactions with individual children, building flexibly on students' thinking and mathematics from everyday experiences. The study showed that Swedish discourse on classroom teaching can be characterized by constructivism and student-centered teaching that focuses on problem-solving. Moreover, the study revealed that the Swedish discourse was missing the importance of a clear and logical presentation of mathematics and mathematical connections to students' prior learned skills and content, as well as the importance of well-thought-out lesson plans with clear goals and instruction. Hemmi and Ryve (2015) also stated that the discourse represented in the Swedish teacher education context is coherent with the national curriculum document for compulsory school that emphasizes, among other things, everyday connections in mathematics teaching.

The study of Asami-Johansson et al. (2020) focused on teacher education mathematics lessons. The results indicated that the structure of mathematics courses may vary between Swedish teacher educators even at the same university, since there seemed to be a lack of a collectively shared and generally adapted view of pedagogics. In line with Alvunger and Wahlström (2018), as well as Hemmi and Ryve (2015), Asami-Johansson et al. (2020) also stated that Swedish teacher education usually builds on constructivist learning theories and that content-specific mathematical learning theories are presented on a general level, which is difficult to relate to actual mathematical teaching tasks. Moreover, mathematics teaching is connected to group work techniques and psychological ideas such as self-efficacy (Asami-Johansson et al., 2020). Asami-Johansson et al. (2020) also stated that the unspecified mathematical content and descriptions in the national school curriculum document may explain Swedish teacher educators' limited discussion of the curriculum in their mathematics lessons.

Ebbelind (2020) investigated the role of Swedish teacher education in PTs' professional development as mathematics teachers. His findings revealed that with the open discourses provided in mathematics education courses, teacher educators failed to engage PTs' interest in their learning to teach mathematics and in challenging them to encounter the theoretical perspectives in mathematics teaching.

Further, Christiansen and Erixon (2024) examined Grade 1–12 PTs' experiences in opportunities to learn mathematics pedagogy and learning to teach mathematics in Swedish teacher education. The results showed that last-year PTs in 13 Swedish universities reported good opportunities to learn about different components of mathematics pedagogy, such as assessment, curriculum, planning, and building on learners' thinking. However, the study revealed substantial differences in PTs' opportunities to learn specific mathematical teaching competencies, such as analyzing learners' answers, using examples in teaching, and leading mathematical discussions, as well as to learn from teacher practicum experiences. The results indicated limitations in teacher education practices that should include a more consistent presentation of inclusive teaching and a clearer and a more coherent grounding in relevant research (Christiansen & Erixon, 2024).

The findings in the literature review above can be interpreted as indicating deficiencies in the practices within the present Swedish teacher education model. Moreover, previous research has also revealed that student outcomes from their teacher education programs may not correspond to desired outcomes as knowledgeable teachers, which highlights the relevance of the topic and the aim of this thesis.

2.2 Fractions in Swedish elementary school mathematics teaching

When referring to the national curriculum document, this thesis refers to the curriculum version established in 2011, and the excerpts are taken from the official English translation *Curriculum for the compulsory school, preschool class and school-age educare 2011* (Skolverket, 2018). The curriculum document was used in Swedish elementary schools until the end of the school year 2021–2022, and it covers the period in which the participating PTs in this thesis attended their teacher education programs. A new curriculum, *Curriculum for compulsory school, preschool class and school-age educare: Lgr22* (Skolverket, 2024), was implemented in 2022. The national curriculum document for compulsory school specifies the core content that needs to be taught in each subject during different grade levels. The document also provides knowledge requirements and general aims for teaching the subjects. However, since 2011 the goals are described in the form of different abilities and connected to the learning of the subjects (Karlsson, 2015). Table 1 summarizes the abilities presented in the mathematics syllabus included in the curriculum document.

Table 1. Mathematical abilities as described in the syllabus (Skolverket, 2018, p. 56).

Teaching in mathematics should essentially give pupils the opportunities to develop their ability to:
• formulate and solve problems using mathematics and also assess selected strategies and methods, • use and analyse mathematical concepts and their interrelationships, • choose and use appropriate mathematical methods to perform calculations and solve routine tasks, • apply and follow mathematical reasoning, and • use mathematical forms of expression to discuss, reason and give an account of questions, calculations and conclusions.

In the mathematics syllabus in the curriculum document, descriptions of fraction content remained the same in the 2022 version compared to the previous curriculum. Fractions are presented under the knowledge category *Understanding and use of numbers*. Fraction content is first specified for Grades 1–3, and the content area is then broadened and deepened during the upper grades (see Table 2). In connection to the curriculum document, teachers are also provided with a commentary material that is a document intended to help teachers to understand and use the curricular document (Skolverket, 2017). However, in addition to the descriptions related to fractions in the mathematics syllabus, the commentary material only gives some quite brief examples of how to relate fractions to elementary students' daily lives, for example, by using recipes with cooking. The commentary material also states that it is important for elementary students to understand the connections between numbers in the form of percentages, decimals, and fractions, and that their early knowledge of fractions is the basis for developing knowledge of algebra and the concept of percentages (Skolverket, 2017). The practical choices regarding how to teach fractions and what fraction content to include in teaching is thus left to the individual teachers. Since the core content for fractions is described on a general level in the mathematics syllabus in the national curriculum document and in the commentary material, elementary mathematics teachers themselves need to have a robust knowledge of the content area.

Table 2. Fraction content for elementary school Grades 1–6 as described in the mathematics syllabus in the official English version of the national curriculum document (Skolverket, 2018, pp. 56–57).

Grades	*Understanding and use of numbers*
1–3	• Parts of a whole and parts of a number. How parts are named and expressed as simple fractions, and how simple fractions are related to natural numbers. • Natural numbers and simple numbers as fractions and their use in everyday situations.
4–6	• Rational numbers and their properties. • Numbers in fractions and decimals and their use in everyday situations. • Numbers in percentage form and their relation to numbers in fraction and decimal form.

Some recent studies have focused on the teaching of fractions in the Swedish elementary school context. For example, Sveider (2021) investigated how Swedish Grade 4–6 elementary school teachers use manipulatives and different forms of representations when teaching fractions. The study revealed that teachers did not consciously create such contrasts and variation that enabled their students to discern the fraction content to be learned, but the lessons were rather more like repetition. Moreover, while the teachers in the study used several different fraction representations in diverse ways, starting with concrete representations and then switching to more abstract ones, they did not always make links between the used representation forms. Further, fractions were mainly taught by using one single example, focusing on the procedure that should be implemented when solving a specific fraction task. Thus, the way teachers gave fraction instruction and used manipulatives and other representation forms for fractions limited their students' opportunities to generalize the studied fraction content and understand where to focus on in their learning (Sveider, 2021).

Further, Björkhammer et al. (2023) examined the effects of a fraction intervention on Swedish Grade 5 students' conceptual knowledge of fractions. The participating mathematics teachers in the intervention classes focused on the measurement interpretation of fractions instead of the part-whole model. The number line was introduced to the elementary students as a whole supported by rectangular area models, and fraction magnitudes were concretized by comparing and ordering fractions and placing them on a number line. Teachers in the control condition classes used teaching materials that are usually employed for fraction instruction in the given grade level. The results of the study showed that the participating students in the intervention classes outperformed the control group on fraction concept as a measure and fraction arithmetic. Thus, Björkhammer et al. (2023) concluded that effective teaching of fractions should focus on the measurement interpretation supported by number lines.

2.3 On fractions in general

Before discussing PTs' knowledge of fractions, it is relevant to provide a brief general overview of different aspects of fractions. Several researchers (e.g., Ball, 1993; Behr et al., 1983; Kieren, 1976, 1993; Lamon, 2007, 2020) have agreed that there are different interpretations for fractions. Kieren (1976) was among the first researchers to conceptualize fractions as a set of interrelated constructs that includes the interpretations of part-whole, ratio, operator, quotient, and measure. Later, Behr et al. (1983) developed Kieren's conceptualization further, proposing a theoretical model that links the different interpretations of fractions to operations and equivalence of fractions and problem solving. Moreover, Behr et al. considered the part-whole interpretation, which requires the ability to partition a continuous quantity or a set of discrete objects into equal-sized subparts or sets, as a fundamental interpretation compared to other fraction interpretations. Ball (1993) summarized previous research analyses in the rational number domain stating that fractions can be interpreted

> *(a) in part-whole terms, where the whole unit may vary, (b) as a number on the number line, (c) as an operator (or scalar) that can shrink or stretch another quantity, (d) as a quotient of two integers, (e) as a rate, or (f) as a ratio.* (p. 168)

Moreover, fractions can be represented using different verbal, visual, and symbolic representations. Verbal representations are expressed in spoken and written language, such as "three fourths" or "one third," whereas drawings, pictures, and manipulatives are used to illustrate and concretize fractions in visual representations. Different representations expressing fractions symbolically make fractions more difficult to understand than whole numbers (Lortie-Forgues et al., 2015). In symbolic representations, fractions are written using two numerals (integers), but a fraction stands for one number in the form of $\frac{a}{b}$ or a/b. Fractions can also be represented in the form of mixed numbers, percentages, and decimals.

Mastering the different interpretations and representations for fractions contributes to proficiency in different fraction operations and fraction equivalence (Charalambous & Pitta-Pantazi, 2007). Developing a fraction number sense and performing fraction arithmetic with addition, subtraction, multiplication, and division operations also requires knowledge of extending, reducing, and simplifying fractions and converting fractions to, for example, mixed numbers, as well as comparing fractions' size and order (e.g., Lamon, 2020; Lortie-Forgues et al., 2015).

In addition to the different interpretations, representations, and procedures, several core notions and other concepts and mathematical constructions also have a relation to fractions. This conceptual discourse includes, for example, the meaning and use of different and same numerators and denominators, and knowledge of the concepts of equivalent fractions, unit fractions, and inverted fractions (e.g., Skolverket, 2021). Fractions should also be understood as numbers in the rational number set having a connection to natural numbers and promoting proficiency with ratios and proportional reasoning, as well as algebra and more advanced mathematics (Ball et al., 2005; Booth & Newton, 2012; Siegler et al., 2012).

2.4 Previous research on prospective teachers' knowledge of fractions

Internationally, Ma's study that was originally published in 1999 raised a general interest in elementary teachers' knowledge for teaching mathematics and provided a basis for research on PTs' knowledge of fractions. Ma (2010) considered elementary mathematics as fundamental, comprising not only a collection of number facts and calculational algorithms but also a foundation on which to build further mathematics. Ma also pointed out the importance of teacher education in developing PTs' fraction knowledge.

In the mathematics subject matter knowledge domain, numerous researchers have over the years been interested in examining PTs' proficiency with basic fraction procedures and their conceptual understanding of fractions (e.g., Alenazi, 2016; Ball, 1990b; Chinnappan & Forrester, 2014; Ibañez & Pentang, 2021; Lin et al., 2013; Lovin et al., 2018; Marchionda, 2006). Previous research reveals that division and multiplication are the most challenging fraction operations for many PTs (e.g., Ball, 1990a; Borko et al., 1992; Sahin et al., 2020; Son & Lee, 2016; Tirosh, 2000). Newton (2008) investigated 85 American elementary PTs' knowledge in routine fraction tasks including all basic operations—that is, addition, subtraction, multiplication, and division. The study showed that the PTs had difficulties with all the fraction operations. Most errors were found for fraction division, and these error patterns concerned (a) finding a common denominator and keeping it in the product, (b) leaving the task blank, (c) reciprocals, (d) flipping the dividend instead of the divisor, (e) making mistakes with whole number facts, (f) cross-dividing or cancelling, and (g) adding or subtracting numerators or denominators. In general, the most common error in the operations was related to the use of denominators. The study of Young and Zientek (2011) also revealed PTs' varying competence with fraction operations. The PTs' knowledge of fraction operations was shown to be rule-based, involving incorrect memories of the use of fraction algorithms and difficulties for the PTs in judging their own abilities in performing the operations.

PTs often relate their procedural fraction knowledge to standard algorithms that are learned in elementary school (Bansilal & Ubah, 2020; Jóhannsdóttir & Gíslandóttir, 2014). However, difficulties occur when PTs try to solve and create fraction word problems (e.g., Ball, 1990b; Jakobsen et al., 2014; López-Martín et al., 2022; Osana & Royea, 2011; Tirosh, 2000; Toluk-Uçar, 2009). Lo and Lou (2012) found that fraction word problems are challenging even for PTs who are regarded as highly proficient in elementary mathematics. Many PTs also lack flexibility in moving away from procedures and using other representations for fractions, for example, when converting fractions to decimal forms (Muir & Livy, 2012; Olanoff et al., 2014).

Several studies have shown that PTs often perform better with fraction procedures than in understanding the meanings behind the procedures (e.g., Ball, 1990b; Ma, 2010; Olanoff et al., 2014). Findings in previous studies show a weak relationship between PTs' procedural and conceptual knowledge related to fraction operations, and many PTs seem to have a more robust procedural than conceptual fraction knowledge (e.g., Lin et al., 2013; Tirosh, 2000). Many PTs also enter teacher education with fraction knowledge that mainly concerns procedures (Chinnappan & Forrester, 2014). Stohlmann et al. (2014) found that PTs' primarily focus on procedural fluency and that they consider conceptual understanding of mathematics as remembering mathematical procedures. PTs in Osana and Royea's study (2011) faced several obstacles when trying to construct meaningful fraction solutions and represent their solutions symbolically. Osana and Royea concluded that the PTs' weak understanding of fraction concepts hindered them in constructing meaningful solutions and that their lack of fraction number sense and their prior procedural knowledge of fractions prevented them from making sense of problem situations and using non-algorithmic solution methods.

Previous research has also found deficiencies in PTs' conceptual knowledge of different interpretations of fractions. PTs tend to prefer the part-whole interpretation and struggle with other fraction interpretations (Lamon, 2020; López-Martín et al., 2022; Lovin et al., 2018; Olanoff et al., 2014). Rizvi and Lawson (2007) showed that the PTs in their study did not have a well-developed multiplicative reasoning to be able to use the rate or ratio interpretation of fractions. Further, Luo et al. (2011) found significant differences between U.S. and Taiwanese PTs in their knowledge of area, linear, and set models for fractions. Lee and Lee (2023) also concluded that PTs' use of invalid fraction models indicated a misunderstanding of fraction key concepts.

PTs also struggle with using and understanding correct mathematical language with fractions (Fauskanger & Mosvold, 2017) and especially distinguishing the questions *how much* and *how many* and defining a whole for a fractional number (Tobias, 2013). Osana and Royea (2011) also argued that PTs "would likely benefit from discussions on the language used in word problems and how the concept images they evoke align with more formal models of the operations" (p. 350).

Previous research also points out limitations in PTs' pedagogical knowledge of fractions. Many PTs have difficulties interpreting elementary students' fraction solutions and making sense of solutions different from their own (Borko et al., 1992; Jakobsen et al., 2014). PTs' pedagogical limitations also concern difficulties in identifying sources for elementary students' incorrect responses and understanding their misconceptions, as well as using different instructional strategies and representations in the fractional number domain (e.g., Depaepe et al., 2015; Tirosh, 2000). Şahin et al. (2016) argued in their study that PTs' pedagogical content knowledge was not adequate to identify and correct student mistakes in fractions. Further, Shaughnessy et al. (2021) found that PTs were more fluent in eliciting a student's procedural thinking in an arithmetic process than the student's conceptual understanding. Thus, the PTs in the study were more familiar with asking *what* questions concerning students' methods and solution processes than *why* questions concerning students' understanding and reasoning for their solutions. The study of Son and Crespo (2009) also revealed indications of PTs' insufficient pedagogical fraction knowledge when providing few opportunities for their students to explain their reasoning in fraction division.

The limitations identified in PTs' fraction knowledge do not seem to be remedied much during teacher education (Lin et al., 2013; Lovin et al., 2018; Muir & Livy, 2012; Young & Zientek, 2011). Many PTs also possess fraction knowledge that reflects the misconceptions elementary school students have with fractions (e.g., Perry, 2023; Tirosh & Graeber, 1989; Van Steenbrugge et al., 2014). Natural number bias—that is, a tendency to apply and overgeneralize natural number reasoning and rules in the fractional number domain—is a typical source of elementary students' fraction misconceptions (Ni & Zhou, 2005). A common mistake based on natural number bias is to add across numerators and denominators in fraction addition (Siegler et al., 2011), which is also identified as a typical error among PTs (e.g., Newton, 2008; Young & Zientek, 2011). Halme et al. (2024) stated that students with natural number bias may not recognize their misconceptions and incorrect answers on fractions, which may then reduce their fraction state anxiety.

In the Swedish mathematics education context, research on fractions has mainly focused on the view of school students and mathematics teachers (e.g., Björkhammer et al., 2023; Nagy, 2017; Sveider, 2021), while PTs' fraction knowledge has not been thoroughly investigated. However, some teacher education research projects (see, e.g., Karlsson, 2015) examining PTs' mathematical knowledge have also addressed the topic of fractions. Karlsson (2015) found that Swedish PTs attending Grade 4–6 teacher education programs displayed mathematical knowledge that was mainly on procedures (procedural knowledge) and that they perceived mathematics as counting with different formulas and methods. Many of the 54 participating PTs showed a lack of knowledge of basic arithmetic and algebra. For example, only 12 and 15 of them, respectively, could correctly solve the tasks $\frac{a}{2} \cdot \frac{2}{a}$ and $\frac{a}{2} / \frac{a}{2}$. Even though Grade 6 elementary students should master the tasks according to the requirements in the mathematics syllabus, over 40% of the participating PTs expressed that they did not have any idea how to solve the tasks. Moreover, the PTs did not demonstrate flexibility in using varying solution methods and correcting their obvious misconceptions, which Karlsson (2015) considered was connected to their anxiety in mathematics. In general, Karlsson pointed out as problematic in the Swedish teacher education context that many PTs do not seem to understand that academic fulltime studies cover 40 hours work per week.

As the literature review above shows, previous research has identified and revealed several limitations and deficiencies in PTs' knowledge of fractions, which also seem to persist. Moreover, the fraction difficulties found in previous research not only relate to the content of the discipline but also concern the pedagogical knowledge domain. However, Tröbst et al. (2018) argued that teacher education programs delivering early and explicit instruction on pedagogical content knowledge may contribute to the formation of PTs' pedagogical content knowledge, even when their knowledge of the content is limited. Numerous previous research (e.g., Fauskanger, 2015; Jakobsen et al., 2015; Jóhannsdóttir & Gísladóttir, 2014; Lo & Luo, 2012; Olanoff et al., 2014) has focused on PTs' fraction knowledge based on the mathematical tasks of teaching and the knowledge categories described in the Mathematical Knowledge for Teaching framework (Ball et al., 2008), which are also used in this thesis and presented next along with other theoretical aspects related to the teacher knowledge base.

3 Theoretical framework

The presentation of the theoretical framework in this chapter begins with an overview of different ways of discussing the teacher knowledge base in general and the field of mathematics in particular. Then, the Mathematical Knowledge for Teaching framework by Ball et al. (2008) is presented before a description is given of teachers' mathematical identities.

3.1 Teacher knowledge base

Over time, numerous models for the teacher knowledge base have been generated. The seminal classification of the teachers' professional knowledge base was developed by Shulman (1986, 1987), based on the idea of pedagogical content knowledge. In his seminal work, Shulman (1986) suggested three categories for teachers' content knowledge (CK): (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. For Shulman, subject matter knowledge (SMK) involves not only the knowledge of the content of a subject, such as mathematical facts and concepts, but also knowledge of how the subject is structured. Thus, teachers need to be able to define for their students what is included in a domain, as well as to explain why certain topics are particularly central to a discipline and how the different topics are related to other topics. Pedagogical content knowledge (PCK) refers to knowledge for teaching, and within this category Shulman (1986) includes "the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (p. 9). PCK requires teachers to have knowledge of how to make a subject understandable to their students and to understand students' conceptions and misconceptions, which can make their learning either easy or difficult.

By curricular knowledge, Shulman (1986) referred to knowledge of the full range of materials and alternatives available for teaching a subject. This category includes aspects of lateral and vertical knowledge, where the former aspect consists of teachers' ability to connect the content of their teaching to topics being taught simultaneously in other subjects. Vertical curriculum knowledge relates to teachers' familiarity with the content taught within their own subjects during previous and coming school years. Altogether, Shulman (1987) identified seven categories for the teacher knowledge base, which are summarized in Table 3.

Table 3. Shulman's categories of the teacher knowledge base (Shulman, 1987, p. 8).

- Content knowledge.
- General pedagogical knowledge (principles and strategies of classroom management and organization that appear to transcend subject matter).
- Curriculum knowledge (materials and programs that serve as "tools of the trade" for teachers).
- Pedagogical content knowledge (amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding).
- Knowledge of learners and their characteristics.
- Knowledge of educational contexts (workings of the group or classroom, the governance and financing of school districts, the character of communities and cultures).
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.

Since Shulman (1986) introduced the term PCK and proposed the unique subject-matter domain in teachers' professional knowledge, a large number of researchers have assimilated, developed, reformulated, and critiqued his ideas, and different components have been included in the teacher knowledge base. However, researchers differ in their definitions, and the content and relationships between various components in teacher knowledge have remained unclear. Grossman (1990), for example, suggested four general areas as cornerstones of the professional knowledge for teaching: general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge, and knowledge of context. Of these, the general pedagogical knowledge, which includes general knowledge, beliefs, and skills related to learning and learners, has long been the predominant focus in research on teaching. PCK includes, according to Grossman (1990), four central subcomponents: (a) knowledge of purposes for teaching subject matter at different grade levels, (b) knowledge of students' understanding in a subject, (c) curricular knowledge, and (d) knowledge of instructional strategies and representations for teaching a topic. While Shulman (1987) did not address the connections between his knowledge categories, Fennema and Franke (1992) pointed out that the teacher knowledge base is a system where the different categories integrate.

In the field of mathematics, Rowland et al. (2009) developed the Knowledge Quartet (KQ) framework, highlighting the importance of teachers' mathematical

CK including both SMK and PCK. The KQ framework provides a tool for elementary teachers for developing their teaching, discussing four dimensions of knowledge: (1) Foundation—with a focus on what teachers know about mathematics; (2) Transformation—with a focus on representing mathematics to learners through examples, analogies, illustrations, and demonstrations; (3) Connection—with a focus on helping learners to make sense of mathematics through understanding how ideas and concepts are linked to each other; and (4) Contingency—with a focus on what to do when the unexpected happens during mathematics teaching. The first stage, making sense of foundation knowledge, involves teachers' possessed mathematical knowledge, beliefs, understanding, use of terminology, and identifying of errors, and it coincides with Shulman's (1987) first stage of pedagogical reasoning (Rowland, 2013).

Teachers' knowledge can also be stated as knowledge *about* and *of* mathematics (see, e.g., Ball, 1990b), where the latter can be divided into two distinct approaches, conceptual and procedural knowledge (Hiebert & Lefevre, 1986). While it is not always easy to draw a strict line between conceptual and procedural knowledge, conceptual knowledge is considered to be the knowledge that is rich in relationships (Hiebert & Lefevre, 1986), for example, understanding the definition of fractions and how the fundamental facts of different number sets and other mathematical concepts and constructions are related to fractions. In the context of fractions, procedural knowledge concerns computational skills related to fraction tasks, using established rules and notations and proper ways to denote fractions and their operations, for example, to perform the division operation of fractions (Hiebert & Lefevre, 1986).

Tchoshanov (2011) suggested that the development of teachers' knowledge of concepts and connections should be emphasized when seeking to improve student performance, and he referred to three types of cognitive knowledge that are required for successful mathematics teaching: knowledge of facts and procedures (type 1), knowledge of concepts and connections (type 2), and knowledge of models and generalizations (type 3). The first knowledge type is connected to the "memorization of facts, definitions, formulas, properties, and rules, performing procedures and computations; making observations, conducting measurements, and solving routine problems" (Tchoshanov, 2011, p. 148). The second type of knowledge includes, for example, understanding concepts, making connections, selecting and using multiple representations, transferring knowledge to a new situation, and solving non-routine problems. Type 3 knowledge refers to teachers' ability to make generalizations regarding mathematical statements, design mathematical models, test conjectures, and prove theorems. Tchoshanov concluded that teachers with strong conceptual knowledge employ effective teaching practices and that differences in teachers' performance are related to differences in their knowledge of concepts and connections. Moreover, this type 2 knowledge has a significant relation to student achievement and lesson quality.

Gorgorió et al. (2019) elaborated on the notion of Fundamental Mathematical Knowledge (FMK) as the required minimum mathematical CK when entering elementary teacher education. FMK was specified in terms of evidence of being competent in five key areas of elementary mathematics, where fractions were included in the topic *Number and arithmetic* or *Number and operations*. Gorgorió et al. (2019) considered that

> *[m]athematical competence goes beyond knowledge of procedures and is manifested in the use of conceptual knowledge in different situations; it requires the knowledge of rules, definitions and connections and domain structure, knowing why certain procedures work for certain problems, the purpose of the steps of procedures and connecting these steps to their conceptual foundations.* (p. 4)

Conceptual and procedural knowledge can also be seen as building upon each other (Thurtell et al., 2019), and researchers have suggested that mathematical concept formation develops through a procedural approach to conceptual understanding (e.g., Gray & Tall, 1994; Sfard, 1991). When characterizing the participating PTs' fraction knowledge, this thesis also uses the approaches of conceptual and procedural knowledge as essential parts in the PTs' subject matter knowledge domain, which will be described in more detail in connection to the Mathematical Knowledge for Teaching framework.

Kuhs and Ball (1986) discussed in their literature review four dominant approaches for good teaching of mathematics and characterized these distinctive approaches as (a) a learner-focused view emphasizing the learner's personal construction of mathematical knowledge, (b) a content-focused view emphasizing conceptual understanding, (c) a content-focused view emphasizing student performance and mathematical rules and procedures, and (d) a classroom-focused view emphasizing research knowledge about effective classrooms. Son and Crespo (2009) used Kuhs and Ball's approach when examining PTs' reasoning and responses to a student's non-traditional strategy for dividing fractions and synthesized the approaches into student-focused and teacher-focused categories. For Son and Crespo, the approaches (a) and (b) represent a student-focused view of teaching, while (c) and (d) indicate a teacher-focused view "in terms of who does the explaining and justifying and who ultimately decides on the adequacy of the ideas presented" (Son & Crespo, 2009, p. 241). The student-focused approach "is evident when a prospective teacher provides the students with opportunities to explain and justify their strategy," while in the teacher-focused approach, "the prospective teacher tells, explains, or shows" whether the solution method works or not (Son & Crespo, 2009, p. 245). In this thesis, the approaches of student-focused and teacher-focused are used in the analysis of PT interview data in the meaning that within the student-focused approach PTs' reflected on fraction solutions from the point of view of someone else's mathematical knowledge, and within the latter approach they explained the solutions based on their own understanding and mastery of the procedures used in the solutions.

3.2 Mathematical Knowledge for Teaching

As reported in previous research, the knowledge that mathematics teachers use for their work differs from the mathematical knowledge used in other professions (e.g., Ball et al., 2008; Shulman, 1986, 1987). Even though teachers' mathematical knowledge has a significant relation to student achievement (e.g., Charalambous et al., 2020; Hill et al., 2005), there is no shared conception within mathematics educational research of the mathematical knowledge needed for teaching (Hoover et al., 2016). Ball and her colleagues (Ball et al., 2008; Hill & Ball, 2009; Hill et al., 2008) took the work of mathematics teachers as a basis when defining the needed knowledge base and thus, identified common teaching tasks that require different mathematical knowledge (see Table 4). Teachers' mathematical understanding is realized, for example, when posing questions, interpreting students' answers, providing explanations, and using representations, and an essential part of teachers' mathematical skill is to open themselves to their students' perspectives and to understand what students are doing with mathematical content (Hill & Ball, 2009). This work of teaching mathematics entails mathematical reasoning and skill that are not needed for nonteachers working with mathematics or for mathematicians researching mathematics (Ball et al., 2008; Hill & Ball, 2009).

Table 4. Mathematical tasks of teaching (Ball et al., 2008).

• Presenting mathematical ideas. • Giving or evaluating mathematical explanations. • Responding to students' *why* questions. • Evaluating the plausibility of students' claims. • Asking productive mathematical questions. • Finding an example to make a specific mathematical point. • Modifying tasks to be either easier or harder. • Appraising and adapting the mathematical content of textbooks.	• Recognizing what is involved in using a particular representation. • Linking representations to underlying ideas and to other representations. • Selecting representations for particular purposes. • Explaining mathematical goals and purposes to parents. • Connecting a topic being taught to topics from prior or future years. • Choosing and developing useable definitions. • Using mathematical notations and language and critiquing its use. • Inspecting equivalencies.

When studying the work that effective mathematics teaching entails, Ball and her colleagues (Ball et al., 2008; Hill & Ball, 2009; Hill et al., 2008) differentiated two subdomains in Shulman's (1986) CK: common content knowledge and specialized content knowledge. Building on Shulman's PCK, they also pointed out two empirically discernable subdomains within this domain: knowledge of content and students and knowledge of content and teaching. Further, as an operationalization of Shulman's initial categories of SMK and PCK, Ball et al. (2008) proposed teachers' mathematical knowledge domains in the Mathematical Knowledge for Teaching (MKT) framework (see Figure 1). Within the practice-based MKT framework, many of the mathematical teaching tasks in the SMK domain involve mathematical knowledge that is independent of the content in the PCK domain (Ball et al., 2008). While the KQ framework by Rowland et al. (2009) considers as essential the classification of the situations in which mathematical knowledge is faced in teaching, the MKT framework aims to empirically clarify and make distinct the theoretically developed notions of SMK and PCK for future teaching (Rowland, 2013).

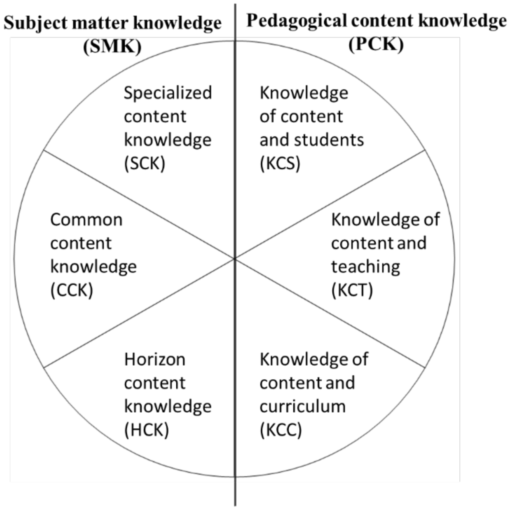

Figure 1. An illustration of the Mathematical Knowledge for Teaching (MKT) framework (a modified figure from Ball et al., 2008, p. 408).

The category labeled "common content knowledge" (CCK) in the MKT framework requires mathematical knowledge that is not unique to teaching and is used in a wide range of settings by professionals when working with mathematics. In the work of mathematics teachers, CCK includes teaching tasks such as knowing whether students' answers for fraction tasks are correct, knowing the definition of fraction concepts and how to carry out fraction procedures and calculations, and knowing the correct notions and notations to use with fractions.

The specialized content knowledge (SCK) category is, according to Ball et al. (2008), a category of "pure" knowledge and mathematical skill needed and used only in settings for mathematics teaching. This mathematical knowledge is more than being able to solve mathematical problems oneself. It includes everyday mathematical tasks that teachers usually routinely do (see Table 4). In the case of fractions, this special work that other professionals do not need entails, for example, understanding the source of a student's fraction error, being able to explain and justify why to invert and multiply when dividing fractions, and to model fractions with different representations. Thus, SCK demands unique understanding and reasoning beyond the mathematics taught to students, making a particular content comprehensible to students.

The third category in the SMK domain, horizon content knowledge (HCK), refers to teachers' larger view of mathematics that helps them to set the foundation for their students' later mathematical studies. It is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). For fractions, this involves, for example, teachers' use of different pictorial representations that can later be connected to different fraction interpretations.

In the PCK domain of the MKT framework (see Figure 1), Ball et al. (2008) and Hill and Ball (2009) placed three categories: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). These are like finer-grained categories of Shulman's (1986, 1987) PCK, and they all relate to the mathematical content in question. In the fractional number domain, KCS combines teachers' knowledge of fraction procedures and their students' common misconceptions and errors when using the procedures. Moreover, teachers need to be aware of their students' mathematical thinking, knowing what students usually do with particular fraction tasks, whether students find the tasks easy or difficult, and what motivates them with fractions.

KCT, in turn, entails pedagogical knowledge of how to design the instruction when teaching fractions. Teachers need knowledge of how to sequence fraction content for instruction and knowledge of appropriate fraction examples and representations, as well as an understanding of how different teaching methods and procedures presented affect their students' learning of fractions. The last category in the PCK domain, KCC, involves teachers' knowledge of both the curricular instructions at a given level and the content of the specific mathematical topic. In the case of fractions, this means that teachers are familiar with the descriptions included in the curriculum and that they understand how the core content, methods, and materials presented in the curriculum are connected to fractions.

Even though the knowledge categories in the MKT framework are presented as their own categories, the distinction between the categories is not always clear. As Ball et al. (2008) stated, detailed knowledge of fraction representations may be considered both as common and specialized content knowledge, and similarly, teachers may analyze students' errors with fractions using their SCK or using their knowledge of the fraction content and students. Moreover, the KCC category is provisionally placed in the MKT framework within the PCK domain, like the HCK category in the SMK domain, but these categories may also be parts of other categories, or they may run across several categories (Ball et al., 2008). However, Ball et al. (2008) put more emphasis on the multidimensional nature of teachers' mathematical knowledge than a sharp distinction of the categories.

Hill et al. (2005, 2008) showed the importance of mathematical knowledge for teaching when involving over 300 mathematics teachers in their studies. The results revealed significant effects between teachers' mathematical knowledge and students' mathematics achievement. Moreover, teachers' MKT was found to be strongly related to the mathematical quality of teachers' instruction, including their use of mathematical explanation, representations, and responses to students' mathematical ideas. Bray (2011) also found that teacher knowledge is the primary determinant of the mathematical and pedagogical quality of teachers' responses to student errors and that teachers should be better aware of students' common errors and how these are related to key concepts in mathematics. Further, Stockero and Van Zoest (2013) argued that strong mathematical knowledge for teaching, as defined by Ball et al. (2008), is a key element when addressing students' responses.

Fauskanger (2015) considered the SCK defined by Ball et al. (2008) as closely related to Tchoshanov's (2011) type 2 knowledge of concepts and connections, since SCK includes responding to students' *why* questions. Based on her analyses of Norwegian mathematics teachers' responses to multiple-choice and open-ended MKT items (e.g., Ball & Hill, 2008; Hill et al., 2004), Fauskanger concluded that teachers can be divided into three categories: (1) teachers who express that they do not understand the content in focus, (2) teachers who possess and emphasize type 1 knowledge of facts and procedures, and (3) teachers whose responses reflect the type 2 knowledge identified by Tchoshanov. However, as stated by Fauskanger, challenges may occur when using the MKT items to make teachers' knowledge visible. Thus, it seems to be beneficial to also use other measurements and different kinds of methods, such as written responses, to capture a better understanding of mathematics teachers' knowledge (Fauskanger, 2015).

