SIMPLE NON-ITERATIVE CALIBRATION FOR TRIAXIAL ACCELEROMETERS

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Abstract. For high precision measurements, accelerometers need recalibration between different measurement occasions. In this paper we derive a simple calibration method for triaxial accelerometers with orthogonal axes. Just like previously proposed iterative methods, we compute the calibration parameters (biases and gains) from measurements of the Earth gravity for six different unknown orientations of the accelerometer. However, our method is non-iterative, so there are no complicated convergence issues depending on input parameters, round-off errors etc.

The main advantages of our method are that only from the accelerometer output voltages it gives a complete knowledge of whether it is possible, with any method, to recover the accelerometer biases and gains from the output voltages, and when this is possible, we have a simple explicit formula for computing them with a smaller number of arithmetic operations than previous iterative approaches. Moreover, we show that such successful recovery is guaranteed if the six calibration measurements deviate with angles smaller than some upper bound from a natural setup with two horizontal axes. We provide an estimate from below of this upper bound that, for instance, allows 5 degree deviations in arbitrary directions for the Colibrys SF3000L accelerometers in our lab.

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1. Introduction

For many different applications each output voltage of a triaxial accelerometer can be well modeled as depending linearly on the measured acceleration via a multiplicative gain factor and a zero offset (bias). Both the gain and the bias change noticeably, for example, with the temperature [1]. They can differ by ±10 % from one accelerometer to the next, or between the axes of the same accelerometer[1]. This gives an error both in the direction and the size of the measured acceleration. For good reliability for instance at different temperatures, and in collaborative use of several accelerometers, it is therefore important to calibrate the accelerometers before measuring or repeatedly during long-time measurements.

This paper is intended for in-field calibration of triaxial accelerometers with no other sensors available for assisting the calibration. A commonly used in-field approach [2, 3, 4, 5, 6] is to assume the accelerometer to measure in three perfectly orthogonal directions and to compute the three gain and three bias parameters from measurements of the local Earth gravity for six different orientations of the accelerometer. This approach requires no explicit knowledge of the actual orientation of the accelerometer with respect to gravity.

For solving the resulting system of six nonlinear equations, different iterative numerical solution methods have been proposed [2, 4, 5, 6]. A linear minimum variance unbiased estimator approach has also been suggested for some slightly different applications [3, 7]. For iterative methods, more work is needed to clarify for which measurement setups there is a unique solution and whether the iterative algorithm converges to this solution. We solve this problem by deriving a simple *non-iterative* calibration method for which we can state explicit conditions on the measurement setups that guarantee successful calibration. In a direct comparison of our solution with the fast converging method proposed in [6], we show that the latter actually always converges in at most two iterations.

We derive our calibration method for accelerometers that measure in three perfectly orthogonal directions with unknown *gains* g_m^{true} and *biases* b_m^{true} for $m = 1, 2, 3$. We assume, contrary to [3, 7], that the accelerometer can be rotated and placed at rest in six different angular orientations. Then we get acceleration measurements $A_{m,n}$ that satisfy the fundamental relation

$$A_{1,n}^2 + A_{2,n}^2 + A_{3,n}^2 = 1 \text{ g}, \qquad n = 1, 2, \ldots, 6. \tag{1a}$$

The corresponding accelerometer voltage outputs can be organized into the matrix

$$V = \begin{pmatrix} V_{1,1} & \cdots & V_{1,6} \\ V_{2,1} & \cdots & V_{2,6} \\ V_{3,1} & \cdots & V_{3,6} \end{pmatrix} \qquad \text{with} \qquad V_{m,n} \stackrel{\text{def}}{=} g_m^{\text{true}} A_{m,n} + b_m^{\text{true}}. \tag{1b}$$

We address three main topics in this paper:

(1) In Section 2.1, we derive simple conditions on the available measurements V, for which we provide simple formulas for *unique* calibration parameters b_m and g_m such that

$$\left(\frac{V_{1,n} - b_1}{g_1}\right)^2 + \left(\frac{V_{2,n} - b_2}{g_2}\right)^2 + \left(\frac{V_{3,n} - b_3}{g_3}\right)^2 = 1, \qquad n = 1, 2, \ldots, 6. \tag{2}$$

From the uniqueness and (1), it then clearly follows that $b_m = b_m^{\text{true}}$ and $g_m = g_m^{\text{true}}$.

[1]Percentages for the LIS3L02AQ accelerometer described at
http://www.sunspotworld.com/docs/AppNotes/AccelerometerAppNote.pdf.

