Safety Regions in Process Capability Plots

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Abstract: A process is usually defined to be capable if the process capability index exceeds a stated threshold value, e.g., $C_{pm} > 4/3$. This inequality can be expressed graphically as a region in the plane defined by the process parameters (μ, σ) . In the obtained plot special regions can be plotted to test for process capability. These regions are similar to confidence regions for (μ, σ) . This idea of using regions in process capability plots to assess the capability is developed further for the capability index C_{pm} . A new circular region is constructed that can be used, in a simple graphical way, to draw conclusions about the capability of the process at a given significance level. Using circular regions several characteristics with different specification limits and different sample sizes can be monitored in the same plot. Under the assumption of normality the suggested method is investigated with respect to power as well as compared to other existing graphical methods for drawing inference about process capability.

Keywords: Capability index, capability region, circular region, graphical methods, process capability plots, rectangular region, safety region.

1. Introduction

When measuring the capability of a manufacturing process some form of process capability index is often used. Such an index is designed to quantify the relation between the actual performance of the process and its specified requirements. For thorough discussions of different capability indices and their statistical properties see, e.g., the books by Kotz and Johnson [5] and Kotz and Lovelace [7] and the review paper with discussion by Kotz and Johnson [6].

The two most widely used capability indices in industry today are

$$C_p = \frac{USL - LSL}{6\sigma} \text{ and } C_{pk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sigma}, \quad (1)$$

where $[LSL, USL]$ is the specification interval, μ is the process mean and σ is the process standard deviation of the in-control process. According to today's modern quality improvement theories, it is important to use target values and to keep the process on target. The indices in (1) and (2) do not take into account that the process mean, μ , may differ from the target value, T . A capability index which does take this proximity to target into account and hence can be used as a measure of process centering, is C_{pm} , where

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}} \quad (2)$$

with $d = (USL - LSL)/2$. The index C_{pm} is suitable when the specification interval is two-sided with the target value T at the mid-point of the specification interval, which is the case that will be studied in this paper.

In order to gain sensitivity with regard to departures of the process mean from the target value, and at the same time generalize the indices above, Vännman [9] defined a family of capability indices, depending on two non-negative parameters, u and v , as

$$C_p(u, v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (3)$$

The indices in (1)-(2) are obtained by setting $u = 0$ or 1 and $v = 0$ or 1 in (3), i.e., $C_p(0, 0) = C_p$, $C_p(1, 0) = C_{pk}$, $C_p(0, 1) = C_{pm}$. The family in (3) also includes the index C_{pmk} introduced by Pearn et al. [8]. We get

$$C_p(1, 1) = C_{pmk} = \frac{d - |\mu - T|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \quad (4)$$

In this paper we will discuss plots, based on the above-mentioned indices, which can be used to assess process capability at a given significance level. When the process parameters μ and σ are known, a kind of contour plot of the index C_{pk} , called process performance chart, was introduced by Gabel [3] to grasp the capability of a process. To compare the indices C_p , C_{pk} , and C_{pm} Boyles [1] used contour plots, called (μ, σ) -plots, of these indices as functions of the process parameters (μ, σ) . Deleryd and Vännman [2] and Vännman [10] used contour plots, called (δ, γ) -plots, as functions of the process parameters to illustrate the restrictions that the different indices in the $C_p(u, v)$ -family in (3) impose on the process parameters (μ, σ) .

When the process parameters μ and σ are unknown and need to be estimated, Deleryd and Vännman [2] and Vännman [10] developed what they called the (δ^*, γ^*) -plot and the confidence rectangle plot. These plots are tools to draw inference about the process capability based on a random sample from the studied quality characteristic. The plots are based on the (δ, γ) -plot and the $C_p(u, v)$ -family of indices in (3). In Vännman [10] the theory behind the plots is described in more detail, while in Deleryd and Vännman [2] the emphasis is on comparing the two plots both theoretically with respect to power and from a practitioner's point of view.

In this paper we will develop the idea behind the confidence rectangle plot. First a short review is given about the plots discussed by Deleryd and Vännman [2] and Vännman [10]. They used the expression process capability plots to comprise the (δ, γ) -plot as well as the (δ^*, γ^*) -plot and the confidence rectangle plot. For clarification we will, in this paper, restrict the name process capability plot to the (δ, γ) -plot only. Furthermore, we will use the name estimated process capability plot for the (δ^*, γ^*) -plot and introduce the name rectangular safety region for the confidence rectangle plot. Here the idea of using regions in process capability plots is developed further by introducing a new region, a circular region, in order to obtain a more efficient graphical tool than the rectangular region. For this circular region, under the assumption of normality, we derive formulas to be used to calculate significance level and power when C_{pm} is used to define a capable process. A comparison is made between the considered graphical methods with respect to power.

Furthermore, a new application of the estimated process capability plot is given. In Section 8 a short summary of the studied methods, serving as a guideline for the practitioner can be found.

2. Process Capability Plots

When using process capability indices a process is defined to be capable if the process capability index exceeds a certain threshold value $k > 0$. Some commonly used values are $k = 1$, $k = 4/3$, or $k = 5/3$. Assume that we will use an index defined by the family $C_p(u, v)$ in (3), e.g., C_{pk} or C_{pm} , and define the process to be capable if $C_p(u, v) > k$, for given values of u, v , and k . Note that, if the process is on target all indices in the family $C_p(u, v)$ reduce to the same index, which equals C_p . Different choices of u, v , and k impose different restrictions on the process parameters (μ, σ) . This can be easily seen in a process capability plot. This plot is simply a contour plot of $C_p(u, v) = k$ as a function of μ and σ , or as a function of δ and γ , where

$$\delta = \frac{\mu - T}{d} \text{ and } \gamma = \frac{\sigma}{d}. \quad (5)$$

The contour curve is obtained by rewriting the index in (3) as a function of δ and γ , solving the equation $C_p(u, v) = k$ with respect to γ , and then plotting γ as a function of δ . We easily find that $C_p(u, v) = k$ is equivalent to

$$\gamma = \sqrt{\frac{(1 - u|\delta|)^2}{9k^2} - v\delta^2}, \quad |\delta| \leq \frac{1}{u + 3k\sqrt{v}}, \quad (u, v) \neq (0, 0). \quad (6)$$

