Pointwise multipliers of Orlicz spaces

LECH MALIGRANDA AND EIICHI NAKAI

Abstract. We show that the result on multipliers of Orlicz spaces holds in general. Namely, under the assumption that three Young functions Φ_1, Φ_2 and Φ , generating corresponding Orlicz spaces, satisfy the estimate $\Phi^{-1}(u) \leq C\Phi_1^{-1}(u)\Phi_2^{-1}(u)$ for all $u > 0$, we prove that if the pointwise product xy belongs to $L^\Phi(\mu)$ for all $y \in L^{\Phi_1}(\mu)$, then $x \in L^{\Phi_2}(\mu)$. The result with some restrictions either on Young functions or on the measure μ was proved by Maligranda and Persson (Indag. Math. **51** (1989), 323–338). Our result holds for any collection of three Young functions satisfying the above estimate and for an arbitrary complete σ -finite measure μ .

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1. Introduction and the result. Let (Ω, Σ, μ) be a complete σ -finite measure space and $L^0(\Omega)$ be the space of all classes of measurable functions on Ω . Let X and Y be ideal Banach spaces in $L^0(\Omega)$ with their norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The space of pointwise multipliers $M(X, Y)$ is defined by

$$M(X, Y) = \{x \in L^0(\Omega) : xy \in Y \text{ for all } y \in X\} \quad (1)$$

and the functional on it

$$\|x\|_{M(X, Y)} = \sup\{\|xy\|_Y, y \in X, \|y\|_X \leq 1\} \quad (2)$$

defines a complete semi-norm. It is a norm and $M(X, Y)$ is an ideal Banach space if $\text{supp } X = \Omega$, that is, X has a *weak unit*, i.e., there is an element $x_0 \in X$ such that $x_0 > 0$ μ -a.e. on Ω (in particular, $X \neq \{0\}$). Several properties and some calculated concrete examples we can find in the paper by

Maligranda and Persson [6] (see also Nakai [7] and Calabuig, Delgado, and Sánchez Pérez [2]). Note that $M(X, Y)$ can be $\{0\}$.

In this note we give an improvement of one result on multipliers of Orlicz spaces given by Maligranda and Persson in [6, Theorem 4]. We need to recall some necessary definitions about Orlicz spaces.

A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a *Young function* (or sometimes also called an *Orlicz function*) if Φ is convex, nondecreasing with $\Phi(0) = 0$; we assume also that Φ is neither identically zero nor identically infinity on $(0, \infty)$ and $\lim_{u \rightarrow b(\Phi)-} \Phi(u) = \Phi(b(\Phi))$ if $b(\Phi) < \infty$, where $b(\Phi) = \sup\{u > 0 : \Phi(u) < \infty\}$.

Note that from the convexity and $\Phi(0) = 0$ it follows that $\lim_{u \rightarrow 0+} \Phi(u) = \Phi(0) = 0$. Furthermore, from the convexity and $\Phi \not\equiv 0$ it follows that $\lim_{u \rightarrow \infty} \Phi(u) = \infty$.

If we denote $a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}$, then $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$ and $a(\Phi) < \infty$, $b(\Phi) > 0$, since Young function is neither identically zero nor identically infinity on $(0, \infty)$. Moreover, $a(\Phi) = 0$ if Φ is 0 only at 0 and $b(\Phi) = \infty$ if $\Phi(u) < \infty$ for $u \in [0, \infty)$. If Φ takes only two values 0 and ∞ , then $0 < a(\Phi) = b(\Phi) < \infty$. The function Φ is continuous and nondecreasing on $[0, b(\Phi))$ and is strictly increasing on $[a(\Phi), b(\Phi))$.

The *Orlicz space* $L^\Phi = L^\Phi(\Omega)$ generated by the Young function Φ is defined as

$$L^\Phi(\Omega) = \{x \in L^0(\Omega) : I_\Phi(x/\lambda) < \infty \text{ for some } \lambda = \lambda(x) > 0\},$$

where $I_\Phi(x) := \int_\Omega \Phi(|x(t)|) d\mu(t)$. It is an ideal Banach space with the *Luxemburg–Nakano norm*

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}.$$

It is also well-known that $\|x\|_\Phi \leq 1$ if and only if $I_\Phi(x) \leq 1$. In particular, if $x \in L^\Phi$ and $x \neq 0$, then $I_\Phi(x/\|x\|_\Phi) \leq 1$ (cf. [4], [5] and [3]). Note that $\text{supp } L^\Phi = \Omega$, that is, the Orlicz space L^Φ has a weak unit.