The developers of the MKT framework state that teacher educators should also be familiar with the framework (Ball et al., 2008; Hill & Ball, 2009). Moreover, the multidimensional mathematical knowledge described in the framework should be presented in teacher education in focused ways to give PTs opportunities to learn mathematical knowledge for teaching and to help them to understand the wide range of knowledge and skills they will need later in their professions as elementary mathematics teachers (Ball et al., 2008; Hill & Ball, 2009). In this thesis, the MKT framework is illustrated with a pie model that connects in the middle of the circle all the MKT categories from the SMK and PCK domains (see Figure 1). Instead of using the original oval-shaped figure, which does not provide an image of closely connected knowledge categories (see Ball et al., 2008, p. 403), this thesis also highlights the knowledge categories as equal-sized. However, when investigating PTs' knowledge for teaching fractions in this thesis, the emphasis is put on the CCK and SCK categories in the SMK domain and on KCS and KCT in the PCK domain. The categories of HCK and KCC, which are also less addressed by Ball et al. (2008), are briefly considered in this thesis in connection with the studies presented in Papers III and IV.

3.3 Mathematical identity

This thesis also takes into account the concept of mathematical identity as an aspect that is connected to PTs' mathematical knowledge. Kaiser et al. (2017) consider that both cognitive abilities and affective-motivational characteristics are essential when describing teachers' mathematical knowledge. While cognitive abilities entail teachers' professional knowledge—that is mathematical CK and PCK as described, for example, by Shulman (1986)—the affective-motivational domain includes teachers' professional beliefs, motivation, and self-regulation (Kaiser et al., 2017). Previous research has indicated significant relationships between teacher beliefs about mathematics teaching and learning and student learning outcomes (e.g., Polly et al., 2013). Further, Kaasila (2007) regards PTs' view of mathematics an important part of their mathematical identities, involving knowledge, beliefs, conceptions, attitudes, and emotions. Previous studies (e.g., Hannula et al., 2007; Lutovac & Kaasila, 2018) have also shown that emotions, and especially mathematics anxiety, may affect PTs' development as mathematics teachers.

The mathematics education literature offers different approaches to examining mathematical identity, and various categorizations are proposed in research on the topic (Graven & Heyd-Metzuyanim, 2019). Mathematics education identity research can be divided into two categories focusing either on learner identities or (prospective) teacher identities. Mathematics identity research is also categorized based on the theoretical perspectives and definitions for identity that are used (Graven & Heyd-Metzuyanim, 2019). Darragh (2016) identified two main theoretical frames on identity within mathematics education: a sociological frame that considers identity as an action, and a psychological frame that sees identity as an acquisition. Within the sociological frame, Darragh distinguished a narrative identity that refers to a view of identity that uses the stories people tell. Kaasila

(2007) represents a researcher that used a narrative inquiry method when analyzing turning points and important episodes in PTs' stories of development as mathematics teachers. Lutovac and Kaasila (2014) defined PTs' mathematical identities as narratives where PTs "tell themselves or others about themselves as mathematics learners and teachers" (pp. 130–131). Mathematical identity or narrative mathematical identity is then realized in PTs' narrative identities when talking about their own relationships to mathematics and its teaching and learning (Kaasila, 2007). Further, Kaasila (2007) concluded that PTs' emotions and attitudes need to be taken seriously in elementary teacher education programs and that it is important to listen to PTs talking about themselves as future mathematics teachers.

Understanding a person's narrative identity may focus on past, present, or future dimensions of identity. Lutovac and Kaasila (2014) found substantial differences in PTs' ways of conducting their future-oriented mathematical identity work during teacher education. The results revealed that despite negative views of mathematics during their own time in school, some PTs were able to rise above their mathematical fears, and they tried to understand mathematics to be able to teach it. However, some other PTs with similar prior negative views of mathematics seemed to carry over their fears and difficulties to their future professions. Lutovac and Kaasila (2014) concluded that the main reasons for the differences in the PTs mathematical identity work were based on different emphases and pedagogical practices in teacher education mathematics courses. Thus, Lutovac and Kaasila suggested that mathematical identity should be addressed in teacher education when preparing PTs for their future professions.

Maasepp and Bobis (2014) also showed in their study that the role of teacher education is critical in contributing to the development of PTs' positive mathematical beliefs that are involved in their mathematical identity (Kaasila, 2007). The results also suggested that PTs' mathematical beliefs affect how they see themselves as teachers and how they later implement their teaching. Poulaki Mandt (2021) investigated PTs emotions and identity, concluding that "teacher education needs to develop greater support and tools for the PTs in terms of identity-building and for dealing with various emotions, especially in the transition between education and practice" (p. 38).

The topic of mathematical identity has also been addressed in studies concerning the Swedish teacher education context. For example, Palmér (2013) investigated the professional identity development of seven novice elementary school teachers, starting their interviews just before the PTs graduated from teacher education. The study showed that even though the PTs expressed that they wanted to reform mathematics teaching in schools, none of the four PTs who were focused on in-depth in the study had developed a professional mathematics teacher identity two years after their graduation. Palmér (2016) also concluded that PTs' image of their future profession should be included in teacher education programs to make it more visible for the PTs, since the image of what it means to be an elementary school teacher directs PTs' actions and becomes the goal of their professional identity development.

Further, Ebbelind (2020) examined how the similarities in the discursive patterns of two PTs framed their processes of becoming mathematics teachers by staying committed to their prior positive experiences of mathematics. The study showed that even though teacher education had an impact on the participating PTs' professional development, the PTs remained in a safe discursive zone that did not stimulate the developmental change as expected by the teacher educators but rather reinforced the PTs' existing perspectives of mathematics teaching. Marschall (2021), in turn, focused on five secondary mathematics PTs' meaning-making of their experiences in their development process and concluded that teacher self-efficacy appraisal is closely connected to the development of professional mathematical identity.

In this thesis, and in the study in Paper IV, the concept of mathematical identity is based on the work of Kaasila (2007). Mathematical identity is therefore considered in this thesis as PTs' narrative stories where they reflect their beliefs and emotions toward mathematics, and the concept of mathematical identity is also a foregrounded concept when characterizing the PTs' fraction knowledge.

4 Methodology

This chapter first describes the procedures and methodological choices in data collection. Then, the main methods in the analysis of the empirical data for the studies in Papers I–IV are presented. The chapter is closed by presenting ethical considerations.

4.1 Data collection

Data collection for the thesis was carried out in three phases to capture PTs' fraction knowledge in different phases during their teacher education: (1) at the start of the second teacher education mathematics course in the third year of the studies, (2) after the last teacher practicum course in the fourth year, and (3) at the end of teacher education. The data that were collected from one cohort of PTs consist of three types of empirical data that were used for the studies presented in Papers I–IV (see Figure 2).

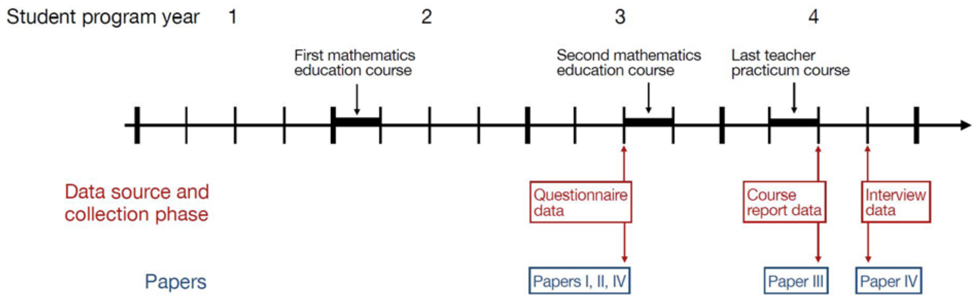

Figure 2. An illustration of the data collection phases connected to PTs' program years and the papers where the data from the different sources were used.

4.1.1 Participants

All the participating PTs studied at one Swedish university. They attended the two teacher education programs, where one prepares teachers for the preschool class and Grades 1–3 and the other focuses on teaching in Grades 4–6 of the compulsory school. In this thesis, the PTs attending the two programs are considered as one cohort since the content of mathematics courses in the teacher education was much the same for both programs. Moreover, during the distance given mathematics courses, the programs were also provided with a couple of joint lectures at the university campus. Altogether, the cohort consists of those 61 PTs who participated in the first data collection by answering a paper-and-pencil questionnaire. The number of PTs selected from the cohort as participants for the studies in Papers I–IV is described later in connection to the analysis of the different sources of data.

In the previous academic year before the first data collection, the PT cohort completed the first mathematics education course of their programs, *Mathematics for compulsory school teachers*, *part 1*, which is described in Chapter 2 of this thesis. The goals for the course expected the participants to have recalled and rehearsed knowledge of fraction concepts, operations, algorithms, and notions studied prior to commencing university studies. The PTs were also expected to have deepened their knowledge of fractions from the point of view of teacher education studies during the course.

4.1.2 Questionnaire data

The first source of empirical data was collected with a paper-and-pencil questionnaire from the PTs who participated in the course *Mathematics for compulsory school teachers*, *part 2*. Altogether, 67 PTs were enrolled in this second mathematics course of their programs, and at the time of the first data collection, 61 of them were at the university campus where they were asked to answer the questionnaire. All the 61 PTs voluntarily agreed, and they were given 90 minutes to answer the questionnaire before their mathematics lecture. After submission, the questionnaires were provided with number codes 1–61, which are used when referring to the participating PTs in the thesis and in Papers I–IV.

The printed questionnaire approach was chosen to capture the PTs' conceptual and procedural knowledge of fractions in their written descriptions. The questionnaire comprised an introduction section describing the research project, three research sections concerning fractions in elementary school mathematics teaching, and a background information section (see Appendix A). The fraction content included in the questionnaire was such content that the responding PTs were familiar with based on their prior studies in mathematics. In the first research section, the instructions asked the respondents to reflect on different concepts and connections that they thought might relate to fractions and then write and draw everything they knew about the concept of fractions. The second section asked the respondents to describe how they might teach the task $\frac{1}{2}+\frac{3}{4}$ to elementary school students. The third research section involved nine fraction tasks similar to tasks found in Swedish elementary mathematics textbooks, national mathematics tests, and support materials for elementary mathematics teaching. For the selected tasks, the PTs were asked to present all steps in their solutions. The last section in the questionnaire consisted of statements focusing on the respondents' experiences on fractions and mathematics teaching, such as "Calculating with fractions is easy for me" and "I would be happy to teach mathematics in elementary school." Other questions in the background section asked about the PTs' expectations concerning their future profession as mathematics teachers and their studies prior to the second teacher education mathematics course. Moreover, at the end of the questionnaire, the respondents could decide whether to answer anonymously or to include their contact information with the questionnaire for a possible interview or a follow-up study.

4.1.3 Course report data

The second source of data concerns the PTs' written course assignment reports. The reports were produced in conjunction with the ten-week teacher practicum course that the PTs completed in the last year of their programs before starting to write their final teacher education bachelor theses (see Figure 2). In this phase of their education programs, the participating PTs should have received comprehensive knowledge both of mathematics and teaching the discipline. Altogether, 49 PTs of the original cohort of 61 PTs participated in the teacher practicum course. After submitting their reports, 42 of the 49 course participants gave permission for their reports to be used for research purposes.

The teacher practicum course assignment was originally constructed for the purpose of being a part of the examination of the practicum course. However, the different PCK categories of the MKT framework were also indirectly included in the assignment, also making the use of the reports relevant for research purposes in this thesis. The instructions for the assignment (see Appendix B) first asked the PTs to plan and conduct mathematics lessons that stimulated elementary students' learning and were based on the elementary school curriculum document, scientific research, and proven experience. Thereafter, they were instructed to write a report describing and discussing their choices concerning different concepts, representation forms, working methods, and teaching materials for their lessons, as well as to reflect on the effects of the choices on their students' learning. The instructions also asked the PTs to describe the connection between their lessons and educational policy documents, such as the national curriculum document, as well as scientific research and different theories and methods related to mathematics teaching and learning.

4.1.4 Interview data

The last data collection comprised four semi-structured interviews conducted at the end of the last term of the PTs' teacher education programs (see Figure 2). In this phase of teacher education, the PTs were expected to be able to make reflections concerning their education, the mathematical knowledge they had received, and their expectations for their professions as mathematics teachers. Semi-structured interviews were chosen as a data collection method to get a deeper view of the PTs' mathematical identities and fraction knowledge compared to their written responses in the questionnaire and the lesson reports, as the interviewed PTs were given opportunities to freely develop their responses (Bryman, 2021). For the interviews, four PTs were selected from among the original cohort of 61 PTs, using purposive sampling to capture data from different types of PT respondents that reflected the variation found in the original cohort (Bryman, 2021). Based on the PTs' expressed positive or negative views and attitudes toward mathematics and its teaching in the questionnaire, the respondents were first divided into two groups. From these groups, the PTs who had provided their contact information with the questionnaire were selected (n = 27). Among these selected PTs were found four PTs who agreed when asked to be interviewed for the study in Paper IV, and they were given the pseudonyms Anna, Julia, Matilda, and Sabina. The selected participants had slightly different backgrounds, such as age and mathematics studies prior to teacher education, and they represented the different groups of questionnaire respondents as follows: Anna and Julia displayed positive and Matilda and Sabina somewhat negative attitudes toward teaching elementary school mathematics. Moreover, Julia correctly solved all the fraction tasks in the research questionnaire, while the others made different kinds of mistakes with the tasks.

The interviews were conducted by two researchers via the video meeting tool Zoom, which both the participating PTs and the researchers were familiar with using. There was one interviewer for each individual interview, which lasted for 23–26 minutes. The interviews were audio-recorded and afterwards transcribed in the original language. An interview guide that was piloted prior to the PT interviews was followed during the semi-structured interviews to ensure comparability between the four interviews (cf. Bryman, 2021). In addition to the interview protocol, the PTs were also asked follow-up questions when necessary (cf. Brinkmann & Kvale, 2018).

To investigate the PTs' mathematical identities, the interviews were opened and also closed by discussions concerning the PTs' expectations about themselves as mathematics teachers, reflections related to the PTs' experiences and knowledge in mathematics, as well as the impact of teacher education on the PTs' development as mathematics teachers (see Appendix C). The main part of the interviews focused on the PTs' analysis and reflections on six different solutions for the fraction division task $\frac{3}{4}/3$ (see Figure 3). The presented solutions were selected for the interviews among the typical solutions that the original cohort of 61 PTs provided in the paper-and-pencil questionnaire. Moreover, the selection of the different solutions involved one solution from each of the interviewed PTs and two solutions from other PT respondents. During the interviews, the solutions were presented to the PTs one at a time, and they were asked to compare and evaluate the solutions, to improve them, and to explain the methods and procedures used in the solutions.

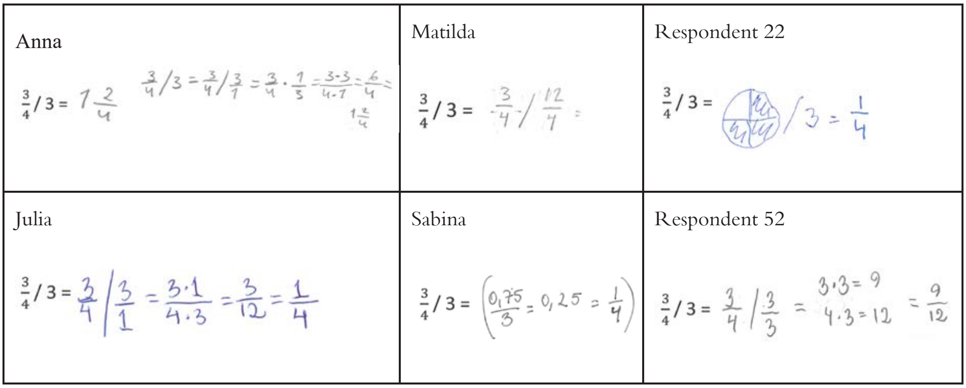

Figure 3. The six fraction division solutions used in the semi-structured interviews.

The PTs were asked to analyze different solutions for the same task to investigate their SMK and PCK related to fractions when explaining mathematical ideas, procedures, errors, and misconceptions connected to the solutions. Moreover, the task $\frac{3}{4}/3$ was chosen to be analyzed based on findings in previous research showing that division and fraction tasks with whole numbers are challenging for many PTs (e.g., Newton, 2008; Tirosh, 2000). The analysis of the questionnaire data for Paper I also revealed the PTs' difficulty with the task involving fraction division by a whole number.

4.2 Data analysis

Different methods were needed to carry out the analyses of the different types of empirical data and to fulfill the aims of the studies in Papers I–IV. Table 5 illustrates how the data and the different knowledge categories of the MKT framework overlap in the four studies. Table 5 also presents the number of PTs from the original cohort of 61 PTs who served as participants in the studies. One PT, given the pseudonym Matilda in Paper IV, participated in all the studies, and she will be presented as a typical case when characterizing the PTs' knowledge for teaching fractions in connection with the main findings of the thesis in Chapter 6. The presentation of the data analysis in the next section follows the order that the analyses were conducted in the research process and, thus, the phases of the analysis process do not correspond to the coding of Papers I–IV.

Table 5. Overview of the studies in Papers I–IV.

	PTs	*Source of data*	*Focus of the study*	*Methods for analysis*
I	59	Questionnaire section 4: solutions for six routine fraction tasks	SMK domain: CCK (procedural knowledge)	Categorization of errors
II	57	Questionnaire section 2: descriptions for conceptions of fractions	SMK domain: CCK (conceptual knowledge)	An analysis framework with four categories for aspects of fractions
III	6	Written course reports: reflections on teaching and learning of fractions	PCK domain: KCS, KCT, and KCC	Theory-guided content analysis
IV	4	Questionnaire sections 4 and 5: fraction solutions and background information Semi-structured interviews: reflections on fraction division solutions and mathematical identity	SMK domain: CCK and SCK PCK Mathematical identity	Narrative inquiry

4.2.1 Analysis of fraction solutions

The first phase of data analysis looked at the PTs' solutions for the nine fraction tasks (a)–(i) included in the questionnaire. First, the answers in the solutions were categorized as correct or incorrect. Then, six tasks (a)–(f) were selected for closer analysis for the study in Paper I, where the focus is on PTs' procedural knowledge in the CCK category. Respondents with the number codes 32 and 36 in the questionnaires did not answer the fraction tasks section, thus resulting in 59 participants for Paper I.

Table 6. The analyzed routine fraction tasks.

Addition with common denominators	***Addition with different denominators***	***Subtraction with different denominators***
a) $\frac{2}{3}+\frac{2}{3}$	b) $\frac{4}{5}+\frac{2}{3}$	c) $\frac{3}{4}-\frac{1}{2}$
Subtraction with a whole number	***Multiplication with different denominators***	***Division by a whole number***
d) $1-\frac{2}{6}$	e) $\frac{3}{4}\cdot\frac{2}{5}$	f) $\frac{3}{4}$ / 3

The six tasks selected represent routine fraction tasks without any text or context, and they include four operations with different types of fraction content (see Table 6). In the analysis, the definition of a correct answer for a fraction task was based on the PTs' prior mathematics education studies. Thus, in the case of the six routine tasks, a correct answer was defined as an answer that was converted to a mixed number form when possible or presented in the simplest fractional number form. It was also investigated whether the PTs provided mathematical steps with their solutions as instructed in the questionnaire and whether they used representations other than mathematical algorithms, such as pictorial representations. Thereafter, the analysis of the tasks entailed categorizing errors in the solution methods. The definition of errors was built on three error types classified by Radatz (1979): (1) lacking knowledge of prerequisite skills, facts, and concepts, (2) incorrect associations or inflexibility in thinking, and (3) application of irrelevant rules or strategies. Moreover, following Young and Zientek (2011), the errors were categorized as either technical or procedural errors. The former type refers to errors that are not directly related to fractions, such as presenting the answer in the form of decimals or using illogical mathematical writing. The latter error type consists of obvious errors in fraction operations, such as adding across numerators and denominators in addition or cross-multiplying in fraction multiplication. Altogether, seven error types were differentiated within the two main error categories and coded as E1–E7 (see Table 7).

Table 7. The main error categories and the seven error types connected to them.

Technical errors in	***EI: presenting the answer***	***E2: mathematical writing***	***E3: mathematical facts***	***E4: leaving the task blank***
Examples	$\frac{2}{3}+\frac{2}{3}=\frac{4}{3}\approx 1{,}33$	$1-\frac{2}{6}=1=\frac{6}{6}$	$12+10=24$	
Procedural errors in	***E5: addition or subtraction***	***E6: multiplication***	***E7: division***	
Examples	$\frac{4}{5}+\frac{2}{3}=\frac{6}{8}$	$\frac{3}{4}\cdot\frac{2}{5}=\frac{8}{15}$	$\frac{3}{4}/3=\frac{9}{12}/3=\frac{3}{12}$	

The analysis revealed several subtypes of errors connected to E1–E7 as results in the study in Paper I. The results are presented in Paper I both with qualitative descriptions of the PTs' solutions and in terms of the number or percentage of PTs who provided correct answers or made errors with their solutions.

4.2.2 Analysis of the interviews

The second phase of data analysis comprised data from the semi-structured interviews and the background section in the questionnaire, where the PTs expressed their views and attitudes toward mathematics and its teaching. The data were used in Paper IV for the study that focuses on the PTs' mathematical identities and mathematical knowledge for teaching fractions at the end of their teacher education. The analysis of the transcribed interview data and the PTs' written descriptions in the questionnaire concerned the use of narrative inquiry, since the analysis aimed to focus both on the content and the form of the PTs' responses (cf. Kaasila, 2007).

When analyzing the content of the interviews, the focus was on the PTs' mathematical SMK and PCK in their analyses and reflections on the fraction division solutions presented to them. The analysis also investigated how the PTs positioned themselves when focusing on the fraction solutions. That is, it was investigated whether they took a student-focused approach, analyzing the solutions from the point of view of someone else's mathematical knowledge, or whether they took a teacher-focused approach, explaining the solutions based on their own understanding and mastery of the procedures used in the solutions (cf. Son & Crespo, 2009). Moreover, the narrative analysis focused on different aspects that the PTs expressed in relation to their mathematical identities, such as factors facilitating changes and turning points, the development of their views of mathematics, their experiences of mathematics during their own school years and teacher education, and their future expectations about mathematics teaching.

When the narrative analysis focused on the form of the PTs' responses, it was analyzed *how* they expressed themselves using different affective and emotional elements in connection to their reflections on the fraction division solutions and their mathematical identities. For example, expressions like "I do not really understand this [method]" and "I do not really know how to calculate the answer to this [task]" as well as "I cannot follow this [solution]" indicated difficulties in the PTs' SMK and PCK. Similarly, expressions like "I will not be able to teach mathematics" and "mathematics is my biggest challenge" displayed negative expectations in relation to teaching mathematics and views of the PTs as mathematics teachers. Instead, the following expressions entailed a more positive mathematical identity and picture of the PTs as mathematics teachers: "I will do my best [when teaching different pupils]" and "mathematics is my favorite subject." The results of the PTs' narratives are presented in Paper IV, providing qualitative examples of two pairs of PTs (Matilda–Sabina, Anna–Julia) that had a similar narrative theme—that is, attitude toward mathematics teaching.

4.2.3 Analysis of fraction conceptions

In the next phase of data analysis, the attention was turned back to the questionnaire data, as the analysis of the semi-structured interviews indicated that the PTs had difficulties when describing the different fraction solutions. The analysis focused on data concerning the PTs' conceptual knowledge of fractions provided in the first research section in the questionnaire. Respondents 32, 36, 42, and 46 did not answer the section, thus resulting in a total of 57 participants for Paper II.

For the analysis of the PTs' descriptions of fractions and their connections to other mathematical constructions, a new analytical framework was devised (see Figure 4). The framework is based on different core aspects of fractions found in previous research, and it includes four categories for fractions: (F1) Interpretations, (F2) Representations, (F3) Procedures, and (F4) Notions. Each category F1–F4 has several subcategories with different content, as illustrated in Figure 4. The analysis of the data focused on the categories and subcategories that the PTs referred to in their descriptions. It was also investigated which content of the different categories did not appear in their answers and what mistakes and misconceptions the answers revealed in their conceptions of fractions. The results of the analysis are reported in Paper II qualitatively as well as quantitatively in terms of the number of PTs providing a description that belonged to a specific category or subcategory.

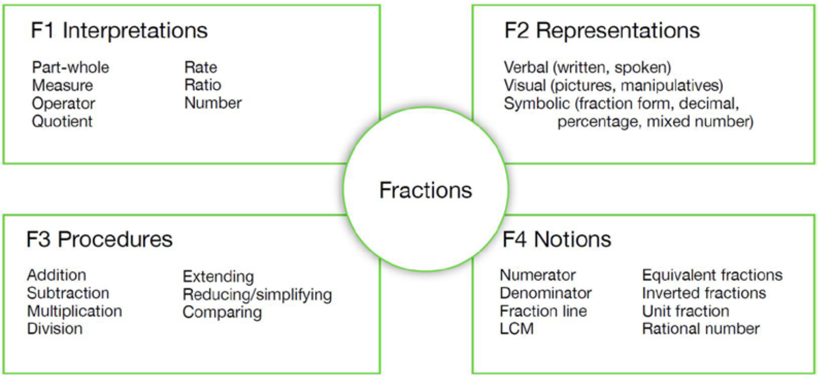

Figure 4. The categories and their subcategories in the framework for analyzing different aspects of fractions (Paper II, Figure 1).

4.2.4 Analysis of the course reports

The last phase of the data analysis process concerned the data for the study in Paper III that focuses on the PTs' PCK related to the teaching and learning of fractions. For that, data from the teacher practicum course assignment reports were analyzed. First, all the reports concerning fraction lessons, that is, 6 reports, were selected from among the 42 reports that the PTs submitted for research purposes after the teacher practicum course. The selected reports were written by the PTs with the codes 28, 39, 47, 49, 58, and 60 in their paper-and-pencil questionnaires, and these PTs represent the six participants in Paper III.

The analysis of the written course report data was conducted by using a theory-guided content analysis (Schreier, 2012). Ball et al.'s (2008) MKT framework was considered as a relevant guiding theory since the aspects of the PCK domain in the framework were also indirectly written in the instructions for the course assignment (see Appendix A in Paper III). Thus, a coding frame was built defining the contents of the categories KCS, KCT, and KCC in the PCK domain. This operationalization of the concepts from Ball et al. (2008) is presented in Table 8.

Each main category in the coding frame was also complemented with two subcategories that differentiated whether the PTs' displayed mathematical knowledge that concerned fractions or other mathematical content and teaching in general. For the qualitative description of the results in Paper III, the PTs' identified PCK was organized within each main category under common themes, such as students' fraction knowledge (KCS), organizing the work for teaching mathematics (KCT), and fractions in the curriculum (KCC).

Table 8. Description of the main knowledge categories in the theory-guided content analysis (Paper III, Table 1).

Knowledge of content and students, KCS
PTs demonstrate knowledge that combines knowing about students and knowing about mathematics, for example, - showing familiarity with common student errors, conceptions, and misconceptions about mathematical content and deciding which of several errors students are most likely to make - anticipating and predicting • what students are likely to think and what they will find confusing, • what students are likely to do with a mathematical task and whether they will find it easy or hard, • what students will find interesting and motivating.
Knowledge of content and teaching, KCT
PTs demonstrate knowledge that combines knowing about teaching and knowing about mathematics, for example, - sequencing the mathematical content for instruction, - choosing suitable examples to start with and examples that take students deeper into the content, - evaluating advantages and disadvantages of different representations and identifying what different methods and procedures afford instructionally.
Knowledge of content and curriculum, KCC
PTs demonstrate knowledge that combines knowing about curriculum and knowing about mathematics, for example, - showing familiarity with the topics that will be taught in mathematics during the school years at a given level - referring to the instructional materials that embody the mathematical topics in curriculum.

4.3 Ethical considerations

As the collected data for the thesis concerned the cohort of 61 PTs in one Swedish university, it was of great importance to consider ethical aspects when carrying out the research project. Thus, the ethical guidelines stated by the Swedish Research

Council (Vetenskapsrådet, 2017) and the Swedish Ethical Review Authority (Etikprövningsmyndigheten, 2023) were followed in all phases of the research project. Before the participation, the PTs were informed about the research project and how the data they provided would be handled. They were also informed that their participation in the empirical studies was voluntary. Moreover, when asked whether they gave permission for their teacher practicum course reports to be used for research purposes in this thesis, the PTs' were also informed that whether they agreed to this or not did not affect the assessment of the report and the practicum course since I, as a researcher, was not connected to the teacher education course. The informed consent was given by the participants for their participation and their data to be used in the studies presented in Papers I–IV.

The confidentiality of the participants was secured by marking their submitted questionnaires with identification codes 1–61 that are used when referring to the PTs in the thesis frame and in Papers I, II, and III. The four interviewed participants in Paper IV were given pseudonyms that are not connected to their real names. Moreover, no personal information that the PTs provided in connection to the questionnaire, the teacher practicum course reports, or the semi-structured interviews is shared outside the research project.

In addition to the ethical aspects concerning the involved participants, the research project concerned the university context that I, as a researcher, had a connection to. Therefore, the ethical considerations also concern the researcher's own role within the specific research context. Floyd and Arthur (2012) refer to this as an internal ethical engagement that relates to the ethical and moral dilemmas that an insider researcher has to deal with. This internal ethical engagement entails personal and professional relationships with the participants, insider knowledge, conflicting professional and researcher roles, and anonymity (Floyd & Arthur, 2012; Trowler, 2011). Even though I was teaching some of the PT participants along with many other teacher educators before starting the research project for the thesis, as a researcher I was not connected to teaching the participating cohort. While an internal researcher may benefit from better access to data and respondents and prior insider knowledge of the research context when carrying out the research (Floyd & Arthur, 2012; Trowler, 2011), misleading assumptions have been avoided in this thesis by discussing the data collection and analysis and the findings with other researchers who were also familiar with the research context. Discussions about the research project have also aimed to avoid the possible conflicting consequences that an insider researcher may face within the context both during and after the research work (Floyd & Arthur, 2012).

Trowler (2011) stated that when researching one's own higher education institution, institutional anonymity cannot be guaranteed. Floyd and Arthur (2012) also argued that institutional anonymity is meaningless for an insider researcher since "[w]hatever efforts are made to preserve anonymity, a simple online search will allow the most novice investigator to identify the institution" (p. 177). Therefore, an insider researcher should work on the assumption that the site of his/her study cannot be anonymous, but the researcher should aim to ensure that the research participants cannot be identified (Floyd & Arthur, 2012). Generally, this thesis aims to be descriptive-analytical rather than normative or critical when presenting and discussing the findings connected to the participants and the teacher education.

5 Results

This chapter first summarizes the results in each Paper I–IV. Then, the collective findings of the thesis are presented in relation to the main research question. The chapter is closed by presenting one participant, Matilda, as a case that reveals some aspects of the outcomes of teacher education connected to PTs' mathematical knowledge for teaching fractions and mathematical identity just before finishing their education.

5.1 Paper I

The study in Paper I investigated the participating 59 PTs' CCK, which is included in the SMK domain in the MKT framework (Ball et al., 2008). More precisely, the study focused on the PTs' procedural knowledge demonstrated when solving stepwise six routine fraction tasks. The research question posed in Paper I was "How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks?"

In general, the results revealed substantial differences between the PTs in their procedural knowledge of fractions. Even though the participants had already completed the first teacher education mathematics course, where the fraction content included in the tasks was studied, only two of the 59 PTs correctly solved all the six fraction tasks. Eight PTs provided only one or no correct answers to the tasks, and their difficulties concerned all the operations that the tasks consisted of—that is, addition, subtraction, multiplication, and division. Many of the provided solution procedures also involved errors from several error categories E1–E7. However, the identified differences in the PTs' procedural fraction knowledge not only concerned *how many* correct or incorrect solutions and different error categories were involved in the solutions, but also *how* the fraction tasks were solved. While some solutions were provided with detailed solution steps using fractional numbers, in some other solutions fractions were converted to decimals or the tasks were solved only through illustrating them with pictures (see Figure 5).

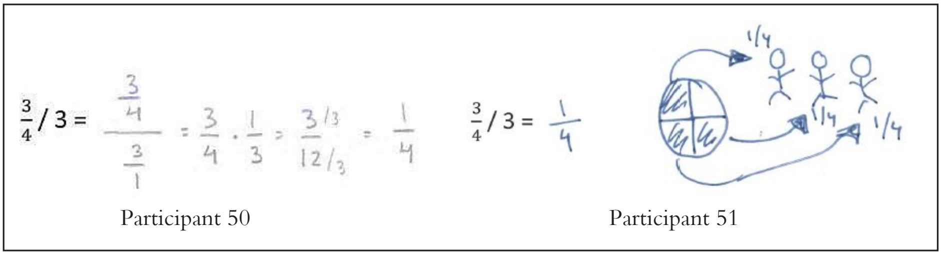

Figure 5. Two examples of correct solutions for a fraction division task showing a difference in the PTs' procedural fraction knowledge.

The results also revealed CCK difficulties in using fraction operations, especially with division and multiplication. While the procedural errors in multiplication (E6) mainly concerned cross multiplying numerators and denominators, the error category for division (E7) involved several subtypes of errors. Errors found in addition and subtraction (E5) were mainly connected to the use of different denominators. Many solutions also reflected a rule-based and rote understanding of the fraction procedures.

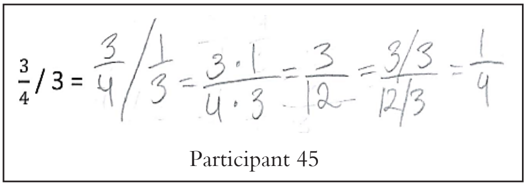

Figure 6. An example of a solution that ends up as a correct answer but involves incorrect use of the procedure dealing with division.

The provided fraction solutions also reflected uncertainty that was displayed by the PTs by producing two alternative solution procedures (usually a correct and an incorrect), giving alternative answers in parenthesis, and leaving the solution procedures unfinished. Moreover, some solutions were provided with question marks, and the tasks were solved with decimals and pictures or just left blank. Uncertainty was expressed in the questionnaire by Participant 55 as follows: "Unfortunately, I do not remember how to divide with fractions."

The results in Paper I also indicated general mathematical CCK limitations that were not only connected to fractions. These general difficulties were realized as technical errors in E1 and E2—that is, errors in presenting the answer and using mathematical writing. Altogether, 35 of the 59 PTs made at least one E1 error and

38 also at least one E2 error. The equal sign was misused in many solutions. It was also typical that symbolic mathematical writing was not used logically throughout the solutions, which led to mathematically incorrect statements and unreasonable answers that the PTs did not recognize.

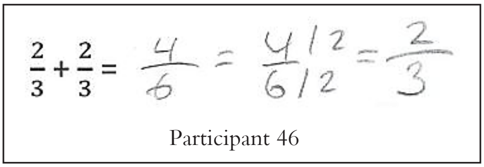

Figure 7. A typical example of a solution with an incorrect use of the procedure for addition and an unreasonable answer.