(2) We show in Section 2.2 that there are nontrivial rotations of the accelerometer for which the measurements V does not satisfy the conditions necessary to compute unique b_m and g_m satisfying (2). On the other hand, we also show that we always get unique b_m and g_m satisfying (2) for all "reasonably small" deviations from a natural measurement setup with 90 degree rotations of the accelerometer and one measurement axis aligned with the Earth gravity field. Here, "reasonably small" means, for example, at most 5° deviation for the Colibrys SF3000L accelerometers in our lab.

(3) In Section 2.3, a direct comparison of our explicit non-iterative solution with the iterative method proposed in [6] shows that the latter always converges in at most two iterations.

2. Main results

2.1. The calibration method. In this section, we will derive a simple calibration method for triaxial accelerometers with perfectly orthogonal axes and voltage outputs satisfying the linear model (1b). It is natural and a simple task to set up the accelerometer and its wiring so that a large positive acceleration corresponds to a positive output voltage. We will therefore also assume the accelerometer to have positive gains.

The accelerometer voltage outputs V in (1b) depend on b_m^{true}, g_m^{true} and $A_{m,n}$, all of which are unknown, but the following theorem shows that if V satisfies a certain easily checked condition, then (6) gives $b_m = b_m^{\text{true}}$ and $g_m = g_m^{\text{true}}$.

Theorem 2.1. *For any* 3×6*-matrix* V *with real-valued entries, define*

$$V^+ \stackrel{\text{def}}{=} \begin{bmatrix} P_2(V^{\mathrm{T}}) & V^{\mathrm{T}} \end{bmatrix} \qquad \textit{with elementwise squaring} \qquad (P_2(M))_{m,n} \stackrel{\text{def}}{=} M_{m,n}^2. \quad (3)$$

There is a unique $\boldsymbol{u} = [u_1\ u_2\ u_3\ w_1\ w_2\ w_3]^T$ *such that (with* $\operatorname{sgn}(0) \stackrel{\text{def}}{=} 0$ *and, for* $x \neq 0$, $\operatorname{sgn}(x) \stackrel{\text{def}}{=} \frac{x}{|x|}$*)*

$$V^+\boldsymbol{u} = \mathbf{1} \quad \textit{and} \quad 0 \neq \operatorname{sgn}(u_m) = \operatorname{sgn}\left(1 + \frac{w_1^2}{4u_1} + \frac{w_2^2}{4u_2} + \frac{w_3^2}{4u_3}\right) \quad \textit{for } m = 1, 2, 3 \quad (4)$$

if and only if there are two unique vectors $\boldsymbol{b}, \boldsymbol{g} \in \mathbb{R}^3$ *with entries* b_m, g_m *such that*

$$\sum_{m=1}^{3} \frac{(V_{m,n} - b_m)^2}{g_m^2} = 1, \qquad g_m > 0 \qquad \textit{and} \qquad \sum_{m=1}^{3} \frac{b_m^2}{g_m^2} \neq 1 \qquad \textit{for } m = 1, 2, 3. \quad (5)$$

When these equivalent statements hold, the unique b_m *and* g_m *satisfying* (5) *are*

$$b_m = -\frac{w_m}{2u_m} \qquad \textit{and} \qquad g_m = \sqrt{\frac{1 + \sum_{k=1}^{3} \frac{w_k^2}{4u_k}}{u_m}}. \quad (6)$$

As explained in Remark 2.4 on p. 5, the condition $\sum_{m=1}^{3} \frac{b_m^2}{g_m^2} \neq 1$ in (5) is always true for accelerometers with $\boldsymbol{b}^{\text{true}}$ having length less than the smallest g_m^{true}, which holds with probability 1 for an arbitrary chosen accelerometer.

Further, we show in Theorem 2.3 how to "get rid of" the condition $\operatorname{sgn}(u_m) = \operatorname{sgn}\left(1 + \sum_{k=1}^{3} \frac{w_k^2}{4u_k}\right)$ in (4), hence reducing Theorem 2.1 to basically saying that we can recover the parameters $b_m = b_m^{\text{true}}$ and $g_m = g_m^{\text{true}}$ from the measurements V if and only if the matrix V^+ in (3) is invertible. Finally, we show in Section 2.2.2 how to choose the accelerometer orientations when measuring for guaranteeing the invertibility of V^+.

Proof of Theorem 2.1. We will prove the equivalence by establishing an invertible mapping of the set of all $\boldsymbol{u}$ satisfying (4) onto the set of all $\boldsymbol{b}, \boldsymbol{g}$ satisfying (5). This shows both that uniqueness is preserved (both sets contain the same number of elements) and that the conditions (4) via the established mapping are equivalent to the conditions (5).

