BRIEF COMMUNICATIONS

Geometry of Cesàro Function Spaces*

S. V. Astashkin and L. Maligranda

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ABSTRACT. Geometric properties of Cesàro function spaces $\text{Ces}_p(I)$, where $I = [0, \infty)$ or $I = [0, 1]$, are investigated. In both cases, a description of their dual spaces for $1 < p < \infty$ is given. We find the type and the cotype of Cesàro spaces and present a complete characterization of the spaces l^q that have isomorphic copies in $\text{Ces}_p[0, 1]$ ($1 \leq p < \infty$).

KEY WORDS: Cesàro space, Köthe dual space, dual space, q -concave Banach space, type and cotype of a Banach space, Dunford–Pettis property.

1. Introduction. The *Cesàro sequence spaces* ces_p ($1 \leq p \leq \infty$) have become widely known in the late 1960s in connection with the problem of describing their duals, which was posed by the Dutch Mathematical Society (see [1]). Recall that the space ces_p consists of all sequences of real numbers $x = (x_k)_{k=1}^\infty$ such that

$$\|x\|_{\text{ces}_p} = \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |x_k| \right)^p \right]^{1/p} < \infty$$

(with a natural modification in the case $p = \infty$).

The Cesàro function space $\text{Ces}_p = \text{Ces}_p(I)$ consists of all Lebesgue measurable functions f defined on $I = [0, 1]$ or $I = [0, \infty)$ such that

$$\|f\|_{\text{Ces}_p} = \left[\int_I \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right]^{1/p} < \infty \quad \text{if } 1 \leq p < \infty$$

and

$$\|f\|_{\text{Ces}_\infty} = \sup_{x \in I, x > 0} \frac{1}{x} \int_0^x |f(t)| dt < \infty \quad \text{if } p = \infty.$$

The space $\text{Ces}_\infty[0, 1]$, which is known as the Korenblyum–Krein–Levin space K , first appeared as early as in 1948 in connection with studying problems of the theory of singular integrals (see [2] and [3]). Later, the spaces $\text{Ces}_p[0, \infty)$ ($1 \leq p \leq \infty$) were considered in [4]–[6].

This note is devoted to studying geometric properties of Cesàro function spaces. In particular, we describe the (Banach) dual space of $\text{Ces}_p(I)$ ($1 < p < \infty$) both for $I = [0, \infty)$ and for $I = [0, 1]$. We should mention that, in the first case, another (implicit) description of this space was obtained in [6]. Moreover, we determine the type and the cotype of Cesàro function spaces and present a complete characterization of the spaces l^q that have isomorphic copies in $\text{Ces}_p[0, 1]$ ($1 \leq p < \infty$).

Below we recall some definitions and notation, which we will need later on. Let $L^0 = L^0(I)$ be the set of all real-valued Lebesgue measurable functions (more precisely, the set of their equivalence classes) defined on I . By a *normed ideal space* $X = (X, \|\cdot\|)$ we understand a linear normed space $X \subset L^0(I)$ with the following property: if $|f| \leq |g|$ almost everywhere on I and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$. If, in addition, X is a complete space, then it is called a *Banach ideal space*.

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For a more detailed information on properties of normed ideal spaces we refer the reader to the monographs [7]–[9].

If X and Y are Banach ideal spaces on I , then $X \hookrightarrow Y$ means that $X \subset Y$ and the embedding is continuous; if, in addition, the inequality $\|x\|_Y \leq C\|x\|_X$ holds for all $x \in X$, then we write $X \xhookrightarrow{C} Y$. By $\text{supp } f = \{t \in I : f(t) \neq 0\}$ we denote the support of a measurable function f . We begin with collecting the most elementary properties of Cesàro function spaces.

Theorem 1. (a) *If $1 < p \leq \infty$, then $\text{Ces}_p(I)$ is a Banach ideal space, $\text{Ces}_1[0, 1] = L_w^1$ with the weight $w(t) = \ln(1/t)$ ($t \in (0, 1]$), and $\text{Ces}_1[0, \infty) = \{0\}$.*

(b) *The space $\text{Ces}_p(I)$ is separable if $1 \leq p < \infty$ and nonseparable if $p = \infty$.*

(c) *If $1 < p \leq \infty$, then $L^p(I) \xhookrightarrow{p'} \text{Ces}_p(I)$, where $p' = p/(p-1)$, and the embedding is strict.*