For two ideal Banach spaces on Ω the symbol $X \xhookrightarrow{C} Y$ means that the embedding $X \subset Y$ is continuous with a norm which is not bigger than C , i.e., $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$.

To state our main result we will also need an inverse of Young function Φ , namely the right-continuous inverse in a generalized sense (cf. O’Neil [8]). For a Young function Φ and $v \in [0, \infty)$ let

$$\Phi^{-1}(v) = \inf\{u \geq 0 : \Phi(u) > v\} \quad \text{and} \quad \Phi^{-1}(\infty) = \lim_{v \rightarrow \infty} \Phi^{-1}(v).$$

Theorem 1. *If Φ, Φ_1, Φ_2 are Young functions and there exist a constant $C > 0$ such that*

$$\Phi^{-1}(u) \leq C\Phi_1^{-1}(u)\Phi_2^{-1}(u) \quad \text{for all } u > 0, \quad (3)$$

then $M(L^{\Phi_1}, L^\Phi) \xhookrightarrow{C} L^{\Phi_2}$.

Maligranda and Persson [6, Theorem 4] (cf. also [5, Theorem 10.5]) proved this theorem for finite-valued strictly increasing Young functions and under the additional assumption that either measure μ is nonatomic or $\Phi(u) = u^r$

and $u^{-r}\Phi_1(u)$ is nondecreasing. We will prove here this theorem in the general case.

Remark 2. If the reverse inequality to (3) holds, that is, $\Phi_1^{-1}(u)\Phi_2^{-1}(u) \leq A\Phi^{-1}(u)$ for some $A > 0$ and for all $u > 0$, then $L^{\Phi_2} \xrightarrow{2A} M(L^{\Phi_1}, L^{\Phi})$ (cf. [1], [8], [5], [6]). Putting then these results together we have that the equivalence $\Phi^{-1} \approx \Phi_1^{-1} \cdot \Phi_2^{-1}$ on the inverses of Young functions gives the equality $M(L^{\Phi_1}, L^{\Phi}) = L^{\Phi_2}$ with equivalent norms, where the considered Orlicz spaces are on arbitrary σ -finite complete measure spaces (Ω, Σ, μ) .

In particular, if $\Phi_2(u) = 0$ for $0 \leq u \leq 1$ and $\Phi_2(u) = \infty$ for $u > 1$, then $L^{\Phi_2} = L^\infty$ and since $\Phi_2^{-1}(u) = 1$ for $0 < u < \infty$ it follows that $M(L^{\Phi}, L^{\Phi}) = L^\infty$ for any Young function Φ . The last equality is true even for more general spaces, that is, $M(X, X) = L^\infty$ for any ideal Banach space X with $\text{supp } X = \Omega$ (cf. [6, Theorem 1]).

2. Proof of the main result. We define three subsets of Young functions $\mathcal{Y}^{(i)}$ ($i = 1, 2, 3$) as

$$\begin{aligned}\mathcal{Y}^{(1)} &= \{\Phi : b(\Phi) = \infty\}, \\ \mathcal{Y}^{(2)} &= \{\Phi : b(\Phi) < \infty, \Phi(b(\Phi)) = \infty\}, \\ \mathcal{Y}^{(3)} &= \{\Phi : b(\Phi) < \infty, \Phi(b(\Phi)) < \infty\}.\end{aligned}$$

If $\Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then Φ is continuous, strictly increasing on $[a(\Phi), b(\Phi))$ and bijective from $[a(\Phi), b(\Phi))$ to $[0, \infty)$.

It is known that $I_\Phi(x/\|x\|_\Phi) = 1$ for all $x \in L^\Phi$ if and only if $\Phi \in \Delta_2$ (condition Δ_2 corresponding to the measure space). We will prove a similar result for simple functions; it is important to note that we don't assume explicitly the Δ_2 -condition.