When $u = v = 0$, i.e., when considering the index $C_p = k$, we have

$$\gamma = \frac{1}{3k}, \quad |\delta| \leq 1. \quad (7)$$

The reason for making the contour plot as a function of (δ, γ) instead of a function of (μ, σ) is to obtain a plot where the scale is invariable, irrespective of the values of the specification limits. This is useful when considering several different characteristics, which is seen in the next section.

To obtain the expression for γ in the case of C_{pk} we let $u = 1$ and $v = 0$ in (6) and we find that the contour curve is composed of two straight lines. To obtain the expression for γ in the case of C_{pm} we let $u = 0$ and $v = 1$ in (6) and we find that the contour curve is a semi-circle. See Figure 1.

Values of the process parameters μ and σ which give (δ, γ) -values inside the region bounded by the contour curve $C_p(u, v) = k$ and the δ -axis will give rise to a $C_p(u, v)$ -value larger than k , i.e., a capable process. We call this region the capability region. Furthermore, values of μ and σ which give (δ, γ) -values outside this region will give a $C_p(u, v)$ -value smaller than k , i.e., a non-capable process.

Figure 1 shows some examples of capability regions. In Figure 1 (b), $C_p = 1$ corresponds to the straight line, $C_{pk} = 1$ to the two lines (dashed) forming a triangle with the δ -axis, $C_{pm} = 1$ to the semi-circle (continuous), and $C_{pmk} = 1$ to the dotted curve closest to the γ -axis. We see clearly the different requirements with regard to closeness to target and

small spread for a capable process using the different indices. From Figure 1 it is clear that a large value of C_{pk} gives no information about closeness to target, which indices like C_{pm} and C_{pmk} do.

For more details regarding process capability plots, see Deleryd and Vännman [2] and Vännman [10].

Figure 1. (a) The process capability plot for $C_{pm} = 1$. (b) The contour curves for the process capability indices C_p , C_{pk} , C_{pm} , and C_{pmk} when $k = 1$. The region bounded by the contour curve and the δ -axis is the corresponding capability region.

3. Estimated Process Capability Plots

We can now state a decision rule, based on sample statistics and an estimated process capability plot, to be used for deciding whether a process can be considered capable or not, when μ and σ are unknown and need to be estimated. We treat the case when the studied characteristic of the process is normally distributed. Let $X_1, X_2, \dots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 as the parametric process quality indicators.

Here we will exemplify the reasoning when C_{pm} is used to define the capability region, i.e., when the studied process is defined to be capable if $C_{pm} > k_0$. For the general case, see Vännman [10]. When using C_{pm} the relation between μ and σ^2 is obtained from (6) using $u = 0, v = 1$, and $k = k_0$. Note that this is a semi-circle with centre in $(0, 0)$ and radius $1/(3k_0)$. See Figure 1 (a).

To obtain an appropriate decision rule we consider a hypothesis test with the null hypothesis $H_0: C_{pm} \leq k_0$ and the alternative hypothesis $H_1: C_{pm} > k_0$. As a test statistic we will use the estimator $\hat{C}_{pm}$, which is obtained by estimating μ and σ^2 with their maximum likelihood estimators, i.e.,

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2. \quad (8)$$

The null hypothesis will be rejected whenever $\hat{C}_{pm} > c_\alpha$, where the constant c_α is determined so that the significance level of the test is α . Vännman [10] showed that the null hypothesis $H_0: C_{pm} \leq k_0$ can be reduced to $H_0: C_{pm} = k_0$. Henceforth in this paper we will consider the null hypothesis $H_0: C_{pm} = k_0$.

The decision rule to be used is then that, for given values of α and n , the process will

be considered capable if $\hat{C}_{pm} > c_\alpha$, where $c_\alpha > k_0$. See Vännman [10] and Hubele and Vännman [4] for details about the distribution of $\hat{C}_{pm}$. The distribution of $\hat{C}_{pm}$ depends on μ and σ but not solely through C_{pm} . Hence for a given value of $C_{pm} = k$, there is not a unique probability $P(\hat{C}_{pm} > c_\alpha | C_{pm} = k)$. This implies that the null hypothesis $H_0: C_{pm} = k_0$ is a composite hypothesis. Hubele and Vännman [4] showed that, when using C_{pm} , the critical value at significance level α is obtained as

$$c_\alpha = k_0 \sqrt{\frac{n}{\chi_{\alpha,n}^2}}, \quad (9)$$

where $\chi_{\alpha,n}^2$ is the α th quantile from a χ^2 -distribution with n degrees of freedom.

Now we can illustrate the decision rule in a plot corresponding to the process capability plot, described in the previous section, to obtain an estimated process capability plot. To do so we introduce the notation

$$\hat{\delta} = \frac{\hat{\mu} - T}{d} = \frac{\bar{X} - T}{d} \quad \text{and} \quad \hat{\gamma} = \frac{\sqrt{\hat{\sigma}^2}}{d} = \frac{\hat{\sigma}}{d} = \frac{1}{d} \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \quad (10)$$

for the estimators of δ and γ , respectively. We will then make a contour plot of the estimated index $\hat{C}_{pm}$, when, $\hat{C}_{pm} = c_\alpha$, as a function of $\hat{\delta}$ and $\hat{\gamma}$. This plot is called the estimated process capability plot. As above we easily find that $\hat{C}_{pm} = c_\alpha$ is equivalent to a semi-circle with centre in $(0, 0)$ and radius $1/(3c_\alpha)$. We can now use the estimated process capability plot to define an estimated capability region in a way corresponding to using the process capability plot. The estimated capability region is then defined as the region bounded by the semi-circle $\hat{C}_{pm} = c_\alpha$ and the $\hat{\delta}$ -axis.