Hence, first assume that (4) holds. We can then write the equation $V^+\boldsymbol{u} = \mathbf{1}$ in the form

$$V_{1,n}^2 u_1 + V_{1,n} w_1 + V_{2,n}^2 u_2 + V_{2,n} w_2 + V_{3,n}^2 u_1 + V_{3,n} w_1 = 1 \qquad \text{for } n = 1, \dots, 6, \tag{7}$$

which can be completed to three full squares by adding $\frac{w_1^2}{4u_1} + \frac{w_2^2}{4u_2} + \frac{w_3^2}{4u_3}$, so that

$$\sum_{m=1}^{3} u_m \left(V_{m,n} - \left(-\frac{w_m}{2u_m} \right) \right)^2 = 1 + \frac{w_1^2}{4u_1} + \frac{w_2^2}{4u_2} + \frac{w_3^2}{4u_3}. \tag{8}$$

Since all $u_m \neq 0$ and $\operatorname{sgn}(u_m) = \operatorname{sgn}(1 + \sum_{k=1}^3 \frac{w_k^2}{4u_k})$ we can let b_m and positive g_m be defined by (6) and rewrite (8) in the form $\sum_{m=1}^3 \frac{(V_{m,n} - b_m)^2}{{g_m}^2} = 1$. It also follows that

$$\sum_{m=1}^{3} \frac{b_m^2}{g_m^2} = \sum_{m=1}^{3} \frac{\frac{w_m^2}{4u_m}}{1 + \sum_{k=1}^3 \frac{w_k^2}{4u_k}} = \frac{\sum_{m=1}^3 \frac{w_m^2}{4u_m}}{1 + \sum_{k=1}^3 \frac{w_k^2}{4u_k}} \neq 1.$$

Consequently, (6) defines a function f that maps $\boldsymbol{u}$ satisfying (4) to $(\boldsymbol{b}, \boldsymbol{g})$ satisfying (5). To show that f is invertible, we choose *arbitrary* $\boldsymbol{b}, \boldsymbol{g}$ satisfying (5), for which we need to find a *unique* $\boldsymbol{u}$ satisfying (4) and such that $f(\boldsymbol{u}) = (\boldsymbol{b}, \boldsymbol{g})$. In other words, we need to find a unique solution to the equations $w_m = -2b_m u_m$ and $g_m^2 = \frac{1 + \sum_{k=1}^3 b_k^2 u_k}{u_m}$, which we can rewrite as a system of linear equations

$$\begin{pmatrix} g_1^2 - b_1^2 & -b_2^2 & -b_3^2 \\ -b_1^2 & g_2^2 - b_2^2 & -b_3^2 \\ -b_1^2 & -b_2^2 & g_3^2 - b_3^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which has exactly the same solutions as

$$\begin{pmatrix} g_1^2 & 0 & -g_3^2 \\ 0 & g_2^2 & -g_3^2 \\ 0 & 0 & g_3^2 \left(1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2} \right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{9}$$

Since all $g_m > 0$ and $\sum_{k=1}^3 \frac{b_k^2}{g_k^2} \neq 1$ by (5), there is exactly one solution of (9), namely

$$u_m = \frac{1}{g_m^2 \left(1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2} \right)} \qquad \text{and} \qquad w_m = -2b_m u_m = -\frac{2b_m}{g_m^2 \left(1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2} \right)}, \tag{10}$$

from which it also follows that $u_m \neq 0$, $\operatorname{sgn}(u_m) = \operatorname{sgn}\left(1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2} \right)$ and

$$\begin{aligned} \operatorname{sgn}\left(1 + \sum_{k=1}^{3} \frac{w_k^2}{4u_k} \right) &= \operatorname{sgn}\left(1 + \sum_{k=1}^{3} b_k^2 u_k \right) = \operatorname{sgn}\left(1 + \sum_{k=1}^{3} \frac{b_k^2}{g_k^2 \left(1 - \sum_{i=1}^3 \frac{b_i^2}{g_i^2} \right)} \right) \\ &= \operatorname{sgn}\left(\frac{1}{1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2}} \right) = \operatorname{sgn}\left(1 - \sum_{k=1}^{3} \frac{b_k^2}{g_k^2} \right) = \operatorname{sgn}(u_m). \end{aligned} \tag{11}$$

Thus, insertion of (6) in $\sum_{m=1}^{3} \frac{(V_{m,n}-b_m)^2}{g_m^2} = 1$ gives (8), which implies (7). Hence, we have found a unique $\boldsymbol{u}$ satisfying (4) such that $f(\boldsymbol{u}) = (\boldsymbol{b}, \boldsymbol{g})$, which completes the proof. □

The following corollary of Theorem 2.1 will be useful in comparison of our results with the results obtained in [6].