Lemma 3. *If $\Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ and $x \neq 0$ is a simple function, then $I_\Phi(x/\|x\|_\Phi) = 1$.*

Proof. Case 1. $\Phi \in \mathcal{Y}^{(1)}$: In this case Φ is strictly increasing and bijective from $(a(\Phi), \infty)$ to $(0, \infty)$. Let x be a simple function. We may assume that $x \geq 0$, i.e.,

$$x = \sum_{k=1}^N c_k \chi_{A_k}, \quad 0 < c_1 < c_2 < \dots < c_N < \infty, \quad 0 < \mu(A_k) < \infty,$$

where A_k are pairwise disjoint. Then every $\Phi(c_k/\lambda)$ is continuous and non-increasing with respect to $\lambda > 0$. Moreover, $\Phi(c_k/\lambda)$ is strictly decreasing on $(0, c_k/a(\Phi))$ (for $a(\Phi) = 0$ we understand $c_k/a(\Phi) = \infty$). Therefore,

$$I_\Phi\left(\frac{x}{\lambda}\right) = \sum_{k=1}^N \Phi(c_k/\lambda) \mu(A_k)$$

is continuous and strictly decreasing on $(0, c_N/a(\Phi))$. Since $\lim_{\lambda \rightarrow 0} I_\Phi(x/\lambda) = \infty$ and $\lim_{\lambda \rightarrow c_N/a(\Phi)} I_\Phi(x/\lambda) = 0$, we obtain that $I_\Phi(x/\cdot)$ is bijective from

$(0, c_N/a(\Phi))$ to $(0, \infty)$. That is, there exists a unique $\lambda \in (0, c_N/a(\Phi))$ such that $I_\Phi(x/\lambda) = 1$.

Case 2. $\Phi \in \mathcal{Y}^{(2)}$: In this case Φ is strictly increasing and bijective from $(a(\Phi), b(\Phi))$ to $(0, \infty)$. Let x be a simple function as in Case 1. Then, in the same way as in Case 1, we obtain that $I_\Phi(x/\cdot)$ is bijective from $(c_N/b(\Phi), c_N/a(\Phi))$ to $(0, \infty)$. That is, there exists a unique $\lambda \in (c_N/b(\Phi), c_N/a(\Phi))$ such that $I_\Phi(x/\lambda) = 1$. $\square$

To prove the theorem we collect some properties of Φ and its inverse:

- (i) $\Phi(\Phi^{-1}(u)) \leq u$ for all $u \in [0, \infty)$ and $u \leq \Phi^{-1}(\Phi(u))$ if $\Phi(u) < \infty$ (Property 1.3 in [8]).
- (ii) $\Phi^{-1}(\Phi(u)) = u$ if $0 < \Phi(u) < \infty$.
- (iii) If $\Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Phi(\Phi^{-1}(u)) = u$ for all $u \in [0, \infty)$.
- (iv) If $\Phi \in \mathcal{Y}^{(3)}$ and $0 < \delta < 1$, then there exists a Young function $\Psi \in \mathcal{Y}^{(2)}$ such that $b(\Phi) = b(\Psi)$ and

$$\Psi(\delta u) \leq \Phi(u) \leq \Psi(u) \quad \text{for all } u \in [0, \infty).$$

To see (iv) we only set $\Psi = \Phi + \Theta$, where we choose $\Theta \in \mathcal{Y}^{(2)}$ such that $a(\Theta) = \delta b(\Phi)$ and $b(\Theta) = b(\Phi)$.

Proof of Theorem 1. **Case 1.** Φ and Φ_2 are in $\mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$: Let $x \in M(L^{\Phi_1}, L^\Phi)$ and $x \neq 0$. Assume first that x is a simple function. Then $x \in L^{\Phi_2}$ and

$$y(t) := \Phi_2 \left(\frac{|x(t)|}{\|x\|_{\Phi_2}} \right) < \infty \quad \text{a.e. in } \Omega.$$

Put

$$z(t) = \begin{cases} \Phi_1^{-1}(y(t)) & \text{if } 0 < y(t) < \infty, \\ 0 & \text{if } y(t) = 0. \end{cases}$$

From the property (i) it follows that $\Phi_1(z(t)) \leq y(t)$ a.e. in Ω and

$$I_{\Phi_1}(z) \leq I_{\Phi_2} \left(\frac{x}{\|x\|_{\Phi_2}} \right) \leq 1,$$

which gives $\|z\|_{\Phi_1} \leq 1$. If $y(t) > 0$, then by the property (ii) and the assumption (3),

$$\begin{aligned} z(t) \frac{|x(t)|}{\|x\|_{\Phi_2}} &= \Phi_1^{-1}(y(t)) \Phi_2^{-1} \left(\Phi_2 \left(\frac{|x(t)|}{\|x\|_{\Phi_2}} \right) \right) \\ &= \Phi_1^{-1}(y(t)) \Phi_2^{-1}(y(t)) \geq \frac{1}{C} \Phi^{-1}(y(t)) \end{aligned}$$

and hence, by (iii),

$$\Phi \left(Cz(t) \frac{|x(t)|}{\|x\|_{\Phi_2}} \right) \geq \Phi(\Phi^{-1}(y(t))) = y(t).$$