5.2 Paper II

In Paper II, the aim was to investigate how the PTs demonstrated their conceptual knowledge of fractions when asked to relate the concept of fractions to other concepts and mathematical constructions. Altogether, 57 PTs provided descriptions to the questionnaire section that focused on the CCK category of the MKT framework. The PTs' conceptions of fractions were investigated using a new devised framework that was built on core aspects of fractions in four categories: (F1) Interpretations, (F2) Representations, (F3) Procedures, and (F4) Notions. Two research questions were asked in Paper II: 1) "Which aspects of fractions do student teachers refer to in their conceptions of fractions?" and 2) "What gaps related to the aspects of fractions can be identified in student teachers' conceptions of fractions?"

The results in Paper II showed that the four categories for the different aspects of fractions were all represented in the participants' conceptions of fractions. However, substantial differences were found when looking at how the PTs' descriptions were spread within the categories. The PTs mainly provided conceptions that were connected to different representations and procedures, and their descriptions involved all the subcategories included in the analytical framework for these categories F2 and F3. Conversely, in categories F1 and F4, several core interpretations and fraction notions were not included in the PTs' conceptions.

The PTs showed awareness of mathematical symbolic representations and different visual representations for fractions. However, they provided several pictorial illustrations that were not divided into equal-sized parts. As in the study in Paper I, some PTs expressed uncertainty concerning different procedures for fractions, and the results revealed difficulties presenting how to work through fraction procedures, especially multiplication and division operations. The most severe gaps were found in relation to interpretations of fractions, where only the part-whole and the quotient interpretations were displayed among the PT respondents; the measure, operator, rate, ratio, and number interpretations were missing completely in their descriptions. Another gap related to interpretations concerned the difficulty in identifying the unit for a fraction, since the PTs only considered one single object as the unit. Moreover, the overrepresentation of the part-whole interpretation and the quite shallow use of proper notions with fractions characterized the PTs' conceptions of fractions in Paper II.

In addition to the identified conceptual limitations among the participating PT group, the analysis also revealed substantial differences between individual PTs, as was also found in connection to their procedural knowledge. While some PTs gave comparatively detailed descriptions and examples of the fraction aspects covering most subcategories, some other participants provided very limited descriptions of the content in different subcategories. For example, the fraction conception provided by Participant 41 (see Figure 8) can be considered as limited even though it includes references to three fraction categories: (F1) Interpretations (the part-whole model; the Swedish terms "*hel*" refer to a whole and "*del*" to a part), (F2) Representations (pictorial circle models and the mathematical symbol representation), and (F4) Notions (the Swedish terms "*täljare*" and "*nämnare*" correspond to terms of numerator and denominator, respectively). The results also revealed that the PTs could demonstrate a comprehensive conception in relation to one fraction category while their descriptions in other categories included severe gaps.

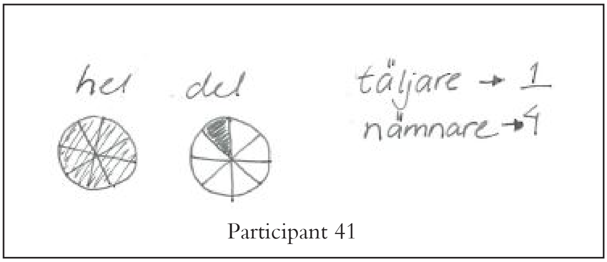

Figure 8. An example of fraction conception provided by one PT participant. The Swedish terms "*hel*" and "*del*" mean *a whole* and *a part*, respectively, and the terms "*täljare*" and "*nämnare*" refer to the terms of numerator and denominator, respectively.

5.3 Paper III

The study in Paper III aimed to expand knowledge about the outcome of teacher education related to PTs' PCK of fractions. Focusing on the three categories of PCK included in the MKT framework (Ball et al., 2008), the study analyzed six PTs' written teacher practicum course reports of fraction lessons that combined their knowledge of fraction content with their knowledge about elementary students, teaching, and the curriculum. The research question addressed in Paper III was "How are different aspects of pedagogical content knowledge related to the teaching and learning of fractions displayed among prospective elementary teachers at the end of their teacher education?"

In their written reports of planning and carrying out fraction lessons, the PTs displayed fraction knowledge relating to each of the categories KCS, KCT, and KCC. However, the results revealed that the teaching and learning of fractions was mainly described on a general pedagogical level, and few reflections were connected to fraction content. Thus, few descriptions differentiated characteristics for elementary students' fraction knowledge (e.g., common student errors and misconceptions related to fractions), teaching of fractions, and fraction core content that is included in the mathematics syllabus in the national elementary school curriculum for different grade levels. Moreover, none of the six PTs referred to their main mathematics course literature, the book *Mathematics for elementary teachers with activities* by Beckmann (2018), which includes a comprehensive presentation of fractions. Neither did they refer to scientific research concerning fractions. Instead, when justifying the rationale behind their fraction lessons, the PTs used literature that focused on mathematics and teaching on a general level.

The results also showed that the PTs did not provide a coherent discussion of fraction knowledge across the PCK categories in their written reports. For example, even though the PTs expressed being aware of elementary students' difficulties with fractions, they did not report using this awareness as a basis for their planning and teaching of fractions. In the KCT category, the PTs' PCK was characterized including general sociocultural aspects of learning based on the use of the EPA model, a Swedish version of the cooperative learning activity Think-Pair-Share (e.g., Lyman, 1981), *genomgång* (a Swedish term for a teacher-led lesson activity, usually at the beginning of a lesson), and group work. Uncertainty related to fraction contents and challenges in the teaching of fractions were also expressed in the PTs' descriptions. Participant 58 stated this as follows: "I was challenged … didactically, and my subject knowledge in mathematics was tested."

When the PTs related their teaching of fractions to the national curriculum document, they emphasized connections between mathematics teaching and students' daily lives. Moreover, the general abilities (i.e., problem-solving, communication, conceptual understanding, reasoning, and procedural ability) included in the curriculum document were highlighted more than the fraction content for Grades 1–6 where the PTs carried out their fraction lessons.

5.4 Paper IV

The aim of the study presented in Paper IV was to find out how prepared for teaching fractions the participating PTs were at the end of their teacher education. Using narrative inquiry, the study focused on four PTs' reflections on their mathematical identities and their analyses of six different solutions for the fraction division task $\frac{3}{4}/3$. The research question was as follows: "How is prospective teachers' mathematical identity and knowledge for teaching fractions at the time of their graduation?"

As in Papers I and II, differences were also found between the PTs in the study in Paper IV. The results showed substantial differences in the interviewed PTs' SMK and PCK related to fractions and in their attitudes toward mathematics teaching just before finishing teacher education. Two of the PTs, Matilda and Sabina, displayed serious difficulties analyzing the fraction solutions presented to them and understanding the mathematical ideas behind the solution procedures, and their interviews were narrated with a teacher-focused approach referring to their own mathematical difficulties. Both Matilda and Sabina reported already in the paper-and-pencil questionnaire that they had experienced mathematics as a difficult subject during their own years at school and that teaching mathematics would be a challenge for them. The two other PTs, Anna and Julia, reported positive attitudes toward mathematics in the questionnaire, and they had chosen topics related to mathematics education for their final bachelor theses. In their interviews, Anna and Julia expressed confidence concerning their mathematical CCK, analyzing the fraction division solutions and making suggestions how to improve them. However, even though Anna was interested in mathematics, she displayed limitations in understanding and accepting solution procedures other than her own, and she mainly preferred mathematical symbol representations as correct ones. Julia, in turn, displayed a strong student-focused approach in her analyses, pointing out the importance of explaining the solutions to others who did not know the methods and finding out children's reasoning as they worked with fraction tasks.

It was also reflected in the PTs' narratives that their mathematical SMK and PCK did not develop much during teacher education. The PTs who were selected for the interviews with different backgrounds as well as different views and attitudes toward mathematics and its teaching expressed varying educational needs in their development as elementary mathematics teachers. The interview responses indicated that teacher education did not fully manage to respond to their educational development needs. The results in Paper IV also revealed that there was not a great positive change in the mathematical identities of Matilda and Sabina, who at the end of their teacher education still narrated their responses mainly with negative views of mathematics. Similarly, even though having a more positive mathematical identity, Anna and Julia expressed that they were missing PCK for teaching mathematics in their education. Julia stated this as follows: "The focus [of teacher education mathematics courses] was very much on our own math skills and perhaps not so much on how we should teach them [elementary school students]."

5.5 Main findings of the thesis

This thesis aimed to characterize PTs' mathematical knowledge for teaching fractions as an outcome of Swedish teacher education. The participating PTs' knowledge of fractions and their mathematical identities were investigated in light of the MKT framework by Ball et al. (2008). In the SMK domain, the focus was mainly on the CCK category, where aspects of procedural and conceptual fraction knowledge were analyzed. In the PCK domain, the three categories, KCS, KCT, and KCC were analyzed, also relating to the SCK and HCK categories of the

SMK domain. The main research question "How can prospective elementary teachers' knowledge for teaching fractions be characterized in light of the Mathematical Knowledge for Teaching framework?" is next answered by presenting the synthesized results in Papers I–IV in relation to the SMK and PCK domains of the MKT framework.

5.5.1 Prospective teachers' subject matter knowledge

In this thesis, serious difficulties and limitations were identified in the participating PTs' SMK of fractions. Several different procedural errors were found in the PTs solutions for routine fraction tasks. Moreover, several gaps were revealed in the PTs' conceptual knowledge of fractions when analyzing their descriptions of the concept of fractions and its relation to other concepts and mathematical constructions (see Figure 9).

The most important finding in the SMK domain is the difference that was found between the participating PTs. Figure 9 illustrates the number of correct answers and the different error categories identified in the PTs' solutions in Paper I, as well as the PTs' references to different conceptions of fractions in Paper II categorized according to the four aspects of fractions. The figure reveals, for example, that Participant 10 correctly solved five of the six fraction tasks, and that the solutions involved only one error category. Moreover, the same PT referred to all four fraction categories when describing different aspects of fractions. Similarly, Participant 50 correctly solved all six fraction tasks without any errors in the solution procedures, and all the four fraction categories were also included in the PT's conception of fractions. Participant 4, in turn, provided a correct answer only to one task, the solutions included five of the seven error categories, and the fraction concept was only connected to one of the four core aspects of fractions. Participant 21 also represents a similar worrying example of a PT whose fraction knowledge involved many errors and gaps both in the procedural and the conceptual knowledge of fractions.

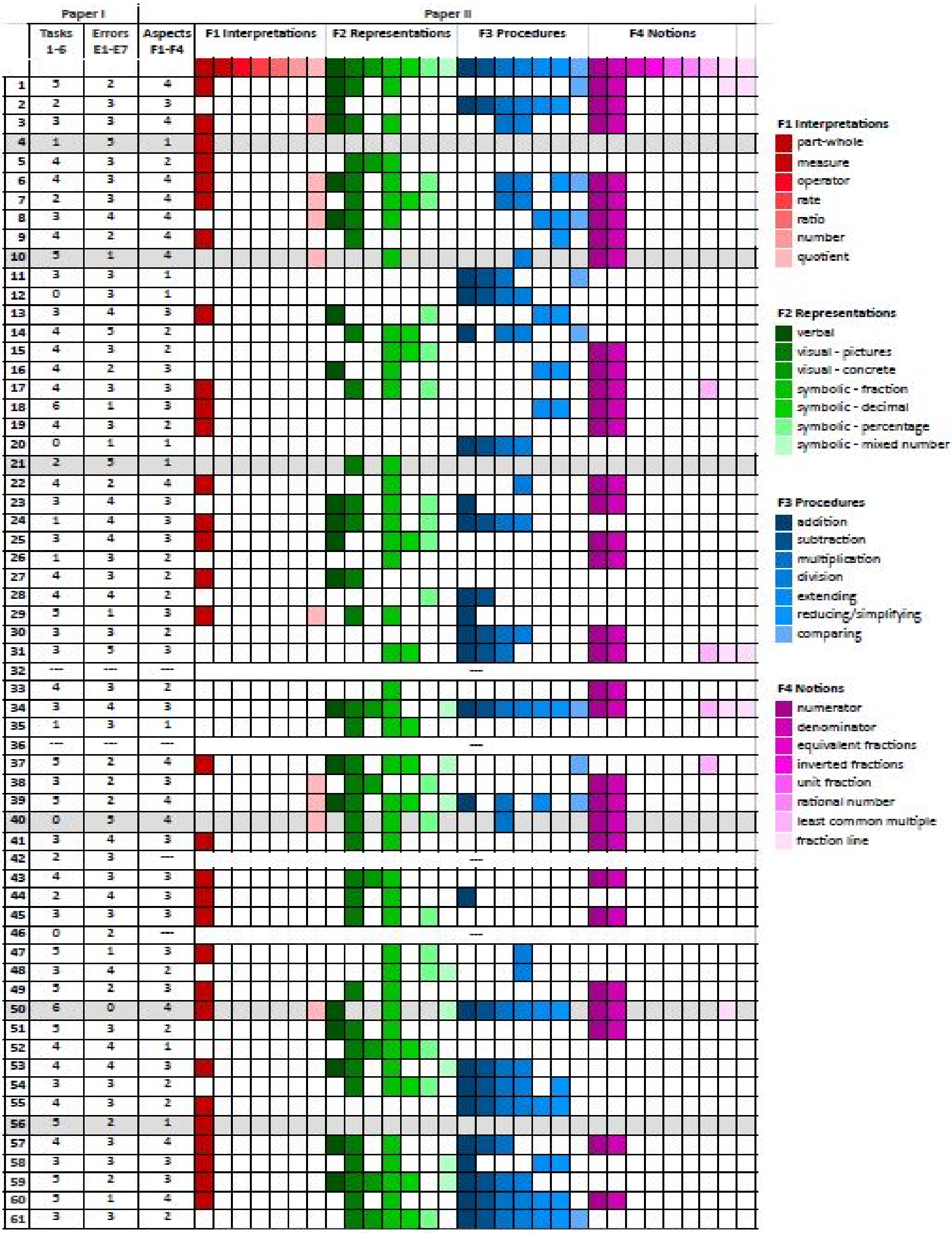

Figure 9. The PTs' codes (1–61), the number of their correct solutions for the six fraction tasks, and the number of different error categories (E1–E7) found in their solutions, as well as the number of referred fraction aspects (F1–F4) and the PTs' fraction descriptions categorized according to F1–F4. The non-colored squares connected to Paper II illustrate gaps in the PTs' conceptions of fractions.

Another important finding that can also be seen in Figure 9 is the apparent inconsistency found between some PTs' procedural and conceptual knowledge of fractions. For example, while Participant 56 could solve five tasks, only making a couple of technical errors, the same PT's conception of fractions only included one aspect of fractions. The conceptual fraction knowledge of Participant 40, in turn, was more comprehensive than his/her procedural knowledge as the PT did not provide any correctly solved tasks and exhibited five different error categories in the solution procedures.

In relation to the CCK category, the findings also indicated general mathematical limitations that were not directly connected to fractions, such as deficiencies in presenting logical mathematical solution steps and the incorrect use of mathematical symbol writing and other representation forms. Uncertainty also characterized the PTs' mathematical knowledge in the SMK domain.

5.5.2 Prospective teachers' pedagogical content knowledge and mathematical identities

The PTs' PCK for teaching fractions and their mathematical identities were demonstrated in the analyzed teacher practicum course reports, semi-structured interviews, and their questionnaire responses. As in the SMK domain, substantial differences were also found between the PTs in relation to their fraction knowledge in the PCK domain. Uncertainty was also identified in connection to PCK of fractions and mathematical identity.

An important result was the PTs' difficulty combining their fraction knowledge with their knowledge of elementary students, teaching, and the national curriculum document. The PTs' descriptions of their fraction lessons highlighted general pedagogical aspects of teaching and learning and general abilities from the national curriculum as a basis for their mathematics teaching, involving few reflections connected to fraction content. The PTs' descriptions also reflected a limitation related to the HCK category in the SMK domain, since few descriptions indicated the idea that the teaching of fractions also sets a foundation for elementary students' later mathematical studies. Moreover, the difficulties identified in the PTs' analyses of the fraction division solutions not only revealed limitations in their PCK but also in the SCK category in the SMK domain. As defined in the MKT framework by Ball et al. (2008), SCK requires mathematics teachers to be able to interpret students' answers, provide explanations to mathematical procedures, use different representations, and put themselves in the student's position.

In general, the findings revealed insufficient mathematical knowledge for teaching fractions and negative mathematical identities among the original cohort of 61 PTs. The interviewed PTs expressed that they would have needed more knowledge from teacher education. Moreover, the negative views of mathematics and its teaching and learning that two of the interviewed PTs reported in the questionnaire in the third year of their teacher education programs still existed at the end of teacher education.

5.5.3 The case of Matilda

Matilda was the only PT who provided questionnaire data, interview data, and course report data that were used in Papers I–IV, and, thus, she participated in all the four empirical studies that this thesis is grounded in. Although Matilda was only one PT among the participating cohort of 61 PTs, she can be considered as a typical PT participant in several senses. First, Matilda represented a typical PT with her background, since she entered teacher education several years after finishing upper secondary school and already had experience in the teaching profession from working as a substitute elementary teacher. Like most participants, she did not show a special interest in mathematics by completing extra mathematics courses during upper secondary education or the time in teacher education. Moreover, Matilda, like many other PTs, reported negative attitudes toward the teaching and learning of mathematics in the research questionnaire.

When working through the six routine fraction tasks in the questionnaire, Matilda correctly solved five of the six tasks. Like many other PT respondents, she displayed difficulties in the task including division by a whole number, where Matilda first correctly converted the divisor 3 to a fraction form $\frac{12}{4}$ but then left the solution unfinished (see Figure 3). In addition to the E7 error in division, Matilda's solution procedures in the other tasks included another type of typical PT error, E2, which is a technical error referring to missing solution steps.

When relating the concept of fractions to other concepts and mathematical constructions, Matilda referred to three of the four fraction aspects included in the analytical framework, namely (F1) Interpretations, (F2) Representations, and (F4) Notions. However, like many other PT respondents, Matilda provided few references and very short answers connected to these categories. In F1, she only referred to the part-whole interpretation of fractions that also dominated in the other PTs' conceptions of fractions. In F2, Matilda provided two non-colored pictures, a circle and a rectangle, which were divided into non-equal-sized parts as visual representations for fractions. She also used the written form $\frac{numerator}{denominator}$ to illustrate the fraction form without giving any examples of actual fractional numbers. The content concerning procedures for fractions (F3) was missing in Matilda's description, and in F4, she only mentioned the two core notions of fractions—numerator and denominator. Also, as typical for the other PTs, Matilda's identified conceptual knowledge of fractions included severe gaps both between the four different categories of fractions and within the categories (see PT 49 in Figure 9).

In her teacher practicum course report, Matilda described the difficulty in teaching fractions as follows: "Fractions can be difficult to understand and teach, largely due to the existence of several different interpretations and representations." However, she did not report using different interpretations and representations for fractions in her lessons with elementary students. Instead, like the other PTs who submitted reports of fraction lessons, Matilda highlighted general sociocultural aspects of learning, such as group work and joint discussions, as a basis for mathematics teaching. She also emphasized the importance of connecting the teaching of fractions to students' everyday lives using practical and motivating examples and concrete materials. However, she provided few descriptions showing how the fraction content was connected to the knowledge of students and curriculum. Moreover, Matilda referred in her report to research literature that concerned mathematics teaching on a general level, and when describing the implementation of her planned fraction lessons, she expressed that she did not manage to support all her elementary students in their learning of fractions. Even though Matilda described that the class involved students with some special needs and difficulties in learning, she did not report taking these different needs into account during her teaching, which was also typical for the other PTs in their report descriptions.

In her interview, Matilda demonstrated comprehensive difficulties analyzing and developing further the different fraction division solutions. Matilda recognized her own unfinished solution, but at the end of her teacher education she was still not able to finish it. Moreover, Matilda stated that the solution with a pictorial representation seemed "logical" but that she did not understand the method. She commented the solution with a correct flip-and-multiply procedure without giving an analysis of it: "I can recognize it [the method], but I still have a little difficulty understanding it." Similarly, she did not analyze the two other solutions that ended up as incorrect answers but rather stated that she could not follow the solution procedures. Neither gave Matilda examples of fraction tasks that are not possible to solve by converting fractional numbers into decimal numbers. Moreover, with her limited attempts to analyze the fraction division solutions, Matilda also represented the many other PT respondents who expressed uncertainty in connection to fraction division in the different sections in the research questionnaire.

When answering the paper-and-pencil questionnaire during her third year of teacher education, Matilda expressed that she did not have enough knowledge for teaching mathematics to elementary school students. She perceived mathematics as difficult and the most challenging school subject for her in her future role as an elementary teacher. At the end of the education, similar negative experiences and attitudes toward mathematics and its teaching were identified in Matilda's interview narrative although she had already worked as a substitute elementary teacher. Matilda expressed that the level of mathematics courses in teacher education had been too advanced for her and that during the last teacher practicum course, she had had difficulties carrying out the fraction lessons that she described in her course report. Thus, even though having experience of being an elementary teacher, no significant positive change could be identified in Matilda's mathematical identity at the time of her graduation. However, Matilda showed a positive attitude, believing that her knowledge for teaching mathematics would grow over the time after starting work as a qualified elementary teacher.

6 Discussion

This chapter first discusses the findings concerning prospective elementary teachers' fraction knowledge and mathematical identities, as well as implications of the findings for teacher education. Methodological choices made for the research project in the thesis and suggestions for further research are also discussed in this last chapter.

6.1 On prospective teachers' fraction knowledge

This thesis aimed to characterize prospective elementary teachers' (PTs') mathematical knowledge for teaching fractions as an outcome of Swedish teacher education. Grounded in four empirical studies, the thesis investigated the participating PTs' fraction knowledge in light of the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008), also focusing on their mathematical identities as an affective-motivational part of the pedagogical knowledge in the teaching of mathematics.

The findings of the thesis highlight substantial differences between individual PTs in relation to their fraction knowledge, which have not been pointed out in previous research. However, the presented differences between the participating PTs in this thesis hardly correspond to desired outcomes of Swedish teacher education. As previous research (Van Steenbrugge et al., 2014) has shown, final-year PTs may not perform better than first-year PTs in their procedural and conceptual knowledge of fractions. Therefore, it is likely that the PTs with most difficulties with fractions in the context of this thesis were not able to attain a comprehensive fraction subject matter knowledge during their teacher education, as indicated in the PT interviews. Matilda's interview revealed that she had not reached the needed level of mathematical knowledge before entering teacher education, which made mathematics studies in the program difficult to her, and in her teacher practicum report, Matilda expressed difficulties carrying out her fraction lessons. In contrast, Julia, with a more robust knowledge of fractions, expressed in her interview that PTs could have studied themselves the needed basic knowledge of mathematics.

Even though the PTs still have opportunities to learn and grow in their practice as novice elementary teachers after graduating from teacher education, the elementary students that they will be teaching should not be expected to be waiting for their teachers' mathematical knowledge to grow little by little. Some

PTs may expect their limited fraction knowledge to grow further over time, as seen in the case of Matilda. However, this will probably require support from their colleagues and opportunities for organized professional development since previous research has shown that teachers' conceptions of mathematics and their mathematical knowledge do not easily change during the first years in the profession (e.g., Kaiser et al., 2017; Thompson, 1992).

Another important finding presented in this thesis that has not been emphasized in previous research concerns the PTs' PCK. It was found that the PTs had difficulty connecting a particular mathematical topic—that is, fractions—to their knowledge of elementary students, teaching, and the curriculum document. However, it is not clear based on the analysis whether this difficulty only concerned fractions or whether the PTs had in the PCK domain general limitations understanding the different aspects that are involved in mathematics teaching. The findings also point out an incoherent fraction knowledge that existed across the knowledge categories both in the SMK and PCK domains in the MKT framework, which is indicated in previous research as well (e.g., Borko, 1992; Depaepe et al., 2015).

In general, the findings presented in this thesis indicate that the participating PTs' knowledge of fractions can be characterized as containing several deficiencies, limitations, errors, and gaps in relation to their procedural and conceptual knowledge of fractions. Similar types of PTs' procedural and conceptual difficulties have also been evidenced in previous research (e.g., Ball, 1990b; Chinnappan & Forrester, 2014; Newton, 2008; Young & Zientek, 2011). Like the substantial differences found between the PTs, the number of different error categories in their fraction solutions and the many gaps in their conceptions of fractions, as well as the large number of PTs displaying the mentioned limitations, cannot be considered as desired outcomes of the teacher education programs. Before answering the research questionnaire, the participating cohort of PTs had already completed the teacher education mathematics course where the relevant content of fraction-related procedural and conceptual aspects was included. Based on the goals for the first mathematics course of their programs, the PTs could have been expected to be able to demonstrate a more comprehensive fraction CCK in their questionnaire responses. The findings indicate similar results to those found by Van Steenbrugge et al. (2014), who concluded that PTs' knowledge of fractions mirrored elementary students' fraction knowledge.

When summarizing reasons for difficulties in the learning of fractions, Moss and Case (1999) proposed that children's difficulties relate to teaching that (a) emphasizes syntactic knowledge (i.e., procedures and rules) more than semantic knowledge (i.e., conceptual meanings), (b) discourages children in their attempts to make sense of rational numbers on their own, encouraging them to adopt a rote understanding of rules, (c) uses representations that do not differentiate rational numbers from whole numbers, especially in the case of using pie charts for introducing fractions, and (d) ignores the problems concerning rational number notations. Interestingly, all the above-mentioned characteristics were also found as typical in the participating PTs' fraction knowledge reported in this thesis. In other words, a rule-based and rote understanding of fraction procedures was highlighted in the PTs' fraction knowledge, as well as the emphasis on using circle models (pie charts) for fractions and difficulties in mathematical symbol writing with fractional numbers. The challenge remains then in teacher education ensuring that PTs are able to transfer a robust fraction-related teaching to their future professions and to effectively carry out the different mathematical tasks of teaching as defined by Ball et al. (2008).

Previous research literature has shown that understanding fractions deeply requires knowledge of different aspects of fractions and, especially, knowledge of the different interpretations of fractions (e.g., Behr et al., 1983; Kieren, 1993; Lamon, 2020). However, the overrepresentation of the part-whole interpretation of fractions was displayed among the PTs in the current research, which is in line with previous research as well (e.g., Olanoff et al., 2014). The predominance of the part-whole interpretation both in school mathematics and in the PTs' fraction knowledge base has long been recognized in the field of mathematics education (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Olanoff et al., 2014). Still, the results presented in this thesis suggest that the teacher education programs in question have not managed to overcome this predominance or improve PTs' mathematical knowledge toward a wider understanding of fractions, which probably also concerns other Swedish universities.

Chinnappan and Forrester (2014) concluded in their study that PTs' procedural-based fraction knowledge has limited value and that it may impede the development of SCK and PCK, which are both needed for quality mathematics teaching. Further, Karlsson (2015) concluded that it is not possible for PTs to discuss central pedagogical aspects in mathematics, alternative solution methods, and how to vary, individualize, and concretize mathematics teaching unless the

PTs themselves are able to analyze and solve the mathematical tasks they will later teach to elementary students. The effect of the PTs' insufficient procedural knowledge in the fraction CCK category was also identified in the current research in relation to their difficulties analyzing the fraction division solutions during the interviews (deficiencies in SCK) as well as in their report descriptions of planning and conducting fraction lessons (deficiencies in PCK). Moreover, the challenges found in connection to the PTs' fraction solution procedures make a hindrance for the quality teaching of fractions. Thus, the findings suggest that PTs' misconceptions and limitations in the CCK category also limit their SCK of fractions and, further, their PCK for teaching fractions. However, mathematics education researchers have not found a consensus whether content knowledge is a prerequisite for effective PCK (e.g., Agathangelou & Charalambous, 2021).

Procedural-based mathematical knowledge can also lead PTs to difficulties in reading and analyzing their course literature and limitations when reasoning and considering pedagogical questions and trying to understand the importance of pedagogical research in teachers' practical work (Karlsson, 2015). In the analyzed teacher practicum course reports, the PTs referred to literature that describes mathematics teaching on a general pedagogical level instead of focusing on the core content of fractions as done, for example, in their main course book *Mathematics for elementary teachers with activities* (Beckmann, 2018), which none of the PTs used as a reference in their reports. Moreover, the preference in describing mathematics teaching and learning on a general pedagogical level not only concerned the analyzed six reports of fraction lessons; a similar tendency was also found in the remaining over thirty reports that the PTs participating in the last teacher practicum course submitted for research proposes. This finding of general descriptions for mathematics teaching that builds on constructivist learning theories and everyday experiences is in line with previous research on Swedish teacher education (Asami-Johansson et al., 2020; Hemmi & Ryve, 2015). The PTs' few references to relevant research and literature on fractions may also reflect the difficulty caused by course literature whose language and format are adapted to PTs (Alvunger & Wahlström, 2018). Further, Karlsson (2015) stated that Swedish PTs may not have understood the importance of mathematics course literature as a part of their education.

Since mathematics teachers need to be able to interpret and adapt policy makers' ideas from the national curriculum and other steering documents to their teaching in practice, it is important that PTs develop a robust understanding of the curricular texts already in teacher education. In their fraction reports, the PTs' few references related to the categories of HCK and KCC concerning mathematical aspects in the curriculum may reflect the low specificity of fraction content in the mathematics syllabus in the Swedish curriculum document. In general, the curriculum, which focuses on different abilities more than specific mathematical content in the syllabus, probably makes it difficult for PTs and novice teachers to relate their mathematics teaching to the curricular descriptions and to decide, for example, the aspects of fractions that should be taught during the different elementary school grade levels. Better support from the curriculum and other teacher guidance materials might also enhance PTs' pedagogical knowledge in mathematics. Van Steenbrugge and Ryve (2018) concluded in their study that Swedish Grade 1 and 2 elementary teachers preferred practical support from the curriculum, such as lesson slides, to structure their mathematics lessons and principles for mathematical discussions. Van Steenbrugge and Ryve also argued that the quest for this kind of explicit support is connected to the lack of similar pedagogical support from the national steering documents, teacher education programs, and colleagues. As Hill and Ball (2009) stated, "[t]ools and resources typically support professionals' work in other fields, yet in teaching we have left most of the reasoning to the individual teacher, based on the view that teaching is a creative act that depends on context" (p. 71). Ibañez and Pentang (2021) also concluded in their study, "[e]nriching mathematics curriculum by incorporating some of the teaching techniques and strategies would assist students develop their conceptual understanding" (p. 42).

It could be argued with the findings of this thesis and previous research that teaching the challenging area of fractions systematically in teacher education might also provide advantages in enhancing PTs' knowledge of mathematics as a whole. As teachers' mathematical knowledge impacts their students' learning (e.g., Hill et al., 2005), the newly qualified elementary teachers' limited fraction knowledge also challenges their students' opportunities to gain quality learning of fractions, which should be taken seriously in teacher education. When aiming to enhance PTs' knowledge for teaching fractions, the focus should be on the two knowledge domains in the MKT framework, SMK and PCK, since a pedagogically powerful representation requires that mathematics teachers also have a comprehensive understanding of the topic in question (Ma, 2010). Interestingly, Ball et al. (2008) argued, "the overwhelming majority of subject matter courses for teachers, and teacher education courses in general, are viewed by teachers, policy makers, and society at large as having little bearing on the day-to-day realities of teaching and little effect on the improvement of teaching and learning" (p. 404). While this argument may not be directly connected to the outcomes of the current Swedish teacher education programs, some new directions for instruction in mathematics education should be found to enhance future elementary teachers' mathematical knowledge for teaching fractions.

6.2 Mathematical identity and teaching of fractions

When investigating the PTs' knowledge of fractions, the findings in this thesis also revealed uncertainty that was connected both to fractions and the PTs' mathematical identities. It is natural that PTs with little experience of mathematics teaching express uncertainty concerning their future role as mathematics teachers. Expressing uncertainty may also be the PTs' way to show that they still did not position themselves on the level of experienced mathematics teachers. However, the uncertainty expressed by the PTs in connection to the CCK category with fractions did not reflect a robust mathematical identity and a desired outcome of teacher education. Since CCK includes knowledge of fraction procedures and conceptions that is common in the sense that is not only unique to teaching (Ball et al., 2008), PTs' uncertainty in this knowledge domain may predict a negative mathematical identity in their future profession in elementary mathematics teaching.

The negative emotions and attitudes toward mathematics and its teaching and learning that were identified in the interviewed PTs' narratives also had an effect on their SCK when trying to analyze the different fraction division solutions. Moreover, the analysis of the PTs' interview narratives and reports of fraction lessons indicated that their unsure mathematical identities did not support their teaching of fractions and their image of themselves as mathematics teachers. As teacher beliefs about mathematics teaching and learning also affects elementary student performance in mathematics (Polly et al., 2013), the topic of mathematical identity should not be ignored and overlooked in teacher education. Moreover, PTs' anxiety in mathematics may even prevent them reaching the knowledge requirements of their programs (Karlsson, 2015). Further, Gellert (2000) argued of a self-fulfilling prophecy according to which PTs with negative attitudes toward mathematics transfer their own mathematics anxiety to their students by insinuating that they are also afraid of mathematics. Thus, some PTs use games and quizzes to motivate students and to make teaching fun even though the entertainment principle easily leads to external learning, where the mathematical content loses its importance (Gellert, 2000). Similar elements of fun and motivating materials and mathematical activities at the cost of reduced fraction content was also identified in the PTs' fraction lesson reports in Paper III.

PTs' identity work in the form of reflecting on their relation to mathematics is essential for their development as mathematics teachers (Kaasila, 2007). Reflections on mathematics are also useful for PTs who "have positive experiences of their own school years but have difficulties in putting themselves in the position of weaker pupils when teaching mathematics" (Kaasila, 2007, p. 213). This kind of identity work might support future mathematics teachers such as the interviewed PT Anna, as she displayed limitations in accepting others' mathematical thinking even though expressing a positive mathematical identity and confidence in her own mathematical knowledge and giving herself an image of "good at math." A challenge for teacher education is to guide PTs such as Anna to improve their beliefs and views of mathematics teaching and learning to be better prepared to carry out the varying mathematical tasks of teaching as described by Ball et al. (2008).

6.3 Implications for teacher education

What matters at the end of teacher education is the pedagogical knowledge that PTs possess before entering their professional careers. Liljedahl et al. (2009) described PTs' developing mathematical knowledge as follows:

> *As pre-service teachers progress through the initial teacher education experience, these different forms of knowledge are wound tighter and tighter together until the content of their experience can best be described as knowledge needed for teaching. […] In ideal circumstances this braid tightens towards the end of the initial teacher experience to form a unified fibre, the content of which is teacher knowledge.* (p. 30)

Although previous research has delved into the challenges that PTs face when working with fractions, these challenges seem to persist (e.g., Perry, 2023). However, previous research has also shown that even small instructional changes in teacher education mathematics courses may enhance PTs' fraction knowledge (Stevens et al., 2020). Perry (2023, p. 11) concluded, "[a] greater focus on concepts, using more representations, and developing a wider range of solution strategies all appear to move PSTs closer to mastery." Bobos and Sierpinska (2017) proposed a measurement approach to fractions to support PTs' development with fractions, stating, however, "whatever the approach taken, fractions will remain a difficult subject to teach and to learn" (2017, p. 235). The results presented in this thesis indicate that teacher education indeed needs systematic instruction for fractions to enhance PTs' knowledge for teaching fractions.