Corollary 2.2. *For arbitrary real-valued* 3×6 *matrices* V *and* Λ *such that* $\Lambda_{m,n} = \frac{V_{m,n}-b_m^{(0)}}{g_m^{(0)}}$, *define* $V^+ \stackrel{\text{def}}{=} [P_2(\Lambda^T)\ \Lambda^T]$ *as in* (3). *Suppose that* $b_m^{(0)}$ *and* $g_m^{(0)}$ *are such that* (4) *holds for a unique* $\boldsymbol{u} = [u_1\ u_2\ u_3\ w_1\ w_2\ w_3]^T$. *Then,* $\sum_{m=1}^{3} \frac{(V_{m,n}-b_m)^2}{g_m^2} = 1$ *for*

$$b_m \stackrel{\text{def}}{=} b_m^{(0)} - \frac{w_m}{2u_m} g_m^{(0)} \qquad \textit{and} \qquad g_m \stackrel{\text{def}}{=} g_m^{(0)} \sqrt{\frac{1+\sum_{k=1}^{3} \frac{w_k^2}{4u_k}}{u_m}}. \tag{12}$$

Proof of Corollary 2.2. We know from (6) that for $b'_m = -\frac{w_m}{2u_m}$ and $g'_m = \sqrt{\frac{1+\sum_{k=1}^{3} \frac{w_k^2}{4u_k}}{u_m}}$,

$$1 = \sum_{m=1}^{3} \frac{(\Lambda_{m,n} - b'_m)^2}{{g'}_m^2} = \sum_{m=1}^{3} \frac{\left(\frac{V_{m,n}-b_m^{(0)}}{g_m^{(0)}} - b'_m\right)^2}{{g'}_m^2},$$

from which (12) follows. □

Now recall that Theorem 2.1 is supposed to be applied to real measurement data from an accelerometer that can be well modeled as having orthogonal axes and voltages given by the linear model (1b). Hence, we typically know from accelerometer data sheets and/or previous calibrations that there exists *at least one* choice of the parameters b_m and g_m (namely $b_m = b_m^{\text{true}}$ and $g_m = g_m^{\text{true}}$), for which $\sum_{m=1}^{3} \left(\frac{V_{m,n}-b_m}{g_m}\right)^2 = 1$ and $\sum_{k=1}^{3} \frac{b_k^2}{g_k^2} \neq 1$. By using this extra knowledge, we can basically remove the condition $0 \neq \operatorname{sgn}(u_m) = \operatorname{sgn}\left(1 + \frac{w_1^2}{4u_1} + \frac{w_2^2}{4u_2} + \frac{w_3^2}{4u_3}\right)$ from (4), so that unique b_m and g_m can be computed if and only if V^+ is invertible. More precisely, we get the main theorem of this paper:

Theorem 2.3 (Calibration method)**.** *For any* 3×6*-matrix* V *with real-valued elements, define the matrix*

$$V^+ \stackrel{\text{def}}{=} \left[P_2(V^{\mathrm{T}}) \quad V^{\mathrm{T}}\right] \textit{ with elementwise squaring } (P_2(M))_{m,n} \stackrel{\text{def}}{=} M_{m,n}^2.$$

Assume that there exist vectors $\boldsymbol{b}, \boldsymbol{g} \in \mathbb{R}^3$ *such that*

$$\sum_{m=1}^{3} \left(\frac{V_{m,n}-b_m}{g_m}\right)^2 = 1 \qquad \textit{and} \qquad \sum_{k=1}^{3} \frac{b_k^2}{g_k^2} \neq 1. \tag{13}$$

Then,

$$V^+\boldsymbol{u} = \boldsymbol{1} \textit{ has a unique solution } \boldsymbol{u} = [u_1\ u_2\ u_3\ w_1\ w_2\ w_3]^T \tag{14}$$

if and only if the vectors $\boldsymbol{b}, \boldsymbol{g} \in \mathbb{R}^3$ *in* (13) *are unique.*

When these equivalent statements hold, (6) *gives the unique* b_m *and* g_m *satisfying* (13).

As before, the inverse of the mapping $\boldsymbol{u} \to (\boldsymbol{b}, \boldsymbol{g})$ in (6) is given by (10).

Proof. Firstly, assume that $V^+\boldsymbol{u} = \mathbf{1}$ has a unique solution. From $\sum_{m=1}^3 \left(\frac{V_{m,n}-b_m}{g_m}\right)^2 = 1$ we have $\sum_{m=1}^3 \frac{V_{m,n}^2}{g_m^2} - \frac{2b_m V_{m,n}}{g_m^2} = 1 - \sum_{k=1}^3 \frac{b_k^2}{g_k^2}$, which for $\sum_{k=1}^3 \frac{b_k^2}{g_k^2} \neq 1$ implies

$$\sum_{m=1}^3 \frac{V_{m,n}^2}{g_m^2\left(1-\sum_{k=1}^3 \frac{b_k^2}{g_k^2}\right)} - \frac{2b_m V_{m,n}}{g_m^2\left(1-\sum_{k=1}^3 \frac{b_k^2}{g_k^2}\right)} = 1.$$

