Interpolation of Cesàro sequence and function spaces

by

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Abstract. The interpolation properties of Cesàro sequence and function spaces are investigated. It is shown that $\text{Ces}_p(I)$ is an interpolation space between $\text{Ces}_{p_0}(I)$ and $\text{Ces}_{p_1}(I)$ for $1 < p_0 < p_1 \leq \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, where $I = [0, \infty)$ or $[0, 1]$. The same result is true for Cesàro sequence spaces. On the other hand, $\text{Ces}_p[0, 1]$ is not an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$.

1. Introduction and preliminaries. The structure of Cesàro sequence and function spaces was investigated by several authors (see, for example, [Be], [MPS], [A], [AM] and references therein). Here we are interested in interpolation properties of these spaces. The main purpose is to give interpolation theorems for the Cesàro sequence spaces ces_p and Cesàro function spaces $\text{Ces}_p(I)$ on $I = [0, \infty)$ and $I = [0, 1]$. In the case of $I = [0, \infty)$ some interpolation results for Cesàro function spaces are contained implicitly in [MS]. Moreover, using the so-called K^+ -method of interpolation it was proved in [CFM] that the Cesàro sequence space ces_p is an interpolation space with respect to the couple $(l_1, l_1(2^{-k}))$.

Our main aim is to give a rather complete description of Cesàro spaces as interpolation spaces with respect to appropriate couples of weighted L_1 -spaces as well as Cesàro spaces. For example, if either $I = [0, \infty)$ or $[0, 1]$ and $1 < p_0 < p_1 \leq \infty$ with $1/p = (1 - \theta)/p_0 + \theta/p_1$ for $0 < \theta < 1$, then

$$(1.1) \quad (\text{Ces}_{p_0}(I), \text{Ces}_{p_1}(I))_{\theta, p} = \text{Ces}_p(I) \quad \text{and} \quad (\text{ces}_{p_0}, \text{ces}_{p_1})_{\theta, p} = \text{ces}_p,$$

where $(\cdot, \cdot)_{\theta, p}$ denotes the K -method of interpolation.

We have a completely different situation in a more interesting and non-trivial case when $I = [0, 1]$ and $p_0 = 1$, $p_1 = \infty$. It turns out that $\text{Ces}_p[0, 1]$ is not an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$, whereas $(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{\theta, p}$ for $1 < p < \infty$ is a weighted Cesàro function space.

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Let us collect some necessary definitions and notations related to the interpolation theory of operators as well as Cesàro, Copson and down spaces.

For two normed spaces X and Y the symbol $X \xhookrightarrow{C} Y$ means that the imbedding $X \subset Y$ is continuous with norm not greater than C , i.e., $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$, and $X \hookrightarrow Y$ means that $X \xhookrightarrow{C} Y$ for some $C > 0$. Moreover, $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, that is, the spaces are the same and the norms are equivalent. If f and g are real functions, then $f \approx g$ means that $c^{-1}g \leq f \leq cg$ for some $c \geq 1$.

For a Banach couple $\bar{X} = (X_0, X_1)$ of two compatible Banach spaces X_0 and X_1 consider the Banach spaces $X_0 \cap X_1$ and $X_0 + X_1$ with their natural norms

$$\|f\|_{X_0 \cap X_1} = \max(\|f\|_{X_0}, \|f\|_{X_1}) \quad \text{for } f \in X_0 \cap X_1,$$

and for $f \in X_0 + X_1$,

$$\|f\|_{X_0 + X_1} = \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\}.$$

For more careful definitions of a Banach couple, intermediate and interpolation spaces with some results introduced briefly below, see [BK, pp. 91–173, 289–314, 338–359] and [BS, pp. 95–116].

A Banach space X is called an *intermediate space* between X_0 and X_1 if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$. Such a space X is called an *interpolation space* between X_0 and X_1 (and we write $X \in \text{Int}(X_0, X_1)$) if, for any bounded linear operator $T : X_0 + X_1 \rightarrow X_0 + X_1$ whose restriction $T|_{X_i} : X_i \rightarrow X_i$ is bounded for $i = 0, 1$, the restriction $T|_X : X \rightarrow X$ is also bounded and $\|T\|_{X \rightarrow X} \leq C \max\{\|T\|_{X_0 \rightarrow X_0}, \|T\|_{X_1 \rightarrow X_1}\}$ for some $C \geq 1$. If $C = 1$, then X is called an *exact interpolation space* between X_0 and X_1 .

An *interpolation method* or *interpolation functor* $\mathcal{F}$ is a construction (a rule) which assigns to every Banach couple $\bar{X} = (X_0, X_1)$ an interpolation space $\mathcal{F}(\bar{X})$ between X_0 and X_1 . The interpolation functor $\mathcal{F}$ is called *exact* if the space $\mathcal{F}(\bar{X})$ is an exact interpolation space for every Banach couple $\bar{X}$. One of the most important interpolation methods is the *K-method*, also known as the *real Lions–Peetre interpolation method*. For a Banach couple $\bar{X} = (X_0, X_1)$ the *Peetre K-functional* of an element $f \in X_0 + X_1$ is defined for $t > 0$ by

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\}.$$

Then the *spaces of the K-method of interpolation* are

$$(X_0, X_1)_{\theta, p} = \left\{ f \in X_0 + X_1 : \right. \\ \left. \|f\|_{\theta, p} = \left(\int_0^\infty [t^{-\theta} K(t, f; X_0, X_1)]^p \frac{dt}{t} \right)^{1/p} < \infty \right\}$$

if $0 < \theta < 1$ and $1 \leq p < \infty$, and

$$(X_0, X_1)_{\theta, \infty} = \left\{ f \in X_0 + X_1 : \|f\|_{\theta, \infty} = \sup_{t>0} K(t, f; X_0, X_1)/t^\theta < \infty \right\}$$

if $0 \leq \theta \leq 1$. Very useful in calculations is the so-called *reiteration formula* showing the stability of the K -method of interpolation. If $1 \leq p_0, p_1, p \leq \infty$, $0 < \theta_0, \theta_1, \theta < 1$ and $\theta_0 \neq \theta_1$, then with equivalent norms

$$(1.2) \quad ((X_0, X_1)_{\theta_0, p_0}, (X_0, X_1)_{\theta_1, p_1})_{\theta, p} = (X_0, X_1)_{\eta, p},$$

where $\eta = (1 - \theta)\theta_0 + \theta\theta_1$ (see [BS, Theorem 2.4, p. 311], [BL, Theorems 3.5.3], [BK, Theorem 3.8.10]) and [Tr, Theorem 1.10.2]).

The space $(X_0, X_1)_{\Phi}^K$ of the *general K -method of interpolation*, where Φ is a *parameter of the K -method*, i.e., a Banach function space on $((0, \infty), dt/t)$ containing the function $t \mapsto \min\{1, t\}$, is the Banach space of all $f \in X_0 + X_1$ such that $K(\cdot, f; X_0, X_1) \in \Phi$ with the norm $\|f\|_{K_\Phi} = \|K(\cdot, f; X_0, X_1)\|_\Phi$. The space $(X_0, X_1)_{\Phi}^K$ is an exact interpolation space between X_0 and X_1 .

In particular, if $L_p = L_p(\Omega, \mu)$, where (Ω, μ) is a complete σ -finite measure space, then for any $f \in L_1 + L_\infty$ we have

$$(1.3) \quad K(t, f; L_1, L_\infty) = \int_0^t f^*(s) ds.$$

Here and below, f^* denotes the non-increasing rearrangement of $|f|$ defined by $f^*(s) = \inf\{\lambda > 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\}) \leq s\}$ (see [BK, Proposition 3.1.1], [KPS, pp. 78–79], [BS, Theorem 6.2, pp. 74–75]). Moreover, for two non-negative weight functions w_0, w_1 and for $f \in L_1(w_0) + L_1(w_1)$ we have

$$(1.4) \quad K(t, f; L_1(w_0), L_1(w_1)) = \|\min(w_0, tw_1)f\|_{L_1}$$

(see [BK, Proposition 3.1.17] and [Ov, p. 391]).

If the inequality $K(t, g; X_0, X_1) \leq K(t, f; X_0, X_1)$ ($t > 0$) with $f \in X$ and $g \in X_0 + X_1$ implies that $g \in X$ and $\|g\|_X \leq C\|f\|_X$ for any $X \in \text{Int}(X_0, X_1)$ and some $C \geq 1$ independent of X , f and g , then (X_0, X_1) is called a *K -monotone* or *Calderón–Mityagin couple*. For every K -monotone couple (X_0, X_1) the spaces $(X_0, X_1)_{\Phi}^K$ of the general K -method are the only interpolation spaces between X_0 and X_1 (see [BK]).

Now, we recall the definitions of Cesàro spaces. The *Cesàro sequence spaces* ces_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{\text{ces}(p)} = \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |x_k| \right)^p \right]^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|x\|_{\text{ces}(\infty)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n |x_k| < \infty \quad \text{for } p = \infty.$$

The *Cesàro function spaces* $\text{Ces}_p = \text{Ces}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0, 1]$ or $I = [0, \infty)$ such that

$$\|f\|_{\text{Ces}(p)} = \left[\int_I \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|f\|_{\text{Ces}(\infty)} = \sup_{0 < x \in I} \frac{1}{x} \int_0^x |f(t)| dt < \infty \quad \text{for } p = \infty.$$

Cesàro spaces are Banach lattices which are not symmetric except when they are trivial, namely, $\text{ces}_1 = \{0\}$, $\text{Ces}_1[0, \infty) = \{0\}$. By a *symmetric space* we mean a Banach lattice X on I with the additional property: if $g^*(t) = f^*(t)$ for all $t > 0$, $f \in X$ and $g \in L^0(I)$ (the set of all classes of Lebesgue measurable real functions on I), then $g \in X$ and $\|g\|_X = \|f\|_X$ (cf. [BS], [KPS]). Moreover, $l_p \xrightarrow{p'} \text{ces}_p$, $L_p(I) \xrightarrow{p'} \text{Ces}_p(I)$ for $1 < p \leq \infty$ (in what follows $1/p + 1/p' = 1$), and if $1 < p < q < \infty$, then $\text{ces}_p \xrightarrow{1} \text{ces}_q \xrightarrow{1} \text{ces}_\infty$. Also for $I = [0, 1]$ and $1 < p < q < \infty$ we have $L_\infty \xrightarrow{1} \text{Ces}_\infty \xrightarrow{1} \text{Ces}_q \xrightarrow{1} \text{Ces}_p \xrightarrow{1} \text{Ces}_1 = L_1(\ln(1/t))$ and $\text{Ces}_\infty \xrightarrow{1} L_1$.

Let $1 \leq p < \infty$. The *Copson sequence spaces* cop_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{\text{cop}(p)} = \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p \right]^{1/p} < \infty,$$

and the *Copson function spaces* $\text{Cop}_p = \text{Cop}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0, \infty)$ or $I = [0, 1]$ such that

$$\|f\|_{\text{Cop}(p)} = \left[\int_0^\infty \left(\int_x^\infty \frac{|f(t)|}{t} dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } I = [0, \infty),$$

and

$$\|f\|_{\text{Cop}(p)} = \left[\int_0^1 \left(\int_x^1 \frac{|f(t)|}{t} dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } I = [0, 1].$$

Sometimes we will use the *Cesàro operators*

$$C_d x(n) = \frac{1}{n} \sum_{k=1}^n |x_k|, \quad C f(x) = \frac{1}{x} \int_0^x |f(t)| dt$$

and the *Copson operators*

$$C_d^* x(n) = \sum_{k=n}^{\infty} \frac{|x_k|}{k}, \quad C^* f(x) = \int_{(x, \infty) \cap I} \frac{|f(t)|}{t} dt$$

related to appropriate spaces. Then ces_p (resp. cop_p) consists of all real sequences $x = \{x_k\}$ such that $C_d x \in l_p$ (resp. $C_d^* x \in l_p$), and $\text{Ces}_p(I)$ (resp. $\text{Cop}_p(I)$) consists of all classes of Lebesgue measurable real functions f on I such that $Cf \in L_p(I)$ (resp. $C^*f \in L_p(I)$) with natural norms. By the Copson inequalities (cf. [HLP, Theorems 328 and 331], [Be, p. 25] and [KMP, p. 159]), valid for $1 \leq p < \infty$, we have $\|C_d^* x\|_{l_p} \leq p\|x\|_{l_p}$ for $x \in l_p$ and $\|C^* f\|_{L_p(I)} \leq p\|f\|_{L_p(I)}$ for $f \in L_p(I)$. Therefore, $l_p \xrightarrow{p} \text{cop}_p$ and $L_p \xrightarrow{p} \text{Cop}_p$.