As a direct implication for teacher education, a new fraction-related analytical framework is provided in this thesis. The framework, with four core aspects of fractions, can be used as a tool for the assessment of PTs' knowledge of fractions, helping teacher educators to structure their instruction for fractions. The framework can also be presented to PTs as such when focusing on the challenging topic of fractions from the point of view of SMK and PCK and discussing the wide mathematical scope that fractions cover and the several connections that fractions have to other mathematical constructions. As in Liljedahl et al.'s (2009) description above, PTs should be supported to develop an image of mathematical knowledge categories that are tightly bound together, constructing a coherent network rather than isolated islands of mathematical understanding (cf. Thompson, 2013). The MKT framework could, hence, be presented in teacher education with the figure provided in this thesis (see Figure 1), highlighting an image of the connection of all the knowledge categories, which are also all needed in the development of effective mathematics teaching and carrying out the different mathematical tasks of teaching (Ball et al., 2008).

The topic of mathematical identity also deserves a robust grounding in teacher education as a part of developing PTs' pedagogical knowledge for their future roles as elementary mathematics teachers. In line with Hadley and Dorward (2011), more pedagogical discussions with PTs are also suggested to make PTs more comfortable about teaching mathematics and especially the challenging content of fractions. Moreover, a coherent grounding in relevant research literature is needed in teacher education, as also suggested by Christiansen and Erixon (2024). This also concerns a clear focus on the compulsory school curriculum and especially the topics included in the mathematics syllabus. As found by Ball et al. (2008, p. 399), "an understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction." Further, while mathematics teaching requires knowledge beyond the topics taught to elementary students (Ball et al., 2008), PTs need to learn to identify a progression in the teaching of different mathematical content, including fraction content, across all the elementary school grade levels.

Outside the constraints of this thesis, but for future work in teacher education, new ways may need to be found to make distance-based studies more attractive so that PTs choose to participate in all lectures and other mathematical activities provided by their programs. It is known in the teacher education context concerned in this thesis that many PTs choose to work, for example, as substitute elementary teachers while carrying out their distance-based teacher education studies (cf. Karlsson, 2015).

6.4 Methodological discussion

When discussing the quality of the research presented in this thesis, the criteria proposed by Lester and Lambdin (1998) can be applied to assess the designing, conducting, and reporting of the research. The suggested seven general criteria for evaluating different aspects of the research process in mathematics education are worthwhileness, coherence, competence, openness, ethics, credibility, and other qualities of good research reports (Lester & Lambdin, 1998). These criteria have also been followed throughout the research project for this thesis.

Lester and Lambdin (1998) consider worthwhileness as the most important criterion, which indicates that the study (1) generates good research questions, (2) contributes to the development of rich theories of mathematics teaching and learning, (3) is clearly situated in the existing body of research on the question under investigation, and (4) informs or improves mathematics education practice. In this thesis, it was aimed to address an important research question to investigate PTs' mathematical knowledge for teaching fractions as an outcome of Swedish teacher education. Reviewing previous research literature, a new framework for the analysis of different aspects of fractions was devised in the study presented in Paper II, which can be considered as a new theoretical contribution of this thesis. The new framework can also be regarded as a proper methodological tool assessing PTs' conceptions of fractions and a frame for the teaching of fractions and, therefore, a relevant tool for improving mathematics teaching in teacher education practice. This thesis also suggests implications for teacher education in relation to enhancing PTs' fraction knowledge and mathematical identities and making them familiar with the MKT framework.

Through the selected methods for data collection and data analysis, the aim was also to follow the criterion of coherence when answering the addressed research questions (Lester & Lambdin, 1998)—that is, the research design was formed to be appropriate for the asked questions both in the thesis frame and in the four empirical studies that the thesis is grounded in. Thus, when collecting the data, the PT respondents were given clear instructions and ample time to answer the questionnaire and the interview questions. Since the collected data were from different sources, different types of methods were needed to analyze the data. Even though the characterization of the PTs' knowledge for teaching fractions revealed many errors, gaps, and limitations, and in this sense also deficient outcomes of teacher education, the intent of the methodological choices was to take a descriptive-analytical approach without normative or critical efforts. Moreover, the different phases in the conducting of the research project were carefully carried out to fulfill the criterion of competence. To ensure the consistency of the research project, the procedures and methods used in the different phases of data collection and data analysis are precisely described in the thesis frame and in the appended studies in Papers I–IV.

According to Lester and Lambdin (1998), openness means that the researcher is aware of the personal biases and assumptions connected to the research and makes them public. In this thesis project, the researcher's connection to the research context and objectivity in relation to the methodological choices and research findings can be considered from the point of view of an insider researcher, as already discussed in Chapter 4, and also from the perspective of a cultural outsider (cf. Andrews & Larson, 2017). Having background in another Nordic teacher education context, as a cultural outsider I perceived that I was able to distance myself from the role of an insider researcher during the research project. At the same time, the interference of the background context was eliminated by reflecting on discussions with other researchers who were more familiar with the Swedish teacher education context. Openness of the research also includes that the research methods are described precisely so that other scholars can analyze them (Lester & Lambdin, 1998), which in this thesis also means that a clear sense of how the data were collected, analyzed, and used for conclusions were provided. Similar studies to those presented in Papers I–IV can, therefore, be reproduced by other researchers as well.

Openness is also connected to the credibility criterion that refers to research claims and conclusions that are justified in an acceptable and believable way (Lester & Lambdin, 1998). In the study presented in Paper I, which investigated the PTs' CCK for solving routine fraction tasks, answers that were converted to a mixed number when possible or presented in the simplest fraction form were accepted as correct answers. However, providing an answer in the form of a decimal or a percentage might also have been mathematically correct in contexts other than the one defined for the study in Paper I. Nevertheless, excluding the technical errors concerning the form when presenting the answers in the fraction solutions would not have increased substantially the number of correct answers in the PTs' solutions. Moreover, if considering all the nine fraction tasks that were included in the questionnaire and first classified as correct or incorrect before selecting the six tasks for the study in Paper I, it can be stated that the conclusions concerning the PTs' difficulties in the CCK category are representative, since most of the PT respondents also made errors with the three tasks that were not taken into deeper analysis.

It is also relevant to consider whether the presented findings of this thesis really represent the PTs' fraction knowledge and mathematical identity that the thesis was aimed at characterizing. The aim of the chosen methods for the data collection and data analysis was to provide deeper knowledge about the PTs' mathematical knowledge for teaching fractions. This was done by first distributing the paper-and-pencil questionnaire to the whole cohort of 61 PTs and then, after collecting the written reports of fraction lessons from PTs in the same cohort, the semi-structured interviews were carried out, focusing on four PTs from the cohort. Even though only four PTs were interviewed, their questionnaire answers were representative of the other PT respondents' answers concerning similar attitudes toward mathematics teaching and learning. In that sense, the conclusions made in the thesis based on the data provided by the six PTs in their fraction reports and the four PTs in their interviews can be regarded as representative of the other PTs' answers in the original cohort of 61 PTs.

Kaasila (2007) stated that PTs' narrative mathematical identity is a context-bound concept that may differ depending on to whom PTs are telling the stories of their relationship to mathematics. In the semi-structured interviews, the four PTs were interviewed by researchers that the PTs could connect to teacher education, which may have affected what and how they were telling their narratives. However, Kaasila (2007) also stated that people usually do not want to give a negative picture of themselves. The PT narratives whose outcome was not a great positive change can hence be interpreted as reflecting the interviewed PTs' mathematical identities in a reliable manner.

Considerations of ethics, such as informed consent and confidentiality concerning the participating PTs and the university context, has already been discussed in

Chapter 4 in this thesis. Ethical considerations also concern acknowledgment of the contributions of others in the research project (Lester & Lambdin, 1998). Therefore, the participating PTs who contributed with their data and scholars whose previous research has formed the basis for the current research, as well as the scientific supervisors of the research project, have been acknowledged. Lester and Lambdin (1998) also referred to the criterion of other qualities of good research reports that involves qualities such as lucidity, clarity, conciseness, and directness of the report, and well-organized research. Since these qualities are difficult to identify and evaluate (Lester & Lambdin, 1998), it remains up to the readers of the present research to decide whether this criterion is achieved in this thesis.

There are also some limitations connected to this thesis. For example, using written assessment such as questionnaire responses and course reports to collect information about PTs' mathematical knowledge has its limitations. Even though the participation was voluntarily, not all the PTs provided data that concerned the questionnaire sections that were used for the studies in Papers I and II. Moreover, it was not clear in the analysis of the six routine fraction tasks why the PTs had left some of the tasks blank—that is, whether it was that they did not know how to solve the tasks or whether they just did not want to solve them. Similarly, even though the respondents were asked to write and draw everything they knew about the concept of fractions, it is not clear whether their descriptions fully represented their conceptual knowledge of fractions. Since the questionnaire contained three research sections about fractions and a background information section, it is much more likely that some of the respondents chose not the complete the sections with all their fraction knowledge. It was therefore considered as important in the research project to also collect data that was originally not intended for research purposes, such as the teacher practicum course assignment, and data where the PTs were able to freely develop their responses and the researcher could ask further questions based on their answers, as was the case in the semi-structured interviews (Bryman, 2021). However, also investigating the PTs' CKK in their work with fraction tasks and conceptions of fractions after their participation in the second teacher education mathematics course would have increased the reliability of the presented results. In doing so, it would have been possible to examine whether and how the PTs' mathematical knowledge developed in relation to fractions in the SMK domain when completed all the teacher education mathematics courses.

As already mentioned, the course reports of fraction lessons were originally developed for the assessment of the teacher practicum course. Thus, the PTs submitting the reports were not aware of the research that focused on the three PCK categories, which were indirectly included in the instructions for the assignment. It can therefore be questioned whether the PCK identified in the six PT reports is representative. It also likely that the PTs were writing their reports in the way that was expected of them by their teacher practicum course teachers. Another limitation when characterizing the PTs' knowledge for teaching fractions is related to the use of the MKT framework categories and, more precisely, the lesser focus given to the categories of HCK and KCC compared to the other MKT categories. If the two categories HCK and KCC, which refer to mathematical aspects included in the curriculum, would have been addressed more in the paper-and-pencil questionnaire and in the PT interviews, an even more comprehensive view of the PTs' fraction knowledge could have been provided with the thesis.

Another limitation in the research project concerns the semi-structured interviews, where purposive sampling was used to select the respondents, since this kind of sampling may lead to sampling bias. It is possible that the PTs who did not volunteer to be interviewed might have provided different data than the four PTs who volunteered. For example, it cannot be taken as guaranteed that other PTs with positive attitudes toward mathematics and robust content knowledge, like Julia, would have expressed that the PTs could have studied the basic mathematical knowledge themselves and that the teacher education mathematics courses did not focus enough on the PCK domain. According to the description of the latter teacher education mathematics course, the focus of the course was on mathematics teaching—that is, PCK.

The data and results presented in this thesis only provide a portrait of a group of 61 prospective elementary teachers in one Swedish university and may not be generalized to other populations. Hopefully, other scholars and teacher educators may still take into consideration the relevance of the characteristics for mathematical knowledge for teaching fractions that were presented in this thesis and address them in their own efforts to support PTs' development as future mathematics teachers.

It should also be noted that the new analytical framework of different aspects of fractions was devised to correspond to the Swedish context. For example, the content for the category F4, Notions, was defined practically with notions, terms, and expressions for fraction-related concepts that did not directly fall into any of the other categories and that were also common in the Swedish school context. Even though discussed in Paper II, the different models for fractions such as area models, linear and number line models, and discrete or set models, were not included in the framework as such. The inclusion of the models might, however, be necessary in educational contexts other than the Swedish context. Moreover, the study in Paper II did not intend to investigate the PTs' conceptions of formal definitions for fractions, since a formal fraction definition is rarely needed in elementary mathematics teaching. However, knowledge of choosing and developing useable definitions can be regarded as a mathematical task of teaching (Ball et al., 2008) and, hence, a category focusing on definitions for the fraction concept might provide an even richer view of fraction-related conceptions when using the analytical framework.

6.5 Considerations for future research

As already mentioned above, the new framework of different aspects of fractions needs further development and testing in different contexts. The framework can be used as a tool for continued research on PTs' fraction knowledge, for example, when examining whether and how the presentation of the framework improves PTs' conceptions of fractions. Moreover, an interesting aspect for further research would be investigating how the MKT framework is perceived when implementing the visual image as presented in Figure 1 in this thesis.

An interesting question for future research would also be to develop and test specific interventions targeting the differences and individual needs that PTs bring with them to teacher education. It was not intended in this thesis project to examinate the sources for the mathematical knowledge differences between the PTs or the actual impact of teacher education on the development of the PTs' mathematical knowledge for teaching fractions and their mathematical identities, which would also be topics of wider interest.

The PTs in this thesis were not observed in connection with their actual work with fraction tasks or their teaching of fractions with elementary students in classroom settings. To get a more comprehensive view of PTs' mathematical knowledge and mathematical identities, further research is needed to take a deeper look at PTs' knowledge for teaching fractions in practice, as also suggested by Stevens et al. (2020). A follow-up study among some of the participating PTs, for example, among the interviewed PTs, might provide important information for teacher education about its outcomes concerning novice elementary teachers' preparedness for working with fractions. Moreover, as already mentioned in connection to the methodological discussion, the categories HCK and KCC of the MKT framework were addressed less in the current research. However, when expanding knowledge about the outcomes of teacher education, the mathematical knowledge that PTs attain related to the curricular aspects of fractions also deserves more attention in future research.

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Appendices

Appendix A: The paper-and-pencil questionnaire

The protocol is in the original language. The research information section and the spaces for the respondents' written answers as well as references to the university in questions 5.13 and 5.15 are removed.

Bråkbegrepp i matematikundervisning

2. Tal i bråkform

Tillbringa några minuter för att reflektera över alla olika begrepp som du tycker har samband med bråk i matematikundervisning i grundskolan. **Skriv och rita** därefter allt det som du har lärt dig om bråkbegrepp.

3. Att undervisa bråk

Tillbringa några minuter för att reflektera över hur du som matematiklärare skulle förklara följande bråkuppgift till en klass av elever i grundskolan: $\mathbf{\frac{1}{2} + \frac{3}{4}}$

Beskriv därefter din genomgång så noggrant som möjligt.

4. Att räkna med bråk

Lös följande uppgifter så bra som du kan utan att använda miniräknare. Visa alla mellanled som du använder.

a) $\frac{2}{3} + \frac{2}{3} =$

b) $\frac{4}{5} + \frac{2}{3} =$

c) $\frac{3}{4} - \frac{1}{2} =$

d) $1 - \frac{2}{6} =$

e) $\frac{3}{4} \cdot \frac{2}{5} =$

f) $\frac{3}{4} / 3 =$

g) Skriv talet som saknas i rutan så att likheten stämmer: $\frac{1}{3} + \frac{8}{\square} = 1$

h) Vad är hälften av $\frac{1}{3}$? Skriv svaret i bråkform.

i) Ringa in det största talet $\frac{3}{7}$ $\frac{5}{9}$ och resonera därefter varför det är så.

5. Bakgrundsinformation

Följande frågor fokuserar på dina personliga erfarenheter för att ta reda på vad som anses vara en väsentlig del av undervisning i matematik.

Nedan hittar du ett antal påståenden. Använd skala 1–5 och berätta hur mycket du håller med eller inte håller med respektive påstående genom att markera ett kryss för varje rad.

	1 = starkt oense	**2 = något oense**	**3 = inte oense, inte överens**	**4 = något överens**	**5 = starkt överens**
5.1. Jag har erfarenhet av att undervisa matematik i grundskolan.					
5.2. Jag vill gärna undervisa matematik i grundskolan.					
5.3. Att planera undervisning i matematik är lätt för mig.					
5.4. Jag kan använda varierande undervisningsmetoder i matematik.					
5.5. Jag kan stimulera alla elevers lärande i matematik.					
5.6. Jag har bekantat mig med kursplanens centrala innehåll för matematikundervisning i grundskolan (Lgr11).					
5.7. Jag förstår vad bråkbegreppet innebär.					
5.8. Att räkna med bråk är lätt för mig.					

5.9. Angående dina erfarenheter och kunskaper i matematik hur ser du din framtid som matematiklärare i grundskolan?

5.10. Har du läst mer än bara de obligatoriska matematikkurserna i gymnasiet?

Nej ______ Ja ______

5.11. Har du läst andra matematikkurser på universitetsnivå utöver de som tillhör ditt nuvarande program?

Nej ______ Ja ______

5.12. Hur många högskolepoäng i matematik har du hittills klarat av? Markera med kryss.

0 – 4 hp ______
5 – 10 hp ______
11 – 15 hp ______
16 – 20 hp ______
över 20 hp ______

5.13. Vilket program läser du? ____________________________________

5.14. Vilken termin har du börjat med det programmet? _______________________

5.15. Har du läst något annat program?

Nej _____

Ja _____ Vilket program? __________________________________

5.16. Hur många terminer har du studerat på universitet sammanlagt? _________________

5.17. Vilket år är du född? ____________________________________

Till sist efterfrågas ditt namn och din e-mailadress för att jag ska kunna nå dig i ett senare skede för en eventuell intervju eller en uppföljningsstudie. Precis som för de övriga frågorna är det frivilligt att uppge detta.

5.18. Namn: ____

5.19. Min e-mailadress: ________________________________

Appendix B: Description of the teacher practicum course assignment

The description of the assignment is in the original language. The instructions for submission and assessment of the course assignment are removed.

Syfte: Arbetet med denna uppgift handlar om att du ska få erfarenhet av att ta ansvaret över, genomföra, dokumentera och analysera elevers visade kunnande i en sammanhängande undervisningssekvens i ämnet matematik som ska syfta till att stimulera alla elevers lärande.

I detta arbete ska du skaffa dig erfarenhet av att:

- omsätta kursplanens innehåll i praktiken,
- planera en sammanhängande undervisningssekvens som utvecklar barns matematiska kunnande samt kunna förankra dina val i planeringen och kunna argumentera för dem,
- bilda dig en uppfattning om barns matematiska kunnande och dokumentera detta.

Övergripande beskrivning av uppgiften

- Kom överens med din VFU-handledare om vilket matematiskt innehåll och vilka förmågor din undervisningssekvens (minst fyra lektioner) ska handla om. Om möjligt koppla din undervisning till förståelse av tal som helhet och delar.
- Gör en didaktisk analys av innehållet. *(Denna ska inte lämnas in)*
    - En didaktisk analys omfattar fyra delar,
      - **A** (Arbetet startar med en innehållsanalys.)
      - **B** (Beslut om vilka mål, både vad gäller centralt innehåll och förmågor, som ska uppnås.)
      - **C** (Här tänker ansvariga igenom hur arbetet med innehållet kan se ut och fattar beslut om hur arbetet ska genomföras denna gång.)
      - **D** (Ansvariga bildar sig en uppfattning om elevernas förståelse av det bearbetade innehållet.)
- Planera din undervisningssekvens baserad på skollagens (2010:800) text där det betonas att utbildningen ska vila på vetenskaplig grund och beprövad erfarenhet.
- Gör din planering så att du kan koppla din undervisning till matematikämnets syfte och centralt innehåll i läroplan (Lgr 11) liksom åtminstone följande krav som finns i läroplan: ”*Undervisningen ska anpassas till varje elevs förutsättningar och behov. Den ska främja elevernas fortsatta lärande och kunskapsutveckling med utgångspunkt i elevernas bakgrund, tidigare erfarenheter, språk och kunskaper.*” (Lgr11, s. 6) --- ”*Skolan ska erbjuda eleverna strukturerad undervisning under lärarens ledning, såväl i hel klass som enskilt. Lärarna ska sträva efter att i undervisningen balansera och integrera kunskaper i sina olika former.*” (Lgr11, s. 11–12)
- Genomför undervisningssekvensen.

Redovisningen lämnas in som en text där du väljer ut ett undervisningstillfälle ur din sekvens som du fokuserar på. Dela in din text enligt rubrikerna nedan.

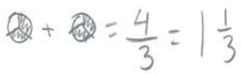
- ***Inledning*** (där du placerar in uppgiften i sitt sammanhang och ger en förklaring till varför du valde just det aktuella lektionstillfället),
- ***Bakgrund*** (där du ger en generell, teoretisk beskrivning av begreppen, arbetsformer och arbetssätt och olika representationsformer som kan användas i arbetet med matematiken),
- ***Metod*** (där du beskriver och motiverar de arbetsformer och arbetssätt du valde att använda),
- ***Resultat och diskussion*** (där du ger *en beskrivning* av och *reflekterar över* hur de val du beskrivit i ”Metoddelen” fungerade, både i förhållande till de uppsatta målen, men också i relation till alla elevers möjligheter till lärande - hjälpte dina val av arbetsformer och arbetssätt eleverna att nå de uppsatta målen? Varför/varför inte?). Där resonerar du också att på vilket sätt det syns i din undervisning att din planering har baserats på de olika styrdokumenten, vetenskaplig forskning och olika (matematiska) lärandeteorier och undervisningsmetoder.

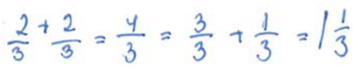
Appendix C: Description of the interview guide

The protocol for the semi-structured interviews is in the original language.

1. Du har varit med i en studie som syftar till att utveckla matematikundervisningen i lärarutbildningen.
    a. I den enkätundersökningen skrev du att *(studentens eget svar på frågan 5.9 i enkäten)* - känner du igen det?
    b. Kommer du ihåg hur du tänkte då kring din framtid som lärare i matematik? Hur tänker du idag när du är nästan klar med din utbildning? Vad är det som fått dig att se annorlunda på din framtid som matematiklärare?
    c. Har dina erfarenheter och kunskaper i matematik förändras? - På vilket sätt? Vad tror du har bidragit till förändringen?
    d. Är det något särskilt/annat i utbildningen som du tycker har bidragit till din utveckling som blivande matematiklärare?
2. Här är en uppgift i enkäten - hur skulle du beskriva den uppgiften? $\frac{2}{3}+\frac{2}{3}$

    Här är två olika lösningar på samma uppgift - på vilket sätt skiljer sig dessa lösningar? Kan du beskriva vad du ser i dessa lösningar? Är det skillnad i kvalitet mellan dessa lösningar?

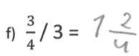
    

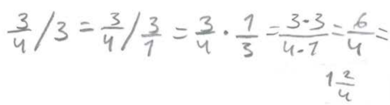
    

3. Analys och reflektion kring uppgift f:

    I. En matematisk lösning med flera fel

    Vad säger du om din lösning? Hur har du tänkt här? Skulle du förändra någonting i din lösning?

    II. Lösning med bildstöd

    Här finns en annan lösning till samma bråkuppgift. På vilket sätt skiljer sig den jämfört med din lösning? Vad tänker du om att använda bilder i lösningar?

III. Lösningsförsök utan svar

Här är en annan lösning. Vad ser du här? På vilket sätt kan du hjälpa eleven att fortsätta lösa uppgiften?

f) $\frac{3}{4} / 3 = \frac{3}{4} / \frac{12}{4} =$

IV. En matematisk lösning som går fel

Här finns också en lösning på samma uppgift men med ett annat svar. På vilket sätt kan du analysera den här lösningen?

f) $\frac{3}{4} / 3 = \frac{3}{4} / \frac{3}{3} = \frac{3 \cdot 3 = 9}{4 \cdot 3 = 12} = \frac{9}{12}$

V. En korrekt matematisk lösning

Vad har eleven gjort här?

f) $\frac{3}{4} / 3 = \frac{3}{4} / \frac{3}{1} = \frac{3 \cdot 1}{4 \cdot 3} = \frac{3}{12} = \frac{1}{4}$

VI. Lösning med decimaler

f) $\frac{3}{4} / 3 = \left(\frac{0{,}75}{3} = 0{,}25 = \frac{1}{4} \right)$

Varför har eleven skrivit på det här sättet?
Vad tänker du om att använda decimaler för att lösa en bråkuppgift?
Finns det situationer där man inte kan använda decimaler?
Om uppgiften hade varit t.ex. $\frac{2}{3}/3$ skulle du ha löst också det med decimaler?

4. Vad har du valt för ämne i ditt examensarbete? - Varför just det?

5. Är det något som du saknar i utbildningen som hade kunnat bidra till din utveckling som blivande matematiklärare?

Paper I

Student teachers' common content knowledge for solving routine fraction tasks

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This study focuses on the knowledge base that Swedish elementary student teachers demonstrate in their solutions for six routine fraction tasks. The paper investigates the student teachers' common content knowledge of fractions and discusses the implications of the findings. Fraction knowledge that student teachers bring to teacher education has been rarely investigated in the Swedish context. Thus, this study broadens the international view in the field and gives an opportunity to see some worldwide similarities as well as national challenges in student teachers' fraction knowledge. The findings in this study reveal uncertainty and wide differences between the student teachers when solving fraction tasks that they were already familiar with; two of the 59 participants solved correctly all tasks, whereas some of them gave only one or not any correct answer. Moreover, the data indicate general limitations in the participants' basic knowledge in mathematics. For example, many of them make errors in using mathematical symbol writing and different representation forms, and they do not recognize unreasonable answers and incorrect statements. Some participants also seemed to guess at an algorithm to use when they did not remember or understand the correct solution method.

Keywords: common content knowledge, elementary school, fractions, student teacher, teacher education

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1 Introduction

Teaching and learning of fractions has shown to be a challenging area in mathematics (e.g., Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Löwing, 2016; Ma, 2010; Newton, 2008). As Lamon (2007, p. 629) expresses, fractions like ratios and proportions are "the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science." Nevertheless, fractions are an essential part of school mathematics and an important part in the development of algebra and proportional reasoning. Elementary school students' knowledge of fractions and division can even predict their algebraic skills and performance in mathematics several years later (Siegler et al., 2012).

A deep understanding of rational numbers requires knowledge of different fraction interpretations such as the operator model and linear models (see e.g., Kieren, 1993; Lamon, 2007, 2020). However, student teachers seem to favor the part-whole model that has traditionally been connected to fractions and taught in elementary schools, and they struggle with other fraction interpretations (Lamon, 2020; Olanoff et al., 2014). Developing skills with fractions also requires the ability to perform fraction operations and to build up some degree of fraction sense. According to Lamon,

> This means that students should develop an intuition that helps them make appropriate connection, determine size, order, and equivalence, and judge whether answers are or are not reasonable. Such fluid and flexible thinking is just as important for teachers who need to distinguish appropriate student strategies from those based on faulty reasoning. (Lamon, 2020, p. 143)

In Sweden, the national curriculum for the compulsory school states the core content related to fractions first as parts of a whole and as parts of whole numbers, which should be compared and named as simple fractions in grades 1-3 (Skolverket, 2011). Further, in grades 4-6, the knowledge requirements include an understanding of rational numbers in fraction, decimal and percentage form. The main calculation methods for fractions are included in the curriculum for grades 7-9. Even though efforts have been made to improve learning results in mathematics, studies show that Swedish elementary school students still have deficiencies in fulfilling the above knowledge requirements (Löwing, 2016; Skolverket, 2016, 2019). Therefore, it is also important to focus on student teachers and to study their knowledge of fractions thoroughly.

Previous studies (e.g., Ma, 2010; Tirosh et al., 1998; Zhou et al., 2006) have shown the important role of teacher education in developing student teachers' fraction knowledge and the need for further research and international comparisons in this topic (Olanoff et al., 2014). The present study is a part of a more comprehensive research project that seeks to respond the research needs in this field by expanding the view to the Swedish teacher education context. The aim of this paper is to investigate Swedish elementary student teachers' common content knowledge (CCK) of fractions by analyzing errors and difficulties in their solutions for routine fraction tasks. The research question of this study is:

> How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks?

2 Previous research on student teachers' fraction knowledge

A number of studies investigating different aspects of student teachers' fraction knowledge have been published in mathematics education research. Olanoff et al. (2014) present a summary of 43 research articles focusing on student teachers' mathematical content knowledge in the area of fractions. These studies conducted, e.g., in Australia, Taiwan, Turkey and in the USA between the years 1989 and 2013, show that student teachers' fraction knowledge is relatively strong in performing fraction procedures. However, when including all basic operations of arithmetic and using basic fraction tasks that can be found in elementary school mathematics textbooks some studies also show limitations in student teachers' knowledge of fraction operations (e.g., Newton, 2008; Young & Zientek, 2011).

For example, Newton (2008) identified several error patterns when studying elementary student teachers' knowledge of routine fraction tasks in the USA. For addition, and especially when the denominators were different, the most common error was adding across numerators and denominators. In the subtraction of fractions, student teachers had difficulties changing forms, they subtracted across and left blank. In multiplication, they made whole-number errors with mixed numbers, cross-multiplied fractions instead of multiplying across, kept the common denominator in the answer, added numerators or denominators, and made errors in changing forms as well. Student teachers in Newton's study were most uncertain about dividing fractions, and even more error patterns were found for that operation: (a) finding a common denominator and keeping it in the product, (b) leaving blank, (c) reciprocals, (d) flipping the dividend instead of the divisor, (e) making mistakes with whole number facts, (f) cross-dividing or cancelling, and (g) adding or subtracting numerators or denominators. Newton (2008) concluded that the most common error in the operations with the routine fraction tasks was keeping the denominator the same even though it was not suitable.

A few years later, Young and Zientek (2011) showed that student teachers' competence vary by fraction operation; division and multiplication are the most difficult operations for student teachers. Moreover, student teachers' knowledge of fraction operations was partly rule-based and, for example, they tended to overgeneralize the rule of converting fractions to have like denominators for multiplication as well. Many of the student teachers' error patterns seemed to be based on incorrect memories of algorithms they had learned before which led them to inappropriate use of procedures; in some tasks they used correct procedures and in some other tasks with the same operation they chose the incorrect ones. Thus, Young and Zientek (2011) concluded that student teachers in their study were not able to accurately judge their abilities to correctly perform the fraction operations.

Previous research have also reported on student teachers' difficulties understanding the meanings behind fraction procedures and why the procedures work (e.g., Ma, 2010; Marchionda, 2006; Olanoff et al., 2014). Tirosh (2000) concludes that many student teachers in Israel are not capable of explaining the fraction division procedure even though they are able to use it. Similarly, the American final-year student teacher in Borko et al.'s study (1992) showed a weak understanding of both multiplication and division of fractions at the end of her teaching practice after completed a mathematics methods course; her knowledge of fraction division was based on a rote understanding of the invert-and-multiply algorithm and she lacked any knowledge of other representations such as visual representations of fractions she could use to demonstrate the division solution. Moreover, student teachers seem to lack flexibility in moving away from procedures and using fraction number sense, for example, when converting a fraction to a decimal (Muir & Livy, 2012; Olanoff et al., 2014). This may be one reason many student teachers have difficulties solving fraction story problems and creating their own fraction word problems (e.g., Ball, 1990; Tirosh, 2000; Toluk-Uçar, 2009).

Researchers in previous studies have also concluded that the relationship between student teachers' conceptual and procedural knowledge of fraction operations is weak, and that their fraction knowledge reflects the misconceptions that children have when working with fractions (e.g., Lin et al., 2013; Van Steenbrugge et al., 2014; Young & Zientek, 2011). Similar to children, many student teachers make errors based on prior knowledge of whole numbers, and when misapplying algorithms, especially the multiplication algorithm, student teachers' errors can also relate to their prior knowledge of fractions, e.g., to cross-multiplying which can be used when comparing fractions (Newton, 2008).

Student teachers are assumed to have a certain level of competence in using fractions when they are admitted to teacher education. However, Van Steenbrugge et al. (2014) concluded that one reason Flemish student teachers perform at a low level with fractions is the limited time spent on fractions in teacher education. Teacher education does not seem to have an impact on student teachers' common content knowledge of fractions, which reveals a need to develop mathematics teaching in this area (Van Steenbrugge et al., 2014).

Even though the multiple challenges related to the teaching and learning of fractions are widely recognized in many international studies as shown in the examples above, there seem to be few recent studies focusing on student teachers' fraction knowledge in the Nordic countries. One such study focuses on Icelandic student teachers' mathematical content knowledge showing that they have considerable difficulty with fractions; their knowledge is procedural and relates to "standard algorithms" learned in elementary school (Jóhannsdóttir & Gíslandóttir, 2014). A study of Norwegian student teachers (Jakobsen et al., 2014) shows that they have difficulties when solving fraction word problems; the student teachers seem to lack familiarity with mathematical notions of fractions, and they have difficulties interpreting elementary students' solutions and giving sense to fraction solutions different from their own. Furthermore, a study conducted in Finland indicates that a large number of those applying for teacher education have challenges in solving fraction algorithms (Häkkinen et al., 2011). As stated in many previous studies in the field, researchers in these Nordic studies as well highlight teacher educators' responsibility in ensuring the quality of student teachers' fraction knowledge and the need for further research in this area. The present study contributes to the field by taking the topic to Swedish teacher education and presenting an analysis of student teachers' CCK of fractions in a Swedish context. This gives an opportunity to see some worldwide similarities and national challenges in student teachers' fraction knowledge.

3 Theoretical framework

Over the last few decades, an increasing research interest has been given to subject matter knowledge as an important part of teaching (e.g., Shulman, 1986). In his original work, Shulman (1986) suggests three categories of teacher knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Subject matter knowledge includes not only the knowledge of the content of a subject area but also knowledge of substantive and syntactic structures. By these, Shulman refers to the varying ways the basic concepts, principles and facts of a discipline are organized and identifies the legitimate rules in that domain. Successful teaching requires also pedagogical content knowledge, what Shulman (1986) calls "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

In mathematics, there has been a lack of agreement about definitions, language, and basic concepts within teaching-specific mathematical knowledge (Hoover et al., 2016). Ernest states that

> the teacher's knowledge of mathematics is a complex conceptual structure which is characterized by a number of factors, including its extent and depth; its structure and unifying concepts; knowledge of procedures and strategies; links with other subjects; knowledge about mathematics as a whole and its history. (Ernest, 1989, p. 16)

Many studies concerning student teachers' knowledge base have focused on the differences between their conceptual and procedural knowledge (e.g., Lin et al., 2013; Marchionda, 2006). Conceptual knowledge is knowledge that is rich in relations (Hiebert & Lefevre, 1986). When it comes to fractions, it includes the understanding of the definition of fractions and other relevant number sets, fundamental facts about these numbers, and how the essential facts are related in the context of fraction tasks. Procedural knowledge about fractions concerns computational skills that are needed for solving fraction tasks and familiarity with the proper ways to denote fractions and their operations, for example, how to use appropriate rules and notations for the division of fractions (Hiebert & Lefevre, 1986). Maciejewski and Star (2016) conclude that flexible procedural knowledge is a key skill, which can be a way to improve students' conceptual knowledge as well. However, as Newton (2008) states, "dichotomizing mathematical knowledge into procedures and concepts does not account for its complexity" (p. 1105).