Thus, $\boldsymbol{u}$, where $u_m = \frac{1}{g_m^2\left(1-\sum_{k=1}^3 \frac{b_k^2}{g_k^2}\right)}$ and $w_m = -\frac{2b_m}{g_m^2\left(1-\sum_{k=1}^3 \frac{b_k^2}{g_k^2}\right)}$ is a solution of $V^+\boldsymbol{u} = \mathbf{1}$ and by the assumption above it is the only solution. Moreover, for $\sum_{k=1}^3 \frac{b_k^2}{g_k^2} \neq 1$ we have $u_m \neq 0$ and, by (11), $\operatorname{sgn}(u_m) = \operatorname{sgn}(1 + \sum_{k=1}^3 \frac{w_k^2}{4u_k})$ for $m = 1, 2, 3$. Thus, the sufficient condition of Theorem 2.1 is satisfied and we get that $\boldsymbol{b}$ and $\boldsymbol{g}$ are unique.

The inverse statement follows directly from Theorem 2.1. □

Remark 2.4. The condition $\sum_{m=1}^3 \frac{b_m^2}{g_m^2} \neq 1$ in (5) means that (b_1, b_2, b_3) is a point not located on the ellipsoid surface $E_{\boldsymbol{g}} \overset{\text{def}}{=} \left\{(b_1, b_2, b_3) \in \mathbb{R}^3 \,\middle|\, \sum_{m=1}^3 b_m^2/g_m^2 = 1\right\}$. This will always be true for any accelerometer with biases b_m^{true} that are small enough compared to the gains, for example, in the sense that $\boldsymbol{b}^{\text{true}}$ has length $\sqrt{b_1^{\text{true}2} + b_2^{\text{true}2} + b_3^{\text{true}2}} < \min_m g_m^{\text{true}}$, or in the sense that each $|b_m| \leq g_m/2$, as in Theorem 2.7, below. Both these conditions hold for the accelerometer calibrated in Section 2.1.1.

Moreover, for a randomly chosen accelerometer, it should be safe to assume that the probability for the accelerometer to have bias $\boldsymbol{b}$ and gain $\boldsymbol{g}$ is described by an unknown but integrable probability density function $f(\boldsymbol{b}, \boldsymbol{g})$. Then, since the ellipsoid surface $E_{\boldsymbol{g}}$ has Lebesgue measure 0 in $\mathbb{R}^3$ [8], the probability for a randomly chosen accelerometer not to satisfy the condition $\sum_{m=1}^3 \frac{b_m^2}{g_m^2} \neq 1$ in (5) is $\int_{\mathbb{R}_+^3} \int_{E_{\boldsymbol{g}}} f(\boldsymbol{b}, \boldsymbol{g})\, d\boldsymbol{b}\, d\boldsymbol{g} = \int_{\mathbb{R}_+^3} 0\, d\boldsymbol{g} = 0$.

Remark 2.5. If some additional information on the gain factors is presented, then it is possible to reduce the number of measurements in order to compute the true bias and gain parameters in the following way. For example, if it is known in advance that all three gain factors are equal, then it is enough to make four measurements in order to find the unknown parameters. In this case Theorems 2.1 and 2.3 can be applied to the matrix V^+ defined by

$$V^+ = \begin{pmatrix} V_{1,1}^2 + V_{2,1}^2 + V_{3,1}^2 & V_{1,1} & V_{2,1} & V_{3,1} \\ V_{1,2}^2 + V_{2,2}^2 + V_{3,2}^2 & V_{1,2} & V_{2,2} & V_{3,2} \\ V_{1,3}^2 + V_{2,3}^2 + V_{3,3}^2 & V_{1,3} & V_{2,3} & V_{3,3} \\ V_{1,4}^2 + V_{2,4}^2 + V_{3,4}^2 & V_{1,4} & V_{2,4} & V_{3,4} \end{pmatrix} \tag{15}$$

and $\boldsymbol{u} = [u\ w_1\ w_2\ w_3]$. Thus,

$$b_m = -\frac{w_m}{2u}, \qquad g = \sqrt{\frac{1 + \sum_{k=1}^3 \frac{w_k^2}{4u}}{u}}.$$

If instead, it is known that two of three gain factors are equal, then it is enough with five measurements to define the unknown parameters. For example, if the gain factors for the accelerometer in the x- and y-directions are the same, then the