We can define similarly the spaces cop_∞ and Cop_∞ but it is easy to see that $\text{cop}_\infty = l_1(1/k)$ and $\text{Cop}_\infty = L_1(1/t)$. Moreover, for $I = [0, 1]$ we have $L_p \xrightarrow{p} \text{Cop}_p \xrightarrow{1} \text{Cop}_1 = L_1$.

We will also consider more general Cesàro spaces $\text{Ces}_E(I)$, where E is a Banach function space on I with the natural norm $\|f\|_{\text{Ces}(E)} = \|Cf\|_E$.

For a Banach function space E on $I = [0, \infty)$ the *down space* $E^\downarrow$ is the collection of all $f \in L^0$ such that

$$\|f\|_{E^\downarrow} = \sup \int_I |f(t)|g(t) dt < \infty,$$

where the supremum is taken over all non-negative, non-increasing Lebesgue measurable functions g from the Köthe dual E' of E such that $\|g\|_{E'} \leq 1$. Let us recall that the *Köthe dual* of a Banach function space E is defined as

$$E' = \left\{ f \in L^0 : \|f\|_{E'} = \sup_{\|g\|_{E'} \leq 1} \int_I |f(t)g(t)| dt < \infty \right\}.$$

It is routine to check that the space $E^\downarrow$ has the *Fatou property*, that is, if $0 \leq f_n$ increases to f a.e. on I and $\sup_{n \in \mathbb{N}} \|f_n\|_{E^\downarrow} < \infty$, then $f \in E^\downarrow$ and $\|f_n\|_{E^\downarrow}$ increases to $\|f\|_{E^\downarrow}$. Moreover, $E'' \xrightarrow{1} E^\downarrow$, where E'' is the second Köthe dual of E . Recall also that a Banach function space E has the Fatou property if and only if $E = E''$ with equality of norms.

Sinnamon [Si01, Theorem 3.1] proved that if E is a symmetric space on $I = [0, \infty)$, then $\|f\|_{E^\downarrow} \approx \|Cf\|_E$ if and only if the Cesàro operator $C : E \rightarrow E$ is bounded. In particular, then $E^\downarrow = \text{Ces}_E$. Moreover, $(L_1)^\downarrow = L_1$ since

$$\|f\|_{L_1^\downarrow} = \sup_{0 \leq g} \frac{\int_0^\infty |f(t)|g(t) dt}{\|g\|_{L_\infty}} \geq \sup_{0 \leq g \downarrow} \frac{\int_0^\infty |f(t)|g(t) dt}{\|g\|_{L_\infty}} \geq \frac{\int_0^\infty |f(t)| dt}{\|1\|_{L_\infty}} = \|f\|_{L_1}$$

(cf. [MS, p. 194]).

The paper is organized as follows. In Section 2 we prove that the Cesàro and Copson sequence and function spaces on $[0, \infty)$ are interpolation spaces obtained by the K -method from weighted L_1 -spaces. At the same time, in the case of $I = [0, 1]$, only the Copson spaces can be described as interpola-

tion spaces with respect to the analogous couple of weighted L_1 -spaces (see Theorem 1(iii)). In particular, we obtain a new description of the interpolation spaces $(L_1, L_1(1/t))_{1-1/p, p}$ in the off-diagonal case both for $I = [0, \infty)$ and $I = [0, 1]$.

In Section 3 it is shown that the Cesàro function spaces $\text{Ces}_p[0, \infty)$, $1 < p < \infty$, can be obtained by the K -method of interpolation also from the couple $(L_1[0, \infty), \text{Ces}_\infty[0, \infty))$. Hence, applying the reiteration theorem, we conclude that $\text{Ces}_p[0, \infty)$ is an interpolation space with respect to the couple $(\text{Ces}_{p_0}[0, \infty), \text{Ces}_{p_1}[0, \infty))$ for arbitrary $1 < p_0 < p_1 \leq \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$.

In Section 4 interpolation of Cesàro function spaces on $[0, 1]$ is investigated. We prove that for $1 < p < \infty$,

$$(L_1(1-t)[0, 1], \text{Ces}_\infty[0, 1])_{\theta, p} = \text{Ces}_p[0, 1] \quad \text{with } \theta = 1 - 1/p.$$

As a consequence of this result and the reiteration equality (1.2), we infer

$$(1.5) \quad (\text{Ces}_{p_0}[0, 1], \text{Ces}_{p_1}[0, 1])_{\theta, p} = \text{Ces}_p[0, 1]$$

for all $1 < p_0 < p_1 \leq \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$.

We are also interested in description of interpolation spaces between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$. In Section 5, in Theorem 3, we find an equivalent expression for the K -functional with respect to this couple and then in Section 6 we prove that the real interpolation spaces

$$(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, p}$$

for $1 < p < \infty$ can be identified with the weighted Cesàro function spaces $\text{Ces}_p(\ln(e/t))[0, 1]$.

Finally, in Section 7, we show in Theorem 6 that $\text{Ces}_p[0, 1]$ for $1 < p < \infty$ are not interpolation spaces between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$.

2. Cesàro and Copson spaces as interpolation spaces with respect to weighted L_1 -spaces. We start with the main result in this part.

THEOREM 2.1.

(i) If $1 < p < \infty$, then

$$(l_1, l_1(1/k))_{1-1/p, p} = \text{ces}_p = \text{cop}_p.$$

(ii) If $I = [0, \infty)$ and $1 < p < \infty$, then

$$(L_1, L_1(1/t))_{1-1/p, p} = \text{Ces}_p = \text{Cop}_p.$$

(iii) If $I = [0, 1]$ and $1 < p \leq \infty$, then

$$(L_1, L_1(1/t))_{1-1/p, p} = \text{Cop}_p.$$

Moreover, $\text{Cop}_p \xrightarrow{p'} \text{Ces}_p$ and the reverse imbedding does not hold.

Proof. (i) If $f \in l_1 + l_1(1/k)$, then $K(t, x; l_1, l_1(1/k)) = t \sum_{k=1}^{\infty} |x_k|/k$ for $0 < t \leq 1$, and

$$K(t, x; l_1, l_1(1/k)) = \sum_{k=1}^{\infty} |x_k| \min(1, t/k) = \sum_{k=1}^{[t]} |x_k| + t \sum_{k=[t]+1}^{\infty} \frac{|x_k|}{k}.$$

for $t \geq 1$. Therefore, for $n \leq t < n+1$ ($n \geq 1$), we have

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \leq \frac{1}{n} \sum_{k=1}^n |x_k| + \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1).$$

Since

$$\begin{aligned} C_d C_d^* x(n) &= \frac{1}{n} \sum_{m=1}^n \left(\sum_{k=m}^{\infty} \frac{|x_k|}{k} \right) \\ &= \frac{1}{n} \left[\sum_{k=1}^n \left(\sum_{m=1}^k \frac{|x_k|}{k} \right) + \sum_{k=n+1}^{\infty} \left(\sum_{m=1}^n \frac{|x_k|}{k} \right) \right] \\ &= \frac{1}{n} \sum_{k=1}^n |x_k| + \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1), \end{aligned}$$

it follows that, for $n \leq t < n+1$ ($n \geq 1$),

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \leq C_d C_d^* x(n).$$

Using the classical Hardy inequality (cf. [HLP, Theorem 326] or [KMP, Theorem 1]), we obtain

$$\begin{aligned} \|x\|_{1-1/p, p} &= \left(\int_0^{\infty} \left(\frac{K(t, x; l_1, l_1(1/k))}{t} \right)^p dt \right)^{1/p} \\ &= \left[C_d^* x(1)^p + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{K(t, x)}{t} \right)^p dt \right]^{1/p} \\ &\leq \left[C_d^* x(1)^p + \sum_{n=1}^{\infty} (C_d C_d^* x(n))^p \right]^{1/p} \\ &\leq C_d^* x(1) + \|C_d C_d^* x\|_{l_p} \leq C_d^* x(1) + p' \|C_d^* x\|_{l_p} \\ &\leq (p' + 1) \|C_d^* x\|_{l_p} = (p' + 1) \|x\|_{\text{cop}(p)}. \end{aligned}$$

This means that $\text{cop}_p \hookrightarrow (l_1, l_1(1/k))_{1-1/p, p}$. On the other hand, for $n \leq t < n+1$ ($n \geq 1$), we have

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \geq \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d^* x(n+1)$$

and

$$\begin{aligned} \|x\|_{1-1/p,p} &= \left(\int_0^\infty \left(\frac{K(t, x; l_1, l_1(1/k))}{t} \right)^p dt \right)^{1/p} \\ &\geq \left(C_d^* x(1)^p + \sum_{n=1}^\infty C_d^* x(n+1)^p \right)^{1/p} = \|C_d^* x\|_{l_p} = \|x\|_{\text{cop}(p)}, \end{aligned}$$

which gives the reverse imbedding $(l_1, l_1(1/k))_{1-1/p,p} \xrightarrow{1} \text{cop}_p$. The equality $\text{ces}_p = \text{cop}_p$, $1 < p < \infty$, was proved by Bennett (cf. [Be, Theorems 4.5 and 6.6]).

(ii) For $f \in L_1 + L_1(1/s) = L_1(\min(1, 1/s))$ we have

$$\begin{aligned} K(t, f; L_1, L_1(1/s)) &= \int_0^\infty |f(s)| \min(1, t/s) ds \\ &= \int_0^t |f(s)| ds + t \int_t^\infty \frac{|f(s)|}{s} ds. \end{aligned}$$

Thus,

$$\frac{K(t, f; L_1, L_1(1/s))}{t} = Cf(t) + C^*f(t), \quad t > 0,$$

and therefore

(2.1)

$$\|f\|_{1-1/p,p} = \left(\int_0^\infty \left(\frac{K(t, f; L_1, L_1(1/s))}{t} \right)^p dt \right)^{1/p} = \|Cf + C^*f\|_{L_p(0,\infty)}.$$

Since, by the Fubini theorem,

$$\begin{aligned} C^*Cf(t) &= \int_t^\infty \left(\frac{1}{u^2} \int_0^u |f(s)| ds \right) du \\ &= \int_0^t \left(\int_t^\infty \frac{1}{u^2} du \right) |f(s)| ds + \int_t^\infty \left(\int_s^\infty \frac{1}{u^2} du \right) |f(s)| ds \\ &= \frac{1}{t} \int_0^t |f(s)| ds + \int_t^\infty \frac{|f(s)|}{s} ds = Cf(t) + C^*f(t), \end{aligned}$$

from the Copson inequality (cf. [HLP, Theorem 328]) it follows that

$$\begin{aligned} \|f\|_{\text{Ces}(p)} &= \|Cf\|_{L_p(0,\infty)} \leq \|Cf + C^*f\|_{L_p(0,\infty)} \\ &= \|C^*Cf\|_{L_p(0,\infty)} \leq p \|Cf\|_{L_p(0,\infty)} = p \|f\|_{\text{Ces}(p)}. \end{aligned}$$

Combining this with (2.1), we obtain $\|f\|_{1-1/p,p} \approx \|f\|_{\text{Ces}(p)}$.