Even though teacher knowledge base has been regarded as an essential part of effective teaching, scholars have argued whether and how it contributes to students' learning. Thus, several studies have been conducted to examine the extent to which, for example, the mathematical knowledge for teaching framework (MKT) relates to learning (e.g., Charalambous et al., 2020). When analyzing the mathematical demands of teaching, Ball et al. (2008) identified the mathematical knowledge that is needed for teachers to effectively perform their work. They present the MKT framework based on Shulman's (1986) knowledge categories by using domains of subject matter knowledge and pedagogical content knowledge, and suggest that many teaching tasks included in the subject matter knowledge domain require mathematical knowledge that is not dependent on the content in the pedagogical domain.

This study focuses on the common content knowledge (CCK) category of the subject matter knowledge domain. Ball et al. (2008) define CCK "as the mathematical knowledge known in common with others who know and use mathematics" (p. 403). This knowledge and skill are used in a wide variety of settings in day-to-day work, and is thus not unique to teaching. CCK can be regarded as a basic competence in mathematics since it includes, e.g., performing calculations correctly, carrying out mathematical procedures, recognizing wrong answers, and using definitions, terms and notations correctly as well as understanding fractions (Ball et al., 2008). CCK covers mathematical tasks and questions that can be answered by anyone with a general knowledge of mathematics.

A robust CCK is a requirement for specialized content knowledge (SCK), which contains mathematical knowledge and skills that are used in teaching settings and are typically not needed for purposes other than teaching. As Ma (2010) concludes, "in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it" (p. 71). SCK includes abilities like explaining why common denominators are used when adding fractions and what is the procedure behind the invert-and-multiply algorithm in dividing fractions or determining whether a nonstandard approach would work in general to solve a given problem (Ball et al., 2008). In other words, this is knowledge of how to make mathematics understandable to students. However, in some cases it can be difficult to differ CCK from SCK. For example, detailed knowledge of different fraction representations such as symbolic and pictorial representations can be regarded as specialized knowledge, but it can also be common knowledge for others in their daily work (Ball et al., 2008).

Based on Ball et al.'s (2008) description of CCK, the present study investigates elementary student teachers' CCK by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. The concept error is chosen for this study instead of, e.g., misconception or misunderstanding, and its definition for this study is presented later in this paper. The concept difficulty is also used since it was assumed that not all findings in the analyzed fraction solutions could be categorized as obvious errors. However, this paper does not intend to explain why specific errors appear. As Radatz (1979) states, "errors in the learning of mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is such a close interaction among causes" (p. 164).

4 Methodology

4.1 Participants

The participants in this study were 59 university students in Swedish elementary school teacher programs, which are meant for to prepare teachers for the preschool class and grades 1-6 of compulsory school. Most of the participants were in the third academic year of their four-year programs, and they had already passed their first mathematics course in teacher education. One of the key aims of this mandatory course is to deepen student teachers' mathematical knowledge and strengthen their computational skills. During the first mathematics course, fraction content that is studied before entering teacher education and included in the curriculum for the compulsory school, e.g., calculating with fractions by using all operations, simplifying, reducing and extending fractions, and converting fractions to decimal, percent and mixed number forms, is recalled and repeated with all student teachers. At the time of the present study, the participating student teachers were starting their second mathematics course, which had a focus on the didactics of mathematics.

4.2 Data collection

Data for this study were collected by using a printed questionnaire. The voluntary participants were given 90 minutes to answer it before the first lecture of their mathematics didactics course at the university campus. They were asked for some background information (part 4 in the questionnaire), to write about the concept of fraction (part 1), and to describe how they might teach a fraction addition task to elementary school students (part 2). This paper focuses on six routine fraction tasks that were included in the questionnaire as well (part 3, see Appendix A). The instruction for the tasks was presented as follows: 'Calculating with fractions. Solve the following tasks as well as you can without using a calculator. Show all the steps you use.' With the instruction 'show all the steps', the participants were indirectly guided to show their fraction knowledge using mathematical algorithms, which they had been repeating in the previous mathematics course in teacher education and which can be regarded as CCK for mathematics teachers. It was also possible to use other representations such as pictures or decimal forms since the instruction was written: 'Solve the following tasks as well as you can'. Detailed knowledge of fractions and their correspondence to different representations is also knowledge that mathematics teachers need in their daily work (Ball et al., 2008).

The six fraction tasks used in this study were based on similar tasks that can be found in Swedish mathematics books and support materials for grades 4-6 mathematics. All four operations, i.e. addition, subtraction, multiplication and division, were included in the tasks with different types of fraction content: (a) addition with common denominators, (b) addition with different denominators, (c) subtraction with different denominators, (d) subtraction with a whole number, (e) multiplication with different denominators, and (f) division by a whole number. The participating student teachers were already familiar with this kind of tasks, and the tasks were defined as routine tasks since the operations were written without any context (c.f. Newton, 2008).

4.3 Data analysis

In this study, elements from Radatz's (1979) information-processing classification were used to categorize the errors in the participants' solutions. Three error types were of interest in the analyzed routine tasks: errors that are due to (1) lacking knowledge of prerequisite skills, facts, and concepts, (2) incorrect associations or inflexibility in thinking, and (3) application of irrelevant rules or strategies. Radatz (1979) states that category (1) "includes all deficits in the content- and problem-specific knowledge necessary for the successful performance of a mathematical task" (pp. 165-166), and he continues by elaborating "Deficits in basic prerequisites include ignorance of algorithms, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of necessary concepts and symbols" (p. 166). The error type (2) includes negative transfer from similar tasks even though the conditions for the tasks are different. In the last category, the errors are mainly based on successful experiences when applying comparable rules or strategies in other content areas. However, making a clear distinction between those error types mentioned above is often difficult because many of the causes interact during the learning process (Radatz, 1979).

When analyzing the participants' solutions, the answers were first coded as correct or incorrect. As a correct answer, it was assumed in this study that the answer was converted to a mixed number when possible or that it was presented in the simplest fraction form. This decision was based on the instructions and examination of the previous mathematics course, which the participants had passed in their teacher education. In Sweden, simplifying and extending fractions are considered to be prerequisite skills for the addition and subtraction of fractions (Löwing, 2016). Thus, giving the answers for fraction tasks in the simplest fraction form or as a mixed number is also encouraged in Swedish compulsory school mathematics books, and it is often expected that the answers are primary given as a fraction and not as a decimal or percent, which might be mathematically correct as well. However, providing an answer in these forms was not mentioned in the instruction since in another part of the questionnaire it was examined whether the participants were able to provide these fraction-related concepts themselves.

After the first round of coding, a qualitative analysis focusing on the solution methods was conducted. It was investigated whether there were solution methods used other than mathematical symbol representations and what kind of errors were included in the solutions. However, using other methods than mathematical algorithms was not classified as an error. Following Young and Zientek (2011), errors were defined as technical and procedural errors, where the latter consist of obvious errors in using fraction operations. This was in the cases where the participants were misusing the procedures, for example, adding across numerators and denominators in addition. This refers to Radatz's (1979) first error type. Also, if their methods seemed inefficient or misleading when used in teaching settings, and if there seemed to be a lack of number sense or negative transfer from similar tasks in the solutions, the operations were classified as including errors in this study. For example, this was done in the cases where the participants were using unnecessary long solution methods or big common denominators, or they were using common denominators when unnecessary. This classification has a connection to Radatz's error type (2) presented above.

Before deciding on the final error categories, the errors were coded several times to ensure the reliability of the coding. The rating of the errors was also discussed with an additional researcher and after that, the primary errors were coded by using symbols E1, E2, E3 etc. (see Appendix B). The errors were categorized altogether as seven error types. Three of them are related to fraction operations (procedural errors): errors in addition or in subtraction (E5), errors in multiplication (E6), and errors in division (E7). These categories include several subtypes of errors that were made by individual or multiple students.

Technical errors in this study are related to presenting the answer (E1), mathematical writing (E2), mathematical facts (E3), and leaving the task blank in the research questionnaire (E4). These errors include also solutions that can be regarded as correct in contexts other than this study. For example, E1 category consists of five subcategories that describe the solutions, which were counted to be incorrect in the context of this study even though the answers may otherwise be mathematically correct, i.e. presenting the answer as decimal or not as a simplified fraction form. E2 includes partial computation and missing solution steps as well as illogical mathematical symbol writing, and E3 consists of minor errors in calculation. Despite mathematical writing errors in the procedures, the participants' solutions have been counted as correct in the analysis if they produced a correct answer for the fraction task.

The findings of the study were analyzed in terms of the number or percentage of participants who successfully performed the fraction tasks by providing correct answers or those who made errors in their solutions. Otherwise, the main data analysis was based on a qualitative description of the student teachers' solutions for the tasks. When analyzing the solutions, the anonymous participants were given number codes according to the order their questionnaires were analyzed. These number codes are used as references in the figures presented in the next section.

5 Results

The research question of the study 'How is CCK reflected in student teachers' fraction solutions and especially in their errors and difficulties with routine fraction tasks' will be answered next. This section begins by describing the participants' fraction solutions in general; their errors and difficulties with the different tasks will then be described in more detail.

5.1 On student teachers' solutions for the routine fraction tasks

Table 1 shows the number of student teachers giving correct answers for the fraction tasks, using pictorial representations and making the most common technical errors E1, E2 and E4. As can be seen in Table 1, there is a wide difference between the student teachers when solving the routine fraction tasks. Two of the 59 participants gave correct answers to all six tasks, whereas on the other end of the spectrum, there were participants that gave only one or not any correct answer. The participants with all correct answers used mathematical symbol representations and wrote their solution steps in the algorithms in such a way that it was easy to follow the procedures they used. The participants with the least correct answers made errors with all operations, and they had difficulties in simplifying the fractions and converting them to mixed numbers. Only one of these student teachers seemed to demonstrate knowledge in using the different algorithms and writing the mathematical steps; otherwise, the participants with the least correct answers did not seem to notice the errors they made with the operations.

Table 1. A summary of the participants' solutions for the fraction tasks

Number of correct answers	Number of participants	Number of participants using pictures	Number of participants making errors in presenting the answer (E1)	Number of participants making mathematical writing errors (E2)	Number of participants leaving blank (E4)
6 (all correct)	2	0	0	0	0
5	11	2	4	8	2
4	17	3	8	13	3
3	16	2	14	10	3
2	5	0	3	3	4
1	4	0	4	3	1
0	4	0	2	1	3
Total	**59**	**7**	**35**	**38**	**16**

In general, the participating student teachers did not show a robust CCK in presenting mathematical algorithms and solutions steps. Almost a half of the participants failed to follow the instruction to show all their solution steps at least with one of the tasks. This may indicate that they had difficulties in mathematical symbol writing or that they did not notice where or how to write more details in their solutions. This was most common in the case of division where only six participants presented a logical mathematical solution by using fractions. For example, the step showing how to do the change to common denominators is missing in the next solution even though the mathematical writing is done correctly and the right answer is found: $\frac{4}{5}+\frac{2}{3}=\frac{12}{15}+\frac{10}{15}=\frac{22}{15}=1\frac{7}{15}$. Furthermore, the participants using pictorial representations did not present any steps with their solutions. However, they provided more often the correct answers for the tasks than those who used mathematical algorithms incorrectly in their solutions.

Moreover, the participants' CCK in using different representations in their fraction solutions seemed limited. Some participants used decimals but they made errors in giving correct answers; one of them used decimals for all the tasks without ending to any correct answer. Pictorial representations were used most often to solve the division task. The multiplication task $\frac{3}{4} \cdot \frac{2}{5}$ was not solved with pictures, which may indicate that the multiplication procedure is more challenging to present with pictures than the other fraction operations in the analyzed tasks. Also, it seemed that the participants used pictures for the tasks that were easier to visualize with pie charts; for example, not for the addition task with different denominators 5 and 3. Moreover, when the participants used two separate circles for subtraction, the circles (pie charts) in their solutions seemed to represent the fractions rather than the subtraction procedure (see Figure 1). In the case of addition, two circles can easier be used to illustrate the procedure as well (see Figure 2). To summarize, it seemed that the participating student teachers' CCK knowledge for using pictorial fraction representations to demonstrate solution procedures was limited.

Figure 1. A subtraction solution with pie charts (participant 16)

Figure 2. An addition procedure illustrated with pie charts (participant 22)

Many participants also made different kinds of obvious E3 errors in their solutions. These errors in mathematical facts did not seem to be directly related to fractions but were rather simple mistakes in calculation, like 12+10=24 and 3·3=6. Some participants also made multiple error types in their solutions, e.g., they used illogical mathematical writing for a wrong solution method and made calculation mistakes as well (see Figure 3).

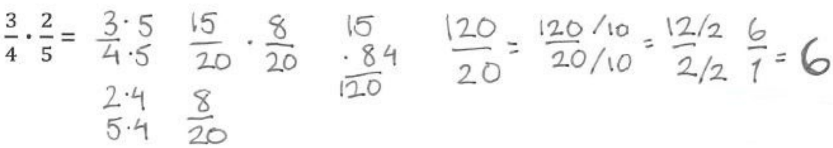

Figure 3. A solution with multiple errors (participant 41)

The most common technical error types were E1, E2 and E4 (see Table 1). Most of the participants who had difficulties in mathematical writing (E2) did not use mathematical notations correctly throughout their solutions; many of them used the equal sign incorrectly presenting their solutions often as separate calculations and ignoring whether the equal sign was written between the solution steps or not. The thinking model behind these solutions can often be understood, but mathematically, this kind of partial writing results in illogical statements (see Figures 3 and 4).

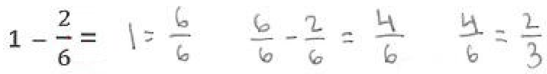

Figure 4. A solution with illogical writing (participant 27)

Several participants also used the division sign incorrectly and, in particular, they seemed to have difficulties in making a distinction between dividing and simplifying the fractions with their notations (see Figure 5).

Figure 5. Examples of errors in using the division sign (participants 47 and 48)

It seems that the participants who provided the solutions above were using division while meaning to simplify the fraction $\frac{6}{20}$, which should have led to an answer that was different from the one they provided. However, some participants were able to use the mathematical notations correctly, writing, for example: $\frac{3}{4} \cdot \frac{2}{5} = \frac{6/2}{20/2} = \frac{3}{10}$.

More than half of the participating student teachers made errors concerning the proper form for the answer (E1), and their uncertainty and illogical use of different fraction forms could be found in many solutions: in some tasks they provided the answer as a simplified fraction or a mixed number while in other similar cases, they did not. If neglecting these technical E1 errors, the total number for correct answers in the tasks would have been greater; still, it would not have led to all answers correct in any of these routine fraction tasks, and only seven participants would have correctly solved all tasks.

Several participants also left at least one of the tasks blank. This leaving blank error (E4) was made in all types of the fraction tasks except addition with common denominators, and it was most common for multiplication and division, which were both left blank by 10 students. Leaving blank may indicate uncertainty in using the procedures when the participants did not remember the correct algorithms.

5.2 Student teachers' errors and difficulties with the routine fraction tasks

The number of participants giving correct answers and making different error types E1-E7 in the analyzed six fraction tasks are summarized in Appendix B. An analysis of their errors and difficulties with the fraction tasks is presented below in the same order as the tasks existed in the questionnaire.

Addition with common denominators: $\frac{2}{3}+\frac{2}{3}$. Altogether, 42 participants (71%) gave the correct mixed number answer for this task. Five of them showed detailed steps in their solutions, writing, for example, $\frac{2}{3}+\frac{2}{3}=\frac{2+2}{3}=\frac{4}{3}=1\frac{1}{3}$. Some participants may have perceived this task so simple that there was no need to show detailed solution steps, and four participants used a pictorial representation (circles or rectangles) as a method to find the correct answer.

Most errors here were technical E1 errors. Six participants gave the answer as an improper fraction $\frac{4}{3}$ instead of converting it to a mixed number, and one participant gave the answer as a decimal, i.e. 1.33. Four student teachers seemed uncertain and wrote their mixed number answers within parentheses or as an unfinished answer in two parts $1+\frac{1}{3}$ or they gave even two alternative answers, $\frac{4}{3}$ or $1+\frac{1}{3}$. Moreover, nine participants made a procedural E5 error by adding across the numerators and denominators. After adding incorrectly, three of them also simplified the fraction $\frac{4}{6}$ to $\frac{2}{3}$ without noticing that this was not a reasonable answer when adding $\frac{2}{3}+\frac{2}{3}$.

Addition with different denominators: $\frac{4}{5}+\frac{2}{3}$. Compared with the first addition task, a smaller amount of participants, 37 of 59 (62%), performed correctly this task. Those who had difficulties in the previous task made similar E1 errors in presenting the answer here as well. One participant converted his/her improper fraction solution again to a decimal number (1.466). However, all these participants as well as those with the answer in the correct mixed number form, showed their mathematical solution steps: they found the common denominator for the given fractions and used a proper solution method. Some participants made minor computational errors (E3), and two student teachers left this task blank (E4). Interestingly, the other of them solved correctly the previous task (addition with same denominators) and the next one (subtraction with different denominators). Thus, it seemed that he/she was uncertain about the role and use of denominators in these fraction tasks.

Technical error E2 was the most common error type here since several participants used incorrect mathematical notations, and had partial computations or missing solution steps. However, there were even more procedural E5 subtype errors in the addition operation. Ten participants used a total of seven different faulty methods for the addition operation, which led to as many different incorrect answers. Three of these student teachers found the common denominator 15, but they multiplied only the denominators, adding the fractions as follows: $\frac{4}{15}+\frac{2}{15}=\frac{6}{15}$. One participant used an unnecessarily large common denominator, 30, instead of 15. Even though mathematically correct, this method seemed inefficient and it can also be interpreted as a lack of number sense. Two participants added across numerators and denominators; the other of them did this even though he/she did not add the like denominators in the first addition task. Four participants used varying multiplicative methods, for example, they cross-multiplied or multiplied across the numerators and denominators. One student teacher cross-added twice and ended up with the solution presented in Figure 6. In the solution, the participant added across the common denominators, which he/she did with the previous addition task as well.

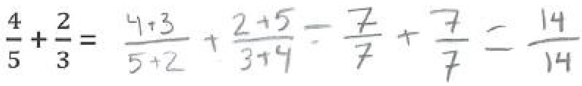

Figure 6. An incorrect solution for addition (participant 40)

One participant seemed to demonstrate uncertainty when presenting two alternative solutions. The other solution procedure and the resulting answer $1\frac{7}{15}$ were correct, but he/she had marked the following method as the correct one: $\frac{4}{5}+\frac{2}{3}=\frac{4+3}{5+3}+\frac{2+5}{3+5}=\frac{7}{8}+\frac{7}{8}=\frac{14}{8}=1\frac{6}{8}$. In general, the participants who made errors with their addition solutions did not seem to notice that their answers were unreasonable. For example, when adding $\frac{4}{5}+\frac{2}{3}$, it is not possible to get $\frac{1}{5}$ as an answer because it is smaller than $\frac{4}{5}$. The number of different incorrect solution methods in this task may indicate that when the participants did not remember or understand the operation procedure they seemed to guess an algorithm to use for the solution.

Subtraction with different denominators: $\frac{3}{4}-\frac{1}{2}$. This task was correctly performed by 47 participants (80%); four of them used pie charts to present their solutions while the others showed their solutions with some mathematical steps. One participant converted the fractions first to percent and then after calculating the answer it was converted to the correct fraction form, which was an example of using different representation forms to find the solution. Moreover, two participants used decimals, and one of them arrived at a right decimal form answer. Three participants left this task blank.

Several participants made technical E2 writing errors also with this task. Procedural E5 errors were made as well, and the most common of them was the use of unnecessarily large common denominators: seventeen participants multiplied both fractions in order to get 8 as the common denominator. This may indicate a lack of number sense related to whole numbers or a poor understanding of the subtraction operation since it was not necessary to multiply both fractions since the denominators were 4 and 2. Moreover, one participant found the common denominator 8 but kept multiplying the numerators following the same logic as he/she did in the latter addition task as well. Two student teachers who added across in addition used a similar method here as well. Thus, they subtracted across the numerators and denominators and wrote the problem out as: $\frac{3}{4}-\frac{1}{2}=\frac{2}{2}=1$. Here, again, it can be seen that the participants did not seem to notice that it was impossible to give 1 as a reasonable answer.

Subtraction with a whole number: $1-\frac{2}{6}$. Unlike the first subtraction task, only 27 participants (less than 50%) gave the correct answer for this task. However, the most common error (E1) occurred when 29 participants left their answer as $\frac{4}{6}$ without simplifying it. Thus, most of the participants were able to work through the subtraction procedure, but they did not present the answer in such a form, which was defined as correct in this study. One student teacher simplified the fraction first from $\frac{2}{6}$ to $\frac{1}{3}$, but after subtracting $1-\frac{1}{3}$, he/she gave the answer in decimal form (0.666). Two participants used colored circles, and one of them arrived at the correct answer. One participant left the task blank.

Mathematical writing errors E2 were also common with this task; ten participants used mathematical symbol writing incorrectly, and some had missing steps in their solutions. Moreover, three participants used a procedurally correct but an unnecessarily long solution method (E5): $1 - \frac{2}{6} = \frac{1}{1} - \frac{2}{6} = \frac{1 \cdot 6}{1 \cdot 6} - \frac{2 \cdot 1}{6 \cdot 1} = \frac{6}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$. After converting the whole number 1 to a fraction form, they multiplied both fractions to get 6 as a common denominator, even though there was no need to multiply the latter fraction by 1. This seemed inefficient, and the participants seemed to do this routinely without thinking about the meaning of multiplying by 1.

Multiplication with different denominators: $\mathbf{\frac{3}{4} \cdot \frac{2}{5}}$. Only 22 student teachers (37%) gave the correct answer by showing some mathematical steps in this task. Three participants used decimals, but they arrived at three different incorrect answers. Ten students left this task blank, which may indicate that they were more uncertain with multiplication than with the operations in the previous tasks.

The difficulty with the multiplication operation was seen also with the number of participants making procedural E6 errors. Eleven participants cross-multiplied the numerators and denominators, which they did in two different ways: $\frac{3}{4} \cdot \frac{2}{5} = \frac{4 \cdot 2}{3 \cdot 5} = \frac{8}{15}$ or $\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 5}{4 \cdot 2} = \frac{15}{8} = 1\frac{7}{8}$. Interestingly, one participant used a correct multiplication algorithm first but then crossed it out and used the latter of the faulty methods presented in previous the example.

Another E6 error in the multiplication operation was the use of common denominators, even though this was unnecessary. Altogether, seven participants multiplied both fractions to get 20 as the common denominator. One of them gave $\frac{120}{400}$ as an answer; the others kept 20 as the denominator after multiplying the numerators and arrived at a procedure as follows: $\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 5}{4 \cdot 5} \cdot \frac{2 \cdot 4}{5 \cdot 4} = \frac{15}{20} \cdot \frac{8}{20} = \frac{120}{20} = \frac{60}{10} = 6$. Again, the participants seemed to be uncertain about the role and use of denominators, and they did not notice that a whole number solution was an impossible answer for this task.

Interestingly, none of the participants who correctly solved the multiplication $\frac{3}{4} \cdot \frac{2}{5}$ used the option of simplifying the numbers 2 and 4 before multiplying across the numerators and denominators. This can be interpreted as a rote understanding of the algorithm or a limited number sense when seeing multiple numbers.

Division by a whole number: $\mathbf{\frac{3}{4} / 3}$. Similar to the results in multiplication, 22 participants gave the correct answer for the division task. Six of them used the mathematical invert-and-multiply procedure and showed the steps that led to the correct solution. Two participants first converted the divisor 3 to fraction form and then wrote the correct answer. However, it was not possible to find out whether they followed the correct division procedure or whether they just divided across since they wrote as follows: $\frac{3}{4} / 3 = \frac{3}{4} / \frac{3}{1} = \frac{1}{4}$. Moreover, five participants used decimals in their solutions; two of them arrived at the correct answer in fraction form and one gave a right answer as decimals. Like in the previous tasks, the participants using pictures were more successful in finding the correct answer than those who used mathematical symbol representations but made errors in them. However, it was difficult to find out the mathematical thinking model behind the correct answer in these pictorial representations as well. For example, it is unclear whether the answer in Figure 7 refers to one of the colored parts in the rectangle or to the remaining white part.

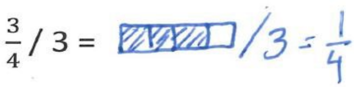

Figure 7. A pictorial solution for the division task (participant 11)

In general, solving the fraction division task by showing their solution steps seemed challenging for the student teachers. A total of eighteen participants made mathematical writing errors (E2), and similar to the multiplication task, ten participants left the division task blank; four of them did this in the case of multiplication as well. In addition to these technical errors, even six different error subtypes that were made altogether by twenty participants were found for the division operation. The most common of these procedural E7 errors occurred when the whole number divisor 3 was converted to fraction form. Some participants seemed to prefer having the same denominators for both the dividend and divisor even though it was unnecessary, and thus, eight of them converted the divisor to $\frac{12}{4}$ and one incorrectly to $\frac{4}{4}$; four participants also changed the divisor 3 to the form $\frac{3}{3}$. Interestingly, only two of those who used the form $\frac{12}{4}$ went further in their solutions but they arrived at the different incorrect answers presented in Figure 8.

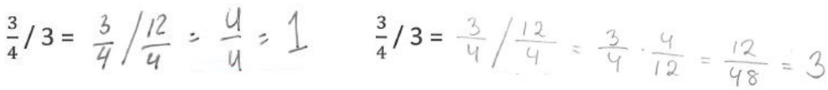

Figure 8. Incorrect solutions for division (participants 15 and 49)

As can be seen in the examples above, the participants made multiple errors in their solutions; in the example on the left, the student teacher has obviously divided the numerator 3 by 12 and kept the denominators to get $\frac{4}{4}$, whereas the other student teacher seems to use the invert-and-multiply procedure, but then incorrectly divides 48 by 12. Other procedural errors for the division operation were (a) dividing the numerator or both the numerator and denominator by the whole number divisor, (b) first multiplying the numerator and denominator by the divisor and then dividing the new fraction by it, (c) dividing across by a fraction form divisor, and (d) cross-multiplying by the inverted divisor. Similar to addition with different denominators, the number of different incorrect solution methods in the division task seems to indicate that the participants are guessing the solution methods when they do not remember or understand the correct algorithm; some participants even wrote on the research questionnaire that they did not remember how to divide fractions.

In this section, the participating student teachers' solutions for fraction tasks were described in general and in terms of their errors and difficulties with the six routine fractions tasks. The analysis revealed several limitations in their CCK on fractions and also some other limitations in their basic knowledge of mathematics; these findings were not directly connected to their knowledge of fractions. In the next section, the most important results of this study will be summarized and discussed.

6 Discussion and conclusions

In this study, student teachers' CCK on fractions was investigated by analyzing their fraction solutions and their errors and difficulties with routine fraction tasks. Many of the findings concerning their procedural errors in fraction operations are in line with findings in previous studies (e.g., Newton, 2008; Van Steenbrugge et al., 2014; Young & Zientek, 2011). In other words, the participants in this study had difficulties with all fraction operations and especially with division and multiplication. Many of them seemed to have a rule-based and rote understanding of the algorithms, and they used several incorrect methods for their solutions. Moreover, they seemed to lack knowledge of using other representations when not being able to use a correct algorithm. It was also seen in this study that student teachers have difficulties in using fraction number sense.

Different problems concerning the teaching and learning of fractions have been reported for decades, and the need to develop student teachers' knowledge of fractions has also been reported earlier (e.g., Van Steenbrugge et al., 2014). This study is consistent with the previous findings about student teachers' limited CCK of fractions. In addition, the study reveals some other limitations in their mathematical CCK.

In general, it was surprising that so many of the participating student teachers made several types of errors and that there was so wide difference between the participants when solving the fraction tasks. The participants were expected to be familiar with the routine tasks and the fraction content included in the tasks, since they had recalled and repeated this content in their previous mathematics course in teacher education. The uncertainty that many participants demonstrated in their CCK was seen in the number of tasks left blank and, for example, in their lack of using different fraction forms coherently throughout the solutions. Moreover, showing how to solve a routine task step-by-step seemed to be challenging for most of the student teachers; the more steps needed to find a solution, the more difficult it became to write out the procedures and the more errors the participants made. Like student teachers in Jakobsen et al.'s study (2014), many participants used in their solutions incorrect mathematical notations and moreover, they used separate solution steps that formed illogical statements without constructing a logical solution procedure.

The participants in this study also demonstrated limitations in their basic knowledge concerning mathematical symbol writing and the use of different representation forms. This is an important finding since these errors did not seem to be directly connected to fractions but rather they seemed to be general limitations in student teachers' CCK, which may have an effect when student teachers work with fraction as well. For example, some of the student teachers were misusing the equal sign, and they made errors in differentiating the symbols to simplify a fraction and to divide it. Making this kind of errors in their mathematics teaching might be confusing for elementary school students. Unlike Newton's study (2008), where none of the 85 participants used pictures to solve routine fraction tasks, seven participants in the present study used pictures to find the correct answers. However, it seemed that pictorial representations were used with tasks where the participants were uncertain about the correct algorithm, and many of the pictures that they presented could be seen as they mental images of the fractions and not as representations of the solution procedures needed for the tasks. As Moss et al. (1999) have stated, especially the use of pie charts may be misleading in elementary mathematics teaching. Thus, it seems that the becoming teachers need to learn how to better use pictorial representations to visualize abstract mathematical procedures. Moreover, a robust knowledge of correct mathematical algorithms is needed as well since pictorial illustrations with simple fractions such as $\frac{3}{4}$ and $\frac{1}{2}$ work well, but the use of pictures becomes complicated for fractions like $\frac{13}{41}$ and $\frac{11}{21}$. Some participants in this study used also decimals throughout the fraction tasks but they did not seem to notice the errors that occurred in their solutions when they converted improper fractions to decimals (c.f. Muir & Livy, 2012).

Moreover, many student teachers in this study did not notice their incorrect statements and unreasonable answers even in the simplest cases. However, determining equivalence and judging the reasonability of answers are essential parts of fraction number sense (Lamon, 2020) and CCK for mathematics teachers in their daily work (Ball et al., 2008). This finding like the previous one concerning mathematical symbol writing and using different representation forms may not be connected to fraction tasks only and should therefore be researched further.

Further, an interesting finding was that the participating student teachers seemed to guess at which algorithm to use when they did not remember or understand the correct solution method. Often, they seemed to remember some separate steps of the algorithms instead of understanding the procedures as a whole. Also, as Newton (2008) states, it seems that even though student teachers remember many procedures, they use them in inappropriate ways with fractions. For a mathematics teacher, a robust CCK goes beyond rote learning and memorization of algorithms since "teaching requires knowledge beyond that being taught to students" (Ball et al., 2008, p. 400).

Although student teachers do not need to hold a level of expertise equivalent to that of an experienced elementary mathematics teachers, they should not be regarded as novices in their mathematical CCK. However, student teachers may enter their studies in teacher education with different prior mathematical knowledge and with different kinds of experiences in mathematics teaching and learning. As seen in this study and in previous research (e.g. Newton, 2008), not all student teachers are competent in their basic knowledge of fractions, and the limitations found in their CCK may not predict success in teaching of fractions in their future profession as elementary mathematics teachers (Van Steenbrugge et al., 2014). Thus, teacher educators need to pay attention to student teachers' individual differences and to be aware of their different error patterns (Young & Zientek, 2011). Especially, the results in this study reveal that student teachers need a deep knowledge of fractions and mathematical symbol writing and the meaning of the procedures as well; it is not enough to be able to produce correct answers for mathematical tasks. To enhance this knowledge and student teachers' ability to interpret others' mathematical solutions as well student teachers should be given fraction tasks to be solved in different ways like Jakobsen et al. (2014) and Maciejewski and Star (2016) conclude in their studies.

The present study, conducted in the Swedish context, confirms the results from other countries during recent decades. Thus, it can be stated that there is still much to do when developing student teachers' CCK on fractions and other mathematical content as well. Since the present study concerned only a group of student teachers in one Swedish university, a limitation of the study is the inability to generalize the results beyond this population. However, some errors did occur across the participants, and this may rise questions about general difficulties in student teachers' CCK. For example, student teachers' use of mathematical symbol writing and mathematical representations for topics other than fractions could be addressed in further research. Moreover, maybe the biggest challenge in teacher education is how to address student teachers' individual differences and their various difficulties in mathematics.

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Appendices

Appendix A. The fraction tasks in the research questionnaire

Calculating with fractions

Solve the following tasks as well as you can without using a calculator. Show all the steps you use.

a) $\frac{2}{3}+\frac{2}{3}=$

b) $\frac{4}{5}+\frac{2}{3}=$

c) $\frac{3}{4}-\frac{1}{2}=$

d) $1-\frac{2}{6}=$

e) $\frac{3}{4}\cdot\frac{2}{5}=$

f) $\frac{3}{4}/\,3=$

Appendix B. Categorization of errors and the number of participants giving correct answers and making different error types in the fraction tasks

	Addition		Subtraction		Multiplication	Division
	Denominators					
	Common	Different	Different Denominator	Whole Number		
Correct answer	42	37	47	27	22	22
Correct answer with mathematical steps	40	37	43	27	22	6
Used pictures	4		4	2		5
E1 Errors in presenting the answer						
Converted the answer to decimal	1	1	1	1	1	2
Not simplified the answer		1	4	29	6	1
Not converted to a mixed number	6	3				
Converted to a mixed number incorrectly		2				
Showed uncertainty when converting to a mixed number	4	3	1	1		
E2 Errors in mathematical writing						
Partial computation or missing solution steps	3	5	12	3		14
Illogical mathematical symbol writing	1	12	5	10	8	4
E3 Errors in mathematical facts		3	1	1	2	6
E4 Left blank		2	3	1	10	10
E5 Errors in addition or in subtraction						
Added across the numerators and the denominators	9	2				
Found a common denominator without multiplying numerators		3	1			
Inverted the later fraction and then multiplied across		1				
Cross-added and then added the new inverted fraction		1				
Multiplied across		1				
Cross-multiplied		1				
Added the denominator of the other fraction with numerator and denominator in both original fractions and then added the fractions keeping the new like denominators		1				
Subtracted across numerators and denominators			2			
Used unnecessarily large common denominator		1	17			
Used unnecessarily long method				3		

Appendix B (continued)

	Addition		Subtraction		Multiplication	Division
	Denominators					
	Common	Different	Different Denominator	Whole Number		
E6 Errors in multiplication						
Cross-multiplied					11	
Used common denominators when unnecessary					7	
E7 Errors in division						
Converted the divisor to a fraction form including same denominators						8
Converted the divisor to a fraction form incorrectly						5
Divided the numerator or both the numerator and the denominator by the divisor						3
First multiplied the numerator and the denominator by the divisor and then divided the new fraction by the divisor						2
Divided across by a fraction form divisor						1
Cross-multiplied by the inverted divisor						1

Note. E1 – E4 refer to technical errors and E5 – E7 to procedural errors.