To illustrate the estimated process capability plot we will use the example presented in Vännman [10], where the process is defined capable if $C_{pm} > 1$ and the significance level is 5%. The example consists of two data sets, each of size 80, from a process where $T = 8.70$ and $d = 0.24$. When $n = 80$ we get the critical value $c_{0.05} = 1.151$ from (9). First a random sample was taken from a process, giving rise to estimates of μ and σ equal to 8.6234 and 0.0519, respectively. Later on a second random sample was taken from the same process, after it had been adjusted as a result from the capability analysis based on the first sample. Then, from the second sample, the obtained estimates of μ and σ were equal to 8.6628 and 0.0516, respectively.

Based on the sample data we find the observed values of $\hat{\delta}$ and $\hat{\gamma}$, using (10), to be -0.319 and 0.216 , respectively, for the first sample, and -0.155 and 0.215 , respectively, for the second sample. In the estimated capability plot defined by $\hat{C}_{pm} = 1.151$ we then plot the points with coordinates $(-0.319, 0.216)$ and $(-0.155, 0.215)$, for the first and second sample, respectively. The results are found in Figure 2 (a) and (b).

We can see that for the first sample, the point $(-0.319, 0.216)$ is outside the estimated capability region and hence the process cannot be considered capable. For the second sample, the point $(-0.155, 0.215)$ is now inside the estimated capability region and hence the process can be considered capable at 5% significance level, according to the definition of a capable process based on $C_{pm} > 1$.

What is obvious from the graphical method used here is that the non-capability in Figure 2 (a) is, to a large extent, due to the deviation of the process mean from the target

value. By looking at the estimated process capability plot we get information instantly about both the deviation from the target value and the process spread as well as information about the capability. We can also see how much the deviation from target must be reduced in order to achieve a capable process, given that the standard deviation is not increased.

Figure 2. The estimated process capability plot defined by $\hat{C}_{pm} = c_{0.05} = 1.151$. In (a) the first sample with the observed value of $(\hat{\delta}, \hat{\gamma})$ outside the estimated capability region, illustrate a non-capable process. In (b) the second sample, with observed value of $(\hat{\delta}, \hat{\gamma})$ inside the estimated process capability region, illustrate a capable process.

The estimated capability region, defined by $\hat{C}_{pm} > c_{\alpha}$, corresponds to a rejection region when testing the null hypothesis $H_0: C_{pm} = k_0$ against the alternative hypothesis $H_1: C_{pm} > k_0$ at significance level α . Hence the size of estimated capability region will depend on the significance level (type I error) and the sample size n . The radius of the semi-circle, which bounds the estimated capability region, is equal to $1/(3c_{\alpha})$, where c_{α} is given in (9). From this we can conclude, for fixed value of n , that the smaller the significance value α the smaller the estimated capability region. Furthermore, for fixed value of α , the smaller the sample size the smaller the estimated capability region. Note that, since $c_{\alpha} > k_0$ the radius of the semi-circle, which bounds the estimated capability region is always shorter than the radius of the semi-circle for corresponding theoretical capability region.

The above reasoning is illustrated in Figure 3, where Figure 3 (a) gives an example of how the estimated capability region changes with the significance level α for fixed sample size. Figure 3 (b) illustrates how the estimated capability region changes with the sample size n for fixed significance level. In both plots in Figure 3 we have also added the theoretical capability level curve corresponding to $C_{pm} = 1$. This is the dashed semi-circle, with the largest radius. Furthermore, the continuous semi-circle defines the estimated capability region at 5% significance level when $n = 80$ and is defined by $\hat{C}_{pm} = c_{0.05} = 1.151$ in both (a) and (b). In Figure 3 (a) the dotted semi-circle corresponds to 1% significance level and is defined by $\hat{C}_{pm} = c_{0.01} = 1.222$. In Figure 3 (b) the dotted semi-circle corresponds to $n = 30$ and is defined by $\hat{C}_{pm} = c_{0.05} = 1.274$.

Note that, the point with coordinates $(\hat{\delta}, \hat{\gamma})$ has to lie inside the *estimated* capability region to be considered capable at a stated significance level. If the point with coordinates $(\hat{\delta}, \hat{\gamma})$ lies inside the theoretical capability region but outside the corresponding estimated capability region, we cannot reject the null hypothesis and hence cannot conclude that the process is capable at significance level α .

Figure 3. The dashed semi-circle defines the theoretical capability region for $C_{pm} = 1$ and the continuous semi-circle defines the estimated capability region at 5% significance level, when $n = 80$. In (a) the dotted semi-circle corresponds to 1% significance level. In (b) the dotted semi-circle corresponds to $n = 30$.

4. A New Application of the Estimated Process Capability Plot

The studied process capability plots are invariable, irrespective of the values of the specification limits. Hence we can monitor, in the same plot, several characteristics of a process and at the same time retain the information on the location and spread of the process. As an example consider the following situation. Three different quality characteristics A , B , and C are of interest. They have the following different specification limits: A : [200, 210]; B : [12.0, 16.0]; C : [1.40, 2.20], with target values equal to the midpoints of each interval. They are assumed to be normally distributed and in statistical control. The process under investigation will be considered capable when all three characteristics are considered capable according to the definition that $C_{pm} > 1$.