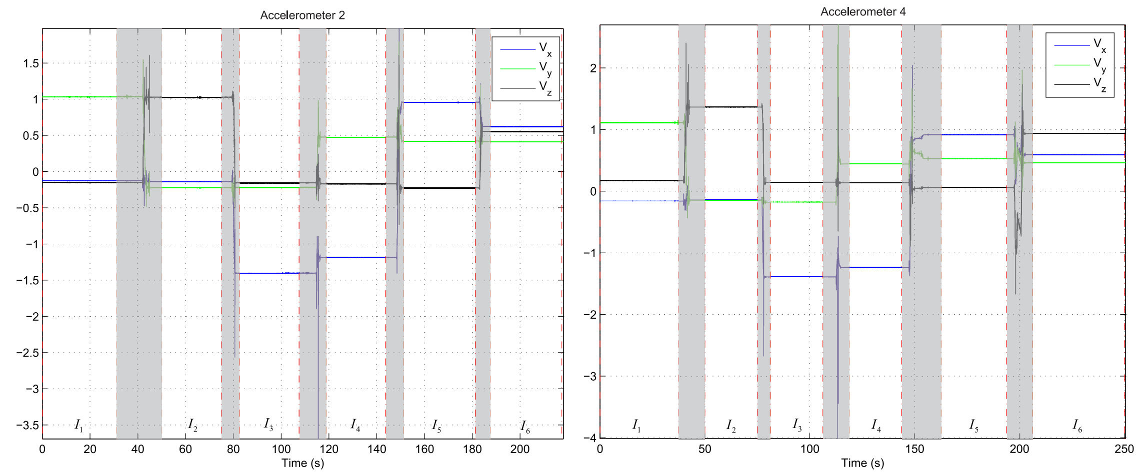

FIGURE 1. Voltage measurements for two of the accelerometers in Table 1.

following matrix can be considered

$$V^{+} = \begin{pmatrix} V_{1,1}^2 + V_{2,1}^2 & V_{3,1}^2 & V_{1,1} & V_{2,1} & V_{3,1} \\ V_{1,2}^2 + V_{2,2}^2 & V_{3,2}^2 & V_{1,2} & V_{2,2} & V_{3,2} \\ V_{1,3}^2 + V_{2,3}^2 & V_{3,3}^2 & V_{1,3} & V_{2,3} & V_{3,3} \\ V_{1,4}^2 + V_{2,4}^2 & V_{3,4}^2 & V_{1,4} & V_{2,4} & V_{3,4} \\ V_{1,5}^2 + V_{2,5}^2 & V_{3,5}^2 & V_{1,5} & V_{2,5} & V_{3,5} \end{pmatrix} \tag{16}$$

and $\boldsymbol{u} = [u_1\ u_3\ w_1\ w_2\ w_3]$. Thus, assigning $u_2 = u_1$

$$b_m = -\frac{w_m}{2u_m}, \qquad g_m = \sqrt{\frac{1 + \sum_{k=1}^{3} \frac{w_k^2}{4u_k}}{u_m}}.$$

We note, that in these two cases if no reduction in measurements is required Theorems 2.1 and 2.3 produce the correct result directly by considering six measurements.

2.1.1. *Numerical examples.* For four triaxial Colibrys SF3000L accelerometers, we have measured the output voltages in room temperature after six different rotations of the accelerometer. Some of the obtained voltages are plotted in Figure 1. The accelerometer has been rotated to a new orientation in the greyed out areas and is considered to be at rest in the remaining intervals I_n. We use the average voltage in each such interval to reduce the noise and thus get higher precision measurements. Then, Theorem 2.3 gives the gains and biases listed in Table 1.

TABLE 1. Biases and gain factors computed using Theorem 2.3.

	Accelerometer 1	**Accelerometer 2**	**Accelerometer 3**	**Accelerometer 4**
g_1	1.2745	1.2717	1.2574	1.2528
g_2	1.3061	1.2731	1.2838	1.2753
g_3	1.2742	1.2462	1.2423	1.2308
b_1	-0.17476	-0.13368	-0.14488	-0.13434
b_2	-0.16946	-0.23839	-0.089937	-0.16311
b_3	-0.014594	-0.22281	-0.10304	0.13309

2.2. **Choosing accelerometer orientations for safe recovery of b_m^{true}, g_m^{true}.** For successful application of Theorems 2.1 and 2.3, it is important to do the calibration measurements with the six orientations of the accelerometer chosen in such a way that the matrix $V^+ \stackrel{\text{def}}{=} \begin{bmatrix} P_2(V^{\mathrm{T}}) & V^{\mathrm{T}} \end{bmatrix}$ is invertible. This is not the case, for example, if the measured accelerations are the column vectors of the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{17}$$