(d) *If $1 < p < \infty$, then $\{f \in \text{Ces}_p[0, 1] : \text{supp } f \subset [0, a]\} \hookrightarrow L^1[0, a]$ for every $a \in (0, 1)$ and $\{f \in \text{Ces}_p[0, \infty) : \text{supp } f \subset [0, a]\} \hookrightarrow L^1[0, a]$ for all $0 < a < \infty$. Moreover, $\text{Ces}_\infty[0, 1] \xhookrightarrow{1} L^1[0, 1]$. At the same time, $\text{Ces}_p[0, 1] \not\subset L^1[0, 1]$ and $\text{Ces}_p[0, \infty) \not\subset L^1[0, \infty)$ ($1 < p < \infty$).*

(e) *If $1 < p < q \leq \infty$, then $\text{Ces}_q[0, 1] \xhookrightarrow{1} \text{Ces}_p[0, 1]$ and the embedding is strict.*

(f) *The space $\text{Ces}_p(I)$ is nonreflexive but strictly convex if $1 < p < \infty$. The latter means that $\|f\|_{\text{Ces}_p} = \|g\|_{\text{Ces}_p} = 1$ and $f \neq g$ imply $\|(f+g)/2\|_{\text{Ces}_p} < 1$.*

2. Description of dual spaces. Let X be a normed ideal space on I . Recall that its Köthe dual (or associated space) X' is the set of all functions $f \in L^0(I)$ such that

$$\|f\|_{X'} := \sup_{g \in X, \|g\|_X \leq 1} \int_I |f(x)g(x)| dx < \infty.$$

This is a Banach ideal space on I . Let X^* denote the (Banach) dual of X ; then $X' \subset X^*$. Moreover, $X' = X^*$ if and only if X is separable.

The following two theorems give a description of the dual (and Köthe dual) space for the space $\text{Ces}_p(I)$ ($1 < p < \infty$) in the cases $I = [0, \infty)$ and $I = [0, 1]$, respectively. The appearance of the weight $(1 - x^{p-1})^{-1}$ in the second case was quite unexpected.

Let $1 < q < \infty$. Then $D(q)$ is the space of functions defined on $[0, \infty)$ with the norm $\|f\|_{D(q)} = \|\tilde{f}\|_{L^q}$, where $\tilde{f}(x) = \text{ess sup}_{t \in [x, \infty)} |f(t)|$.

Theorem 2. *If $I = [0, \infty)$ and $1 < p < \infty$, then*

$$(\text{Ces}_p)^* = (\text{Ces}_p)' = D(p'), \quad \text{where } p' = p/(p-1);$$

the norms on these spaces are equivalent.

For every $1 < q < \infty$, we define the Banach ideal space $U(q)$ on $[0, 1]$ as the set of all functions with finite norm

$$\|f\|_{U(q)} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1 - x^{1/(q-1)}} \right)^q dx \right]^{1/q}, \quad \text{where } \tilde{f}(x) = \text{ess sup}_{t \in [x, 1]} |f(t)|.$$

It is easy to see that an equivalent norm on $U(q)$ (with the constant of equivalence depending on q) can be defined as follows:

$$\|f\|_{U(q)}^0 = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1 - x} \right)^q dx \right]^{1/q}.$$

Theorem 3. *If $I = [0, 1]$ and $1 < p < \infty$, then*

$$(\text{Ces}_p)^* = (\text{Ces}_p)' = U(p'), \quad p' = p/(p-1);$$

the norms on these spaces are equivalent.

The proofs of these theorems use, in particular, some factorization relations similar to those considered in the discrete situation in [10]. We confine ourselves to the case of the semi-axis. Let

$G(p)$ ($1 \leq p < \infty$) be the p -convexification of the space $\text{Ces}_\infty[0, \infty)$ endowed with the norm

$$\|f\|_{G(p)} = \| |f|^p \|_{\text{Ces}_\infty}^{1/p} = \sup_{x>0} \left(\frac{1}{x} \int_0^x |f(t)|^p dt \right)^{1/p}.$$

Proposition 1. *If $1 < p < \infty$, then $\text{Ces}_p = L^p \cdot G(p')$, i.e., $f \in \text{Ces}_p$ if and only if $f = gh$, where $g \in L^p$, $h \in G(p')$, and*

$$\|f\|_{\text{Ces}_p} \approx \inf \{ \|g\|_{L^p} \|h\|_{G(p')} : f = gh, g \in L^p, h \in G(p') \}.$$

Proposition 2. *If $1 < p < \infty$, then $D(p) \cdot G(p) = L^p$ and*

$$\|f\|_{L^p} = \inf \{ \|g\|_{D(p)} \|h\|_{G(p)} : f = gh, g \in D(p), h \in G(p) \}.$$

Remark 1. Theorems 2 and 3 show, in particular, that, in both cases $I = [0, \infty)$ and $I = [0, 1]$, the dual space $(\text{Ces}_p)^*$ is nonseparable.