A sample of size 50 is taken from each of the three characteristics A , B , and C and using (10) the point $(\hat{\delta}, \hat{\gamma})$ is calculated for each of A , B , and C . Note that we have for A : $T = 205$ and $d = 5$; for B : $T = 14$ and $d = 2$; for C : $T = 1.80$ and $d = 0.40$. Since the process will be considered capable when all three characteristics are considered capable, we will choose the significance level for the estimated capability curve to be 1%. The decision rule is that the process will be considered capable if all three points $(\hat{\delta}, \hat{\gamma})$ fall inside the estimated capability region, where the significance level for the estimated capability curve is 1%. Then we know, according to Bonferroni's inequality, that the significance level for the decision rule will be at most 3%. The critical value when $n = 50$ and $\alpha = 0.01$ is found to be $c_{0.01} = 1.199$ according to (9). In Figure 4 we can see the results from a simulated case, where we have the following observed values of $(\hat{\delta}, \hat{\gamma})$ for the three characteristics: A : (0.05, 0.20); B : (0.25, 0.24); C : (0.15, 0.17).

In Figure 4 we cannot consider the process capable, at 3% significance level, although two of the points fall inside the semi-circle, since characteristic B falls outside. Figure 4 shows that from one single plot we can get, for each of the three characteristics A , B , and C , information about the deviation from target and the standard deviation, as well as information of how to adjust the process to make it capable, if it is non-capable. If we study the same single characteristic over a longer period of time we can plot, in a similar way as in Figure 4, the estimated indices from different time periods in one process capability plot and follow the change over time in a lucid way.

Figure 4. The estimated process capability plot defined by $\hat{C}_{pm} = c_{0.01} = 1.199$. A , B , and C indicate the observed value of $(\hat{\delta}, \hat{\gamma})$ of the three characteristics A , B , and C , respectively.

5. A Rectangular Safety Region

As an alternative to the estimated process capability plot, when drawing conclusions about process capability, we can use the theoretical process capability plot together with a suitable rectangular region for (δ, γ) , as shown by Deleryd and Vännman [2]. They suggested the following rectangular region $I_\delta \times I_\gamma$ based on the traditional confidence interval for μ and σ , respectively, in a normal distribution.

$$I_\delta = \hat{\delta} \pm t_{\theta, n-1} \frac{\hat{\gamma}}{\sqrt{n-1}} \quad \text{and} \quad I_\gamma = \left[\sqrt{\frac{n\hat{\gamma}^2}{\chi_{\theta, n-1}^2}}, \sqrt{\frac{n\hat{\gamma}^2}{\chi_{1-\theta, n-1}^2}} \right], \quad (11)$$

where $t_{\theta, n-1}$ is the θ th quantile from the t -distribution, and $\chi_{\theta, n-1}^2$ is the θ th quantile from the χ^2 -distribution, both with $n-1$ degrees of freedom. Note that, in order to obtain a rectangle, both an upper and lower bound is used in the confidence interval for μ and σ .

They defined a process as capable if the whole rectangular region for (δ, γ) is inside the capability region in the theoretical process capability region defined by $C_p(u, v) = k_0$. The θ th quantile from the t - and χ^2 -distribution, used in (11), are chosen so that the probability that the process is considered capable, given that $C_p(u, v) = k_0$, equals the significance level α . This means that the probability that the rectangular region is inside the capability region defined by $C_p(u, v) = k_0$, has to be at most α , for all possible δ -values along the semi-circle defined by $C_p(u, v) = k_0$.

The rectangular region can be considered as a safety region, which is built around the estimate $(\hat{\delta}, \hat{\gamma})$ to take the randomness of the estimator $(\hat{\delta}, \hat{\gamma})$ into account. The size of the region is determined in such a way that it corresponds to significance level α .

It was found, through a simulation study that, when using the index C_{pm} to define a capable process, the value $\theta = 0.94$ will give the significance level $\alpha = 0.05$. This result seems to be independent of the values of n and k . For more details, see Deleryd and Vännman [2]. If we apply this method to the example in the previous section, where $n = 80$ and $\alpha = 0.05$ and the process is defined as capable when $C_{pm} > 1$ we get the results in Figure 5.

Figure 5. The process capability plot defined by $C_{pm} = 1$ with the rectangular region. In (a) the process cannot be considered capable, but in (b) the whole rectangle is inside the capability region and hence the process is considered capable at 5% significance level.

The two methods considered so far have been compared with respect to power by Deleryd and Vännman [2]. The results from this comparison show that for values of δ not far from 0, i.e., when the process output is close to target, the estimated process capability plot gives the largest power. But for values of δ further away from 0, where γ -values are close to 0, the rectangular safety region plot is more powerful.

6. A Circular Safety Region

A drawback of using the rectangular safety region is that there is, so far, no available analytical expression for calculating the probability to reject the null hypothesis. Hence we have to rely on simulations both for determining the significance level and the power. For the estimated process capability plot an analytical expression for the probability to reject the null hypothesis for a given value of an index is derived. See Vännman [10]. Furthermore, the power for the rectangular region is much smaller than for the estimated process capability plot unless the expected process output is away from target and the standard deviation is small. In order to overcome some of these drawbacks and further develop the idea of using a safety region in the process capability plot, we suggest using a circle instead of the rectangle.

One reason for the small power of the rectangular safety region is that the region has corners. Then it seems natural, as a first approximation, to substitute the rectangle with a circle with its centre in $(\hat{\delta}, \hat{\gamma})$, i.e., to put a circular safety region around the estimated point. To find a suitable radius of the circle we consider the variability in the two estimators $\hat{\delta}$ and $\hat{\gamma}$. Under the assumption of normality the standard deviation of each of the estimators $\hat{\delta}$ and $\hat{\gamma}$ is proportional to γ . Hence we will let the radius be proportional to the estimator of γ . This means that the circle to be studied is defined to have its centre in $(\hat{\delta}, \hat{\gamma})$ and its radius equal to $R\hat{\gamma}$, where R is a suitably chosen constant. Also note that the length of the confidence interval for δ as well as for γ is proportional to $\hat{\gamma}$.