For an accelerometer with gains $g_m^{\text{true}} = \sqrt{2}$ and biases $b_m^{\text{true}} = 0$, the corresponding measured voltages would be

$$V = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}. \tag{18}$$

Then it is easy to check that V^+ has linearly dependent rows ($\text{row}_1 = \text{row}_2 - \text{row}_4 + \text{row}_5$). Hence, by Theorem 2.3 there is no *unique* choice of gains and biases giving an acceleration matrix $(A_{\boldsymbol{b},\boldsymbol{g}})_{m,n} \stackrel{\text{def}}{=} \frac{V_{m,n}-b_m}{g_m}$ with column vectors of length 1. In this particular example, one choice is $g_m = g_m^{\text{true}} = \sqrt{2}$ and $b_m = b_m^{\text{true}} = 0$, but the alternative choice $b_m = 0$, $g_1 = 2/\sqrt{3}$ and $g_2 = g_3 = 2$ will also give an acceleration matrix with column vectors of length 1.

A natural and simple calibration setup for a rectangular shaped accelerometer casing is to do the measurements with the accelerometer resting on each of its six different sides. The matrix A in (17) is then "close" to the matrix $A_0 = \begin{pmatrix} I_{3\times 3} & -I_{3\times 3} \end{pmatrix}$, or to any matrix with the same column vectors but in a different order. A_0 has the largest possible minimal angle 90° between two column vectors, and for $b_m^{\text{true}} = 0$ and $g_m^{\text{true}} = 1$ it gives an orthogonal matrix V^+ in Theorems 2.1 and 2.3. Thus, for A close to A_0, corresponding to small variations of the rotations, gains and biases, we could expect some robustness in preserving the invertibility of V^+. Indeed, for *any* b_m^{true} and g_m^{true} such that $\sum_{m=1}^{3} \frac{b_m^{\text{true}2}}{g_m^{\text{true}2}} \neq 1$, Theorem 2.7 guarantees V^+ to be invertible for any A with column vectors deviating from the corresponding column vector of A_0 with arbitrary angles less than a certain upper bound α.

Other A also works, such as the ones chosen for the measurements in Figure 1, so for practical use, one can just do some arbitrary measurements and then Theorem 2.3 immediately tells that unique b_m and g_m can be computed if and only if V^+ is invertible. The role that Theorem 2.7 plays here is to suggest "safe" calibration measurement setups for which V^+ is invertible. In short, Theorem 2.3 is the calibration method and Theorem 2.7 is about robust calibration measurement setups.

2.2.1. *Preliminaries.* For a matrix M, we write M^{-1} for the inverse of M. The ∞-*norm* for an $m \times n$ matrix M is

$$\|M\|_\infty = \max_{1\le k\le m} \sum_{l=1}^{n} |M_{k,l}|. \tag{19}$$

It shares the *submultiplicative* property with other operator norms, that is, for all $n \times n$-matrices A, B

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty.$$

The following theorem is a special case of the more general result [9, Theorem 2.3.4].

Theorem 2.6. *If Λ is invertible and $\|\Lambda^{-1}E\|_\infty < 1$, then $\Lambda + E$ is invertible.*

2.2.2. *Invertibility of the voltage matrix V^+ in Theorem 2.3.* In the following theorem, we will consider the measured accelerations to be some approximation A of the matrix $A_0 = \begin{pmatrix} I_{3\times3} & -I_{3\times3}\end{pmatrix}$, that is, $A_{m,n} = A_{0m,n} + \varepsilon_{m,n}$. For an accelerometer with bias and positive gain factors stored in the vectors $\boldsymbol{b}$ and $\boldsymbol{g}$, respectively, the measured voltages will be

$$V_{A,\boldsymbol{b},\boldsymbol{g}} \stackrel{\text{def}}{=} \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} A + \begin{pmatrix} \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} \end{pmatrix}. \tag{20}$$

Theorem 2.7 (Robust calibration measurement setups)**.** *For $m = 1, 2, 3$ and $b_m, g_m \in \mathbb{R}$ such that $g_m > 0$, suppose that $\frac{|b_m|}{g_m} \le \beta \le \frac{1}{2}$. Then, $S \stackrel{\text{def}}{=} \frac{b_1^2}{g_1^2} + \frac{b_2^2}{g_2^2} + \frac{b_3^2}{g_3^2} < 1$ and $V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}$ is invertible for $A_0 \stackrel{\text{def}}{=} \begin{pmatrix} I_{3\times3} & -I_{3\times3}\end{pmatrix}$. Define*

$$\sigma \stackrel{\text{def}}{=} \min_{1\le m\le 3} g_m, \qquad \gamma \stackrel{\text{def}}{=} \max_{1\le m\le 3} g_m \qquad \text{and} \qquad \Omega = \max\left\{1 + 3\beta - 3\beta^2, \gamma\left(1 + 3\beta^2\right)\right\}.$$