On the other hand, since

$$\begin{aligned}
 CC^*f(t) &= \frac{1}{t} \int_0^t \left(\int_u^\infty \frac{|f(s)|}{s} ds \right) du \\
 &= \frac{1}{t} \int_0^t \left(\int_0^s du \right) \frac{|f(s)|}{s} ds + \frac{1}{t} \int_t^\infty \left(\int_t^s du \right) \frac{|f(s)|}{s} ds \\
 &= \frac{1}{t} \int_0^t |f(s)| ds + \int_t^\infty \frac{|f(s)|}{s} ds = Cf(t) + C^*f(t),
 \end{aligned}$$

by the Hardy inequality,

$$\begin{aligned}
 \|f\|_{\text{Cop}(p)} &= \|C^*f\|_{L_p(0,\infty)} \leq \|Cf + C^*f\|_{L_p(0,\infty)} \\
 &= \|CC^*f\|_{L_p(0,\infty)} \leq p' \|C^*f\|_{L_p(0,\infty)} = p' \|f\|_{\text{Cop}(p)},
 \end{aligned}$$

and, applying (2.1) once more, we conclude that $\|f\|_{1-1/p,p} \approx \|f\|_{\text{Cop}(p)}$.

(iii) For $I = [0, 1]$ and $f \in L_1 + L_1(1/s) = L_1$ we have $K(t, f; L_1, L_1(1/s)) = \|f\|_1$ if $t \geq 1$, and

$$K(t, f; L_1, L_1(1/s)) = \int_0^t |f(s)| ds + t \int_t^\infty \frac{|f(s)|}{s} ds = tCf(t) + tC^*f(t)$$

if $0 < t \leq 1$. Therefore, for $1 < p < \infty$,

$$\begin{aligned}
 \|f\|_{1-1/p,p} &= \left(\int_0^1 [Cf(t) + C^*f(t)]^p dt + \int_1^\infty t^{-p} \|f\|_1^p dt \right)^{1/p} \\
 &= \left(\|Cf + C^*f\|_p^p + \frac{1}{p-1} \|f\|_1^p \right)^{1/p}.
 \end{aligned}$$

Firstly, the last expression is not smaller than $\|C^*f\|_p = \|f\|_{\text{Cop}(p)}$. On the other hand, since again $CC^*f(t) = Cf(t) + C^*f(t)$, by the Hardy inequality, it follows that

$$\begin{aligned}
 \|f\|_{1-1/p,p} &= \left(\|CC^*f\|_p^p + \frac{1}{p-1} \|f\|_1^p \right)^{1/p} \leq \|CC^*f\|_p + (p-1)^{-1/p} \|f\|_1 \\
 &\leq p' \|C^*f\|_p + (p-1)^{-1/p} \|f\|_{\text{Cop}(p)} = (p' + (p-1)^{-1/p}) \|f\|_{\text{Cop}(p)}.
 \end{aligned}$$

Thus, $(L_1, L_1(1/t))_{1-1/p,p} = \text{Cop}_p$ with equivalent norms for $1 < p < \infty$. For $p = \infty$ we have $(L_1, L_1(1/t))_{1,\infty} = L_1(1/t) = \text{Cop}_\infty[0, 1]$.

The imbedding $\text{Cop}_p \xrightarrow{p'} \text{Ces}_p$ for $1 < p \leq \infty$ follows from the inequality

$$\|f\|_{\text{Ces}(p)} = \|Cf\|_p \leq \|Cf + C^*f\|_p = \|CC^*f\|_p \leq p' \|C^*f\|_p = p' \|f\|_{\text{Cop}(p)}.$$

Moreover, $\text{Ces}_p[0, 1] \cap L_1[0, 1] \xrightarrow{p+1} \text{Cop}_p[0, 1]$ for $1 \leq p < \infty$. In fact, observe that in the case of $I = [0, 1]$ the composition operator C^*C has an additional

term. More precisely,

$$C^*Cf(t) = Cf(t) + C^*f(t) - \int_0^1 |f(s)| ds.$$

Therefore,

$$\begin{aligned} \|f\|_{\text{Cop}(p)} &= \|C^*f\|_p \leq \|Cf + C^*f\|_p \\ &= \|C^*Cf + \int_0^1 |f(s)| ds\|_p \leq \|C^*Cf\|_p + \|f\|_1 \\ &\leq p\|Cf\|_p + \|f\|_1 \leq (p+1) \max(\|f\|_{\text{Ces}(p)}, \|f\|_1). \end{aligned}$$

Finally, let us show that $\text{Ces}_p \leftrightarrow \text{Cop}_p$ by comparing norms of the functions $f_h(t) = \frac{1}{\sqrt{1-t}}\chi_{[h,1)}(t)$, $0 < h < 1$, in these spaces. We have

$$C^*(f_h)(t) = \begin{cases} \int_h^1 \frac{1}{s\sqrt{1-s}} ds & \text{if } 0 < t \leq h, \\ \int_t^1 \frac{1}{s\sqrt{1-s}} ds & \text{if } h \leq t \leq 1, \end{cases}$$

and

$$\begin{aligned} \|f_h\|_{\text{Cop}(p)}^p &= \|C^*(f_h)\|_p^p \geq \int_0^h \left(\int_h^1 \frac{1}{s\sqrt{1-s}} ds \right)^p dt = h \left(\int_h^1 \frac{1}{s\sqrt{1-s}} ds \right)^p \\ &\geq h \left(\int_h^1 \frac{1}{\sqrt{1-s}} ds \right)^p = 2^p h(1-h)^{p/2}. \end{aligned}$$

Also,

$$C(f_h)(t) = \begin{cases} 0 & \text{if } 0 < t \leq h, \\ (2/t)(\sqrt{1-h} - \sqrt{1-t}) & \text{if } h \leq t \leq 1, \end{cases}$$

and

$$\begin{aligned} \|f_h\|_{\text{Ces}(p)}^p &= \|C(f_h)\|_p^p = 2^p \int_h^1 \left(\frac{\sqrt{1-h} - \sqrt{1-t}}{t} \right)^p dt \\ &\leq 2^p \int_h^1 \frac{(1-h)^{p/2}}{t^p} dt = 2^p(1-h)^{p/2} \frac{1-h^{p-1}}{(p-1)h^{p-1}}. \end{aligned}$$

Thus,

$$\frac{\|f_h\|_{\text{Cop}(p)}^p}{\|f_h\|_{\text{Ces}(p)}^p} \geq \frac{2^p h(1-h)^{p/2}(p-1)h^{p-1}}{2^p(1-h)^{p/2}(1-h^{p-1})} = (p-1) \frac{h^p}{1-h^{p-1}} \rightarrow \infty \text{ as } h \rightarrow 1^+,$$

and the proof is complete. ■

REMARK 2.2. Alternatively, the space ces_p for $1 < p < \infty$ can be obtained as an interpolation space with respect to $(l_1, l_1(2^{-n}))$ by the so-called K^+ -method, a version of the standard K -method, precisely, $\text{ces}_p = (l_1, l_1(2^{-n}))_{l_p(1/n)}^{K^+}$ (cf. [CFM, proof of Theorem 6.4]).

REMARK 2.3. The results in Theorem 2.1 give a description of the real interpolation spaces $(L_1, L_1(1/t))_{1-1/p, p}$ in the off-diagonal case. Before it was only known that they are intersections of weighted $L_1(w)$ -spaces with the weights w from certain sets (cf. [Gi, Theorem 4.1], [MP, Theorem 2]) or some block spaces (cf. [AKMNP, Lemma 3.1]).

The following corollary follows directly from Theorem 2.1, the reiteration formula (1.2) and the equalities $\text{Cop}_\infty[0, 1] = L_1(1/t)$ and $\text{Cop}_1[0, 1] = L_1$.

COROLLARY 2.4. *If $1 < p_0 < p_1 < \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, then*

$$(2.2) \quad (\text{ces}_{p_0}, \text{ces}_{p_1})_{\theta, p} = \text{ces}_p, \quad (\text{Ces}_{p_0}[0, \infty), \text{Ces}_{p_1}[0, \infty))_{\theta, p} = \text{Ces}_p[0, \infty).$$

If $1 \leq p_0 < p_1 \leq \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, then

$$(2.3) \quad (\text{Cop}_{p_0}[0, 1], \text{Cop}_{p_1}[0, 1])_{\theta, p} = \text{Cop}_p[0, 1].$$

REMARK 2.5. A different proof of the second equality in (2.2) was given by Sinnamoni [Si91, Corollary 2].

3. Cesàro spaces on $[0, \infty)$ as interpolation spaces with respect to (L_1, Ces_∞) . All the spaces considered in this part are on the interval $I = [0, \infty)$. By [MS, p. 194] we have $D^\infty := (L_\infty)^\downarrow = \text{Ces}_\infty$ isometrically. On the other hand, for a Banach lattice F with the Fatou property we have $F \in \text{Int}(L_1, D^\infty) = \text{Int}(L_1, \text{Ces}_\infty)$ if and only if $F = E^\downarrow$ with equality of norms for some $E \in \text{Int}(L_1, L_\infty)$ (see [MS, Theorem 6.4]). Then, in particular, $L_p^\downarrow \in \text{Int}(L_1, \text{Ces}_\infty)$. Since the operator C is bounded in L_p for $1 < p \leq \infty$, by [Si01, Theorem 3.1] it follows that

$$\|f\|_{L_p^\downarrow} = \| |f| \|_{L_p^\downarrow} \approx \|Cf\|_{L_p} = \|f\|_{\text{Ces}(p)}.$$

Thus, for any $1 < p < \infty$ we have $\text{Ces}_p \in \text{Int}(L_1, \text{Ces}_\infty)$ and $\text{Ces}_p = L_p^\downarrow$.

Moreover, by using Theorem 6.4 from [MS], it is easy to prove the following more precise and general assertion.