Hence the circular safety region to be studied is defined as the region bounded by the circle

$$(\delta - \hat{\delta})^2 + (\gamma - \hat{\gamma})^2 = R^2 \hat{\gamma}^2 \quad (12)$$

The constant R will depend on the sample size n and the significance level α . But to keep notations simple we will write R instead of $R(n, \alpha)$.

We will define a process as capable if the whole circular safety region for (δ, γ) is inside the capability region in the theoretical process capability plot defined by $C_{pm} = k_0$. See Figure 6 (b) for an example. The constant R will be chosen so that the probability that the process is considered capable, i.e., that the circular safety region is inside the capability region, given that $C_{pm} = k_0$, equals the significance level α . As above the circular safety region corresponds to a given significance level α .

Under the assumption that the studied characteristic of the process is normally distributed, we can derive the probability that the circular region is inside a capability region defined by $C_{pm} = k_0$. For details, see the Appendix. Let ξ denote a random variable distributed according to a central χ^2 -distribution with $n-1$ degrees of freedom and let f_ξ denote its probability density function. Let τ denote a random variable distributed according to a non-central χ^2 -distribution with 1 degree of freedom and non-centrality parameter

$$\lambda = \frac{(\mu - T)^2 n}{\sigma^2} = \frac{\delta^2 n}{\gamma^2}. \quad (13)$$

Furthermore, let F_τ denote the cumulative distribution function of τ . Then the probability, $\Pi(\delta, k)$, that the circular region for (δ, γ) is inside the capability region defined by $C_{pm} = k_0$, given that (δ, γ) is a point on the semi-circle defined by $C_{pm} = k$, can be expressed as follows.

$$\Pi(\delta, k) = \int_0^{Q(\delta, k)} F_\tau \left(\left(S(\delta, k) - R\sqrt{y} \right)^2 - y \right) f_\xi(y) dy, \quad (14)$$

where

$$S(\delta, k) = \sqrt{\frac{9k^2 n}{9k_0^2(1 - 9k^2\delta^2)}} \quad \text{and} \quad Q(\delta, k) = \frac{S(\delta, k)^2}{(R+1)^2}. \quad (15)$$

The probability in (14) is defined for $|\delta| < 1/(3k)$. Instead of using the cumulative distribution function of the non-central χ^2 -distributed random variable τ when calculating the probability in (14) we can use the cumulative distribution function Φ of a $N(0, 1)$ distributed random variable. We have the following relation between F_τ and Φ .

$$F_\tau(t) = \Phi(\sqrt{t} - \sqrt{\lambda}) - \Phi(-\sqrt{t} - \sqrt{\lambda}) \quad (16)$$

We see from (14) that the probability that the circular region for (δ, γ) is inside the capability region defined by $C_{pm} = k_0$ depends on δ and k . If we want to calculate the significance level we put $k = k_0$ in (14). But the probability in (14) still depends on δ . Hence we have to find the value of δ that maximizes the probability in (14) when $k = k_0$. See the Appendix.

In order to have a decision rule with known significance level α , we have to determine the radius $R = R(n, \alpha)$, so that the probability in (14) is at most α , when $k = k_0$. So far this has to be done numerically utilizing the expressions (31)-(32) in the Appendix.

If we apply this circular safety region to the example in the previous section, where $n = 80$ and $\alpha = 0.05$ and the process is defined as capable when $C_{pm} > 1$ we find $R = 0.1903$. In

Figure 6 we see the circular regions plotted. The conclusions about capability are the same as before.

Figure 6. The process capability plot defined by $C_{pm} = 1$ together with the circular region. In (a) the process cannot be considered capable, but in (b) the whole circle is inside the capability region and hence the process is considered capable at 5% significance level.

In Figure 7 we see the rectangular and circular safety regions plotted together for the example in the previous section. We see that the upper corners of the rectangle lie outside the circle. This indicates that the power might be better when using the circular region instead of the rectangular region. Notice that the rectangular region does not have the observed value of $(\delta, \hat{\gamma})$ as its centre point since the confidence interval for γ is not symmetric around $\hat{\gamma}$.

Figure 7. The process capability plot defined by $C_{pm} = 1$ together with the circular region from Figure 6 and the rectangular region from Figure 5.

Previously we saw how the estimated process capability plot can be used to monitor, in the same plot, several characteristics of a process and at the same time retain the information on the location and spread of the process. However, this is based on the assumption of the same sample size for each characteristic. If the sample sizes are unequal we will get different estimated capability regions and hence cannot monitor, in the same plot, several characteristics of a process. But in such cases we can use the safety regions instead, since the theoretical capability region does not depend on the sample size. Instead the size of areas of the regions will differ.

Consider the example discussed in the Section 4. There, three different quality

characteristics A , B , and C with different specifications limits were studied and a sample of size 50 was taken for each of them. Let us now instead assume that the sample size for characteristic A is 50, for B is 30, and for C is 70. Then we cannot use the estimated process capability plot to monitor all three characteristics in one plot. However, we can use the circular region plot. Let us assume that we have the same observed values of $(\hat{\delta}, \hat{\gamma})$ as above for the three characteristics. At 1% significance level for each circular region we obtain the following values for R in the three cases using the result from the Appendix.

Table 1. The information needed for plotting the three circles, each at 1% significance level.

Characteristic	Sample size	R -value	Radius	Centre point
A	50	0.3567	0.07134	(0.05, 0.20)
B	30	0.4863	0.11671	(0.25, 0.24)
C	70	0.2945	0.05007	(-0.15, 0.17).