Paper II

Student Teachers' Conceptions of Fractions: A Framework for the Analysis of Different Aspects of Fractions

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Fractions are core content of elementary school mathematics, and conceptual knowledge of fractions is essential when developing a comprehensive understanding of fractions. Previous research, however, has indicated limitations in student teachers' fraction knowledge. This study investigated 57 Swedish elementary school student teachers' conceptions of fractions. The data were collected using a paper-and-pencil questionnaire and analysed with an analytical framework building on previous research on four core components of fractions. Using the devised analytical framework, we were able to characterise the conceptual content shown in the student teachers' answers and identify gaps in their fraction knowledge. The most severe gaps were identified in relation to interpretations of fractions, where only the part-whole and the quotient interpretations were identified; the measure, operator, rate, ratio, and number interpretations were missing completely. Aspects of fractions related to representations and procedures were better represented in the participants' conceptions of fractions, but we also illustrate substantial differences between the student teachers. In addition to this quantitative description, we provide qualitative examples. The results raise some questions and implications to be addressed in teacher education programs when developing student teachers' fraction knowledge.

Keywords · conceptions of fractions · conceptual knowledge · elementary school · student teachers · teacher education

Background and Aim

It is expected that student teachers acquire the mathematical knowledge necessary to effectively teach elementary school mathematics, including the core content of fractions, during their teacher education. Research, however, has suggested that fractions is among the most challenging areas in school mathematics to learn and teach (Ibañez & Pentang, 2021; Lamon, 2007). Effective instruction of fractions requires teachers to be aware of common student errors and to understand how the errors relate to fundamental mathematical concepts (Bray, 2011). Despite the importance of this knowledge, numerous studies have highlighted limitations in student teachers' fraction knowledge (e.g., Marchionda, 2006; Newton, 2008; Young-Loveridge et al., 2012). While student teachers may demonstrate proficiency in performing fraction procedures, their knowledge of fractions' procedural and conceptual aspects is often deficient (Lin et al., 2013; Lovin et al., 2018; Muir & Livy, 2012; Young & Zientek, 2011). Procedural knowledge of fractions concerns computational skills related to fraction tasks and familiarity with the proper ways to denote fractions and their operations, whereas conceptual fraction knowledge includes understanding the connections between fractions and other mathematical constructions (Hiebert & Lefevre, 1986).

Student teachers often enter their teacher education with fraction knowledge that mainly concerns procedures and is limited in its conceptual basis (Chinnappan & Forrester, 2014). Moreover, their fraction knowledge does not seem to improve much during teacher education (Lin et al., 2013; Lovin et al., 2018; Van Steenbrugge et al., 2014). Stohlmann et al. (2014) found that at the beginning of a mathematics and pedagogy content course, student teachers focus on procedural fluency, and they believe that conceptual understanding of mathematics is not more powerful or generative than remembering mathematical procedures. Also, final-year student teachers have difficulties creating connections among mathematical concepts that might be needed for the creation of meaningful conceptual representations when teaching fractions (e.g., Tirosh, 2000). Previous research has shown that those already working as mathematics teachers also have difficulties with fractions, for example, when conceptualising the division of fractions (Lamberg & Wiest, 2015). Juter (2022) found that student teachers have difficulties with their arguments for mathematical structures and representations of concepts in real number contexts, stating that this kind of incoherent mathematical content knowledge and concept image can hinder teaching mathematics meaningfully. Thus, procedural-based fraction knowledge may have limited value in mathematics teaching, and it may even impede the development of what Ball et al. (2008) call specialised content knowledge and pedagogical content knowledge needed to teach fractions effectively (Chinnappan & Forrester, 2014).

Differences in mathematics teachers' teaching performance are often due to differences in their knowledge of concepts and connections, which may lead to rule-based procedural teaching (Tchoshanov, 2011). Many student teachers' fraction knowledge is rule-based and includes incorrect memories of algorithms they have learned before attending teacher education (Bansilal & Ubah, 2020; Jóhannsdóttir & Gísladóttir, 2014). When enhancing elementary school students' mathematics learning, however, teachers are supposed to use effective teaching practices that also improve their students' conceptual thinking. This kind of effective teaching consists of practices that make connections and generate conceptual discourses (Anghileri, 2006). Developing effective teaching practices in mathematics indicates to student teachers the importance of gaining knowledge of fraction concepts and fraction connections during teacher education. Moreover, student teachers' conceptual knowledge needs to be investigated further to ensure their limitations are addressed so that they do not pass on their conceptual misunderstandings to elementary students later in their professions (Juter, 2022).

The study reported in this article responds to the research needs in the field of mathematics education focusing on elementary school student teachers' conceptions of fractions. In the context of this study, elementary school consists of Grades 1–6, that is children 6–12 years of age. While Charalambous and Pitta-Pantazi (2007) used Behr et al.'s (1983) theoretical model to investigate elementary school students' understanding of fraction interpretations, the current study contributes to the field by addressing the topic in a Swedish teacher education context with a wider range of fraction-related aspects than what was considered by Behr et al. (1983). The study aimed to investigate how student teachers demonstrate their conceptual knowledge of fractions in a paper-and-pencil questionnaire when asked to relate the concept of fractions to other concepts and mathematical constructions. To support the research, a new framework for analysing different aspects of fractions was developed. The framework is used in this article to describe the fraction aspects that the participating student teachers refer to and analyse gaps in student teachers' conceptual fraction knowledge. The devised framework and its background will be presented in the Theoretical Perspectives and Methodology sections of this article. The study was designed to answer the following research questions:

RQ1: Which aspects of fractions do student teachers refer to in their conceptions of fractions?

RQ2: What gaps related to the aspects of fractions can be identified in student teachers' conceptions of fractions?

Theoretical Perspectives

This section is organised in two parts. First, we give a basic overview of ideas related to conceptual knowledge and its application to fractions. Then, we present an overview of the research literature on fractions, which we organise into four themes that will later form the basis of our analytical framework.

Conceptual Knowledge and Fractions

Mathematics understanding is often categorised as two distinct approaches, conceptual and procedural knowledge. The former is the knowledge that is rich in relationships (Hiebert & Lefevre, 1986), for example, understanding the definition of fractions and how the fundamental facts of different number sets are related to fractions. While it is not always easy to draw a strict line between conceptual and procedural knowledge, in the context of fractions, procedural knowledge can relate to computational skills and using established rules and notations for fraction operations. When describing the relationship between conceptual and procedural knowledge, however, Hiebert and Lefevre (1986) stated, "… the problem is that the types of knowledge themselves are difficult to define. The core of each is easy to describe, but the outside edges are hard to pin down" (p. 3).

Conceptual and procedural knowledge can also be seen as closely interrelated and intertwined and build on each other (Thurtell et al., 2019). Moreover, many researchers have suggested that concept formation develops through a procedural approach to conceptual understanding (e.g., Gray & Tall, 1994). Sfard (1991) concluded that the transition from computational operations to abstract objects is a difficult process consisting of three hierarchical stages: interiorisation (a learner is acquainted with performing operations of lower-level mathematical objects), condensation (the learner becomes capable in alternating between different representations of a mathematical concept; the new concept is born), and reification (the learner conceives the mathematical concept as a fully-fledged object; the concept is reified). At the first stages of concept formation, Sfard (1991) asserted an operational conception is developed. This type of conception of an abstract mathematical notion is conceived as a product of a certain process, and it is supported by verbal representations and seen as necessary, but may not be sufficient for effective learning. From the operational conception evolves a structural conception, which characterises the mathematical notion as a static structure, an object. Visual images often support this conception and facilitates learning. In the case of fractions, the operational description is about seeing fractions as a result of a division of integers whereas the structural description considers fractions as a pair of integers (Sfard, 1991). Not all learners, however, reach the reification stage—which brings relational understanding—because reification demands much effort and the ability to see something familiar in a new way (Sfard, 1991). As Sfard (1991) stated, "… pupils can be quite successful in computations involving fractions in spite of being unable to treat fractions as numbers" (p. 32). This may also concern student teachers' conceptual knowledge of fractions. For example, Siegler and Braithwaite (2017) noted, "Although written fraction notation is usually introduced in early elementary school, connecting written fractions with the magnitudes that they represent remains challenging even for many adults" (p. 195).

Hallett et al. (2010) suggested that children's use of conceptual and procedural knowledge is based on their individual differences more than on their developmental processes, and that they seem to use a combination of conceptual and procedural knowledge when working with fraction tasks. When learning fractions, however, children who rely on their conceptual knowledge seem to have an advantage compared to those who rely solely on procedural knowledge. Pantziara and Philippou (2012) conducted a study investigating sixth grade students' conceptualisations of fractions using the three stages proposed by Sfard (1991). Pantziara and Philippou focused especially on the part-whole interpretation and the measure interpretation of fractions. They also found that students who rely only on procedural knowledge had lower performance on fraction tasks than those who apply conceptual knowledge as well. Previous research has revealed several deficiencies in student teachers' conceptual knowledge of fractions (e.g., Ibañez & Pentang, 2021). Student teachers' limited conceptual knowledge of fractions can be seen, for example, in their difficulties solving and creating fraction word problems and understanding the meanings behind fraction procedures (López-Martín et al., 2022; Marchionda, 2006; Olanoff et al., 2014; Toluk-Uçar, 2009).

Conceptual understanding in the learning of mathematics is demonstrated by making translations from one mathematical representation to another, which is challenging because representations are something that stands for something else (Duval, 2006). Previous research has shown that student teachers lack flexibility when using conceptual fraction knowledge with representations other than mathematical algorithms to demonstrate fraction operations (e.g., Lee & Lee, 2023; Olanoff et al., 2014). The role of different representations, however, is essential in the teaching of mathematics. For example, when learning complex content such as fractions, there seem to be benefits in using multiple representations and making links among them (Ainsworth, 2006; Graeber, 1999; Thurtell et al., 2019). Cramer et al. (2002) stated that using multiple representations is effective in the learning of rational number concepts and procedural skills with fractions. Moreover, as Ebbelind et al. (2012) showed, the interplay between spoken language, gestures, iconic images (pictures), and concrete materials help elementary school students to solve symbolic fraction tasks. Similarly, a representation-based teaching model seems to develop student teachers' conceptual knowledge of fractions (Thurtell et al., 2019). It has also been shown that teachers' knowledge of drawn representation models of fraction multiplication and division is strongly connected to their motivation and instructional practices for using such models (Jacobson & Izsák, 2015).

Conceptual understanding of fractions can also be regarded as mastering different interpretations of fractions (Behr et al., 1983; Cramer et al., 2002). As Vergnaud (2009) stated, the meaning of a concept originates from various situations based on systems of several concepts. Building on the work of Vergnaud (2009), Ahl and Helenius (2021a, 2021b) discussed the complication that arises when the same concept has several but related meanings and call it conceptual polysemy. Ohlsson (1988) wrote about the difficulty of fractions as follows:

> How should fractions be understood? The complicated semantics of fractions is, in part, a consequence of the *composite nature* of fractions. How is the meaning of 2 combined with the meaning of 3 to generate a meaning for 2/3? The difficulty of fractions is also, in part, a consequence of the bewildering array of *many related but only partially overlapping ideas* that surround fractions. What are the relations between fractions, measures, proportions, quotients, rates, ratios, rational numbers, and so on? (p. 53)

Further, Tall and Vinner (1981) used the term "concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). In the case of fractions, the difficulty is that the concept image of many learners is not coherent, and thus, they focus on memorising separate rules for fraction procedures rather than understanding the fraction concept and its relations to other concepts (e.g., Joutsenlahti & Perkkilä, 2021; Lortie-Forgues et al., 2015). This study focuses on the participants' conceptions that represent their individual cognitive structure of the fraction concept but not the formal definition of the concept, as discussed by Tall and Vinner (1981).

Different Aspects of Fractions

In this section, different aspects of fractions are decribed according to four themes. We start with the selection of interpretations of fractions. Then, we go over the theme of representations of fractions, which involves characterisations of verbal, visual, and symbolic representations. The third aspect is procedures related to fractions involving, among other things, operations such as addition and multiplication, as well as procedures for reducing and extending fractions. Finally, we present an aspect that pragmatically combines additional notions and concepts related to fractions.

Interpretations of fractions

Fractions have been interpreted differently by researchers based on the connections between fractions and other mathematical constructs and the situations where fractions are used. The complexity of learning and teaching fractions is highly connected to the multifaceted construct of fractions (Kieren, 1993; Lamon 2020). Some researchers also call the fraction interpretations subconstructs, and the content for interpretations and subconstructs is defined slightly differently depending on the researcher (e.g., Agathangelou & Charalambous, 2021; Charalambous & Pitta-Pantazi, 2007). This study uses the term interpretation for the interpretations and subconstructs found in previous research.

Charalambous and Pitta-Pantazi (2007) refered to Kieren (1976) as the first to conceptualise the concept of fractions as a set of interrelated constructs that includes the interpretations of part-whole, ratio, operator, quotient, and measure. Understanding these five core interpretations of fractions can be regarded as a prerequisite for solving problems in the fractional number domain (Charalambous & Pitta-Pantazi, 2007). The part-whole interpretation, however, is typically seen as the most essential and a fundamental construct when developing the rational number concept and understanding of the multiple meanings of fraction interpretations, and is related specifically to the other interpretations (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007).

Within school mathematics, fractions are often mixed up with positive rational numbers, even though a full appreciation of the formal construction of the rational number system, as it is done in formal mathematics, is beyond the grasp of school students (Thompson & Saldanha, 2003). Ball (1993) summarised fraction interpretations from several studies and stated that as rational numbers, fractions can be

> interpreted (a) in part-whole terms, where the whole unit may vary, (b) as a number on the number line, (c) as an operator (or scalar) that can shrink or stretch another quantity, (d) as a quotient of two integers, (e) as a rate, or (f) as a ratio. (p. 168)

Some researchers also use the term fraction model in connection to fraction interpretations (e.g., Behr et al., 1983; Lamon, 2020; Lee & Lee, 2023). In area models, one whole (the unit) is a given area, which is partitioned to form fractions. Fraction strips, circles, food such as cakes and pizzas, and Cuisenaire rods are examples of area models whereas candies, coins, and two-colour counting chips are examples of discrete (set) models, where individual objects or sets of objects form the unit whole (Lamon, 2020). Thus, unlike whole numbers, where one means one object, the reference unit in fractions may include more than one object or several objects packaged as one. The part-whole interpretation is considered as a situation where a continuous quantity or a set of discrete objects is partitioned, that is, divided up into equal size parts (Behr et al., 1983; Lamon 2020; Marshall 1993), and a fraction is a relative amount telling how much you have relative to the unit. For example, the fraction $\frac{2}{4}$ can be conceived as a part of a whole, that is, two out of four equal parts. When using the part-whole interpretation, it is important to understand the relationship between the numerator and denominator rather than using them as whole numbers.

The part-whole interpretation is most usually emphasised in school mathematics; it is the interpretation "that students encounter the longest and meet more frequently in their mathematics textbooks" (Charalambous & Pitta-Pantazi, 2007, p. 309). In the context of this study, the part-whole interpretation is also emphasised by the national curriculum, and thus, most probably formed already during the very first school years (Skolverket, 2022a). Moreover, students in elementary grades are supposed to learn how the parts are named and expressed as simple fractions, such as $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$, and $\frac{2}{3}$ (Skolverket, 2022a, 2022b). The part-whole relationship, however, can be a hindrance when dealing with fractions larger than the whole number 1 (Rønning, 2013). Lovin et al. (2018) showed that the focus on using the part-whole model can also be an obstacle for more advanced fraction schemes.

When using a length model, also called a linear model, fractions can be interpreted as numbers on the number line. Lamon (2020) also refered to the linear model as a measure interpretation that conveys how big the fractions are and sets them to a point on a number line. A fraction such as $\frac{3}{4}$ can be seen as a unit fraction that is used repeatedly to determine a distance from a given starting point, $\frac{3}{4}$ then corresponds to the distance of 3 ($\frac{1}{4}$ units) from that point (Charalambous & Pitta-Pantazi, 2007; Marshall, 1993). The linear number line model of fractions, however, is a difficult model for many elementary students because the measure interpretation is not emphasised as frequently as the part-whole contruct in mathematics lessons (Charalambous & Pitta-Pantazi, 2007).

Fractions as operators can be shrinking and stretching other quantities, like $\frac{1}{2}$ x 8 = 4 and $\frac{5}{2}$ x 6 = 15. This interpretation is useful, for example, when investigating fraction equivalence and working with the multiplication operation (Behr et al., 1983). Further, the quotient interpretation is fraction as division. A fraction such as $\frac{2}{4}$ can be conceived as a quotient representing the result of division (two divided by four). The ratio interpretation differs from the part-whole and quotient interpretations since it does not involve the idea of partitioning. Rather, the ratio construct refers to the comparison between two quantities (e.g., two parts to four parts), and the rate interpretation can be used as an extended ratio in situations where two quantities define a new quantity, for example, speed is a relationship between distance and time (Behr et al., 1983; Lamon, 2020).

Understanding the different fraction interpretations decribed above is essential for deep fractional understanding. Getenet and Callingham (2021) concluded that how students think about fractions is deeply influenced by how the fraction interpretations are taught by teachers. Previous research, however, has shown that student teachers have difficulties understanding the various fraction interpretations and that they mostly prefer the part-whole interpretation (Olanoff et al., 2014).

Representations of fractions

In addition to the different fraction interpretations, fractions can be represented in several ways. In school mathematics, using different verbal, visual, and symbolic representations for fractions is common. Verbal representations are expressed in spoken and written language, for example, "two thirds" and "one and a half". Verbal expressions are also dependent on the interpretation used for a fraction. For example, according to the part-whole interpretation, the fraction $\frac{1}{4}$ can be expressed as "one fourth" but when interpreted as a ratio, the same fraction can be expressed several ways like "one for four" and "the ratio of one to four". For visual representations, both drawings and pictures (e.g., pie charts, number lines) and manipulative aids (e.g., Cuisenaire rods, pattern blocks, fraction bars and fraction circles, multilink cubes, folded paper) can be used to illustrate and concretise the notion of fractions.

The different representations used to express fractions symbolically make it more difficult to understand fractions than whole numbers (Lortie-Forgues et al., 2015). In symbolic representations, fractions are written using two numerals, but a fraction stands for one number that includes three parts: a numerator (the top number in the fraction form $\frac{a}{b}$), a denominator (the bottom number), and a fraction line (the viniculum) that separates these two numbers, for example, $\frac{1}{4}$. Different fractions can represent the same fractional quantity, for example, $\frac{1}{2}$ is equal to $\frac{2}{4}$ and $\frac{5}{10}$. Fractions can also be addressed with the form a/b, that is, 1/4 = $\frac{1}{4}$, or used in the form of a mixed number as well. An improper fraction converted to a mixed number includes a whole number part and a fraction part, for example, $\frac{4}{3} = 1\frac{1}{3}$.

Even though decimals and percentages are often seen as separate topics in elementary school mathematics, they are special kinds of fractions with their own forms of notation. For example, $\frac{1}{4}$ = 0.25 = 25%. Formally, 0.25 stands for 2 tenths ($\frac{2}{10}$) and 5 hundredths ($\frac{5}{100}$) or 25 hundreths ($\frac{25}{100}$) in total, which makes 25% an alternative representation of $\frac{25}{100}$. Moreover, mathematics learning can be enhanced if learners are able to make connections among fractions, decimals and percentages, understanding that numbers in these forms may represent the same part of a whole and that some calculations can be proceeded with different solution methods when using fractions, decimals, or percentages (Anghileri, 2006). Previous research has shown that student teachers have difficulties converting fractions to decimals and seeing decimals as also having a meaning that is connected to size and quantity (Muir & Livy, 2012).

Using different representation forms, concrete materials, diagrams, symbols, and the knowledge of the connections between fractions, decimals, and percentage forms can be regarded as core requirements in the learning of mathematics (Skolverket, 2022a). Based on their findings with sixth grade students, Pantziara and Philippou (2011) suggested that the use of different representations and the alternation between the representations are substantial factors for the development of students' conceptual knowledge of fractions. Lee and Lee (2023), however, found that some student teachers' use of invalid representations with fraction addition indicated a misunderstanding of the key concepts of fractions. Moreover, previous research has also revealed student teachers' difficulty in making sense of fraction solutions different from their own, which indicates limitations in their use of different representations (e.g., Jakobsen et al., 2014).

Procedures related to fractions

Fractions can also be used with several mathematical procedures. Mastering the different fraction interpretations and representations presented above usually contributes to performance in fraction operations with addition, subtraction, multiplication, and division, and in fraction equivalence (Charalambos & Pitta-Pantazi, 2007). For example, multiplication and division in fraction tasks can be illustrated using an area model and also solved with decimals (Lamon, 2020). The division and multiplication operations for fractions, however, have shown to be the most challenging for many student teachers (e.g., Newton, 2008; Son & Lee, 2016; Tirosh, 2000).

Procedures like reducing and extending fractions can be considered as prerequisite skills for the addition and subtraction of fractions (Löwing, 2016). Moreover, performing fraction arithmetic requires that a learner can make appropriate fraction connections, determine fraction size, order, and equivalence, as well as master simplifying fractions, converting fractions to mixed numbers and mixed numbers to fractions (Lamon, 2020; Lortie-Forgues et al., 2015). Understanding fraction order and equivalence requires understanding of the relation between the size and number of equal parts in a partitioned unit (Behr et al., 1984). The ability to demonstrate the size of fractions is fundamental in developing rational number concepts, relations, and operations (Behr et al., 1983), and requires knowledge of the multiplicative relationship between numerators and denominators. Further, comparing the size of fractions can be done in different ways, for example, using different ordering strategies: (1) same-size parts, (2) same number of parts, and (3) comparison to a benchmark (Lamon, 2020). Even though the presented procedures can be seen as basic tools when working with fractions, many student teachers possess similar misconceptions of the fraction-related procedures as the children they might be teaching (e.g., Jakobsen et al., 2014; Muir & Livy, 2012; Van Steenbrugge et al., 2014). Moreover, student teachers have difficulties judging their abilities to correctly perform fraction operations (Young & Zientek, 2011).

Further notions related to fractions

The last theme in our framework is formed pragmatically since research literature dealing with fractions also includes notions, terms, and expressions for important fraction-related concepts that do not directly fall into any previous themes. In the case of fractions, this conceptual discourse includes the meaning and use of different and same numerators and denominators, knowledge of equivalent fractions, as well as understanding fraction notations and the use of representations such as fraction line, a unit fraction, a mixed number, and inverted fractions (Löwing, 2016). A common misconception among student teachers, however, is related to the use and meaning of numerators and denominators (Newton, 2008; Young & Zientek, 2011). Moreover, the notion of least common multiples is needed when processing different fraction procedures, especially when extending fractions with addition and subtraction. Fractions should also be understood as numbers in the rational number set.

Tobias (2013) examined student teachers' conceptions and development of language use for describing fractions. Her findings showed that student teachers struggle with understanding the language for defining a whole, and that they have difficulties distinguishing between the questions, *How much …?* and *How many …?* in relation to fractional numbers.

The four aspects of fractions presented above can also be considered as a prior knowledge and foundation for algebra and other areas in mathematics. In this study, the different aspects, that is, interpretations, representations, procedures, and notions, form the categories for the analytical framework, which will be described in detail in the next section.

Methodology

Context of the Study

The current study was conducted in connection to one Swedish teacher education institute. In Sweden, elementary teacher education prepares teachers for the preschool class and Grades 1–6 of the nine-year compulsory school. Elementary teachers teach several subjects to children between the ages of 6 and 12, but they are usually, not specialists in their subjects. The core school subjects studied in elementary teacher education are Swedish, English and Mathematics.

The participants of the study were 57 student teachers in the third year of their four-year academic studies. They were at the start of their second mandatory mathematics course that had a focus on mathematics teaching. In the previous year, they had passed successfully a mathematics content course where, for example, fraction content included in the national compulsory school curriculum was addressed. At the time the study, the participants were expected to have recalled and rehearsed fraction concepts, operations, algorithms, and notions studied prior to commencing university studies and also to have deepened their knowledge of fractions from the point of view of teacher education studies.

Data Collection and Analysis

Altogether, 67 student teachers were enrolled in the second teacher education mathematics course. At the time of the data collection, 61 of them were at the university campus where they were asked to answer a paper-and-pencil questionnaire before a mathematics lecture. The voluntary respondents were given 90 minutes to anonymously answer the questionnaire that was comprised of three sections concerning fractions in elementary mathematics teaching. None of the respondents needed the whole reserved time, but four of them chose not to complete the section concerned in this study, thus resulting in 57 participants for the study.

The study used data from the first section where the instructions asked the respondents to reflect on different concepts and connections that they thought might relate to fractions and then write and draw everything they knew about the concept of fractions. In the second section, the participants were asked to describe how they might teach the task $\frac{1}{2}+\frac{3}{4}$ to elementary school students. The last section in the questionnaire involved six routine fraction tasks without context to be solved by presenting all solution steps (for more information, see Tossavainen, 2022). The short background information section was comprised of statements focusing on the participants' experiences on fractions and mathematics teaching. Other questions asked about the participants' expectations concerning their future profession as mathematics teachers and their studies prior to the second teacher education mathematics course. Only the data related to fractions are reported in this paper.

The data were analysed using an analytical framework formed to operationalise the core aspects of fractions. These aspects were found in previous studies and described above in the theoretical section. The aspects are the basis of the four fraction categories in the analytical framework: (F1) Interpretations, (F2) Representations, (F3) Procedures, and (F4) Notions. Each category F1–F4 has several subcategories with different content, described below (see Figure 1).

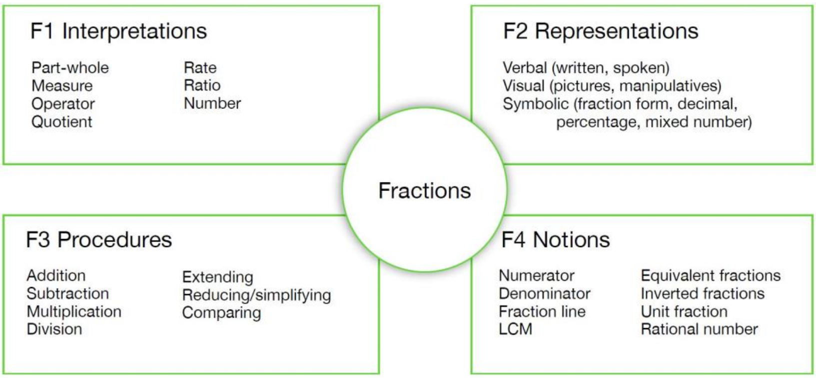


Figure 1. The categories and their subcategories in the framework for analysing different aspects of fractions.

The F1 category includes interpretations of what fractions are. The category F2 focuses on how fractions can be "seen" in different representations when using verbal, visual and symbolic forms of expressions. The F3 category is about how to work with fractions using different mathematical procedures, and the last category, F4, includes core notions that characterise fractions.

To conduct the analysis using the framework, the subcategories and their content in each category were defined more precisely. Thus, F1 includes fraction descriptions that refer to the interpretations presented by Ball (1993) and Lamon (2020): part-whole, measure, operator, quotient, rate, ratio, and number. Descriptions categorised into F2 include written notions for a fraction form, a decimal form, or a percentage, as well as symbolic forms for these representations such as $\frac{3}{4}$, 0.25, and 50%. Moreover, the forms $\frac{\mathrm{a}}{\mathrm{b}}$ and a/b provided with some procedures or with some written text, for example, $\frac{\text{numerator}}{\text{denominator}}$, were also defined as illustrating the fraction form. The written fraction form in the example, however, includes two core notions of fractions, and thus, such fraction descriptions were defined to belong to the category F4 as well. Both drawings and written descriptions of concrete things and manipulatives, such as circles, cubes, and pizzas were counted as visual representations in F2. The F3 category was defined to include mention of different fraction operations in written texts and descriptions that give examples of these operations with addition, subtraction, multiplication, or division, for example, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. The same concerns the procedures of extending, reducing, simplifying, and comparing fractions. F4 includes written texts expressing the following notions: numerator, denominator, fraction line, least common multiple (LCM), rational number, unit fraction, equivalent fractions, and inverted fractions. In this study, these notions are considered such core concepts that elementary school mathematics teachers are expected to be able to understand and use them in their teaching of fractions. Moreover, the choice of these notions is also justified by their appearance in the Swedish school context. For example, the notion of unit fraction is not used in all languages. Further, the focus of the study is not on the formal definition of the fraction concept; thus, the participants were not asked to give that kind of definition for fractions. Although the categories F1–F4 in the analytical framework can be regarded as relatively distinct, some fraction descriptions may include content for several categories as shown above in the text-based example, $\frac{\text{numerator}}{\text{denominator}}$.

After the analysis, the participants' questionnaire answers were compared to the categories in the analytical framework and a summary of their conceptions of fractions was illustrated with a figure similar to the framework (see Figure 2). When answering the first research question, the analysis focused on the categories and subcategories that the student teachers referred to in their descriptions. For the second research question about the gaps in the student teachers' conceptions of fractions, the analysis focused on the content of the different categories that did not appear in the participants' answers. Moreover, the participants' answers were also analysed qualitatively to investigate their mistakes and misconceptions. The results of the study are reported qualitatively as well as quantitatively in terms of the number of participating student teachers providing a description that belonged to a specific category or subcategory. In the results section, the participants are referred to with the number code in their paper-and-pencil questionnaire, and the excerpts of their written texts have been translated into English.

Results

In this section, we first present some general findings and also give a detailed presentation of the results related to the two research questions in connection to the categories F1–F4. We illustrate the findings with a figure similar to the figure presented in the methods section and we also include some qualitative examples of how the participants referred to the different subcategories. Then, we give an overview of the results summarising them in a more detailed figure at the end of the section.

In general, the four categories for the different aspects of fractions were all represented in the participants' conceptions of fractions (see Figure 2). The difference between the categories that the participants referred to when describing fractions was not substantial, ranging from mentions from 33 participants for F4 to 48 participants for F2. Nonetheless, the picture is different when looking at how their descriptions were spread within the categories. All the subcategories in F2 and F3 were represented in the descriptions, although not consistently. For those categories, Comparing fractions in F3 was mentioned the least, whereas Symbolic Representations in F2 was mentioned the most. Conversely, in categories F1 and F4, several core interpretations and fraction notions were not included in the participants' conceptions. For example, only two of the seven interpretations in F1, that is, part-whole and quotient, were mentioned as fraction interpretations. Moreover, while some participants gave comparatively detailed descriptions and examples of the fraction aspects also covering most subcategories, some other participants provided very limited descriptions of the content in different subcategories.

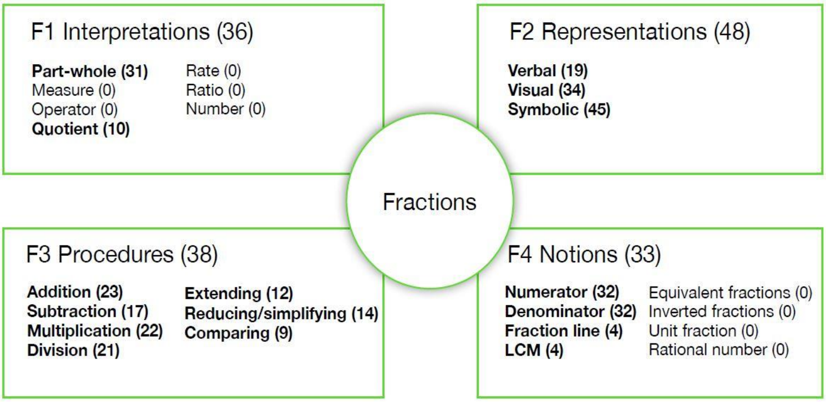


Figure 2. An illustration of the participating student teachers' conceptions of fractions. Numbers in parentheses represent the number of the participants referring to a category or a subcategory.

F1 Interpretations

The part-whole model of fractions dominated as a fraction interpretation in the participants' descriptions. More than half of the student teachers (*n* = 31) referred to this interpretation, describing, for example, that “a fraction is parts of something”, “a part of the whole” or “different parts”. The notion part-whole model was not used as such, but it was referred to by many participants using formulas like $\frac{\text{part}}{\text{the whole}}$ that also refer to the form of fractions. Ten participants also referred to the quotient interpretation, that is, a fraction as the result of division.

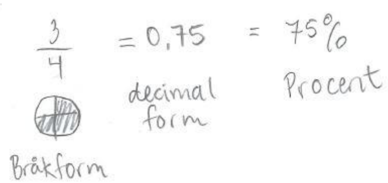
Altogether, 21 participants did not provide any connections to this F1 category of what fractions are. Moreover, the fundamental gap identified in F1 was that none of the participants provided an interpretation of fractions as a measure, operator, rate, ratio, or a separate number. Some identified gaps also concerned the participants' use of the part-whole model and the unit used with fractions. In the participants' descriptions, fractions were mainly interpreted as a quantity that was divided into some number of equal-sized parts of which some number of parts were then taken, for example, $\frac{2}{4}$ of a rectangle. The participants expressed that this quantity is “the whole”, and it was related to the whole number 1. Since fractions were associated with the whole number 1, only this number or one figure was used as the unit in their examples and visual representations when referring to the part-whole interpretation.

F2 Representations

Most participants (48 of 57) connected different representations and forms to fractions. Mathematical symbolic representations, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{10}$, were most common in their descriptions. Some participants also used written forms like $\frac{\text{numerator}}{\text{denominator}}$ to present the fraction form without giving any examples of an actual fraction. Written verbal representations, such as a half, a third, a fourth, and one-fifth were also used, and a couple of participants also provided examples of how to pronounce a fraction in different ways. Fractions were mainly considered as unit fractions. Non-unit fractions that have a number other than 1 in the numerator, such as $\frac{3}{4}$ and $\frac{2}{5}$ were only provided with some pictorial representations and when illustrating fraction procedures. Moreover, examples of more complicated fractions like a fraction with a two-digit number in the numerator and the denominator were not presented, which can be interpreted as a gap in this category.

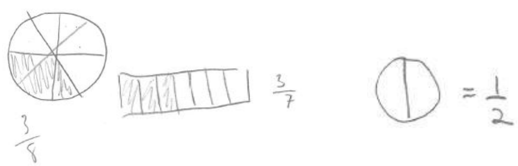
Another limitation identified in F2 was the participants' preference for constructing all the fractions using the form $\frac{a}{b}$; the form a/b was not represented in their descriptions. In Swedish school mathematics, both forms denote fractions and division. The whole number 1 was presented as different fractions, such as $\frac{1}{1}$ and $\frac{2}{2}$. Other whole numbers, however, were not converted to fractions, showing, for example, $4 = \frac{4}{1} = \frac{8}{2}$. One participant described the connection between fraction forms and whole numbers stating incorrectly: "$\frac{1}{1}$ or $\frac{3}{3}$ is the whole, that is, 1 or 3." These findings indicate limitations and gaps in the participants' conceptions concerning the connection between fractions and natural numbers.