PROPOSITION 3.1. *Let $E, F \in \text{Int}(L_1, L_\infty)$ and Φ be an interpolation Banach lattice with respect to the couple $(L_\infty, L_\infty(1/u))$ on $(0, \infty)$. Then*

$$(3.1) \quad (E^\downarrow, F^\downarrow)_\Phi^K = [(E, F)_\Phi^K]^\downarrow.$$

In particular, if $1 < p < \infty$, then

$$(3.2) \quad (L_1, \text{Ces}_\infty)_{1-1/p, p} = \text{Ces}_p.$$

Proof. Firstly, since the Banach couple (L_1, L_∞) is K -monotone [KPS, Theorem 2.4.3], by the assumption and the Brudnyi–Krugljak theorem (cf. [BK, Theorem 4.4.5]), $E = (L_1, L_\infty)_{\Phi_0}^K$ and $F = (L_1, L_\infty)_{\Phi_1}^K$ with some interpolation Banach lattices Φ_0 and Φ_1 with respect to the couple $(L_\infty, L_\infty(1/u))$ on $(0, \infty)$. Applying the reiteration theorem for the general K -method (see [BK, Theorem 3.3.11]), we obtain

$$(E, F)_\Psi^K = ((L_1, L_\infty)_{\Phi_0}^K, (L_1, L_\infty)_{\Phi_1}^K)_\Psi^K = (L_1, L_\infty)_\Psi^K,$$

where $\Psi = (\Phi_0, \Phi_1)_\Phi^K$. Moreover, from the proof of Theorem 6.4 in [MS] and the equality $L_1^\downarrow = L_1$ (see Section 1) it follows that

$$E^\downarrow = [(L_1, L_\infty)_{\Phi_0}^K]^\downarrow = (L_1, D^\infty)_{\Phi_0}^K, F^\downarrow = [(L_1, L_\infty)_{\Phi_1}^K]^\downarrow = (L_1, D^\infty)_{\Phi_1}^K$$

and

$$[(E, F)_\Psi^K]^\downarrow = [(L_1, L_\infty)_\Psi^K]^\downarrow = (L_1, D^\infty)_\Psi^K.$$

Therefore, using the reiteration theorem once again, we obtain

$$(E^\downarrow, F^\downarrow)_\Phi^K = ((L_1, D^\infty)_{\Phi_0}^K, (L_1, D^\infty)_{\Phi_1}^K)_\Phi^K = (L_1, D^\infty)_\Psi^K = [(E, F)_\Psi^K]^\downarrow.$$

and equality (3.1) is proved. In particular, from (3.1) and the well-known identification formula $(L_1, L_\infty)_{1-1/p, p} = L_p$ [BL, Theorem 5.2.1] it follows that

$$(L_1, \text{Ces}_\infty)_{1-1/p, p} = (L_1^\downarrow, L_\infty^\downarrow)_{1-1/p, p} = L_p^\downarrow = \text{Ces}_p,$$

and equality (3.2) is also proved. ■

For a given symmetric space E on $I = [0, \infty)$ the Cesàro function space Ces_E is defined by the norm $\|f\|_{\text{Ces}(E)} = \|Cf\|_E$. If the operator C is bounded in E , then, by [Si01, Theorem 3.1], $\text{Ces}_E = E^\downarrow$. Therefore, applying Proposition 3.1, we obtain

COROLLARY 3.2. *Let the operator C be bounded in symmetric spaces E and F on $[0, \infty)$ and let Φ be an interpolation Banach lattice with respect to the couple $(L_\infty, L_\infty(1/u))$ on $(0, \infty)$. Then*

$$(\text{Ces}_E, \text{Ces}_F)_\Phi^K = \text{Ces}_{(E, F)_\Phi^K}.$$

In particular, for any $1 < p_0 < p_1 \leq \infty$,

$$(3.3) \quad (\text{Ces}_{p_0}, \text{Ces}_{p_1})_{\theta, p} = \text{Ces}_p, \text{ where } 0 < \theta < 1 \text{ and } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

REMARK 3.3. If $1 < p < \infty$, then the restriction of the space $\text{Ces}_p[0, \infty)$ to the interval $[0, 1]$ coincides with $\text{Ces}_p[0, 1] \cap L_1[0, 1]$ (cf. [AM, Remark 5]). Therefore, if we “restrict” (3.3) to $[0, 1]$ we obtain only

$$(\text{Ces}_{p_0}[0, 1] \cap L_1[0, 1], \text{Ces}_{p_1}[0, 1] \cap L_1[0, 1])_{\theta, p} = \text{Ces}_p[0, 1] \cap L_1[0, 1],$$

where $1 < p_0 < p_1 < \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

4. Cesàro spaces on $[0, 1]$ as interpolation spaces with respect to $(L_1(1-t), \text{Ces}_\infty)$. In contrast to the case of $[0, \infty)$, $\text{Ces}_p[0, 1]$ for $1 \leq p < \infty$ is not even an intermediate space between $L_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$. In fact, $\text{Ces}_\infty[0, 1] \hookrightarrow L_1[0, 1]$, but it is easy to show that $\text{Ces}_p[0, 1] \not\subset L_1[0, 1]$ for every $1 \leq p < \infty$.

On the other hand, from the inequality $1 - u \leq \ln(1/u)$ ($0 < u \leq 1$) it follows that $\text{Ces}_p[0, 1]$, $1 \leq p < \infty$, is an intermediate space between $L_1(1-t)[0, 1]$ and $\text{Ces}_\infty[0, 1]$, because

$$\text{Ces}_\infty[0, 1] \xhookrightarrow{1} \text{Ces}_p[0, 1] \xhookrightarrow{1} \text{Ces}_1[0, 1] = L_1(\ln(1/t))[0, 1] \xhookrightarrow{1} L_1(1-t)[0, 1].$$

THEOREM 4.1. *If $1 < p < \infty$, then*

$$(4.1) \quad (\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, p} \xhookrightarrow{1} \text{Ces}_p[0, 1]$$

and

$$(4.2) \quad (L_1(1-t)[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, p} = \text{Ces}_p[0, 1].$$

Proof. All function spaces in this proof are considered on $I = [0, 1]$ unless indicated otherwise.

First, for any $f \in \text{Ces}_1$ and all $0 < t \leq 1$ we have

$$(4.3) \quad K(t, f) := K(t, f; \text{Ces}_1, \text{Ces}_\infty) \geq \int_0^t (Cf)^*(s) ds.$$

In fact, we can assume that $f \geq 0$. If $f = g + h$, $g \geq 0$, $h \geq 0$, $g \in \text{Ces}_1$, $h \in \text{Ces}_\infty$, then $Cf = Cg + Ch$, and therefore, by (1.3),

$$\begin{aligned} \|g\|_{\text{Ces}(1)} + t\|h\|_{\text{Ces}(\infty)} &= \|Cg\|_{L_1} + t\|Ch\|_{L_\infty} \\ &\geq \inf\{\|y\|_{L_1} + t\|z\|_{L_\infty} : Cf = y + z, y \in L_1, z \in L_\infty\} \\ &= K(t, Cf; L_1, L_\infty) = \int_0^t (Cf)^*(s) ds. \end{aligned}$$

Taking the infimum over all suitable g and h we get (4.3).

Next, by the definition of real interpolation spaces, we obtain

$$\begin{aligned} \|f\|_{1-1/p, p}^p &\geq \int_0^1 [t^{1/p-1} K(t, f)]^p \frac{dt}{t} = \int_0^1 t^{-p} K(t, f)^p dt \\ &\geq \int_0^1 t^{-p} \left[\int_0^t (Cf)^*(s) ds \right]^p dt \geq \|Cf\|_{L_p[0, 1]}^p = \|f\|_{\text{Ces}(p)}^p, \end{aligned}$$

and the proof of (4.1) is complete.

Before proceeding with the proof of (4.2) we introduce the following notation: for a Banach function space E on $I = [0, \infty)$ or $[0, 1]$ and any set

$A \subset I$, by $E|_A$ we denote the subspace of E which consists of all functions f such that $\text{supp } f \subset A$. Let also $X_p := (L_1(1-t), \text{Ces}_\infty)_{1-1/p, p}$. Since

$$\|f\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{X_p} + \|f\chi_{[1/2,1]}\|_{X_p},$$

to prove (4.2) it is sufficient to check that

$$(4.4) \quad \|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{\text{Ces}(p)},$$

$$(4.5) \quad \|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{\text{Ces}(p)}.$$

Firstly, since $L_1(1-t)|_{[0,1/2]} = L_1[0, \infty)|_{[0,1/2]}$ and

$$\text{Ces}_\infty|_{[0,1/2]} = \text{Ces}_\infty[0, \infty)|_{[0,1/2]},$$

by Proposition 3.1 (see (3.2)) we obtain

$$(4.6)$$

$$\|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{(L_1[0, \infty), \text{Ces}_\infty[0, \infty))_{1-1/p, p}} \approx \|f\chi_{[0,1/2]}\|_{\text{Ces}_p[0, \infty)}.$$

Note that

$$(4.7) \quad \text{Ces}_p[0, \infty)|_{[0,1/2]} = \text{Ces}_p[0, 1]|_{[0,1/2]}$$

with equivalence of norms. In fact, by [AM, Remark 5], $\text{Ces}_p[0, \infty)|_{[0,1]} = \text{Ces}_p \cap L_1$. If $\text{supp } g \subset [0, 1/2]$, then

$$\|g\|_{L_1} = \int_0^{1/2} |g(s)| ds \leq 2^{1/p} \left(\int_{1/2}^1 \left(\frac{1}{t} \int_0^t |g(s)| ds \right)^p dt \right)^{1/p} \leq 2^{1/p} \|g\|_{\text{Ces}(p)}.$$

Combining this with the previous equality, we obtain (4.7). Clearly, (4.4) is an immediate consequence of (4.7) and (4.6).

Now, we prove (4.5). Since $(L_1(1-s)|_{[1/2,1]}, \text{Ces}_\infty|_{[1/2,1]})$ is a complemented subcouple of $(L_1(1-s), \text{Ces}_\infty)$, the well-known result of Baouendi and Goulaouic [BG, Theorem 1], valid for all interpolation methods (see also [Tr, Theorem 1.17.1]), yields

$$\|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{Y_p},$$

where $Y_p := (L_1(1-s)|_{[1/2,1]}, \text{Ces}_\infty|_{[1/2,1]})_{1-1/p, p}$. To prove (4.5) it is sufficient to show that

$$(4.8) \quad Y_p = \text{Ces}_p|_{[1/2,1]}.$$

On the one hand, since $1-u \leq \ln(1/u) \leq 2(1-u)$ for all $1/2 \leq u \leq 1$ and $\text{Ces}_1 = L_1(\ln(1/s))$, we have $\text{Ces}_1|_{[1/2,1]} = L_1(1-s)|_{[1/2,1]}$, and, by the imbedding (4.1), we obtain

$$Y_p = (\text{Ces}_1|_{[1/2,1]}, \text{Ces}_\infty|_{[1/2,1]})_{1-1/p, p} \subset \text{Ces}_p|_{[1/2,1]}.$$

To prove the opposite imbedding we note, firstly, that for any function h

with $\text{supp } h \subset [1/2, 1]$ we have

$$\|h\|_{\text{Ces}(\infty)} = \sup_{1/2 \leq x \leq 1} \frac{1}{x} \int_{1/2}^x |h(s)| ds,$$

whence

$$\|h\|_{L_1} = \int_{1/2}^1 |h(s)| ds \leq \|h\|_{\text{Ces}(\infty)} \leq 2 \int_{1/2}^1 |h(s)| ds = 2\|h\|_{L_1}.$$

Therefore, using the formula for the K -functional with respect to a couple of weighted L_1 -spaces (see (1.4)), we obtain

$$(4.9) \quad G(t, h) \leq K(t, h; L_1(1-s)|_{[1/2, 1]}, \text{Ces}_\infty|_{[1/2, 1]}) \leq 2G(t, h),$$

where

$$G(t, h) = K(t, h; L_1(1-s)|_{[1/2, 1]}, L_1|_{[1/2, 1]}) = \int_{1/2}^1 \min(1-s, t) |h(s)| ds.$$

Furthermore, let $h \in L_1|_{[1/2, 1]}$. Then

$$Ch(s) = \frac{1}{s} \int_{1/2}^s |h(u)| du \geq \int_{1/2}^s |h(u)| du,$$

whence

$$(Ch)^*(s) \geq \int_{1/2}^{1-s} |h(u)| du, \quad 0 < s \leq 1.$$

Therefore, for all $0 \leq t \leq 1$, we obtain

$$\begin{aligned} \int_0^t (Ch)^*(s) ds &\geq \int_0^t \left(\int_{1/2}^{1-s} |h(u)| du \right) ds \\ &= \int_{1/2}^{1-t} \left(\int_0^t |h(u)| ds \right) du + \int_{1-t}^1 \left(\int_0^{1-u} |h(u)| ds \right) du \\ &= t \int_{1/2}^{1-t} |h(u)| du + \int_{1-t}^1 (1-u) |h(u)| du = G(t, h). \end{aligned}$$

From this inequality and the definition of $G(t, h)$ it follows that

$$\int_0^{\min(1, t)} (Ch)^*(s) ds \geq G(t, h)$$

for all $t > 0$. Hence, by (4.9) and the classical Hardy inequality, for every

$h \in \text{Ces}_p$ with $\text{supp } h \subset [1/2, 1]$ we have

$$\begin{aligned}
\|h\|_{Y_p} &= \left(\int_0^\infty t^{-p} K(t, h; L_1(1-s)|_{[1/2,1]}, \text{Ces}_\infty|_{[1/2,1]})^p dt \right)^{1/p} \\
&\leq 2 \left(\int_0^\infty t^{-p} G(t, h)^p dt \right)^{1/p} \\
&\leq 2 \left(\int_0^\infty t^{-p} \left(\int_0^{\min(1,t)} (Ch)^*(s) ds \right)^p dt \right)^{1/p} \\
&\leq 2 \left(\int_0^1 t^{-p} \left(\int_0^t (Ch)^*(s) ds \right)^p dt \right)^{1/p} + 2 \left(\int_1^\infty t^{-p} \left(\int_0^1 (Ch)^*(s) ds \right)^p dt \right)^{1/p} \\
&\leq 2 \left[\frac{p}{p-1} \|Ch\|_{L_p[0,1]} + \frac{1}{(p-1)^{1/p}} \|Ch\|_{L_1[0,1]} \right] \leq \frac{4p}{p-1} \|h\|_{\text{Ces}_p[0,1]}.
\end{aligned}$$

Thus, $\text{Ces}_p|_{[1/2,1]} \subset Y_p$, (4.8) holds, and the proof is complete. ■

The following result is an immediate consequence of (4.2) and the reiteration equality (1.2).