In Figure 8 (a) we see all circular regions for the three characteristics. Characteristic B has the smallest sample size and the largest standard deviation among the three and hence will have the largest circle. As before we can conclude that, at 1% significance level, both A and C are capable, since their circles are inside the capability region defined by the semi circle. However, B cannot be considered capable since its circle is not completely inside the semi circle. Since the process under investigation will be considered capable, when all three characteristics are capable according to the definition that $C_{pm} > 1$, we cannot claim that the process is capable at 3% significance level.

Figure 8. Process capability plots with circular regions for each of the characteristics A , B , and C , each at 1% significance level. In (a) the sample sizes differ according to Table 1. In (b) all have sample size 50. The observed values of $(\hat{\delta}, \hat{\gamma})$ are the same in (a) and (b).

We can, of course, use circular regions when the sample sizes are equal. In Figure 8 (b) we see the case when all sample sizes equal 50. Figure 8 (b) can be compared with Figure 4. Both Figures describes the same situation. Using circular regions the differences in standard deviations between the characteristics is stressed even more since the radius of the circle is proportional to the standard deviation. Note that in Figure 4 the estimated capability region is plotted and its semi-circle has shorter radius than the theoretical semi-circle in the capability plot in Figure 8 (b).

7. Power Comparisons

Using the probability in (14) we can calculate the power for the circular safety region for different values of $C_{pm} = k > k_0$. To be able to make comparison with the power of the rectangular safety region we will consider the situation studied by Deleryd and Vännman [2] where the null hypothesis studied is $H_0: C_{pm} = 4/3$ and the alternative hypothesis is $H_1: C_{pm} > 4/3$. We consider the power for $C_{pm} = 5/3, 1.8$ and 2.0 and the sample size $n = 50$. Using (31)-(32) in the Appendix we numerically determine the R -value so that the probability in (31) is at most $\alpha = 0.05$, for all possible α -values, when $k_0 = 4/3$. We find $R = 0.2457$.

The power, for a given value of $C_{pm} = k$, will depend on δ and hence we will plot, for a given value of $C_{pm} = k$, the power as a function of δ . In Figure 9 we see the power, as a function of δ , for $C_{pm} = 5/3$ (bottom curve), $C_{pm} = 1.8$ (middle curve), $C_{pm} = 2$ (top curve). Since the power is symmetric around $\delta = 0$ we only plot it for $\delta \geq 0$.

The corresponding power plots for the estimated process capability plot and the rectangular safety plot are given in Deleryd and Vännman [2] and presented in Figure 10.

Figure 9. The power for the circular safety region, as a function of δ , for $C_{pm} = 5/3$ (bottom curve), $C_{pm} = 1.8$ (middle curve), $C_{pm} = 2$ (top curve), when $H_0: C_{pm} \leq 4/3$.

Comparing Figure 9 and Figure 10 (b) we see that when $\delta = 0$ the power for the circular region and rectangular region is approximately the same. But for $\delta > 0$ the power for the circular region will always be larger than the power for the rectangular region.

Since the power for both the circular region and the estimated process capability plot can be calculated using analytical formulas we plot the powers for these two cases in one figure for easier comparison. Furthermore, since the power is close to 1 when $C_{pm} = 2$ we exclude this case in the comparison (see Figure 11). We see from Figure 11 that when δ is not too far away from 0 the estimated process capability plot is more powerful, but when δ gets further away from 0 the power of the circular region gets larger than the power of the estimated process capability plot. Hence if a safety region plot is wanted to decide on the capability of a process the circular safety region plot is recommended. However, if there is indication that the process is fairly close to target, the estimated process capability plot is more powerful than the circular safety region plot and hence to be preferred.

Figure 10. The power, as a function of δ , for $C_{pm} = 5/3$ (bottom curve), $C_{pm} = 1.8$ (middle curve), $C_{pm} = 2$ (top curve). In (a) the power for estimated process capability plot is shown and in (b) the power for the rectangular safety region. The null hypothesis is $H_0: C_{pm} \leq 4/3$.

Figure 11. The power, as a function of δ , for two values of C_{pm} belonging to the alternative hypothesis. (a) $C_{pm} = 5/3$. (b) $C_{pm} = 1.8$. The continuous line shows the power for the circular safety region. The dotted line shows the power for the estimated process capability plot.

8. Summary

In Table 2 we summarize and shortly comment on the studied graphical methods described in Sections 3-7. The basic assumption is that the studied characteristic of the process is normally distributed. Furthermore, the methods imply that decisions can be drawn at a stated significance level α . We will exemplify by using the index C_{pm} and comment whether it can be generalized to the general $C_p(u, v)$ -family of indices defined in (3). The process is defined as capable if $C_{pm} > k_0$.

Table 2. Summary of decision rules for using estimated process capability plot, rectangular safety region plot, and circular safety region plot.

Method	Decision rule at significance level	Comments
Estimated process capability plot. See Sections 3-4.	Consider the process as capable when the observed value of $(\hat{\delta}, \hat{\gamma})$ falls inside the <i>estimated</i> capability region defined by $\hat{C}_{pm} > c_\alpha$. For an example see Figure 2.	$(\hat{\delta}, \hat{\gamma})$ is defined in (10). This method works for the general $C_p(u, v)$ -family. For C_{pm} the critical value c_α can be expressed analytically, see (9). For other indices c_α has to be found by numerical calculations, see Vännman [10]. When the expected process output is on target this method has the largest power.
Rectangular safety region plot. See Section 5.	Consider the process as capable when the whole rectangular region, defined in (11), is inside the capability region defined by $C_{pm} = k_0$. For an example, see Figure 5.	This method works for the general $C_p(u, v)$ -family. However, the θ -value needed in (11) has to be found by simulations. For C_{pm} the value $\theta = 0.94$ will give the significance level $\alpha = 0.05$.
Circular safety region plot. See Section 6.	Consider the process as capable when the whole circular region, defined in (12) is inside the capability region defined by $C_{pm} = k_0$. For an example, see Figure 6.	The radius in the circle can be determined numerically by using the expression in (14). When the process is on target the power of rectangular region and the circular region, respectively, is approximately the same. But the power for the circular region will always be larger than the power for the rectangular region when the process is off target. This method is the most powerful one when the process is somewhat off target.