Fractions were also connected to percentage and decimal forms, which were presented with written and mathematical symbolic representations (see Figure 3). Only six participants, however, connected fractions to both a percentage form and a decimal form. The percentage form was described as a support when calculating fraction tasks, and one participant described the relation between fractions and decimals stating that fractions "can represent something more precise than what decimal numbers can do, for example, $\frac{1}{3} \approx 0.33$." Some participants also mentioned the mixed number form, but there were few descriptions of the connection between fractions and mixed numbers. Mixed numbers were mainly provided with examples of adding and multiplying fractions, and not any pictorial examples of mixed numbers were given.



Participant 52

Figure 3. A fraction presented with different representations and forms ("Bråkform" is the Swedish expression for fraction form).



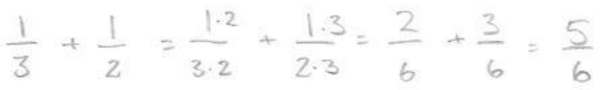
Participant 21 Participant 44

Figure 4. Pictorial representations with non-equal size and non-coloured parts.

Altogether, 34 participants provided drawings as visual representations for fractions, and seven of them also mentioned manipulatives and some concrete materials, such as fraction bars and circles, cubes, pizzas, and cake slices. The drawings that illustrated fractions were mainly separate coloured pie diagrams and rectangles. The participants did not provide sets of drawn objects like a group of four apples. In their written descriptions, the participants highlighted same-sized parts for fractions, but many of their drawings did not include equal-sized parts (see Figure 4). Moreover, in some participants' illustrations, the drawings and the fractions connected to them did not seem to be equivalent as seen in Figure 4. The discrepencies identified in the participants' pictorial representations might cause confusion when teaching elementary school students.

F3 Procedures

In the F3 category, all four operations, that is, addition, subtraction, multiplication, and division, were connected to fractions. Addition was most often mentioned in the participants' descriptions, and it was typically used in the examples of fraction calculations (see Figure 5).



Participant 50

Figure 5. An example of extending fractions in addition.

Some participants referred to the extending and reducing procedures mentioning the need to have the same denominators for fraction addition and subtraction, which was also described as the meaning of the extending process. Few participants, however, provided the procedures for extending or reducing fractions, like in the example in Figure 5. Some participants also referred to reducing in the sense of getting the simplest form for a fraction, writing, for example, $\frac{2}{4} = \frac{1}{2}$. The relation between different fractions was often described as follows: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$. Comparing fractions, however, was usually connected to the extending and reducing procedures that do not change the value of the fractions. Thus, the participants did not provide any procedures or descriptions of how to compare the sizes of different fractions.

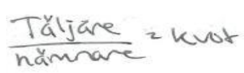
Concerning the F3 category in general, the participants mentioned different procedures for fractions rather than presented how to actually work through the procedures with fractions. Some participants even described their uncertainty using different fraction procedures, especially multiplication and division were challenging for many student teachers as can be seen in the mistakes in Figure 6. Moreover, 19 participants did not refer to any fraction procedures defined for this category.

Participant 14

Figure 6. Gaps in multiplication and division with fractions.

F4 Notions

The F4 category, how to characterise fractions with different notions, was the least represented in the participating student teachers' conceptions of fractions. Altogether, 33 of the 57 participants referred to F4. Their descriptions did not cover the notions of a rational number, unit fraction, or equivalent and inverted fractions, which were included in the framework for the category. Thirty-two participants connected the core notions numerator ("täljare") and denominator ("nämnare") to fractions, and some of them also used the Swedish term "kvot" that refers to the result of a division (see Figure 7). Moreover, when interpreting fractions as quotients of two integers, two participants also used the notions dividend and divisor.



Participant 40

Figure 7. An example of notions connected to the fraction form.

Fraction line was used in the participants' descriptions when illustrating the fraction form (see e.g., Figure 7), but the notion itself was only provided by four participants. Similarly, four participants connected the notion of least common multiple (LCM) to fractions when adding fractions with different denominators. In general, few correct mathematical notions were provided in connection to fractions. For example, few participants used the notions of a mixed number and a simplified fraction even though these fraction forms were represented in their examples of different procedures. Some participants also used informal expressions that refer to some mental images, which are given to fraction notions and often used in Swedish school mathematics, such as the Swedish words "tak" (roof) and "nedre våning" (lower floor) that illustrate the places for the numerator and denominator in the fraction construction $\frac{a}{b}$.

Summary of the Results

As a summary, Figure 8 presents a graphical illustration of how individual participant fraction descriptions were spread both between and within the four categories F1–F4. The participants' references to each subcategory are presented in Figure 8 with different shades of a particular colour per category, and the gaps in their conceptions of fractions are illustrated as white squares. Thus, it can be seen in the white columns that the categories focusing on interpretations (F1) and notions (F4) both contain a lot of gaps whereas the categories of representations (F2) and procedures (F3) are better represented in the participants' conceptions. Note that Respondents 32, 36, 42, and 46 did not answer the questionnaire section concerned in this study and are hence not represented in Figure 8.

Moreover, when looking at Figure 8 by its 57 rows (where each row represents a participating student teacher), more observations can be made concerning individual participants' conceptions of fractions. For example, only 14 participants provided descriptions representing content relating to all the fraction aspects, whereas 21 of the 57 participants referred to three categories. Seven participants only referred to one category, like Participant 20 (procedures) and Participant 52 (representations). Further, the horizontal gaps also illustrate differences between the participants in their conceptions of fractions within the subcategories. For example, in F3, only Participant 34 and Particpant 61 referred to all the seven subcategories (i.e., addition, subtraction, multiplication, division, extending, reducing, and comparing fractions), while 11 participants only desribed one fraction procedure in the same category. Figure 8 also reveals that a student teacher may demonstrate a comprehensive conception related to one fraction category while the same participant's descriptions in other categories may include severe gaps (cf., Participant 61 in F3 and other categories).

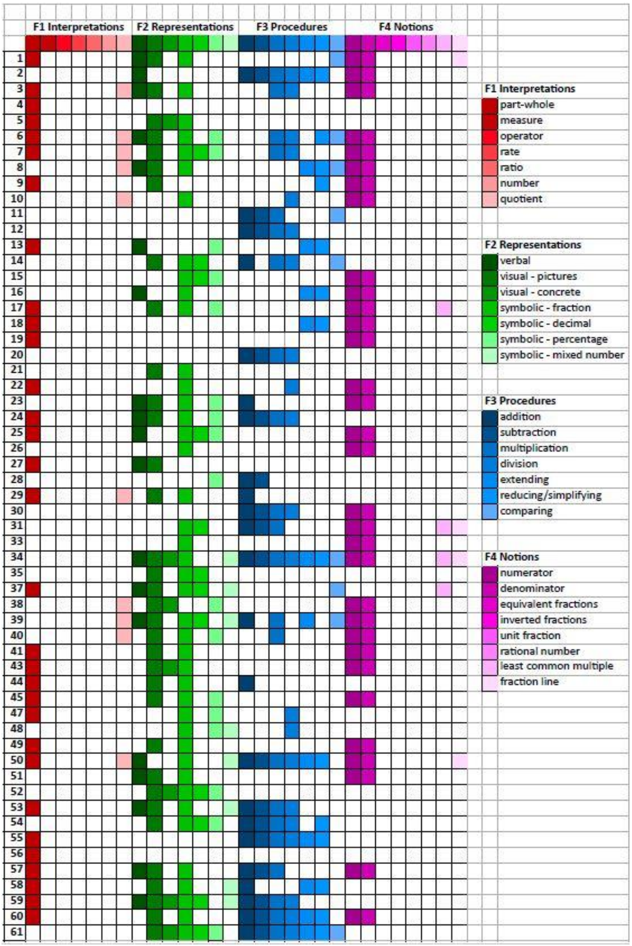


Figure 8. The ID codes for the 57 participants and each participant's fraction descriptions categorised according to the subcategories in F1–F4.

Discussion and Implications

This study investigated elementary school student teachers' conceptions of fractions by analysing their written fraction descriptions. The overall aim was to investigate how student teachers demonstrate their conceptual knowledge of fractions when asked to relate the concept of fractions to other concepts and mathematical constructions. While previous studies (e.g., Charalambous & Pitta-Pantazi, 2007) have mainly focused on fraction interpretations and operations, this study broadens the view of the topic by considering four aspects of fractions with a new analytical framework. Using the devised framework in an interpretive manner, we focused on the student teachers' conceptual knowledge by giving them ample time to describe concepts, terms, and notions associated with fractions. We were also able to identify gaps in their conceptions of fractions. Next, we will discuss the main findings related to the two reseach questions. Then, we will present some implications of the results.

Student Teachers' References to Different Aspects of Fractions

In their descriptions, the participants referred to all four aspects of fractions. They mainly provided conceptions that demonstrated different representations and procedures for fractions, and their descriptions involved all the subcategories included in the analytical framework for these categories. Instead, a deeper conceptual knowledge of what fractions are (interpretations) and what characterises fractions (notions) was not identified to a similar extent in their conceptions. It is hence reasonable to assume that the participating student teachers' conceptions of fractions consist of more procedural than conceptual aspects of fractions, which is also in line with previous research (e.g., Chinnappan & Forrester, 2014).

The participants showed awareness of mathematical symbolic representations and different visual representations of fractions, that is, pictorial and concrete models to illustrate fractions. It was also common for them to mention the core notions of a numerator and denominator and to refer to the part-whole interpretation of fractions. The findings also reveal substantial differences between the student teachers in their conceptions of fractions. Even though attending the same teacher education, some participants' descriptions indicated more comprehensive conceptions than other student teachers' few references and short answers, which is an interesting finding that pinpoints some challenges teacher education may have when developing student teachers with different backgrounds and prior knowledge in mathematics.

Gaps in Student Teachers' Conceptions of Fractions

In general, the participating student teachers' conceptions of fractions included several gaps. Their limited references within each of the F1–F4 subcategories constitute the main gaps in knowledge and understanding. Even though all the subcategories in F2 and F3 were referred to by some participants, no subcategory within F1–F4 was mentioned by all participants. The gaps were identified by the absence of mentions of several core interpretations and notions.

The overrepresentation of the part-whole interpretation in F1, and associated underrepresentation of other interpretations, can be seen as a gap in the participants' conceptions of fractions. Previous research (e.g., Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Olanoff et al., 2014) has also discussed the dominance of the part-whole model, which can hinder the learning of fractions (Lovin et al., 2018; Rønning, 2013). For example, Thomson and Saldanha (2003) stated that having access to only a part-whole conceptualisation of fractions makes multiplicative reasoning with fractions intuitively impossible. Another gap related to interpretations concerns the difficulty in identifying the unit for a fraction, which has also been shown in previous research (e.g., Lamon, 2020). In this study, student teachers only used the whole number 1 or one object as the unit. Thus, when they only drew part-whole figures, the conceptual idea of improper fractions, such as $\frac{7}{4}$, is hard to realise since the fraction seems to consist of more parts than what is included in the unit. Taking some number other than 1 as the unit and dividing the set of objects (e.g., eight circles) into same size groups was also missing from the participants' descriptions. Moreover, the participants associated the meaning of the denominator with the number of parts that the whole unit was divided into. This does not make sense in the case of simplified fractions where the original unit (e.g., a set of objects) is reduced. As Lamon (2020) stated, students should quickly learn "that the unit may be something different in every new context and that the first question they [children] should always ask themselves is, 'What is the unit?'" (pp. 102–103).

Some participants also expressed their uncertainty concerning different procedures for fractions. They had difficulties presenting how to work through fraction procedures, especially the multiplication and division operation. This result aligns with previous research (e.g., Son & Lee, 2016; Tirosh, 2000). Moreover, when making mistakes with the procedures, the student teachers did not notice the unreasonable answers they provided with their examples. These findings also align with previous research findings (e.g., Newton, 2008; Tossavainen, 2022; Young & Zientek, 2011). Further, the participants provided pictorial illustrations that were not divided into equal-sized parts, which might cause misconceptions later in their teaching of fractions. Similarly, their limited use of proper notions with fractions might be a hindrance to effective teaching and learning of fractions. As Kaiser et al. (2017) have shown, teachers' mathematical knowledge does not change much during the first years in the profession. Thus, it is a challenge to teacher education to help student teachers to overcome the gaps and limitations in their fraction knowledge and to gain the knowledge needed.

Implications

The researchers involved in this study did not intend to evaluate the learning processes that have resulted in the identified conceptions and gaps or to compare the participating student teachers. However, the substantial differences found between the individual participants raise some concerns regarding the way teacher education addresses student teachers' individual differences. An interesting direction for future research would therefore be to develop and test specific interventions targeting these differences and individual needs that student teachers bring with them to their teacher education studies. Stolmann et al. (2014) provided an insight into one possibility, showing that by focusing on mathematical topics that student teachers understand mainly procedurally, for example, fraction division, is a way to change their conceptual understanding. Moreover, when enhancing student teachers' conceptual knowledge of fractions, the importance of multiple representations in the teaching of the challenging contents of fractions needs to be realised (Stolmann et al., 2014). For example, teacher education should provide student teachers with opportunities to solve fraction tasks in different contexts with multiple visual representations, explore connections between fraction representations and make transitions from one representation to another, analyse good and poor-quality representations, as well as to interpret and respond to elementary students' thinking in different contexts to overcome misconceptions concerning fractions (Son & Lee, 2016; Stevens et al., 2020; Thurtell et al., 2019). We also think that this should be done constructing a coherent understanding so that the meaning of all contexts used and representations together, build a coherent network rather than isolated islands of understanding (Thompson, 2013).

Further, an interesting question is how teacher education can ensure that student teachers are able to later transfer the enhanced conceptual knowledge of fractions to their teaching practices. Various scholars (e.g., Lamberg & Wiest, 2015; Van Steenbrugge et al., 2014) pointed out the need for sufficient time when working with fractions, which should also be addressed in teacher education when improving student teachers' conceptual knowledge. In line with the arguments by Juter (2022) concerning student teachers' knowledge of real numbers, highlighted in the current study is the need for student teachers to become aware of their mathematical knowledge and limitations in the fractional number domain.

Conclusion and Limitations

Sfard (1991) stated that conceptual knowledge develops through procedural knowledge. Taking the findings presented here into account, it can also be concluded that if student teachers do not have a comprehensive procedural fraction knowledge, they also have difficulties in the conceptual knowledge domain (Chinnappan & Forrester, 2014). This study also confirms the conclusion of Charalambous and Pitta-Pantazi (2007), namely that there is a need of emphasising other fraction interpretations than the part-whole model, and that the use of core notions and proper language with fractions should also be highlighted in teacher education (Stevens et al., 2020). In general, teacher education should focus more on the conceptual knowledge of fractions than on different algorithms to execute operations on fractions (Charalambous & Pitta-Pantazi, 2007; Copur-Gencturk, 2021). As Chinnappan and Forrester (2014) stated, student teachers' procedural-based fraction knowledge can be an obstacle to the development of specialised content knowledge and pedagogical content knowledge that are needed in the teaching of fractions. If our results are representative, there may be a need to revise the ways fractions are introduced in teacher education.

A limitation of the current study is that we only investigated one cohort of student teachers attending one Swedish university. Moreover, we do not know whether their answers in the paper-and-pencil questionnaire really represented their conceptions of fractions. The way our results coincide with previous research, however, might indicate that the reported findings of student teachers' conceptions of fractions can also be considered in other teacher education contexts. Thus, as a consequence of our results, we would encourage teacher educators to examine carefully their student teachers' fraction knowledge before assuming they have the knowledge needed to teach fractions in their future professions as mathematics teachers. The F1–F4 framework developed for this study may be a useful tool for the assessment of that fraction knowedge.

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Ethics Declarations

Ethical approval

The research reported in this paper complied with the *Guidance On Ethical Review of Research on Humans* published by the Swedish Ethical Review Authority. Informed consent was given by all participants for their data to be published.

Competing interests

The authors declare there are no competing interests.

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Paper III

Paper III is accepted to be published in *Scandinavian Journal of Educational Research* and the appended manuscript *Pedagogical content knowledge in prospective elementary teachers' descriptions of teaching and learning of fractions* is in press.

Pedagogical content knowledge in prospective elementary teachers' descriptions of teaching and learning of fractions

This study investigates the pedagogical content knowledge identified in six prospective elementary teachers' written teacher practicum reports. Focusing on the three categories of pedagogical content knowledge included in the Mathematical Knowledge for Teaching framework, the study analyzes the prospective teachers' descriptions and combines their knowledge of fraction content with knowing about students, teaching, and curriculum. The results indicate that prospective teachers describe teaching and learning of fractions mainly on a general pedagogical level. Moreover, they do not differentiate characteristics for elementary students' fraction difficulties and misunderstandings as well as fraction core contents in the curriculum. The findings are discussed in relation to the Swedish teacher education context in which the study was conducted. Implications for teacher education are also discussed.

Keywords: fractions, mathematical knowledge for teaching, pedagogical content knowledge, prospective elementary teacher, teacher education

Introduction

Knowledge of fractions is a core content of elementary school mathematics, which serves proficiency with ratios, proportions, and percentages, and provides a foundation for algebra and more advanced mathematics (Ball et al., 2005; Booth & Newton, 2012; Siegler et al., 2012). However, teaching and learning fractions is difficult (e.g., Copur-Genturk, 2021; Diputra et al., 2022; Getenet & Callingham, 2021; Ma, 2010; Van Steenbrugge et al., 2014). In Sweden, the goal-oriented national curriculum document for elementary school provides teachers with mathematical core contents, knowledge requirements, and general aims for mathematics teaching. However, it does not provide support for instructional practices and comprehensive knowledge of the mathematical contents to be taught. The core content related to fractions is stated as parts of a whole and as parts of a number (Skolverket, 2018). The sparse descriptions in the curriculum document require mathematics teachers to have a well-developed content knowledge of different aspects of fractions, such as interpretations, representations, and core notions, as well as a robust pedagogical knowledge for teaching fractions.

Teacher education is a crucial time for prospective elementary teachers (PTs) to gain the needed knowledge for teaching fractions. However, this is a challenge for teacher education since PTs enter their programs with different experiences, conceptions, and attitudes, and many PTs have deficient prior knowledge of fractions (e.g., Chinnappan & Forrester, 2014). In Swedish teacher education, PTs study 30 credits of mathematics, but the implementation of different mathematical core contents, including fractions, varies between the universities. It is hence essential to expand knowledge about the outcome of teacher education related to PTs' gained pedagogical content knowledge of fractions, which is the aim of the current study.

This study is conducted in a Swedish teacher education context, where PTs' pedagogical content knowledge of fractions has not been on the focus of previous studies. Analyzing six written course assignment reports of fraction lessons conducted by the PTs during their last teacher practicum course, the study addresses the following research question:

RQ: How are different aspects of pedagogical content knowledge related to the teaching and learning of fractions displayed among prospective elementary teachers at the end of their teacher education?

The aspects mentioned in the research question are theoretically grounded, and they will be introduced in the theory section and methodologically operationalized in the methods section.

Previous research on prospective teachers' fraction knowledge

Previous research has shown that many PTs enter teacher education with fraction knowledge that is mainly procedural (e.g., Chinnappan & Forrester, 2014; Stohlmann et al., 2014). However, both their procedural and conceptual knowledge of fractions have limitations that do not seem to be remedied during their time in teacher education (Lin et al., 2013; Lovin et al., 2018; Muir & Livy, 2012; Van Steenbrugge et al., 2014; Young & Zientek, 2011).

Much of the research conducted in the field has focused on PTs' procedural content knowledge when solving fraction tasks. For example, Newton (2008) found several error patterns related to fraction operations (i.e., addition, subtraction, multiplication, and division) when investigating elementary PTs' knowledge of fractions in routine tasks. Many difficulties identified in previous studies concern especially division and multiplication operations and the use of denominators (e.g., Tossavainen, 2022). Moreover, PTs' knowledge of fraction operations is often rule-based and includes incorrect memories of algorithms that they have learned before (Bansilal & Ubah, 2020; Häkkinen et al., 2011; Jóhannsdóttir & Gísladóttir, 2014). Many PTs also have challenges in judging their abilities to perform fraction operations and recognizing their unreasonable answers and incorrect statements connected to fraction tasks (Tossavainen, 2022; Young & Zientek, 2011).

Previous studies have also reported a lack of flexibility among PTs when moving away from procedures and using conceptual fraction knowledge with representations other than mathematical algorithms to demonstrate fraction operations (Borko et al., 1992; Lee & Lee, 2023; Olanoff et al., 2014). When working with different types of fraction tasks, many PTs have difficulties solving and creating fraction word problems (e.g., Ball, 1990; Jakobsen et al., 2014; López-Martín et al., 2022; Toluk-Uçar, 2009). PTs' limited conceptual fraction knowledge is also shown in their difficulties to understand the meanings behind fraction procedures and their reasoning about why the procedures work (e.g., Ma, 2010; Marchionda, 2006; Olanoff et al., 2014; Tirosh, 2000). Understanding different fraction interpretations (see e.g., Kieren, 1993) is also challenging for many PTs. They prefer the part-whole interpretation (Olanoff et al., 2014; Tossavainen & Helenius, 2024), which may be because the part-whole model of fractions "transcends the elementary mathematics curriculum: it is the subconstruct that students encounter the longest and meet more frequently in their mathematics textbooks" (Charalambous & Pitta-Pantazi, 2007, p. 309).

Previous research also indicates that PTs' difficulties in content knowledge of fractions prevent them from having the needed pedagogical content knowledge for teaching fractions (López-Martín et al., 2022). Tirosh (2000) showed over 20 years ago that PTs have difficulties identifying sources of elementary students' incorrect responses in the fractional number domain. Further, PTs in Depaepe et al.'s study (2015) performed better with content knowledge tasks than with the related pedagogical content knowledge tasks, "which suggests that they hold limited knowledge about students' misconceptions and difficulties as well as about instructional strategies and representations, even after having taken a course on teaching rational numbers" (Depaepe et al., 2015, p. 88). PTs' fraction knowledge also reflects elementary students' misconceptions about fractions (Van Steenbrugge et al., 2014). Like elementary students, PTs' prior knowledge of whole numbers often leads to whole number bias (Ni & Zhou, 2005), for example, when dealing with numerators and denominators.

As the literature review examples above reveal, PTs' knowledge of fractions has been investigated over the years from several perspectives but much of the research is done prior to 2020. Moreover, the challenge to overcome PTs' difficulties remains in teacher education. Recently, few studies have focused on PTs' pedagogical content knowledge related to fractions, considering it from the point of view of different aspects, as done in this study. The current study considers the topic as an outcome of teacher education in the Swedish context using the three categories of pedagogical content knowledge developed by Ball et al. (2008) that will be presented in the next section.

Theoretical framework

When conceptualizing teachers' professional knowledge base, Shulman (1987) proposed seven categories, where pedagogical content knowledge (PCK) identifies what is characteristic knowledge for teaching. For Shulman (1987), PCK represents an understanding of how particular content is pedagogically organized, represented, and presented for instruction to learners. PCK entails formulating the subject comprehensibly to others since successful teaching demands an understanding of what makes learning easy or difficult for different learners (Shulman, 1986).

Another key category for teachers' knowledge base is content knowledge (CK), which refers to the amount and organization of teachers' knowledge concerning a specific subject (Shulman, 1986) and that is considered a pre-requisite for teachers' PCK (Agathangelou & Charalambous, 2021). As a basic mathematical knowledge, CK of fractions concerns, for example, performing fraction calculations correctly, recognizing wrong answers, carrying out different procedures, such as reducing, extending, and comparing fractions, as well as using correct terms (e.g., numerator, denominator and unit fraction) and notations with fractions (Ball et al., 2008). Proceeding fraction operations and being able to teach fractions effectively also require making appropriate connections, determining fraction size, order, and equivalence, as well as judging whether answers are reasonable or not (Lamon, 2020).

In mathematics education, Ball et al. (2008) developed the Mathematical Knowledge for Teaching (MKT) framework (see Figure 1) based on tasks involved in mathematics teaching. They divided Shulman's (1986) CK into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK), and his PCK into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). Even though presented as separate categories in the MKT framework, it may be difficult to discriminate, for example, SCK from KCS, and similarly, KCC may overlap with several of the other categories (Ball et al., 2008).

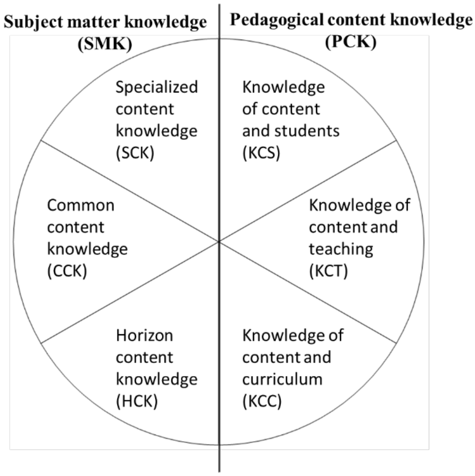

Figure 1. Knowledge categories in the MKT framework (A modified figure compared to Ball et al. 2008, p. 403).

This study focuses on the PCK domain of the MKT framework, where KCS connects specific mathematical knowledge and familiarity with students' mathematical thinking (Ball et al., 2008). KCS includes, for example, knowledge about common student mistakes and the cognitive mathematical processes students use as well as knowledge about the effects of these processes. KCT requires teachers to connect their mathematical knowledge to the pedagogical issues that affect students' learning (Ball et al., 2008), for example, how mathematical and everyday language are connected, how mathematical representations are used, and how numerical examples are selected (Hill et al., 2005). Concerning the KCC category, Ball et al. (2008, p. 399) stated that "an understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction." They referred to Shulman (1986) who considered curricular knowledge as knowledge of different programs and instructional materials meant for teaching particular subjects at given levels. KCC then requires teachers to connect their specific mathematical knowledge to their curricular knowledge. In this study, fractions are considered as the specific mathematical content that PTs should be able to connect within their PCK to their knowledge of students, teaching and curriculum.

Methods

Context and Data Collection

This study is conducted in the context of one Swedish university. There are two separate four-year academic teacher education programs, where one prepares teachers for the preschool class and elementary school grades 1–3 while the other focuses on teaching in grades 4–6. Altogether, the programs concern teaching children between the ages of 6 and 12. This study considers PTs attending the two programs as one cohort since the content of their teacher education mathematics courses is the same to a large extent. For the participating cohort, fractions were handled in their two 15-credit mathematics courses. The first course mainly focused on mathematical concepts, symbols, representation forms, rules, procedures, and algorithms while the other course focused on different theoretical and pedagogical aspects of mathematics teaching.

The study investigates PTs' PCK related to the teaching and learning of fractions using written course assignment reports of fraction lessons. The reports were produced in conjunction with a ten-week teacher practicum course that the PTs completed in the last year of their programs before writing their final bachelor theses and graduating. After submitting the reports, all of the 49 PTs from the PT cohort that participated in the teacher practicum course were asked whether they permit that their reports are used for research purposes in the current study. The PTs were also informed that whether they agreed to it or not did not affect the assessment of the report and the practicum course since the researcher was not connected to the teacher education course. Altogether, 42 PTs agreed, and all the reports concerning fraction lessons, that is six reports, were selected among their reports.

The six selected reports were written by PTs who had also participated in another fraction-related study. Since both studies belong to a larger research project, the PTs behind the six reports are referred to with their number codes in the project, that is, 28, 39, 47, 49, 58, and 60. During the teacher practicum course, they were teaching mathematics in different elementary school grade levels: Grade 1 (PT_{39}), Grade 2 (PT_{49} and PT_{58}), Grade 3 (PT_{47} and PT_{60}), and Grade 6 (PT_{28}).

The instructions for the teacher practicum course assignment (see Appendix A) first asked the PTs to plan and conduct mathematics lessons that stimulated elementary students' learning and were based on the elementary school curriculum. Then, the PTs were instructed to write a report describing and discussing their choices concerning different concepts, representation forms, working methods, and teaching materials for their lessons, as well as to reflect upon the effects of the choices on their students' learning. The PTs were also asked to describe the connection between their lessons, educational policy documents, scientific research, and different theories and methods related to mathematics teaching and learning.

Data Analysis

The selected reports were analyzed using theory-guided content analysis (e.g., Schreier, 2012). First, a coding frame was built defining the contents for the categories KCS, KCT, and KCC of the PCK domain in the MKT framework. This operationalization of the concepts from Ball et al. (2008) is presented in Table 1.

Table 1. The main categories and their contents for analyzing the PTs' reports.

Knowledge of content and students, KCS
PTs demonstrate knowledge that combines knowing about students and knowing about mathematics, for example, - showing familiarity with common student errors, conceptions, and misconceptions about mathematical content and deciding which of several errors students are most likely to make - anticipating and predicting • what students are likely to think and what they will find confusing, • what students are likely to do with a mathematical task and whether they will find it easy or hard, • what students will find interesting and motivating.
Knowledge of content and teaching, KCT
PTs demonstrate knowledge that combines knowing about teaching and knowing about mathematics, for example, - sequencing the mathematical content for instruction, - choosing suitable examples to start with and examples that take students deeper into the content, - evaluating advantages and disadvantages of different representations and identifying what different methods and procedures afford instructionally.
Knowledge of content and curriculum, KCC
PTs demonstrate knowledge that combines knowing about curriculum and knowing about mathematics, for example, - showing familiarity with the topics that will be taught in mathematics during the school years at a given level - referring to the instructional materials that embody the mathematical topics in curriculum.

Further, since the study focuses on PCK related to fractions, each main category in the coding frame above was complemented with two subcategories that differentiated whether the PTs' knowledge concerned fractions or other mathematical contents and teaching in general (see Table 2). Similar color codes that illustrate the PCK categories and the connected subcategories in Table 2 also visualize the inclusion of the main categories in the instructions for the teacher practicum course assignment (see Appendix A). The color codes were also used when analyzing the reports. Finally, the identified PCK was organized within each main category under common themes for presenting the findings.

Table 2. The focus of the subcategories in the operationalized coding frame.

	KCS – focus on students and fractions
	KCS – focus on students and mathematics in general
	KCT – focus on teaching fractions
	KCT – focus on teaching mathematics in general
	KCC – focus on fraction contents in the curriculum
	KCC – focus on curricular contents in general

Results

The results will be presented in three sections connected to the main PCK categories, following the order of the categories in Table 1. In the PTs' reports, however, most emphasis was on the KCT category while KCC was the least represented category. The subheadings in each section concern the identified common themes in their descriptions within each main PCK category. The quotation examples are translated into English.

Knowledge of content and students

Students' general mathematical knowledge. In their reports, the PTs described elementary students' mathematical knowledge mainly without relating it to fractions. They expressed that students regard mathematics as an abstract, complex and difficult subject: "'I do not understand mathematics' is a quite common student response, and often students think that if they are weak in some domain, they are weak in all aspects of mathematics" (PT_{28}). PT_{58} referred to Grade 2 students' mathematical knowledge stating: "Through social interaction, play and discussion, children form mathematical concepts and develop understanding." Only PT_{60} mentioned mathematically talented students whose difficulty was describing and reasoning their solutions to others. In general, the PTs perceived that elementary students lack knowledge about problem-solving and different strategies for mathematical tasks.

Moreover, all the PTs emphasized the importance of spoken language and social interaction with peers for elementary students' mathematics learning. They described that students enhance their mathematical knowledge when helping each other in groups and giving peer feedback and that concrete learning materials and mathematical games motivate students.

Students' fraction knowledge. Concerning fractions, all the PTs except PT_{58} expressed that elementary students have difficulties: "Fractions as a part of a number were … too abstract for most students" (PT_{60}). PT_{28} and PT_{60} who were teaching in Grades 6 and 3 respectively went a little deeper into these difficulties, referring to specific student misunderstandings, such as adding numerators and denominators in fraction addition, and difficulties when comparing fractions with different denominators. PT_{28} also stated that students have difficulty understanding methods for fraction operations and that they "often have not made connections between fractions and, for example, percentages, which could help them to understand [fractions]."

All the PTs connected elementary students' fraction knowledge to the use of concrete circle models, such as pizzas and pies, which they considered enabled students to understand fractions. PT_{39} and PT_{49} provided examples of simple fractions that their students in Grades 1 and 2 were able to work with (i.e., 1/2, 1/3, 1/4, 1/5, and 1/6), and PT_{39} explained students' understanding as follows: "They quickly figured out how to write unit fractions and could argue why these [unit fractions] should be written in that way." PT_{60} also stated that after her first introduction to fractions, students "could now work with the tasks individually." Despite knowing that fractions are difficult for many students, the PTs reported that their students quickly understood different fraction contents.

Knowledge of content and teaching

Teaching of fractions. All the PTs except PT_{28} (Grade 6) highlighted the use of different representation forms, such as written and spoken verbal representations, pictorial representations and mathematical symbol representations. They tried to get their students to use different representations, demonstrating, for example, how to discuss fractions and illustrate fractions with drawings. PT_{49} (Grade 2) described the importance of making connections between different representations:

> Reading out (e.g., two quarters) and writing fractions (using symbols, e.g., 2/4) and being able to see connections between concrete models and pictorial models and being able to express this verbally … is also important in the work of developing students' understanding of fractions.

Different pictorial representations were described as typical illustrations for fractions. Still, in their fraction lessons, the PTs mainly used circle models as PT_{39} in Grade 1: "For all introductions, I used the representation form circles." Only PT_{28} reported using other pictorial representations than circles and reflected upon some advantages and disadvantages related to drawings: "Which one is greater of six-sevenths and seven-eighths becomes quite clear if one has learned to draw it" but if relying too much on the help of pictorial representations "it may not be something that they [elementary students] get very far with" (PT_{28}).

PT_{39} reported a particular focus on fraction content when teaching Grade 1 students: "I supported the students by directing their attention to the numbers in the fraction forms concerning the meaning of the numbers, what the numbers three and one stood for [in the fraction form]." Few fraction core notions were mentioned in the PTs' reports, and only PT_{58} (Grade 2) described concepts that elementary students were supposed to use, that is numerator, denominator, and quotient. Referring to quotients, PT_{58} emphasized that "[a] fraction can be calculated as a division, and then you can proceed to decimal numbers." Reducing and extending fractions were also mentioned as procedures that elementary students should learn. Still, in their reported fraction lessons in different grade levels, the PTs mainly focused on comparing fractions within the part-whole interpretation of fractions.

The PTs also emphasized the importance of connecting the teaching of fractions to students' everyday life using practical examples and concrete materials, such as apples, fraction rods, cubes and magnets: "Concrete models are also easy to relate to reality, for example, comparing a circle to a cake or a pizza" (PT_{49}). PT_{60} stated: "The students [in Grade 3] could have found the teaching more enjoyable if the calculation of fractions had been visualized, for example, with physical [real] cakes."

In their teaching of fractions, the PTs described a focus on mathematically weaker students. They described giving these students more possibilities for practical work with concrete materials, specially formed exercises, and extra support from the teacher or an assistant and peers. On the other hand, students who needed more challenges were reported to be able to work quite independently with fraction tasks. Whole-class discussions mainly supported them, as PT_{28} described: "At the end of the introduction, I went through how to extend or reduce fractions, which they [Grade 6 students] had not had time to work with, so even the students who understood fractions before learned something new." Moreover, the PTs described that they had been teaching fractions like a new topic even though most of their students already had some fraction knowledge. PT_{49} (Grade 2) explained that "in this way, students who had difficulties understanding fractions before got a new opportunity to understand 'from the beginning' while the other students got a repetition."