COROLLARY 4.2. *If $1 < p_0 < p_1 \leq \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, then*

$$(\text{Ces}_{p_0}[0, 1], \text{Ces}_{p_1}[0, 1])_{\theta, p} = \text{Ces}_p[0, 1].$$

REMARK 4.3. An inspection of the proof of Theorem 4.1 shows that

$$(\text{Ces}_1|_{[1/2,1]}, \text{Ces}_\infty|_{[1/2,1]})_{1-1/p, p} = \text{Ces}_p|_{[1/2,1]}$$

for every $1 < p < \infty$ with equivalence of norms.

REMARK 4.4. Comparison of formulas from Remark 3.3 and Corollary 4.2 shows that the real method $(\cdot, \cdot)_{\theta, p}$ “well” interpolates the intersection of Cesàro spaces on $[0, 1]$ with $L_1[0, 1]$ or, more precisely,

$$\begin{aligned}
&(\text{Ces}_{p_0}[0, 1] \cap L_1[0, 1], \text{Ces}_{p_1}[0, 1] \cap L_1[0, 1])_{\theta, p} \\
&= (\text{Ces}_{p_0}[0, 1], \text{Ces}_{p_1}[0, 1])_{\theta, p} \cap L_1[0, 1]
\end{aligned}$$

for all $1 < p_0 < p_1 \leq \infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

REMARK 4.5. We will see further that the imbedding (4.1) is strict for every $1 < p < \infty$ and, even more, $\text{Ces}_p[0, 1]$ is not an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$. Thus, the weighted L_1 -space $L_1(1-t)[0, 1]$ is in a sense the “proper” end of the scale of Cesàro spaces $\text{Ces}_p[0, 1]$, $1 < p \leq \infty$.

5. The K -functional for $(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])$. In this section we will find an equivalent expression for the K -functional

$$K(t, f) = K(t, f; \text{Ces}_1, \text{Ces}_\infty) = K(t, f; \text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1]).$$

We start with a lemma giving its lower estimate. Let us introduce two functions defined on $(0, 1]$ by

$$(5.1) \quad \tau_1(t) = t/\ln(e/t) \quad \text{and} \quad \tau_2(t) = e^{-t} \quad \text{for } 0 < t \leq 1.$$

It is easy to see that there exists a unique $t_0 \in (0, 1)$ such that $\tau_1(t_0) = \tau_2(t_0)$ and $\tau_1(t) < \tau_2(t)$ if and only if $0 < t < t_0$.

LEMMA 5.1 (lower estimates). *Let $f \in \text{Ces}_1[0, 1]$, $f \geq 0$ and $0 < t \leq 1$.*

(i) *If $f_0 = f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}$, then*

$$(5.2) \quad K(t, f) \geq \frac{1}{4} \|f_0\|_{\text{Ces}(1)}.$$

(ii) *If $f_1 = f\chi_{[\tau_1(t), \tau_2(t)]}$, then*

$$(5.3) \quad K(t, f) \geq \frac{1}{e^2} t \|f_1\|_{\text{Ces}(\infty)}.$$

Proof. (i) Firstly, let us prove that

$$(5.4) \quad K(t, f) \geq \frac{1}{3} \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} \quad \text{for all } 0 < t \leq 1.$$

Let $f \in \text{Ces}_1$, $f = g + h$, where $g \in \text{Ces}_1$, $h \in \text{Ces}_\infty$. We may assume that $f \geq 0$ and $0 \leq g \leq f$, $0 \leq h \leq f$. Then

$$(5.5) \quad \begin{aligned} 3(\|g\|_{\text{Ces}(1)} + t\|h\|_{\text{Ces}(\infty)}) &\geq \|g\|_{\text{Ces}(1)} + 3t\|h\|_{\text{Ces}(\infty)} \\ &\geq \|(f - h)\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} + 3t\|h\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(\infty)} \\ &= \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} - \|h\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} + 3t\|h\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(\infty)}. \end{aligned}$$

Let us show that for any $v \in \text{Ces}_\infty$, $v \geq 0$, with $\text{supp } v \subset [0, \tau_1(t)]$ we have

$$(5.6) \quad \|v\|_{\text{Ces}(1)} \leq 3t\|v\|_{\text{Ces}(\infty)}.$$

In fact, by the assumption on the support of v and the Fubini theorem,

$$\begin{aligned} \|v\|_{\text{Ces}(1)} &= \int_0^{\tau_1(t)} \left(\frac{1}{s} \int_0^s v(u) du \right) ds + \int_{\tau_1(t)}^1 \left(\frac{1}{s} \int_0^{\tau_1(t)} v(u) du \right) ds \\ &= \int_0^{\tau_1(t)} \left(\frac{1}{s} \int_0^s v(u) du \right) ds + \int_0^{\tau_1(t)} \left(\int_{\tau_1(t)}^1 \frac{1}{s} ds \right) v(u) du \\ &= \int_0^{\tau_1(t)} \left(\frac{1}{s} \int_0^s v(u) du \right) ds + \int_0^{\tau_1(t)} v(u) du \ln \frac{1}{\tau_1(t)}. \end{aligned}$$

Since $\tau_1(t) \leq t$ it follows that

$$\int_0^{\tau_1(t)} \left(\frac{1}{s} \int_0^s v(u) du \right) ds \leq \tau_1(t) \sup_{0 < s \leq \tau_1(t)} \frac{1}{s} \int_0^s v(u) du \leq t \|v\|_{\text{Ces}(\infty)}.$$

Moreover,

$$\begin{aligned} \int_0^{\tau_1(t)} v(u) du \ln \frac{1}{\tau_1(t)} &\leq \tau_1(t) \ln \frac{1}{\tau_1(t)} \sup_{0 < s \leq \tau_1(t)} \frac{1}{s} \int_0^s v(u) du \\ &= \frac{\ln \frac{1}{t} + \ln \ln \frac{e}{t}}{\ln \frac{e}{t}} t \|v\|_{\text{Ces}(\infty)} \leq 2t \|v\|_{\text{Ces}(\infty)}, \end{aligned}$$

and (5.6) follows. Combining this estimate for $v = h\chi_{[0, \tau_1(t)]}$ with (5.5) we conclude that

$$3(\|g\|_{\text{Ces}(1)} + t\|h\|_{\text{Ces}(\infty)}) \geq \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)}.$$

Taking the infimum over all decompositions $f = g + h$, $g \in \text{Ces}_1$, $h \in \text{Ces}_\infty$ with $0 \leq g \leq f$, $0 \leq h \leq f$ we obtain (5.4).

Next, since $\text{Ces}_1 = L_1(\ln(1/s))$ and $\text{Ces}_\infty \xrightarrow{1} L_1$, we have

$$K(t, f; L_1(\ln(1/s)), L_1) = K(t, f; \text{Ces}_1, L_1) \leq K(t, f).$$

Therefore, applying the well-known equality

$$K(t, f; L_1(\ln(1/s)), L_1) = \int_0^1 \min(\ln(1/s), t) |f(s)| ds$$

and the elementary inequality

$$\int_0^1 \min(\ln(1/s), t) |f(s)| ds \geq \int_{e^{-t}}^1 \ln(1/s) |f(s)| ds = \|f\chi_{[\tau_2(t), 1]}\|_{\text{Ces}(1)},$$

we obtain

$$K(t, f) \geq \|f\chi_{[\tau_2(t), 1]}\|_{\text{Ces}(1)}.$$

Inequality (5.2) is an immediate consequence of the last inequality and (5.4). The proof of (i) is complete.

(ii) Since (5.3) is obvious for $t \in [t_0, 1]$, it can be assumed that $0 < t < t_0$. Let again $f \in \text{Ces}_1$, $f = g + h$, where $g \in \text{Ces}_1$, $h \in \text{Ces}_\infty$ and $0 \leq g \leq f$, $0 \leq h \leq f$. Then for any $c \in (0, 1)$ we have

$$\begin{aligned} (5.7) \quad &\|g\|_{\text{Ces}(1)} + t\|h\|_{\text{Ces}(\infty)} \\ &\geq \|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(1)} + ct\|(f - g)\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)} \\ &\geq \|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(1)} - ct\|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)} + ct\|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)}. \end{aligned}$$

We want to show that for every positive function $w \in \text{Ces}_1$ with $\text{supp } w \subset [\tau_1(t), \tau_2(t)]$,

$$(5.8) \quad \frac{1}{e^2} t \|w\|_{\text{Ces}(\infty)} \leq \|w\|_{\text{Ces}(1)} \quad \text{for any } 0 < t < t_0.$$

Since

$$\begin{aligned} \|w\|_{\text{Ces}(1)} &= \int_0^1 \frac{1}{s} \left[\int_{\tau_1(t)}^s w(u) du \cdot \chi_{[\tau_1(t), \tau_2(t)]}(s) + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \chi_{[\tau_2(t), 1]}(s) \right] ds \\ &= \int_{\tau_1(t)}^{\tau_2(t)} \left(\frac{1}{s} \int_{\tau_1(t)}^s w(u) du \right) ds + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \int_{\tau_2(t)}^1 \frac{ds}{s} \\ &= \int_{\tau_1(t)}^{\tau_2(t)} \left(\int_u^{\tau_2(t)} \frac{ds}{s} \right) w(u) du + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \ln \frac{1}{\tau_2(t)} \\ &= \int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du + t \int_{\tau_1(t)}^{\tau_2(t)} w(u) du, \end{aligned}$$

to prove (5.8) it suffices to show that for all $t \in (0, t_0)$ and $s \in [\tau_1(t), \tau_2(t)]$ we have

$$(5.9) \quad \frac{1}{e^2} t \int_{\tau_1(t)}^s w(u) du \leq s \left[\int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du + t \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \right].$$

We consider the cases when $s \in [\tau_1(t), \tau_2(t)/e]$ and $s \in (\tau_2(t)/e, \tau_2(t)]$ separately. Define a unique $t_1 \in (0, t_0)$ such that $\tau_1(t_1) = \tau_2(t_1)/e$ and note that the interval $[\tau_1(t), \tau_2(t)/e]$ is non-empty only if $0 < t \leq t_1$. Let

$$\varphi(s) := s \ln \frac{\tau_2(t)}{s} \quad \text{for } s \in [\tau_1(t), \tau_2(t)/e].$$