9. Discussion and Concluding Remarks

The graphical methods discussed here are efficient in the respect that in one single plot we get visual information simultaneously about the location and spread of the studied characteristic, as well as information about the capability of the process at a given significance level. In capability analysis using software easily available today, we get the estimated indices together with confidence limits for the indices C_p , C_{pk} , and C_{pm} and a histogram for the studied characteristic. Combining the information from the estimated indices and the histogram we may see if a process is off target or if the spread is large. However, we cannot at a glance relate the deviation from target and the spread to each other and to the capability index in such a way that we are able to see whether the non-capability is caused by the fact that the process output is off target, or that the process spread is too large, or if the result is a combination of these two factors. Furthermore, we cannot easily see how large a change is needed to obtain a capable process.

The circular safety region plot, which is introduced in this paper, is more powerful than the previously suggested rectangular region plot. Furthermore, the significance level and power can be expressed analytically using the circular region plot, which is not the case for the rectangular region plot. Hence the circular safety region plot is recommended if a safety region plot is of interest. If the process is almost on target the estimated process capability plot is most powerful. However, if the process is further away from target the circular safety region plot is most powerful among the three methods studied in this paper.

The circular safety region plot is useful if we want to monitor more than one characteristic in the same plot, since this plot does not require the same sample sizes as the estimated process capability plot does.

The safety region plots have the advantage that they show the theoretical capability area, which defines the capability and in this plot the result from the estimation is plotted. This might be easier to grasp for a practitioner than the estimated capability plot, where the estimated capability region always will be smaller than the corresponding theoretical region.

The safety regions considered in this paper should not be interpreted as confidence regions since they are not designed to contain (δ, γ) or C_{pm} at a certain confidence level. Instead they are constructed so that the probability that the safety region is inside the capability region, given that $C_{pm} = k_0$, equals the significance level α , when testing $H_0: C_{pm} = k_0$ against $H_1: C_{pm} > k_0$. Note that $H_0: C_{pm} = k_0$ is a composite hypothesis. Hence $C_{pm} = k_0$ is not equivalent to $(\delta, \gamma) = (\delta_0, \gamma_0)$. From Figure 1 it is clear that there is an infinite number of (δ, γ) that correspond to $C_{pm} = k_0$, and that they are located along a semi-circle with centre in $(0,0)$ and radius $1/(3k_0)$.

A natural generalization for a safety region is to use an ellipse instead of a circle. This will probably increase power in the vicinity of the target value. The problem of how to define a suitable ellipse plot and how to calculate its power is still under study.

Using any of the herein discussed plots, compared to using the capability index alone, we will, from one single plot, instantly get visual information simultaneously about the location and spread of the quality characteristic, as well as information about the capability of the process. In this way, the proposed plots give clear directions for quality improvement. The capability indices were introduced to focus on the process variability and closeness to target and relate these quantities to the specification interval and the target value. We believe that the plots discussed here will do this in a more efficient way than the capability index alone. It is also well known that the visual impact of a plot is more efficient than numbers, such as estimates or confidence limits. Furthermore, with today's modern software all the plots proposed here are easy to generate.

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Appendix

Here we will derive the probability, $\Pi(\delta, k)$, that the circular safety region for (δ, γ) defined in (12) is inside the capability region defined by $C_{pm} = k_0$, given that (δ, γ) is a point on the semi-circle defined by $C_{pm} = k$. We treat the case when the studied characteristic of the process is normally distributed. Let $X_1, X_2, \dots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 as the parametric process quality indicators.

Consider the case when the circle defined by (12) is inside the capability region defined by $C_{pm} = k_0$, i.e., the circle defined by

$$(\delta - \hat{\delta})^2 + (\gamma - \hat{\gamma})^2 = R^2 \hat{\gamma}^2 \quad (17)$$

lies inside the semi-circle defined by

$$\delta^2 + \gamma^2 = \left(\frac{1}{3k_0} \right)^2, \gamma > 0. \quad (18)$$

Using Lagrange's theorem and straightforward but cumbersome calculations we can show that the shortest distance between the circle in (17) and the semi-circle in (18) is obtained when the normal through a point on the tangent of the outer semi-circle also goes through the centre of the inner circle and through (0, 0). See Figure 12.

Hence the circle in (17) will fall inside the capability region defined by (18) if

$$\frac{1}{3k_0} - (\sqrt{\hat{\delta}^2 + \hat{\gamma}^2} + R\hat{\gamma}) \geq 0 \quad (19)$$

which leads to that

$$\Pi(\delta, k) = P(\sqrt{\hat{\delta}^2 + \hat{\gamma}^2} + R\hat{\gamma} \leq \frac{1}{3k_0}). \quad (20)$$

Figure 12. A sketch showing the principle idea of how to find the shortest distance between the semi-circle defined by $C_{pm} = k_0$ and the circle defined in (17).