Organizing the work for teaching mathematics. The PTs strongly emphasized the sociocultural learning approach, "where relationships and interactions are strengthened, and students have an opportunity to learn from each other and the teacher through interaction" (PT_{58}). The PTs described using group work that allowed all students to discuss and contribute to the shared development of mathematical knowledge: "By allowing students to describe and explain their mathematical thinking to each other, their feeling of participation is strengthened" (PT_{60}). The PTs also considered that three students per group was optimal so that students "would have a greater opportunity to help each other" (PT_{28}).

Moreover, all the PTs organized their teaching based on a model called EPA (*Enskilt-Par-Alla*) in the Swedish context: first, a teacher-led introduction or repetition (*genomgång*) with the whole class followed by a short moment consisting of individual (E) work. For the students, the main work with mathematical activities was done in pairs (P) or small groups, and the lessons were closed with a whole-class discussion (A) led by the teacher. Within this teaching model, "the teacher needs to give space for mathematical discussions where the students have an active role" ($PT_{39,}$ Grade 1), and teachers "circulate among the groups and help where needed, guiding them [the students] a little to the right direction or explaining if there was something they did not understand" (PT_{28}, Grade 6).

The PTs also highlighted the importance of providing students with different types of tasks (e.g., routine tasks, problem-solving tasks, tasks with text, practical and applied tasks) and varying mathematical activities to motivate and engage students in mathematics learning. PT_{47} described this as follows in Grade 3: "I wanted to create … lessons that were varied, challenging, and at the same time full of passion and fun for the students."

Challenges in teaching. All the PTs also expressed that teaching fractions to elementary students was challenging for them: "I was challenged … didactically, and my subject knowledge in mathematics was tested" (PT_{58}). PT_{47} stated that before her Grade 3 lessons, she had "practiced calculating with numbers in fraction form so that I would feel confident in the tasks." PT_{49} also described the difficulty: "Fractions can be difficult to understand and teach, largely due to the existence of several different interpretations and representations." Moreover, PT_{58} and PT_{60} incorrectly related fractions to natural numbers, and the latter presented "a rich fraction problem" to be solved in Grade 3 with unit fractions. Still, the described task did not concern fractions.

A general challenge in their teaching mentioned by the PTs was elementary students' varying mathematical knowledge and different learning skills. As the PT_{60} described: "I had assumed that the students would master the knowledge and could work on their own, and I was surprised when some [Grade 3 students] needed to listen to the task to understand." PT_{58} expressed awareness of the differences between the Grade 2 learners: "When I planned my activities and instructions, I had to consider the needs that existed." Thus, PT_{58} had tried to keep a clear structure in her teaching, "excitement to motivate the students", and her choice of the working method was "based on the idea that the students should gain a broader understanding of the use of fractions in a fun and interesting way" (PT_{58}). On the other hand, PT_{28} did not refer to any of her Grade 6 students' special needs that she might have considered when teaching them, even though she described that there were "weaker students" in the class and that she could not keep all students motivated: "I could notice that the concentration of the students who understood how to calculate with fractions dropped a little and they thought it was a quite boring repetition" (PT_{28}). Concerning some Grade 2 students' difficulties with the Swedish language, PT_{49} stated: "I should have been there [with the group] to a much greater extent to support both this individual student and the group as a whole." Preferring group work also caused challenges when instructing and trying to observe students' learning: "It takes a lot of training for the students to benefit from collaboration, and … very clear guidance on how discussion and dialogue should take place, which I, unfortunately, have to admit that I did not manage to give them" (PT_{49}).

Knowledge of content and curriculum

Fractions in the curriculum. Table 3 shows all the direct descriptions of fractions written in the official English translation of the Swedish national curriculum document for elementary school grades 1–6. When justifying the rationale behind their fraction lessons, all the PTs referred to at least one of the fraction core contents written in the mathematics syllabus in the curriculum.

Table 3. The PTs' references to fractions as presented in the curriculum document (Skolverket, 2018, pp. 56–57).

Grades	Fraction core content in the curriculum category *Understanding and use of numbers*
1–3	• Parts of a whole and parts of a number. How parts are named and expressed as simple fractions, and how simple fractions are related to natural numbers. • Natural numbers and simple numbers as fractions and their use in everyday situations.
4–6	• Numbers in fractions and decimals and their use in everyday situations. • Numbers in percentage form and their relation to numbers in fraction and decimal form.

The PTs mainly used the direct curriculum quotations presented above. Only PT_{28} and PT_{60} (Grades 6 and 3) went a little bit beyond the texts, providing some reflections on the meaning of the curricular texts in practice: "When reflecting upon these points, it is clear that students need to learn a way of using mathematics that they can relate to everyday life" (PT_{28}). PT_{60} had a view across elementary school grades, stating that understanding the fraction concept as a part of a whole and as parts of a number "will also be of great importance for them [students] to master so that they will be given the opportunity to further develop their knowledge in algebra and the concept of percentages."

General aims for teaching. The PTs also quoted in their reports the general aims for mathematics teaching that are included in the national curriculum document and presented in Table 4. Instead of emphasizing fraction core contents, the PTs highlighted abilities, such as problem-solving, communication, reasoning, and learning to investigate and work with others as central goals for their teaching. These general curricular aspects were also involved in the PTs' descriptions of their teaching of fractions.

Table 4. The PTs' references to the aims of mathematics teaching (Skolverket, 2018, p. 55).

Grades	Aims of mathematics teaching in the curriculum
1–9	• Teaching in mathematics should aim at helping the pupils to develop knowledge of mathematics and its use in everyday life and in different subject areas. • Teaching should help pupils to develop their knowledge in order to formulate and solve problems, and also reflect over and evaluate selected strategies, methods, models and results. • Teaching should help pupils to develop their ability to argue logically and apply mathematical reasoning. • Pupils should through teaching be given the opportunity to develop familiarity with mathematical forms of expression and how these can be used to communicate about mathematics in daily life and mathematical contexts.

Discussion and Conclusions

This study investigated the PCK related to the teaching and learning of fractions that six elementary mathematics PTs displayed in their written teacher practicum reports. Using the three PCK categories included in the MKT framework (Ball et al., 2008), the study analyzed the PTs' knowledge of fractions that combines knowing about students, teaching, and curriculum. The results reveal that the PTs displayed knowledge relating to each of the categories KCS, KCT, and KCC, based on the instructions for the report (see Appendix A). However, they described mathematics teaching and learning mainly on a general pedagogical level, providing few reflections connected to fraction contents. Even though the PTs were teaching in different grade levels, they did not provide descriptions that differentiated characteristics for elementary students' fraction knowledge, teaching of fractions and curricular texts in their given grade levels. Moreover, there was no substantial difference between the descriptions of the PTs teaching in the lower grades compared to PT_{28} in Grade 6. These are interesting findings, indicating that none of the PTs demonstrated a robust fraction related knowledge within the PCK categories at the end of their teacher education. Previous studies have also reported limitations in PTs' PCK of fractions (e.g., Depaepe et al., 2015).

In general, the PTs did not provide a coherent discussion of fraction knowledge across the categories. For example, even though they expressed being aware of elementary students' difficulties with fractions, they did not report using this awareness as a basis for planning and teaching fractions, indicating that their KCS related to fractions was not realized as a part of their KCT in practice. Moreover, the PTs' few descriptions of common student errors and misunderstandings with fractions indicated limitations in their KCS. The PTs' identified knowledge in the KCS category seemed somewhat paradoxical since they could consider that their students easily understood fraction contents. At the same time, the PTs expressed that the students regarded mathematics and fractions as difficult.

In relation to the KCT category, the PTs emphasized general sociocultural aspects of learning and the use of the EPA model, a Swedish version of the cooperative learning activity Think-Pair-Share (e.g., Lyman, 1981). They also based their teaching on so-called *genomgång*, a culturally well-known lesson activity usually at the beginning of a lesson comprising "*a whole class event during which the teacher goes through something*" (Andrews & Larson, 2017, p. 85). Instead of reasoning their pedagogical choices from the point of view of fraction contents, different mathematical learning theories, and their students' needs as instructed for the report, the PTs' focus was on the arrangement of group work and peer discussions, that is, an essential part of teaching was based on active-working students. Asami-Johansson et al. (2020) state that in the Swedish teacher education, content-specific mathematical learning theories are presented on a general level and mathematics teaching is described with group work techniques, which was also identified in the PTs' reports.

In their reports, the PTs also expressed uncertainty in relation to fraction content and challenges in the teaching of fractions. While this may not be unusual for PTs with little prior experience of mathematics teaching, it is worrying since the participating PTs were at the end of their teacher education. Moreover, the fraction content concerned elementary school grades 1–6, which raises the question of whether the PTs had gained an adequate CK of fractions to teach the content (cf., Agathangelou & Charalambous, 2021). Since teachers' mathematical knowledge matters to students' performance even when teaching elementary mathematics (Hill et al., 2005), PTs should gain both a robust CK and PCK during their teacher education. As Ball et al. (2005) stated: "Effective teaching requires an understanding of the underlying meaning and justifications for the ideas and procedures to be taught and the ability to make connections among topics. Fluency, accuracy, and precision in the use of mathematical terms and symbolic notation are also crucial" (p. 9). Even though the PTs expressed awareness of different representations, procedures and interpretations for fractions, this knowledge was not realized in practice. The PTs' insufficient PCK was revealed in their sparse use of fraction core notions, limited use of other pictorial representations than circles, and their focus on the part-whole interpretation of fractions (cf., Olanoff et al., 2014), which they described in the reports.

Quoting curricular texts, the participating PTs emphasized connections between mathematics teaching and students' daily life. As Bråting et al. (2019) state, the Swedish national curriculum for elementary school strongly emphasizes teaching practical and everyday mathematics and addresses sparsely different mathematical contents. The overrepresentation of the part-whole interpretation of fractions in the curriculum and elementary mathematics books (Charalambous & Pitta-Pantazi, 2007) may also explain the PTs' focus on the part-whole interpretation when planning and conducting their fraction lessons. While the description of fraction content is very sparse and does not provide knowledge of other interpretations, such as fraction as an operator, ratio, rate and quotient (Ball, 1993), the Swedish curriculum document emphasizes general abilities like problem-solving, communication, conceptual understanding, reasoning and procedural ability. The PTs in this study showed knowledge of these abilities and they also highlighted the use of varying mathematical activities and tasks. However, they described the abilities in general in teaching, providing few reflections on how fractions could be learned in connection to the abilities and through the chosen activities and tasks. The findings indicate that the PTs had difficulties concretizing the content of the curriculum document in practice in their fraction lessons and identifying a progression in the teaching of fraction core content across different elementary school grade levels.

The results presented here raise the question of whether the findings correspond to the desired and expected outcomes of teacher education concerning PTs' PCK related to teaching and learning of fractions. Clearly, PTs need more knowledge of elementary students' conceptions, misconceptions and learning difficulties with fractions (Depaepe et al., 2015) as well as familiarity with different cognitive processes and their effects on students' thinking (Tirosh, 2000). Very little knowledge was displayed in the KCS category among the six participating PTs. Moreover, while the sparsely specified curriculum gives teachers and teacher education freedom when teaching specific mathematical content, maybe a more specified content progression in the national curriculum document would make it easier for PTs to connect KCC with KCS and KCT. The topic of the curriculum should also be addressed in teacher education.

In addition to the unspecified fraction content in the curriculum, what is the role of teacher education when explaining the presented results? When comparing Finnish and Swedish teacher education discourses, Hemmi and Ryve (2015) found that Finnish teacher educators emphasized a clear presentation of mathematics when discussing effective teaching. Swedish educators focused on aspects like building on students' ideas and connecting mathematics to the everyday life, which the PTs also emphasized in the present study. Hemmi and Ryve (2015) also pointed out that the Swedish teacher educator discourse places less emphasis on mathematical connections and mathematics teaching based on students' previously learned skills and contents, which may explain why the PTs, on the one hand, could identify student difficulties, but on the other hand, did not plan their teaching accordingly. Though it should be stated that the present study did not examine the PTs' actual teaching, the teacher educators responsible for the PTs may have different views than those interviewed by Hemmi and Ryve (2015).

Furthermore, while the PTs in this study emphasized mathematical discussions, the low level of connection to the specific fraction content is in line with the study by Christiansen and Erixon (2024), where Swedish PTs reported limited opportunities to learn about choosing resources and tasks, planning thematic teaching, analyzing learners' answers, leading mathematical discussions, and identifying mathematically talented learners. Mercer and Sams (2006) state that elementary students need explicit guidance to develop the use of language for their mathematics learning in group-based peer activities, which the PTs in the present study perceived as a challenging task for them as teachers.

It cannot be assumed that all PTs have developed a sufficient PCK for teaching fractions at the end of teacher education although they have completed all mathematics courses and teacher practicum courses, which should be addressed in teacher education. More focus should be given to the different PCK aspects to make PTs aware of the connection between the categories of KCS, KCT and KCC when teaching a challenging mathematical content such as fractions. Stevens et al. (2020) showed in their study that even small instructional changes in teacher education mathematics courses may enhance PTs' fraction knowledge, which they need as mathematics teachers.

Ball et al. (2008) state that the connection between the domains of the MKT framework and the cultural context should be considered when discussing research findings. The presented findings only concern teacher education at one Swedish university and it can be questioned whether the PCK identified in the PTs' written reports is representative or whether the PTs were writing in the way that was expected of them by their teacher practicum course teachers. However, at the end of their academic studies, the participating PTs were expected to be able to display different PCK aspects also in a written report, as shown in the instructions for the course assignment that was originally not provided for research purposes. Thus, the findings may also be worth considering in the context of other teacher education institutions.

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Appendix A.

A description of the teacher practicum course assignment excluding instructions for submission and assessment (Note: The color codes and the references to the MKT categories in the table below were not included in the original description of the assignment.)

	Reference to knowledge of content and students (KCS)
	Reference to knowledge of content and teaching (KCT)
	Reference to knowledge of content and curriculum (KCC)

Aim: The work with this course assignment is about gaining experience in taking responsibility for, implementing, documenting, and analyzing students' demonstrated knowledge in a coherent mathematics teaching sequence when aiming to stimulate all students' learning.

In this work, you will gain experience in:

- putting the curriculum content into practice,
- planning a coherent teaching sequence that develops children's mathematical skills,
- arguing for your choices in mathematics teaching based on your planning,
- forming an understanding about children's mathematical skills and documenting this.

Overall description of the assignment

- Together with your teaching supervisor, decide the mathematical content and abilities for your teaching sequence (at least for lessons). If possible, connect your lessons to understanding numbers as a whole and parts.
- Make a didactical analysis of the content for your lessons:
  (This analysis should not be included in your submitted course assignment.)
  - Start with an analysis of the mathematical content.
  - Decide the goals that should be achieved both in terms of core mathematical content and abilities.
  - Decide how the work with the mathematical content will be carried out.
  - Form your conception of students' understanding of the mathematical content.
- Plan your teaching sequence based on the Education Act (2010:800) where it is emphasized that teaching must rest on a scientific basis and proven experience.
- Connect your teaching to the aim and core content of the subject of mathematics in the curriculum (Lgr11) as well as at least the following requirements found in the curriculum:
  "*Teaching should be adapted to each pupil's circumstances and needs. It should promote the pupils' further learning and acquisition of knowledge based on pupils' backgrounds, earlier experience, language, and knowledge.*" (Lgr11, p. 6)
  "*The school should provide pupils with structured teaching under the teacher's supervision, both as a whole class and on an individual basis. Teachers should endeavour in their teaching to balance and integrate knowledge in its various forms*." (Lgr11, p. 11)
- Carry out your teaching sequence.

For your submitted written assignment, choose one of your lessons and focus on the following aspects:

- Describe the context for your lessons and explain the reason to present the particular lesson.
- Describe the concepts, working methods and forms as well as different representation forms that can be used in mathematics teaching in the chosen content area.
- Describe and reason the working methods and forms you chose to use.
- Give a description of and reflect upon how your choices worked both in relation to the set goals and in relation to all students' opportunities for learning - did your choices help the students to reach the set goals; why/why not?
- Explain how it is seen in your teaching that your planning was based on different policy documents, scientific research, and various (mathematical) learning theories and teaching methods.

Paper IV

An insight into prospective elementary teachers' mathematical knowledge for teaching: An example of fraction division

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An insight into prospective elementary teachers' mathematical knowledge for teaching: An example of fraction division

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This study investigates four Swedish prospective elementary mathematics teachers' mathematical knowledge for teaching fractions. Using narrative inquiry, the study focuses on the prospective teachers' reflections upon their mathematical identity and their analyses of six different solutions for a fraction division task. The results show that there are substantial differences between the prospective teachers in their mathematical content knowledge, pedagogical content knowledge, and in their attitudes towards mathematics teaching at the time of their graduation. Moreover, the prospective teachers expressed that teacher education did not respond to all their needs in their development as mathematics teachers. Thus, some implications for teacher education are also discussed in this paper.

Keywords: Fraction division, mathematical identity, mathematical knowledge for teaching, prospective teacher, teacher education.

Introduction

Teacher education is a critical time for prospective teachers to gain the mathematical knowledge needed for effective elementary school mathematics teaching. To effectively teach elementary mathematics, both content knowledge (CK) and pedagogical content knowledge (PCK) of the subject are needed (Depaepe et al., 2015). For teacher education, it is a challenge to respond to prospective mathematics teachers' (PTs') varying educational needs when developing their CK and PCK, for example, in the field of fractions. Several international studies have shown that teaching of fractions is among the most challenging fields in elementary school mathematics (e.g., Ma, 2010), but efforts to increase PTs' understanding of fractions seem to have low effect (e.g., Newton, 2008; Van Steenbrugge et al., 2014).

This study analyses the narratives of four PTs with different backgrounds, experiences, attitudes, and views of mathematics and its teaching. The study focuses on the PTs' mathematical identity and their mathematical knowledge for teaching fractions when reflecting upon their relation to mathematics and analysing different solutions for a fraction division task. The aim is to find out how prepared for teaching fractions the PTs are at the end of their studies. Our research question is:

RQ: How is prospective teachers' mathematical identity and knowledge for teaching fractions at the time of their graduation?

Theoretical framework

Kaiser et al. (2017) point out that both cognitive abilities (i.e., teachers' professional knowledge) and affective-motivational characteristics need to be investigated when describing teachers' mathematical knowledge. The affective-motivational part refers to teachers' beliefs about mathematics and its teaching, their professional motivation, and self-regulation. According to Kaasila (2007), PTs' "view of mathematics is an important part of their mathematical identity, consisting of knowledge, beliefs, conceptions, attitudes and emotions" (p. 206). Further, Kaasila (2007) considers that mathematical identity is a part of a person's narrative identity: "One's mathematical identity is manifested when telling stories about one's relationship to mathematics, its learning and teaching" (p. 206).

Teachers' professional knowledge include Shulman's (1986) domains of CK and PCK (Kaiser et al., 2017). Of these, CK can be regarded as a prerequisite for teachers' PCK (e.g., Depaepe et al., 2015). These two domains are also a basis for the mathematical knowledge for teaching framework (Ball et al., 2008). Hill et al. define the mathematical knowledge for teaching as "the mathematical knowledge used to carry out the *work of teaching mathematics*" (2005, p. 373). This work of mathematics teachers includes, for example, interpreting students' solutions, choosing appropriate representations, using them correctly and linking them to other representations, and recognizing what is involved in a particular representation, as well as giving and evaluating mathematical explanations and students' claims (Ball et al., 2008; Hill et al., 2005).

Fractions are a core content of school mathematics and a basis for the development of other mathematical content areas. Fraction CK concerns, for example, performing fraction calculations correctly, carrying out different procedures with fractions, and using correct terms and notations (Ball et al., 2008). PCK related to fractions entails representing and explaining fractions in comprehensible ways to others as well as understanding what makes the learning of fractions easy or difficult to different learners (Shulman, 1986). Teaching fractions effectively requires that teachers have a robust fraction number sense that includes judging whether answers are reasonable and whether their students' strategies are appropriate or based on wrong reasoning (Lamon, 2020). Thus, if PTs have limited CK and PCK in the fraction domain, they also have difficulties analysing their students' misconceptions and using different instructional strategies and representations (Depaepe et al., 2015). However, many PTs have weak conceptual and procedural fraction knowledge and difficulties especially in fraction division, and their fraction misconceptions seem to mirror elementary school children's misconceptions (e.g., Tirosh, 2000; Tossavainen, 2022; Van Steenbrugge et al., 2014, Young & Zientek, 2011). Further, concerning the affective domain, teachers who are not anxious about mathematics are usually not anxious about teaching the subject (Hadley & Dorward, 2011).

When investigating PTs' mathematical knowledge for teaching fractions, this study follows both Kaiser et al. (2017) and Kaasila (2007) and addresses the topics of CK, PCK, and the participating PTs' mathematical identity that includes their experiences of mathematics, its learning and teaching, and the impact of teacher education.

Method

Participants

Four PTs studying Swedish university programs to become teachers for children between the ages of 6 and 12 were chosen for this study from a group of 59 PTs, which in the previous academic year had completed a paper-and-pencil questionnaire concerning fractions in elementary school mathematics teaching (see Tossavainen, 2022). At the time of this study, the selected PTs were finishing their final bachelor theses in the last term of their four-year teacher education programs. The PTs had different backgrounds (e.g., age, studies prior to teacher education), and they represented different types of PT respondents identified in the questionnaire: Anna and Julia (pseudonyms) demonstrated positive, and

Matilda and Sabina (pseudonyms) negative attitudes towards teaching elementary school mathematics. Further, Julia and Sabina had showed a robust procedural fraction knowledge while Anna and Matilda had made several mistakes when solving routine fraction tasks. Moreover, all the four PTs had provided a different solution procedure for the fraction division task that we use as an example in this study when focusing on their knowledge for teaching fractions.

Data collection and analysis

In addition to the PTs' background information in the questionnaire, the main data of the study were collected through four semi-structured interviews that were conducted late in the spring term in the PTs' final year of teacher education. The individual interviews that the participants volunteered to lasted 23–26 minutes, and the recorded interviews were then transcribed in the original language.

The interviews started with reflections concerning the PTs' prior mathematical identity, changes in their experiences and knowledge about mathematics, and their expectations about themselves as mathematics teachers. Then, they were asked to analyse six different solutions for the fraction division task $\frac{3}{4}/3$, to which they had already given their own solutions in the questionnaire. The PTs were asked to compare and evaluate the solutions, to improve them, and to explain the methods and procedures. In addition to the interviewed PTs' own solutions, two solutions from questionnaire respondents 22 and 52 were added to the selection (see Figure 1). Together these solutions represented typical solutions provided in the questionnaire by the larger original group of PTs. The interviews were closed with discussions about the impact of teacher education on the PTs' development as mathematics teachers.

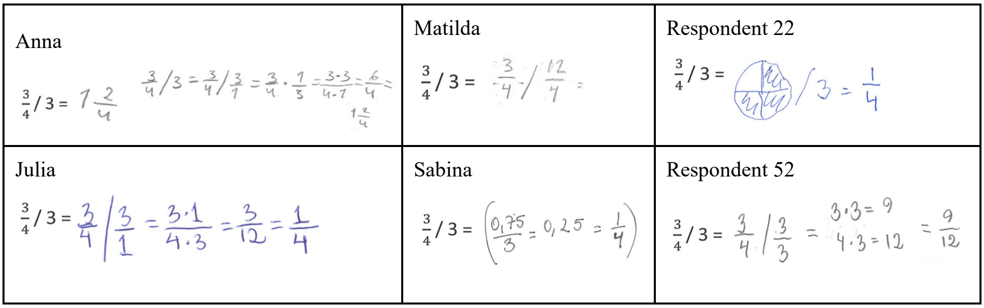

Figure 1: The fraction division solutions used in the interviews

Solutions for the task $\frac{3}{4}/3$ were chosen from the questionnaire as a basis for the interviews since fraction division and especially fraction tasks that include whole numbers are challenging for many PTs (e.g., Newton, 2008; Tirosh, 2000). Further, the PTs were asked to analyse different solutions for the same task since they were expected to be able to solve the task themselves, to explain mathematical ideas and procedures behind the solutions, to analyse others' errors and misconceptions, and to modify the task, i.e., to demonstrate both their CK and PCK that are included in the mathematical knowledge for teaching framework (Ball et al., 2008; Shulman, 1986) and pointed out by Kaiser et al. (2017) as well. We focused on these aspects when analysing the PTs' knowledge for teaching fractions.

In general, we used a narrative inquiry method in our analysis. Following Kaasila (2007), we focused both on the content and the form of the PTs' narratives. That is, we analysed how the PTs demonstrated their fraction knowledge (CK and PCK, as described above and in our theoretical framework) and different themes related to their mathematical identity (e.g., factors facilitating changes or turning points, views of mathematics, their experiences during their own school years and during teacher education, and their future expectations about mathematics teaching). Further, when focusing on the form, we analysed how they were telling their narratives, i.e., we analysed linguistic features and affective and emotional elements in their narratives. In other words, our analysis was focused both on the PTs' knowledge for teaching fractions and their mathematical identity as such and on the ways in which they talked about these. However, in narrative inquiry, it is not always reasonable to separate the form and content but rather to integrate them into a whole (Kaasila, 2007). Thus, in the results section, we present a synthesis of the different aspects and themes found in the PTs' narratives.

Results

Next, as an answer to the research question, a description of the PTs' narratives will be presented as two pairs involving a similar narrative theme, i.e., attitude towards mathematics teaching. We begin with Matilda and Sabina who at the end of their studies still narrated their responses mainly with negative attitudes while the reflections of Anna and Julia were positive. The examples of the interview quotations are translated into English.

Matilda and Sabina

Both Matilda and Sabina hesitated and used several minutes to think each of the different division solutions during their interviews. They seemed to try to memorize separate solution steps instead of focusing on whole solution procedures. Still, both expressed difficulties analysing the solutions and considered that it was easier to solve and understand fraction tasks using decimals. However, neither Sabina nor Matilda was able to give examples of situations where it was not possible to use decimals for fraction tasks, and Sabina concluded that she would also solve a task like $\frac{2}{3}/3$ using decimals. Thus, their CK seemed unrobust, and Sabina commented her own solution with decimals (see Figure 1): "I don't think I really knew how to do it with fractions, so I took it [the solution] into decimal form instead."

When trying to analyse the solution with the pictorial representation (see Figure 1), Sabina stated: "I don't really understand this [method]." Matilda as well, became confused with the solution, and wondered which part of the circle represented the whole that should be divided by three. Neither of them was able to go further with Matilda's unfinished solution procedure, and Matilda concluded: "I have no idea how I would proceed here." As these examples show, Matilda and Sabina were not able to analyse the solutions and to describe alternative methods, which made their PCK to seem low.

In the questionnaire, Matilda and Sabina reported that they had experienced mathematics as a difficult subject at school, and that teaching of mathematics would be a challenge for them. Throughout their interviews as well, Matilda and Sabina narrated their responses with their own mathematical difficulties. Matilda expressed that the level of mathematics studies in teacher education had been too advanced for her. Sabina had demonstrated a robust procedural knowledge with fraction tasks in the questionnaire. In her interview, Sabina considered that she had rehearsed and enhanced her basic knowledge in mathematics during teacher education. However, she stated: “Actually, math is my weak point. I admit that.” Thus, not a substantial positive turn was shown in their fraction knowledge.

Matilda and Sabina also brought up that they would have needed more instructional knowledge for their development as mathematics teachers indicating that not either a positive turn had begun in their views of themselves as mathematics teachers. Obviously, neither Matilda’s experiences as a substitute teacher nor her teacher training courses had changed her attitude towards mathematics teaching: “I somehow paint a picture that tells me that I won't be able to teach mathematics.” Matilda assumed that her knowledge about teaching would increase over the time when started working as a mathematics teacher. Still, she stated: “I don’t feel that I have enough knowledge to be able to explain in such different ways that different pupils would be able to understand.”

Anna and Julia

Anna and Julia were willing to reflect upon the fraction solutions presented to them. They analysed the solutions with mathematical symbol representations, made suggestions for how to make some solutions easier to understand and how to proceed with the unfinished solution (see Figure 1). They also quickly noticed the possible pitfalls of using decimals with fraction tasks. Anna stated that it might be difficult in cases such as thirds or sevenths, and Julia commented: “It’s okey [to use decimals] but you somehow have to make students aware of that it doesn’t always work.”

Both Anna and Julia seemed confident about their CK, and they considered themselves good at mathematics. They were interested in teaching elementary school mathematics, but they also expressed some limitations in their PCK, which might affect their mathematics teaching. Anna described the limitation that she had reported in the questionnaire as well: “One difficulty for me, as a teacher, will be to understand how to get down on their [elementary students’] level when they don’t understand.” Julia, in her interview stated: “I will certainly not be very good at finding all the ways for [teaching] all children, but I will do my best anyway.”

Anna had made several mistakes with routine fraction tasks in the questionnaire, and a clear difference was also found in the narratives of Anna and Julia: When analysing the solutions for the division task, Anna’s confidence in her own CK and thinking models seemed to cause her trouble in understanding and accepting solution procedures other than her own and difficulties in recognizing the mistakes in her own solution (see Figure 1). She seemed to prefer the invert-and-multiple procedure as the correct solution method for fraction division tasks, and in general, Anna highlighted mathematical symbol representations more than, for example, pictorial solutions. When analysing the solution with the pictorial representation (see Figure 1), Anna stated: “I don't think this is a correct way to use a picture, you should rather learn the theory of [the procedure] how to calculate division with fractions.” These examples indicate that Anna’s CK and PCK had not developed further. Julia in turn, still demonstrated a robust procedural fraction CK and a strong PCK as well. In her narrative, Julia had a strong student-focused approach pointing out the importance of explaining the solutions to others who did not know the methods and finding out children’s reasoning as they worked through fraction tasks. She highlighted the need to use varying solution methods as well as concrete learning materials and illustrative pictures to enhance all elementary school students' learning.

In the questionnaire, Anna and Julia had reported positive attitudes towards mathematics. They had experienced mathematics as an easy subject during their own school years, and in teacher education, they had even chosen to write their final bachelor theses on the field of mathematics education. However, they expressed that their mathematical knowledge for teaching did not develop much during their studies in teacher education. At the end of her education, Anna still felt that she did not have the knowledge of how to explain mathematical ideas in different ways. Like Anna, Julia called for more knowledge in methods and didactics of mathematics: "The focus was very much on our own math skills and perhaps not so much on how we should teach them [elementary school students]."

Discussion

The PTs in this study were expected to have received the mathematical knowledge for teaching that was needed to analyse the different solutions for the fraction division task $\frac{3}{4}/3$. It was also expected that they would have developed a positive mathematics teacher identity during the time in teacher education. However, the results show limitations and substantial differences between the PTs in their CK, PCK, and in their mathematical identity just before their graduation. Moreover, the participating PTs expressed that teacher education did not respond to all their needs when developing them to become mathematics teachers.

Even though the multiple challenges concerning PTs' fraction knowledge are widely documented over years in several studies (e.g., Young & Zientek, 2011), an interesting finding in this study is that these difficulties still exist despite of teacher educators' efforts to increase PTs' understanding of fractions. After having completed their mathematics courses and all teaching preparation courses, Matilda, Sabina, and Anna demonstrated limitations in their CK and PCK when analysing the fraction division solutions in this study. It seemed that at the time of their graduation they were still missing such fraction CK that had enabled developing a robust PCK as well. Moreover, they did not seem to trust their own skills. For example, Sabina, who had showed a robust CK with fraction tasks in the questionnaire, was not able to show similar PCK when analysing the different solutions. As Depaepe et al. (2015) state: "prospective teachers should have sufficient CK in order for them to be able to develop their PCK, and … a CK training is not sufficient to develop PCK" (p. 91). An interesting finding was also seen in the case of Anna who appeared to be limited in her mathematical thinking even though she was interested in mathematics teaching and considered mathematics easy.

All the PTs in this study expressed in their narratives that their mathematical knowledge for teaching did not develop much during teacher education, which is an important finding as well. Matilda's and Sabina's uncertainty concerning their CK and PCK and their obvious anxiety in mathematics seemed to affect their attitudes towards mathematics teaching, which is in line with the findings of Hadley and Dorward (2011). Matilda and Sabina did not report a positive turn in their mathematical identity during the time in teacher education, and similarly, there was not a great development and change concerning Anna and Julia who had entered teacher education with positive views of mathematics and its teaching.

The PTs in this study were selected with different backgrounds, experiences, and attitudes towards mathematics and its teaching. Still, it was surprising how clearly they expressed different experiences even though taking the same mathematics courses in teacher education. Julia, who had a strong student-focused approach, felt that the courses had focused too much on basic knowledge in mathematics whereas Matilda felt that she never reached the level of CK that was dealt with in their mathematics studies. Thus, an important finding to be addressed in teacher education is that the PTs in this study clearly demonstrated different needs in their development as mathematics teachers, which teacher education did not manage to fully respond. Moreover, all the PTs pointed out the need for more PCK in mathematics teaching. Van Steenbrugge et al. (2014) state that too much time is devoted to repeating the knowledge that PTs should have acquired before attending teacher education, which means that less time is addressed to the pedagogical knowledge domain in mathematics.

As reflected in the PTs' narratives in this study, there still seems to be a lot to do in teacher education to increase PTs' mathematical knowledge for teaching fractions and to improve their mathematical identity. For example, as Hadley and Dorward (2011) suggest, more pedagogical discussions are needed with PTs so that they could be helped to become more comfortable about teaching mathematics. Kaasila (2007, p. 206) puts it as follows: "During teacher education it is important to listen to the voices of pre-service teachers talking about themselves as future mathematics teachers." As also seen in the findings in this study, self-confidence is an essential component of PTs' view of themselves as learners and teachers of mathematics (Kaasila 2007). Thus, teacher education could address more the affective-motivational domain (c.f. Kaiser et al., 2017) when developing the heterogeneous groups of students to become mathematics teachers. Moreover, a suggestion for teacher education is to find out new ways to ensure that all PTs reach the mathematical knowledge that is needed to effectively carry out the work of teaching mathematics (Ball et al., 2008). Teachers' knowledge of the mathematical content is related to students' mathematical achievement even when teaching very elementary mathematics (Hill et al., 2005), but as Kaiser et al. (2017) conclude, teachers' mathematical knowledge does not develop much during the first years when started working as mathematics teachers. However, elementary school students cannot wait for that knowledge of their mathematics teachers to grow over the time.

The data presented in this paper provide a portrait of four Swedish PTs. More research is then needed for generalization of the results. However, the selected PTs were representative also for the other PT respondents in our background questionnaire. Further, as Kaasila (2007, p. 205) states: "the potential of case studies can be enhanced by applying the narrative approach to data analysis." While it is likely that a larger selection of participants could lead to a more varied selection of narrative themes, we still think that the presented findings can also be representative of many other PTs at other universities and therefore require attention. Thus, we hope that teacher educators in different countries may benefit of the findings presented here in their own work with prospective mathematics teachers.

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