Since

$$\varphi'(s) = \ln \frac{\tau_2(t)}{s} - 1 = \ln \frac{\tau_2(t)}{es} \geq 0 \quad \text{for all } s \in [\tau_1(t), \tau_2(t)/e],$$

it follows that φ increases. Therefore, $\varphi(s) \geq \varphi(\tau_1(t))$ for all $s \in [\tau_1(t), \tau_2(t)/e]$ and so

$$\begin{aligned} (5.10) \quad s \int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du &\geq s \ln \frac{\tau_2(t)}{s} \int_{\tau_1(t)}^s w(u) du \\ &\geq \tau_1(t) \ln \frac{\tau_2(t)}{\tau_1(t)} \int_{\tau_1(t)}^s w(u) du. \end{aligned}$$

We show that

$$(5.11) \quad \tau_1(t) \ln \frac{\tau_2(t)}{\tau_1(t)} \geq \frac{1}{e^2} t \quad \text{for all } 0 < t \leq t_1.$$

The function

$$\psi(t) = \frac{\tau_1(t)}{t} \ln \frac{\tau_2(t)}{\tau_1(t)} = \frac{\ln \frac{\frac{e}{t}}{t} - t}{\ln \frac{e}{t}} \quad \text{for } t \in (0, t_1]$$

is differentiable and its derivative is

$$\psi'(t) = - \left[(t+1) \left(1 + \ln \frac{e}{t} \right) + \ln \tau_1(t) \right] / \left[t \left(\ln \frac{e}{t} \right)^2 \right].$$

It is not hard to check that ψ is increasing on $(0, t_2)$ and decreasing on $(t_2, t_1]$ with $t_2 \in (0, t_1)$. Hence, by the definition of t_1 , for all $t \in (0, t_1]$ we have

$$\psi(t) \geq \min[\psi(0^+), \psi(t_1)] = \min\left(1, \ln^{-1} \frac{e}{t_1}\right) = \ln^{-1} \frac{e}{t_1} = t_1^{-1} e^{-1-t_1} \geq e^{-2}.$$

Thus, we obtain (5.11). Combining it with (5.10), we obtain (5.9) in the case when $0 < t \leq t_1$ and $s \in [\tau_1(t), \tau_2(t)/e]$.

In the second case, when $s \in (\tau_2(t)/e, \tau_2(t)]$, we have $s \geq e^{-1-t} \geq e^{-2}$ and so

$$t \int_{\tau_1(t)}^s w(u) du \leq e^2 t s \int_{\tau_1(t)}^{\tau_2(t)} w(u) du.$$

Hence, (5.9) holds again, and so (5.8) is proved. Combining (5.8) and (5.7) with $c = e^{-2}$, we obtain

$$\|g\|_{\text{Ces}(1)} + t\|h\|_{\text{Ces}(\infty)} \geq \frac{1}{e^2} t \|f_1\|_{\text{Ces}(\infty)} \quad \text{for all } 0 < t < t_0.$$

Taking the infimum over all decompositions $f = g + h$, $g \in \text{Ces}_1$, $h \in \text{Ces}_\infty$ with $0 \leq g \leq f$, $0 \leq h \leq f$ we come to (5.3), and the proof of (ii) is complete. ■

THEOREM 5.2. *For every $f \in \text{Ces}_1[0, 1]$ we have*

$$\begin{aligned} \frac{1}{2e^2} [\|f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}\|_{\text{Ces}(1)} + t\|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)}] &\leq K(t, f; \text{Ces}_1, \text{Ces}_\infty) \\ &\leq \|f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}\|_{\text{Ces}(1)} + t\|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)} \end{aligned}$$

for all $0 < t < 1$, and $K(t, f; \text{Ces}_1, \text{Ces}_\infty) = \|f\|_{\text{Ces}(1)}$ for all $t \geq 1$.

Proof. The first inequality is a consequence of Lemma 5.1 and the definition of the K -functional. The equality $K(t, f; \text{Ces}_1, \text{Ces}_\infty) = \|f\|_{\text{Ces}(1)}$ ($t \geq 1$) follows from the imbedding $\text{Ces}_\infty \xrightarrow{1} \text{Ces}_1$. ■

If a positive function $f \in \text{Ces}_1[0, 1]$ is decreasing, then the description of the K -functional can be simplified.

THEOREM 5.3. *If $f \in \text{Ces}_1[0, 1]$, $f \geq 0$ and f is decreasing, then*

$$(5.12) \quad \frac{1}{3} \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} \leq K(t, f; \text{Ces}_1, \text{Ces}_\infty) \leq \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)}$$

for all $0 < t < 1$, and $K(t, f; \text{Ces}_1, \text{Ces}_\infty) = \|f\|_{\text{Ces}(1)}$ for all $t \geq 1$.

Proof. Taking into account the proof of Lemma 5.1(i) (see (5.4)) it suffices to prove the right-hand inequality in (5.12).

Let $f_0 := [f - f(\tau_1(t))]\chi_{[0, \tau_1(t)]}$ and $f_1 := f - f_0$. Since $f \geq 0$ is decreasing, we have $\|f_1\|_{\text{Ces}(\infty)} = f(\tau_1(t))$. Therefore, by the Fubini theorem,

$$\begin{aligned} & \|f_0\|_{\text{Ces}(1)} + t\|f_1\|_{\text{Ces}(\infty)} \\ &= \int_0^1 \frac{1}{s} \int_0^s \left[f(u) - f(\tau_1(t)) \right] \chi_{[0, \tau_1(t)]}(u) du ds + tf(\tau_1(t)) \\ &= \int_0^1 \frac{1}{s} \int_0^s f(u) \chi_{[0, \tau_1(t)]}(u) du ds - f(\tau_1(t)) \int_0^{\tau_1(t)} \ln \frac{1}{u} du + tf(\tau_1(t)) \\ &= \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} - f(\tau_1(t))\tau_1(t) \left[1 + \ln \frac{1}{\tau_1(t)} \right] + tf(\tau_1(t)) \\ &= \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} + tf(\tau_1(t)) \left[1 - \frac{1 + \ln(\ln \frac{e}{t}/t)}{\ln \frac{e}{t}} \right] \\ &= \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} - \frac{tf(\tau_1(t)) \ln(\ln \frac{e}{t})}{\ln \frac{e}{t}} \leq \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)}, \end{aligned}$$

whence

$$K(t, f; \text{Ces}_1, \text{Ces}_\infty) \leq \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)},$$

and the desired result is proved. ■

6. Identification of the real interpolation spaces $(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, p}$ for $1 < p < \infty$. Let us define the weighted Cesàro function space $\text{Ces}_p(\ln(e/t))[0, 1]$ to consist of all Lebesgue measurable functions f on $[0, 1]$ such that

$$\|f\|_{\text{Ces}(p, \ln)} := \left(\int_0^1 \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p \ln \frac{e}{x} dx \right)^{1/p} < \infty.$$

Clearly, $\text{Ces}_p(\ln(e/t))[0, 1] \xrightarrow{1} \text{Ces}_p[0, 1]$ for every $1 < p < \infty$, and this imbedding is strict.

THEOREM 6.1. *For $1 < p < \infty$,*

$$(6.1) \quad (\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, p} = \text{Ces}_p\left(\ln \frac{e}{t}\right)[0, 1].$$

Proof. Denote $X_p = (\text{Ces}_1, \text{Ces}_\infty)_{1-1/p, p}$, $1 < p < \infty$. Using Theorem 5.2 on the K -functional for the couple $(\text{Ces}_1, \text{Ces}_\infty)$ on $[0, 1]$, we have

$$\begin{aligned} \|f\|_{X_p} &\leq \left[\int_0^{t_0} t^{-p} \|f\chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)}^p dt \right]^{1/p} + \left[\int_0^{t_0} t^{-p} \|f\chi_{[\tau_2(t), 1]}\|_{\text{Ces}(1)}^p dt \right]^{1/p} \\ &\quad + \left[\int_0^{t_0} t^{-p} (t \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)})^p dt \right]^{1/p} + \left[\int_{t_0}^{\infty} t^{-p} \|f\|_{\text{Ces}(1)}^p dt \right]^{1/p} \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \left[\int_0^{t_0} t^{-p} \left(\int_0^{\tau_1(t)} C f(s) ds + \int_{\tau_1(t)}^1 C(f\chi_{[0, \tau_1(t)]})(s) ds \right)^p dt \right]^{1/p} \\ &\leq \left[\int_0^{t_0} t^{-p} \left(\int_0^{\tau_1(t)} C f(s) ds \right)^p dt \right]^{1/p} \\ &\quad + \left[\int_0^{t_0} t^{-p} \left(\int_{\tau_1(t)}^1 C(f\chi_{[0, \tau_1(t)]})(s) ds \right)^p dt \right]^{1/p} = I_{11} + I_{12}. \end{aligned}$$

First of all, we estimate all five integrals from above. Since $\tau_1'(t) = (\ln(e/t) + 1)/(\ln(e/t))^2$ and so $1/\ln(e/t) \leq \tau_1'(t) \leq 2/\ln(e/t)$ for all $0 < t \leq 1$, we get

$$\begin{aligned} I_{11}^p &\leq \int_0^{t_0} t^{-p} \left(\ln \frac{e}{t} \right)^{p-1} \left(\int_0^{\tau_1(t)} C f(s) ds \right)^p dt \\ &\leq \int_0^{t_0} \tau_1(t)^{-p} \left(\int_0^{\tau_1(t)} C f(s) ds \right)^p d\tau_1(t). \end{aligned}$$

Putting $u = \tau_1(t)$ and using the classical Hardy inequality, we obtain

$$\begin{aligned} I_{11} &\leq \left[\int_0^{\tau_1(t_0)} \left(\frac{1}{u} \int_0^u C f(s) ds \right)^p du \right]^{1/p} \leq \|C^2 f\|_{L_p[0, 1]} \\ &\leq p' \|C f\|_{L_p[0, 1]} = p' \|f\|_{\text{Ces}(p)} \leq p' \|f\|_{\text{Ces}(p, \ln)}. \end{aligned}$$

Next, by the estimate $\ln(1/\tau_1(t)) \leq 2 \ln(e/t)$, $0 < t \leq 1$, we get

$$\begin{aligned} I_{12}^p &= \int_0^{t_0} t^{-p} \left(\int_{\tau_1(t)}^1 \left(\frac{1}{s} \int_0^{\tau_1(t)} |f(u)| du \right) ds \right)^p dt \\ &= \int_0^{t_0} t^{-p} \left(\int_0^{\tau_1(t)} |f(u)| du \right)^p \ln^p \frac{1}{\tau_1(t)} dt \\ &\leq 2^p \int_0^{t_0} \tau_1(t)^{-p} \left(\int_0^{\tau_1(t)} |f(u)| du \right)^p dt. \end{aligned}$$

The substitution $t = \tau_1^{-1}(s)$ and the inequalities

$$(6.2) \quad (\tau_1^{-1})'(s) = \frac{1}{\tau_1'(\tau_1^{-1}(s))} \leq \ln \frac{e}{\tau_1^{-1}(s)} \leq \ln \frac{e}{s}$$

show that

$$\begin{aligned} I_{12} &\leq 2 \left[\int_0^{\tau_1(t_0)} \left(\frac{1}{s} \int_0^s |f(u)| du \right)^p \ln \frac{e}{s} ds \right]^{1/p} \\ &\leq 2 \left[\int_0^1 (Cf(s))^p \ln \frac{e}{s} ds \right]^{1/p} = 2 \|f\|_{\text{Ces}(p, \ln)}. \end{aligned}$$