To derive the probability in (20), given that (δ, γ) is a point on the semi-circle defined by $C_{pm} = k$, we introduce the notation

$$\tau = \frac{n\hat{\delta}^2}{\gamma^2} = \frac{n}{\gamma^2} \left(\frac{\bar{X} - T}{d} \right)^2 = \left(\frac{\bar{X} - T}{\sigma/\sqrt{n}} \right)^2 \quad (21)$$

and

$$\xi = \frac{n\hat{\gamma}^2}{\gamma^2} = \frac{n}{\gamma^2} \left(\frac{1}{d} \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (22)$$

In this notation the probability $\Pi(\delta, k)$ becomes

$$\Pi(\delta, k) = P\left(\sqrt{\tau + \xi} + R\sqrt{\xi} \leq \frac{\sqrt{n}}{3k_0\gamma}\right). \quad (23)$$

Under the assumption of normality of the characteristic of the process the random variables τ and ξ are independent, and ξ is distributed according to a central χ^2 -distribution with $n - 1$ degrees of freedom. Let f_ξ denote the probability density function of ξ . Further τ is distributed according to a non-central χ^2 -distribution with 1 degree of freedom and non-centrality parameter λ , where λ is given in (13). Let F_τ denote the cumulative distribution function of τ .

To obtain the probability in (23) we derive the cumulative distribution function of $\sqrt{\tau + \xi} + R\sqrt{\xi}$. By conditioning on ξ we get

$$P(\sqrt{\tau + \xi} + R\sqrt{\xi} \leq z) = \int_0^\infty P(\sqrt{\tau + \xi} + R\sqrt{\xi} \leq z | \xi = x) f_\xi(x) dx$$

$$= \int_0^{z^2/B^2} P\left(\sqrt{\tau+x} \leq z - R\sqrt{x}\right) f_{\xi}(x) dx. \quad (24)$$

The last equality holds since

$$P\left(\sqrt{\tau+x} \leq z - R\sqrt{x}\right) = 0 \text{ for } x > \frac{z^2}{R^2} \quad (25)$$

and hence the values $x > z^2/R^2$ yield no contribution to the integral. Rearranging the last equality in (24) we obtain

$$\begin{aligned} P\left(\sqrt{\tau+\xi} + R\sqrt{\xi} \leq z\right) &= \int_0^{z^2/R^2} P\left(\tau \leq (z - R\sqrt{x})^2 - x\right) f_{\xi}(x) dx \\ &= \int_0^{z^2/(R+1)^2} P\left(\tau \leq (z - R\sqrt{x})^2 - x\right) f_{\xi}(x) dx. \end{aligned} \quad (26)$$

The second equality in (25) is valid since the values $x > z^2/(R+1)^2$ yield no contribution to the integral (being $\tau > 0$). Combining (23) and (26) we obtain

$$P\left(\sqrt{\tau+\xi} + R\sqrt{\xi} \leq \frac{\sqrt{n}}{3k_0\gamma}\right) = \int_0^Q F_{\tau}\left(\left(\frac{\sqrt{n}}{3k_0\gamma} - R\sqrt{x}\right)^2 - x\right) f_{\xi}(x) dx, \quad (27)$$

where

$$Q = \frac{n}{(3k_0\gamma)^2(R+1)^2}. \quad (28)$$

Since the probability in (27) is calculated given that (δ, γ) is a point on the semi-circle defined by $C_{pm} = k$, we can express γ as a function of δ , using the relation in (6) with $u = 0$, $v = 1$, and arrive at the result stated in (14)-(15), i.e.,

$$\Pi(\delta, k) = \int_0^{Q(\delta, k)} F_{\tau}\left(\left(S(\delta, k) - R\sqrt{y}\right)^2 - y\right) f_{\xi}(y) dy, \quad (29)$$

where

$$S(\delta, k) = \sqrt{\frac{9k^2n}{9k_0^2(1-9k^2\delta^2)}} \quad \text{and} \quad Q(\delta, k) = \frac{S(\delta, k)^2}{(R+1)^2}. \quad (30)$$

The probability in (29) is defined for $|\delta| < 1/(3k)$.

We define a process as capable if the whole circular region for (δ, γ) is inside the capability region in the theoretical process capability plot defined by $C_{pm} = k_0$. The constant R will be chosen so that the probability that the process is considered capable, i.e., that the circular region is inside the capability region, given that $C_{pm} = k_0$, equals the significance level α . To obtain this probability we put $k = k_0$ in (29)-(30) and get

$$\Pi(\delta, k_0) = \int_0^{Q(\delta, k_0)} F_r \left(\left(S(\delta, k_0) - R\sqrt{y} \right)^2 - y \right) f_\xi(y) dy, \quad (31)$$

where

$$S(\delta, k_0) = \sqrt{\frac{n}{1 - 9k_0^2 \delta^2}} \quad \text{and} \quad Q(\delta, k_0) = \frac{S(\delta, k_0)^2}{(R+1)^2}. \quad (32)$$

However $\Pi(\delta, k_0)$ in (31) depends on δ , so we have to find the δ -value that maximizes the function $\Pi(\delta, k_0)$ in order to find the significance level. Let δ_0 denote the δ -value, where $|\delta| \leq 1/3k_0$, which gives the supremum of $\Pi(\delta, k_0)$ in (31). Then we have the significance level $\alpha = \Pi(\delta_0, k_0)$. The values of δ_0 and the constant R have to be found numerically using (31), since so far no analytic expression for δ_0 has been found.

In Figure 13 we can see how $\Pi(\delta, k_0)$ in (31) depends on δ for the cases when (a) $k_0 = 4/3$, $n = 50$ and (b) $k_0 = 1$, $n = 80$, which are studied previously. In both cases this probability has its maximum when δ is far from 0 and close to its border value $1/3k_0$. The maximum is obtained for $\delta = \delta_0 = 0.238$ in (a) and for $\delta = \delta_0 = 0.323$ in (b).

Figure 13. The probability, $\Pi(\delta, k_0)$, that the circular region for (δ, γ) is inside the capability region defined by $C_{pm} = k_0$, given that (δ, γ) is a point on the semi-circle defined by $C_{pm} = k_0$ for the cases when (a) $k_0 = 4/3$, $n = 50$ and (b) $k_0 = 1$, $n = 80$.

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