Remark 2. As already mentioned, the space $\text{Ces}_\infty[0, 1] := K$ was introduced by Korenblum, Krein, and Levin [2]. In [11], Luxemburg and Zaanen showed that $K' = \tilde{L}^1$; in particular, the norm on K' is equal to the norm on $\tilde{L}^1$, which is defined by $\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1}$, where $\tilde{f}(x) = \text{ess sup}_{t \in [x, 1]} |f(t)|$. Somewhat earlier, in [12], Tandori described the dual space for K_0 being the separable part of the space K ; namely, $(K_0)^*$ is the space $\tilde{L}^1$ (with the same norm).

3. Copies of l^q in Ces_p . In [13] it was shown that, for every $1 < p < \infty$, the Cesàro space $\text{Ces}_p(I)$ contains an asymptotically isometric copy of the space l^1 , i.e., there exists a sequence $\{\varepsilon_n\} \subset (0, 1)$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and a sequence of functions $\{f_n\} \subset \text{Ces}_p(I)$ such that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |\alpha_n| \leq \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{\text{Ces}_p} \leq \sum_{n=1}^{\infty} |\alpha_n|$$

for any $\{\alpha_n\} \in l^1$. This property shows the similarity of the spaces $\text{Ces}_p(I)$ and $L^1(I)$; at the same time, the following result indicates a connection between the spaces $\text{Ces}_p(I)$ and $L^p(I)$. Consider the characteristic functions $g_n = \chi_{[2^{-n-1}, 2^{-n}]}$ ($n = 1, 2, \dots$). It is not hard to show that $\|g_n\|_{\text{Ces}_p} \approx \|g_n\|_{L^p} \approx 2^{-n/p}$ ($n = 1, 2, \dots$).

Theorem 4. *For every $1 < p < \infty$, the space $\text{Ces}_p(I)$ contains an order isomorphic and complemented copy of l^p . Namely, if $\bar{g}_n = g_n / \|g_n\|_{\text{Ces}_p}$ ($n = 1, 2, \dots$), then the sequence $\{\bar{g}_n\}_{n=1}^{\infty}$ is equivalent to the canonical basis in l^p and generates a complemented subspace in $\text{Ces}_p(I)$.*

Recall that a Banach space X has the *Dunford–Pettis property* if, for any weakly null sequences $\{x_n\}$ in X and $\{f_n\}$ in the dual space X^* , we have $f_n(x_n) \rightarrow 0$. Classical examples of Banach spaces with the Dunford–Pettis property are the AL- and AM-spaces [9, Sec. 1b].

Corollary 1. *The spaces $\text{Ces}_p(I)$ ($1 < p < \infty$) do not have the Dunford–Pettis property.*

It is well known [14, Theorem 6.4.19] that if $1 \leq p \leq 2$ ($2 < p < \infty$), then the space l^q is isomorphically embedded in $L^p[0, 1]$ if and only if $p \leq q \leq 2$ ($q = p$ or $q = 2$, respectively). A similar description of isomorphic copies of l^q can be given also for the Cesàro spaces.

Theorem 5. (a) *If $1 \leq p \leq 2$, then the space l^q is isomorphically embedded in $\text{Ces}_p[0, 1]$ if and only if $q \in [1, 2]$.*

(b) *If $2 < p < \infty$, then the space l^q is isomorphically embedded in $\text{Ces}_p[0, 1]$ if and only if $q \in [1, 2]$ or $q = p$.*

The proof of necessity in the last theorem in the case $p > 2$ uses the well-known Kadec–Pełczyński procedure [15] for constructing a sequence of disjointly supported functions equivalent to a given basis sequence of normalized functions in the space $\text{Ces}_p[0, 1]$ converging to zero in Lebesgue measure. The difference between this theorem and the result for $L^p[0, 1]$ mentioned above is explained mainly by the fact that the space $\text{Ces}_p[0, 1]$ contains a copy of $L^1[0, 1]$; namely, the norms of these spaces are equivalent on the subspace $\{f \in \text{Ces}_p[0, 1] : \text{supp } f \subset [h, 1 - h]\}$ for each $h \in (0, 1/2)$.