From the equality $\text{Ces}_1[0, 1] = L_1(\ln(1/u))$ and the inequalities $\ln(1/u) \leq e(1-u)$ ($1/e \leq u \leq 1$) and $\tau_2(t) = e^{-t} \geq 1-t$ ($0 < t \leq 1$) it follows that

$$\begin{aligned} I_2^p &= \int_0^{t_0} t^{-p} \|f\chi_{[\tau_2(t), 1]}\|_{\text{Ces}(1)}^p dt = \int_0^{t_0} t^{-p} \left(\int_{\tau_2(t)}^1 |f(u)| \ln \frac{1}{u} du \right)^p dt \\ &\leq e^p \int_0^{t_0} t^{-p} \left(\int_{\tau_2(t)}^1 |f(u)|(1-u) du \right)^p dt \\ &\leq e^p \int_0^{t_0} t^{-p} \left(\int_{1-t}^1 |f(u)|(1-u) du \right)^p dt. \end{aligned}$$

Arguing in the same way as in the second part of the proof of Theorem 4.1, for $g = f\chi_{[e^{-1}, 1]}$ and $0 < s \leq 1$ we have

$$Cg(s) = \frac{1}{s} \int_{e^{-1}}^s |f(u)| du \geq \int_{e^{-1}}^s |f(u)| du,$$

whence $(Cg)^*(s) \geq \int_{e^{-1}}^{1-s} |f(u)| du$ and

$$\begin{aligned} \int_0^t (Cg)^*(s) ds &\geq \int_0^t \left(\int_{e^{-1}}^{1-s} |f(u)| du \right) ds \\ &= \int_{e^{-1}}^{1-t} \left(\int_0^t |f(u)| ds \right) du + \int_{1-t}^1 \left(\int_0^{1-u} |f(u)| ds \right) du \geq \int_{1-t}^1 |f(u)|(1-u) du. \end{aligned}$$

Therefore, again by the Hardy inequality,

$$\begin{aligned} I_2 &\leq e \left[\int_0^{t_0} t^{-p} \left(\int_0^t (Cg)^*(s) ds \right)^p dt \right]^{1/p} \leq e \|C[(Cg)^*]\|_{L_p[0, 1]} \\ &\leq ep' \|(Cg)^*\|_{L_p[0, 1]} = ep' \|Cg\|_{L_p[0, 1]} = ep' \|f\chi_{[e^{-1}, 1]}\|_{\text{Ces}(p)} \\ &\leq ep' \|f\|_{\text{Ces}(p)} \leq ep' \|f\|_{\text{Ces}(p, \ln)}. \end{aligned}$$

For the third integral, we have

$$\begin{aligned}
I_3 &= \left[\int_0^{t_0} \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{\text{Ces}(\infty)}^p dt \right]^{1/p} \\
&\leq \left[\int_0^{t_0} \sup_{\tau_1(t) < s \leq 1/2} \left(\frac{1}{s} \int_0^s |f(u)\chi_{[\tau_1(t), \tau_2(t)]}(u)| du \right)^p dt \right]^{1/p} \\
&\quad + \left[\int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left(\frac{1}{s} \int_0^s |f(u)\chi_{[\tau_1(t), \tau_2(t)]}(u)| du \right)^p dt \right]^{1/p} \\
&= \left[\int_0^{t_0} \sup_{\tau_1(t) < s \leq 1/2} \left(\frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \right]^{1/p} \\
&\quad + \left[\int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left(\frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \right]^{1/p} = I_{31} + I_{32}.
\end{aligned}$$

If $\tau_1(t) < s \leq 1/2$, then $2s \leq 1$ and

$$\begin{aligned}
&\int_{\tau_1(t)}^{2s} \left(\frac{1}{v} \int_0^v |f(u)| du \right) dv \\
&= \int_0^{\tau_1(t)} \left(\int_{\tau_1(t)}^{2s} \frac{1}{v} dv \right) |f(u)| du + \int_{\tau_1(t)}^{2s} \left(\int_u^{2s} \frac{1}{v} dv \right) |f(u)| du \\
&= \int_0^{\tau_1(t)} |f(u)| du \ln \frac{2s}{\tau_1(t)} + \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} du \\
&\geq \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} du \geq \ln 2 \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^s |f(u)| du.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{\tau_1(t) < s \leq 1/2} \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du &\leq \frac{2}{\ln 2} \sup_{\tau_1(t) < s \leq 1/2} \frac{1}{2s - \tau_1(t)} \int_{\tau_1(t)}^{2s} Cf(v) dv \\
&\leq \frac{2}{\ln 2} MCf(\tau_1(t)),
\end{aligned}$$

where M is the maximal Hardy–Littlewood operator on $[0, 1]$. The above estimates show that

$$I_{31} \leq \frac{2}{\ln 2} \left(\int_0^{t_0} MCf(\tau_1(t))^p dt \right)^{1/p}.$$

Using once again the substitution $t = \tau_1^{-1}(s)$ and (6.2), we obtain

$$I_{31} \leq \frac{2}{\ln 2} \left[\int_0^{\tau_1(t_0)} [M Cf(s)]^p \ln \frac{e}{s} ds \right]^{1/p} \leq \frac{2}{\ln 2} \|M Cf\|_{L_p(\ln(e/s))}.$$

We will show in the next lemma that the maximal operator M is bounded in $L_p(\ln(e/s))[0, 1]$ for $1 < p < \infty$, which implies that for some constant $B_p \geq 1$, which depends only on p , we have

$$I_{31} \leq \frac{2B_p}{\ln 2} \|Cf\|_{L_p(\ln(e/s))} = \frac{2B_p}{\ln 2} \|f\|_{\text{Ces}(p, \ln)}.$$

The second part of I_3 is estimated in the following way:

$$\begin{aligned} I_{32}^p &= \int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left(\frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \leq 2^p \int_0^{t_0} \left(\int_{\tau_1(t)}^{\tau_2(t)} |f(u)| du \right)^p dt \\ &\leq 2^p \int_0^{t_0} \left(\frac{1}{\tau_2(t)} \int_0^{\tau_2(t)} |f(u)| du \right)^p dt, \end{aligned}$$

and, changing variable $s = \tau_2(t) = e^{-t}$, we obtain

$$\begin{aligned} I_{32} &\leq 2 \left[\int_{e^{-t_0}}^1 \left(\frac{1}{s} \int_0^s |f(u)| du \right)^p \frac{ds}{s} \right]^{1/p} \leq 2e^{t_0/p} \left(\int_0^1 C f(s)^p ds \right)^{1/p} \\ &\leq 2e \|f\|_{\text{Ces}(p)} \leq 2e \|f\|_{\text{Ces}(p, \ln)}. \end{aligned}$$

Since $t_0 > 1/2$, for the last integral we have

$$I_4 = \frac{1}{(p-1)^{1/p} t_0^{1-1/p}} \|f\|_{\text{Ces}(1)} \leq \frac{2}{p-1} \|f\|_{\text{Ces}(1)} \leq \frac{2}{p-1} \|f\|_{\text{Ces}(p, \ln)}.$$

Finally, summing up the above estimates, we get $\|f\|_{X_p} \leq C_p \|f\|_{\text{Ces}(p, \ln)}$, where C_p depends only on p . Thus, the imbedding $\text{Ces}(p, \ln) \hookrightarrow X_p$ is proved.

Now, we proceed with estimations from below. Firstly, by (5.4),

$$(6.3) \quad \|f\|_{X_p}^p \geq 3^{-p} \int_0^{t_0} t^{-p} \|f \chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)}^p dt = 3^{-p} I_1^p \geq 3^{-p} I_{12}^p.$$

It is not hard to check that $\ln \frac{1}{\tau_1(t)} = \ln \frac{\ln(e/t)}{t} \geq e^{-1} \ln \frac{e}{t}$ for $t \in (0, t_0]$. Therefore,

$$I_{12}^p = \int_0^{t_0} t^{-p} \left(\int_0^{\tau_1(t)} |f(u)| du \right)^p \ln^p \frac{1}{\tau_1(t)} dt \geq e^{-p} \int_0^{t_0} \tau_1(t)^{-p} \left(\int_0^{\tau_1(t)} |f(u)| du \right)^p dt.$$

Since $\tau_1'(s) \leq 2/\ln(e/s)$, $\tau_1^{-1}(s) \leq s \ln(e/s)$ and $\ln \ln(e/s) \leq e^{-1} \ln(e/s)$

($0 < s \leq 1$), we have

$$\begin{aligned} (\tau_1^{-1})'(s) &= \frac{1}{\tau_1'(\tau_1^{-1}(s))} \geq \frac{1}{2} \ln \frac{e}{\tau_1^{-1}(s)} \geq \frac{1}{2} \ln \frac{e}{s \ln(e/s)} \\ &= \frac{1}{2} \left(\ln \frac{e}{s} - \ln \ln \frac{e}{s} \right) \geq \frac{1}{2} \left(1 - \frac{1}{e} \right) \ln \frac{e}{s}. \end{aligned}$$

Hence, after the substitution $t = \tau_1^{-1}(s)$, we obtain

$$\begin{aligned} I_{12}^p &\geq e^{-p} \frac{1}{2} \left(1 - \frac{1}{e} \right)^{\tau_1(t_0)} \int_0^{\tau_1(t_0)} \left(\frac{1}{s} \int_0^s |f(u)| du \right)^p \ln \frac{e}{s} ds \\ &\geq \frac{1}{4} e^{-p} \int_0^{\tau_1(t_0)} C f(s)^p \ln \frac{e}{s} ds, \end{aligned}$$

and so, taking into account (6.3), we get

$$\|f\|_{X_p}^p \geq 4^{-1} (3e)^{-p} \int_0^{\tau_1(t_0)} C f(s)^p \ln \frac{e}{s} ds.$$

On the other hand, by the definition of t_0 ,

$$\begin{aligned} \int_{\tau_1(t_0)}^1 C f(s)^p \ln \frac{e}{s} ds &\leq \ln \frac{e}{\tau_1(t_0)} \int_{\tau_1(t_0)}^1 C f(s)^p ds \leq (1 + t_0) \|f\|_{\text{Ces}(p)}^p \\ &\leq 2 \|f\|_{\text{Ces}(p)}^p \leq 2 \|f\|_{X_p}^p, \end{aligned}$$

where the last inequality follows from (4.1). Hence,

$$\|f\|_{X_p} \geq 8^{-1/p} (3e)^{-1} \left(\int_0^1 C f(s)^p \ln \frac{e}{s} ds \right)^{1/p} \geq \frac{1}{72} \|f\|_{\text{Ces}(p, \ln)},$$

and the imbedding $X_p \hookrightarrow \text{Ces}(p, \ln)$ is proved. Thus, the proof of Theorem 6.1 will be finished if we prove the lemma below. ■

LEMMA 6.2. *If $1 < p < \infty$, then the maximal Hardy–Littlewood operator M on $[0, 1]$ is bounded in the weighted space $L_p(\ln(e/x))[0, 1] = L_p([0, 1], \ln(e/x)dx)$.*

Proof. Muckenhoupt [Mu, Theorem 2] proved that the maximal operator M on $[0, 1]$ is bounded in $L_p([0, 1], w(x)dx)$ if and only if the weight $w(x)$ satisfies the so-called A_p -condition on $[0, 1]$, that is,

$$\sup_{(a,b) \subset [0,1]} \left(\frac{1}{b-a} \int_a^b w(x) dx \right) \left(\frac{1}{b-a} \int_a^b w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