If $y(t) = 0$, then $z(t) = 0$ and $\Phi \left(Cz(t) \frac{|x(t)|}{\|x\|_{\Phi_2}} \right) = 0$. Thus, by Lemma 3,

$$I_\Phi \left(Cz \frac{x}{\|x\|_{\Phi_2}} \right) \geq I_{\Phi_2} \left(\frac{x}{\|x\|_{\Phi_2}} \right) = 1$$

and so $\|zx\|_{\Phi} \geq \frac{\|x\|_{\Phi_2}}{C}$, that is, $\|x\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \frac{1}{C} \|x\|_{\Phi_2}$.

In the general case, x can be approximated by a sequence of simple functions $\{x_n\}$ such that

$$0 \leq x_1 \leq x_2 \leq \cdots \rightarrow |x| \quad \text{a.e. in } \Omega,$$

since μ is a σ -finite measure. Then

$$\|x\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \|x_n\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \frac{1}{C} \|x_n\|_{\Phi_2}$$

by our first part of the proof. Using the Fatou property of the Luxemburg–Nakano norm $\|\cdot\|_{\Phi_2}$ in the Orlicz space L^{Φ_2} we obtain

$$\|x\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \frac{1}{C} \|x\|_{\Phi_2}.$$

Case 2. $\Phi \in \mathcal{Y}^{(3)}$ or $\Phi_2 \in \mathcal{Y}^{(3)}$: We consider only the case that both Φ and Φ_2 are in $\mathcal{Y}^{(3)}$, since other cases are similar. In this case, by (iv), for all $0 < \delta < 1$, there exist $\Psi \in \mathcal{Y}^{(2)}$ and $\Psi_2 \in \mathcal{Y}^{(2)}$ such that

$$\Psi(\delta u) \leq \Phi(u) \leq \Psi(u), \quad \Psi_2(\delta u) \leq \Phi_2(u) \leq \Psi_2(u) \quad \text{for all } u.$$

It follows that

$$\begin{aligned} \delta \Phi^{-1}(u) &\leq \Psi^{-1}(u) \leq \Phi^{-1}(u), & \delta \Phi_2^{-1}(u) &\leq \Psi_2^{-1}(u) \leq \Phi_2^{-1}(u), \\ \delta \|x\|_{\Psi} &\leq \|x\|_{\Phi} \leq \|x\|_{\Psi}, & \delta \|x\|_{\Psi_2} &\leq \|x\|_{\Phi_2} \leq \|x\|_{\Psi_2}, \end{aligned}$$

and

$$\delta \|x\|_{M(L^{\Phi_1}, L^{\Psi})} \leq \|x\|_{M(L^{\Phi_1}, L^{\Phi})} \leq \|x\|_{M(L^{\Phi_1}, L^{\Psi})}.$$

Using the inequality

$$\Psi^{-1}(u) \leq \frac{C}{\delta} \Phi_1^{-1}(u) \Psi_2^{-1}(u),$$

which follows by (3) and the definitions of Ψ and Ψ_2 , we have

$$\|x\|_{M(L^{\Phi_1}, L^{\Psi})} \geq \frac{\delta}{C} \|x\|_{\Psi_2}$$

by Case 1. Then

$$\|x\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \frac{\delta^2}{C} \|x\|_{\Phi_2}$$

holds for all $0 < \delta < 1$. Therefore,

$$\|x\|_{M(L^{\Phi_1}, L^{\Phi})} \geq \frac{1}{C} \|x\|_{\Phi_2},$$

and the proof is finished. $\square$

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LECH MALIGRANDA
Department of Mathematics,
Luleå University of Technology,
971 87 Luleå, Sweden
e-mail: lech@sm.luth.se

EIICHI NAKAI
Department of Mathematics,
Osaka Kyoiku University,
Kashiwara, Osaka 582-8582, Japan
e-mail: enakai@cc.osaka-kyoiku.ac.jp

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