4. Rademaher type of Cesàro spaces. A Banach ideal space X is called q -concave ($1 \leq q < \infty$) with constant $L > 0$ if

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq L \left\| \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|$$

for any $x_1, \dots, x_n$ from X . In particular, the space $L^p(I)$ ($1 \leq p < \infty$) is p -concave with constant 1. It turns out that the space $\text{Ces}_p(I)$, where $I = [0, \infty)$ or $I = [0, 1]$, possesses the same property.

Using the fact that the space $L^1[0, x]$ is p -concave with constant 1 for each $x > 0$, it is not hard to prove the following assertion.

Theorem 6. *For every p , $1 < p < \infty$, the Cesàro space $\text{Ces}_p(I)$ is p -concave with constant 1.*

Let $r_n: [0, 1] \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) be the Rademacher functions, i.e., $r_n(t) = \text{sign}(\sin 2^n \pi t)$. A Banach space X is said to have *type p* , $1 \leq p \leq 2$, if there is a constant $K > 0$ such that, for arbitrary vectors $x_1, \dots, x_n$ from X , the following inequality holds:

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}.$$

A Banach space X has *cotype q* $q \geq 2$ if there is a constant $K > 0$ such that, for arbitrary vectors $x_1, \dots, x_n$ from X ,

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt.$$

This definition makes sense also in the case $q = \infty$ provided that the left-hand side of the inequality is replaced by $\max_{1 \leq k \leq n} \|x_k\|$. We say that the space X has *trivial type* (*trivial cotype*) if it does not have any type larger than 1 (any finite cotype, respectively).

As already mentioned, the space $\text{Ces}_p(I)$ contains a copy of L^1 . Therefore, Theorems 4 and 6 yield the following result.

Corollary 2. *If $1 < p < \infty$, then the space $\text{Ces}_p(I)$ has trivial type and cotype $\max(p, 2)$. The space $\text{Ces}_\infty(I)$ has trivial type and trivial cotype.*

References

- [1] A. A. Jagers, Nieuw Arch. Wiskund. (3), **22** (1974), 113–124.
- [2] B. I. Korenblyum, S. G. Kreĭn, and B. Ya. Levin, Dokl. Akad. Nauk SSSR (N.S.), **62** (1948), 17–20.
- [3] W. Wnuk, Banach Lattices with Order Continuous Norms, Polish Scientific Publishers PWN, Warsaw, 1999.
- [4] J.-S. Shiue, Tamkang J. Math., **1**:2 (1970), 91–95.
- [5] B. D. Hassard and D. A. Hussein, Tamkang J. Math., **4** (1973), 19–25.
- [6] P. W. Sy, W. Y. Zhang, and P. Y. Lee, Glas. Mat. Ser. III, **22**:1 (1987), 103–112.
- [7] L. V. Kantorovich and G. P. Akilov, Functional Analysis, Pergamon Press, Oxford–Elmsford, New York, 1982.
- [8] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, Interpolation of Linear Operators, Amer. Math. Soc., Providence, RI, 1982.
- [9] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces, II. Function Spaces, Springer-Verlag, Berlin–New York, 1979.
- [10] G. Bennett, “Factorizing the classical inequalities,” Mem. Amer. Math. Soc., **120**:576 (1996).
- [11] W. A. J. Luxemburg and A. C. Zaanen, Math. Ann., **162** (1965/1966), 337–350.
- [12] K. Tandori, Publ. Math. Debrecen., **3** (1954), 263–268.
- [13] S. V. Astashkin and L. Maligranda, Proc. Amer. Math. Soc., **136**:12 (2008), 4289–4294.

- [14] F. Albiac and N. J. Kalton, Topics in Banach Space Theory, Graduate Texts in Math., vol. 233, Springer-Verlag, New York, 2006.
- [15] M. I. Kadec and A. Pełczyński, Studia Math., **21** (1961/62), 161–176.

SAMARA STATE UNIVERSITY
e-mail: astashkn@ssu.samara.ru

LULEÅ UNIVERSITY OF TECHNOLOGY, SWEDEN
e-mail: lech@sm.luth.se

Translated by S. V. Astashkin and L. Maligranda