Therefore, it is enough to show that for all intervals $(a, b) \subset [0, 1]$ we have

$$(6.4) \quad \int_a^b \ln \frac{e}{x} dx \left(\int_a^b \left(\ln \frac{e}{x} \right)^{-1/(p-1)} dx \right)^{p-1} \leq 2(b-a)^p.$$

Note that for $t \in (0, b)$,

$$\int_t^b \ln \frac{e}{x} dx = b \ln \frac{e}{b} - t \ln \frac{e}{t} + b - t$$

and

$$\begin{aligned} \int_t^b \left(\ln \frac{e}{x} \right)^{-\alpha} dx &= b \left(\ln \frac{e}{b} \right)^{-\alpha} - t \left(\ln \frac{e}{t} \right)^{-\alpha} - \alpha \int_t^b \left(\ln \frac{e}{x} \right)^{-\alpha-1} dx \\ &\leq b \left(\ln \frac{e}{b} \right)^{-\alpha} - t \left(\ln \frac{e}{t} \right)^{-\alpha}, \end{aligned}$$

where $\alpha > 0$. Since the functions

$$\varphi_1(t) = \frac{b \ln(e/b) - t \ln(e/t) + b - t}{b - t}$$

and

$$\varphi_2(t) = \frac{b(\ln(e/b))^{-\alpha} - t(\ln(e/t))^{-\alpha}}{b - t}$$

are both decreasing on $(0, b)$ for every $0 < b \leq 1$ it follows that $\max_{0 < t < b} \varphi_1(t) = \varphi_1(0^+) = \ln(e^2/b)$ and $\max_{0 < t < b} \varphi_2(t) = \varphi_2(0^+) = \ln^{-\alpha}(e/b)$. Therefore, setting $\alpha = 1/(p-1)$, for all $0 \leq a < b \leq 1$ we have

$$\begin{aligned} \frac{1}{(b-a)^p} \int_a^b \ln \frac{e}{x} dx \left[\int_a^b \left(\ln \frac{e}{x} \right)^{-1/(p-1)} dx \right]^{p-1} &\leq \ln \frac{e^2}{b} \left[\left(\ln \left(\frac{e}{b} \right) \right)^{-1/(p-1)} \right]^{p-1} \\ &= \frac{\ln(e^2/b)}{\ln(e/b)} \leq 2, \end{aligned}$$

and (6.4) is proved. ■

7. $\text{Ces}_p[0, 1]$, $1 < p < \infty$, is not an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$. We start with two lemmas (it is instructive to compare the first with (4.1)).

LEMMA 7.1. *If $1 < p < \infty$, then*

$$(7.1) \quad \text{Ces}_p[0, 1] \hookleftrightarrow (\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, \infty}.$$

Proof. Let us consider the family of characteristic functions $f_s = \chi_{[0, s]}$, $0 < s < 1$. As we know (cf. Theorem 5.3),

$$K(t, f_s; \text{Ces}_1, \text{Ces}_\infty) \geq \frac{1}{3} \|f_s \chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} \quad \text{for all } t > 0.$$

Since

$$\begin{aligned} \|f_s \chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} &= \|\chi_{[0, \min(s, \tau_1(t))]\|_{\text{Ces}(1)} = \|\chi_{[0, \min(s, \tau_1(t))]\|_{L_1(\ln(1/s))} \\ &= \int_0^{\min(s, \tau_1(t))} \ln \frac{1}{s} ds = \min(s, \tau_1(t)) \left[\ln \frac{1}{\min(s, \tau_1(t))} + 1 \right], \end{aligned}$$

it follows that for all t such that $\tau_1(t) \leq s$ we have $\|f_s \chi_{[0, \tau_1(t)]}\|_{\text{Ces}(1)} \geq \tau_1(t) \ln(1/\tau_1(t))$. Therefore, using the inequality $\tau_1^{-1}(s) \leq s \ln(e/s)$ once again, for $0 < s < e^{-1}$ we obtain

$$\begin{aligned} \|f_s\|_{(\text{Ces}_1, \text{Ces}_\infty)_{1-1/p, \infty}} &= \sup_{t>0} t^{1/p-1} K(t, f_s; \text{Ces}_1, \text{Ces}_\infty) \\ &\geq \frac{1}{3} \sup_{t>0, \tau_1(t) \leq s} t^{1/p-1} \tau_1(t) \ln \frac{1}{\tau_1(t)} \\ &\geq \frac{1}{3} (\tau_1^{-1}(s))^{1/p-1} s \ln \frac{1}{s} \geq \frac{1}{6} \left(s \ln \frac{e}{s} \right)^{1/p-1} s \ln \frac{1}{s} \\ &\geq \frac{1}{6} s^{1/p} \left(\ln \frac{e}{s} \right)^{1/p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f_s\|_{\text{Ces}_p} &= \left[\int_0^s \left(\frac{1}{u} \int_0^u \chi_{[0, s]}(v) dv \right)^p du + \int_s^1 \left(\frac{1}{u} \int_0^u \chi_{[0, s]}(v) dv \right)^p du \right]^{1/p} \\ &= \left(s + s^p \int_s^1 u^{-p} du \right)^{1/p} = \left(s + \frac{s^p}{p-1} (s^{1-p} - 1) \right)^{1/p} \\ &= \left(\frac{p}{p-1} s - \frac{1}{p-1} s^p \right)^{1/p} \leq (p')^{1/p} s^{1/p}. \end{aligned}$$

Therefore, for $0 < s < e^{-1}$,

$$\frac{\|f_s\|_{(\text{Ces}_1, \text{Ces}_\infty)_{1-1/p, \infty}}}{\|f_s\|_{\text{Ces}_p}} \geq \frac{\frac{1}{6} s^{1/p} (\ln \frac{e}{s})^{1/p}}{(p')^{1/p} s^{1/p}} \geq \frac{1}{6p'} \left(\ln \frac{e}{s} \right)^{1/p},$$

whence

$$\sup_{0 < s < 1} \frac{\|f_s\|_{(\text{Ces}_1, \text{Ces}_\infty)_{1-1/p, \infty}}}{\|f_s\|_{\text{Ces}_p}} = \infty,$$

which shows that (7.1) holds. ■

Recall that the *characteristic function* $\varphi(s, t)$ of an exact interpolation functor $\mathcal{F}$ is defined by the equality $\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \varphi(s, t)\mathbb{R}$ for all $s, t > 0$. By the Aronszajn–Gagliardo theorem (see [BL, Theorem 2.5.1] or [BK, Theorem 2.3.15]), for every Banach couple (X_0, X_1) and every Banach space $X \in \text{Int}(X_0, X_1)$ there is an exact interpolation functor $\mathcal{F}$ such that $\mathcal{F}(X_0, X_1) = X$.

LEMMA 7.2. *Let $1 < p < \infty$. Suppose that $\text{Ces}_p[0, 1] \in \text{Int}(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])$ and $\mathcal{F}$ is an exact interpolation functor such that*

$$(7.2) \quad \mathcal{F}(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1]) = \text{Ces}_p[0, 1].$$

Then the characteristic function $\varphi(1, t)$ of $\mathcal{F}$ is equivalent to $t^{1/p}$ for $0 < t \leq 1$.

Proof. To simplify notation set $V_p := \text{Ces}_p|_{[1/2,1]}$ ($1 \leq p \leq \infty$), that is, V_p is the subspace of $\text{Ces}_p[0,1]$ which consists of all functions f such that $\text{supp } f \subset [1/2,1]$. Since (V_1, V_∞) is a complemented subcouple of $(\text{Ces}_1[0,1], \text{Ces}_\infty[0,1])$, by (7.2) and the equality in Remark 4.3, we obtain

$$(7.3) \quad \mathcal{F}(V_1, V_\infty) = V_p = (V_1, V_\infty)_{1-1/p, p}.$$

Consider the sequence of functions $g_k(t) = \chi_{[1-2^{-k}, 1-2^{-k-1}]}(t)$, $k = 1, 2, \dots$, and the linear projection

$$Pf(t) = \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} f(s) ds \cdot g_k(t), \quad f \in V_\infty.$$

We have

$$\begin{aligned} \|Pf\|_{V_\infty} &\leq 2\|Pf\|_{L_1|_{[1/2,1]}} \leq 2 \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot 2^{-k-1} \\ &= 2\|f\|_{L_1|_{[1/2,1]}} \leq 2\|f\|_{V_\infty}, \end{aligned}$$

and, since $1-u \leq \ln(1/u) \leq 2(1-u)$ for $1/2 \leq u \leq 1$,

$$\begin{aligned} \|Pf\|_{V_1} &\leq \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} \ln \frac{1}{t} dt \\ &\leq \sum_{k=1}^{\infty} 2^{k+2} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} (1-t) dt \\ &\leq \sum_{k=1}^{\infty} 2^{k+2} \cdot 2^{-2k-1} \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \leq 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|(1-s) ds \\ &\leq 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| \ln \frac{1}{s} ds = 4\|f\|_{L_1(\ln(1/s))} = 4\|f\|_{V_1}. \end{aligned}$$

Therefore, P is a bounded linear projection from V_∞ onto $\text{Im } P|_{V_\infty}$ and from V_1 onto $\text{Im } P|_{V_1}$. At the same time, it is easy to see that the sequence $\{2^{k+1}g_k\}_{k=1}^{\infty}$ is equivalent in V_∞ (resp. in V_1) to the standard basis in l_1 (resp. in $l_1(2^{-k})$). Hence, $(l_1, l_1(2^{-k}))$ is a complemented subcouple of (V_1, V_∞) and therefore, by (6.3) and by the Baouendi–Goulaouic result [BG, Theorem 1] (see also [Tr, Theorem 1.17.1]),

$$\mathcal{F}(l_1, l_1(2^{-k})) = (l_1, l_1(2^{-k}))_{1-1/p, p}.$$

In particular, from the last relation it follows that

$$\mathcal{F}(\mathbb{R}, 2^{-k}\mathbb{R}) = (\mathbb{R}, 2^{-k}\mathbb{R})_{1-1/p, p} = 2^{-k/p}\mathbb{R}$$

uniformly in $k \in \mathbb{N}$. Since the characteristic function of any exact interpolation functor is quasi-concave [BK, Proposition 2.3.10], this implies the result. ■

THEOREM 7.3. *For any $1 < p < \infty$, $\text{Ces}_p[0, 1]$ is not an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$.*

Proof. Assume that $\text{Ces}_p[0, 1]$ is an interpolation space between $\text{Ces}_1[0, 1]$ and $\text{Ces}_\infty[0, 1]$. Then there is an exact interpolation functor $\mathcal{F}$ such that (7.2) holds. By Lemma 7.2, the characteristic function $\varphi(1, t)$ of $\mathcal{F}$ is equivalent to $t^{1/p}$ for $0 < t \leq 1$. Therefore, for any Banach couple (X_0, X_1) we have

$$(7.4) \quad \mathcal{F}(X_0, X_1) \subset (X_0, X_1)_{\psi, \infty},$$

where $(X_0, X_1)_{\psi, \infty}$ is the real interpolation space consisting of all $x \in X_0 + X_1$ such that $\sup_{t>0} \frac{\psi(t)}{t} K(t, x; X_0, X_1) < \infty$ and $\psi(t) = \min(1, t^{1/p})$ [BK, Proposition 3.8.6]. Since $\text{Ces}_\infty[0, 1] \xrightarrow{1} \text{Ces}_1[0, 1]$, applying (7.4) to the couple $(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])$ we obtain

$$(7.5) \quad \mathcal{F}(\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1]) \subset (\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, \infty},$$

whence $\text{Ces}_p[0, 1] \subset (\text{Ces}_1[0, 1], \text{Ces}_\infty[0, 1])_{1-1/p, \infty}$. But in view of Lemma 7.1 the last imbedding does not hold, and the proof is complete. ■

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