For an arbitrary 3×6-matrix A, define

$$\varepsilon_{m,n} \stackrel{\text{def}}{=} (A - A_0)_{m,n} \qquad \text{and} \qquad \varepsilon \stackrel{\text{def}}{=} \max_{1\le m\le 3, 1\le n\le 6} |\varepsilon_{m,n}| \tag{21}$$

If ε satisfies

$$0 < \varepsilon < \frac{1}{6}\left(-\left(6\beta + 2 + \frac{3}{\gamma}\right) + \sqrt{\left(6\beta + 2 + \frac{3}{\gamma}\right)^2 + \frac{12\sigma^2(1 - 3\beta^2)}{\gamma^2\Omega}}\right), \tag{22}$$

Then (20) defines an invertible matrix $V^+_{A,\boldsymbol{b},\boldsymbol{g}}$.

Moreover, if the matrix A has column vectors of length 1, then a sufficient condition for (21) to hold is that the angle between each column vector in A_0 and the corresponding column vector in A is at most $\alpha = 2\sin^{-1}(\varepsilon/2)$.

In particular, for the calibration results in Table 1, we get the bounds

$$\beta > \frac{0.239}{1.230} \approx 0.20, \quad \gamma > 1.29, \quad \sigma < 1.20, \quad \Omega > 1.48, \quad \varepsilon < 0.0888, \quad \alpha < 5.09°.$$

For the 8 cm casing side lengths of Colibrys SF3000L accelerometers, ±5.09° misalignment corresponds to a ±7.1 mm vertical movement of one side of the casing. This means that it is a relatively simple task to place the accelerometer "close enough" to being horizontal for Theorem 2.7 to guarantee that the accelerator gains and biases can be computed from the accelerator voltage outputs.

Proof of Theorem 2.7. The condition $\frac{|b_m|}{g_m} \le \beta \le \frac{1}{2}$ implies that $S \le 3/4 < 1$. Moreover, for $\mathcal{A}^\pm_m \stackrel{\text{def}}{=} \frac{S-1\pm\frac{b_m}{g_m}-\frac{b_m^2}{g_m^2}}{2g_m^2}$, $\mathcal{B}^\pm_{m,n} \stackrel{\text{def}}{=} -\frac{b_n(b_n\pm g_n)}{2g_m^2g_n^2}$, $\mathcal{C}^\pm_m \stackrel{\text{def}}{=} -\frac{(S-1)\left(b_m\pm\frac{g_m}{2}\right)\mp\frac{b_m^2}{g_m}-\frac{b_m^3}{g_m^2}}{g_m^2}$, $\mathcal{D}^\pm_{m,n} \stackrel{\text{def}}{=} -2b_m\mathcal{B}^\pm_{m,n}$ and

$$M \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{A}_1^+ & \mathcal{B}_{1,2}^- & \mathcal{B}_{1,3}^- & \mathcal{A}_1^- & \mathcal{B}_{1,2}^+ & \mathcal{B}_{1,3}^+ \\ \mathcal{B}_{2,1}^- & \mathcal{A}_2^+ & \mathcal{B}_{2,3}^- & \mathcal{B}_{2,1}^+ & \mathcal{A}_2^- & \mathcal{B}_{2,3}^+ \\ \mathcal{B}_{3,1}^- & \mathcal{B}_{3,2}^- & \mathcal{A}_3^+ & \mathcal{B}_{3,1}^+ & \mathcal{B}_{3,2}^+ & \mathcal{A}_3^- \\ \mathcal{C}_1^- & \mathcal{D}_{1,2}^- & \mathcal{D}_{1,3}^- & \mathcal{C}_1^+ & \mathcal{D}_{1,2}^+ & \mathcal{D}_{1,3}^+ \\ \mathcal{D}_{2,1}^- & \mathcal{C}_2^- & \mathcal{D}_{2,3}^- & \mathcal{D}_{2,1}^+ & \mathcal{C}_2^+ & \mathcal{D}_{2,3}^+ \\ \mathcal{D}_{3,1}^- & \mathcal{D}_{3,2}^- & \mathcal{C}_3^- & \mathcal{D}_{3,1}^+ & \mathcal{D}_{3,2}^+ & \mathcal{C}_3^+ \end{pmatrix},$$

one can check, for example with Maple, that $\frac{1}{S-1} M V^+_{A_0,\boldsymbol{b},\boldsymbol{g}} = V^+_{A_0,\boldsymbol{b},\boldsymbol{g}} \frac{1}{S-1} M = I_{6\times 6}$, so that $V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}$ is invertible and ${V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} = \frac{1}{S-1} M$. Hence, for $E \overset{\text{def}}{=} V^+_{A,\boldsymbol{b},\boldsymbol{g}} - V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}$, Theorem 2.6 tells that if $\left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} E \right\|_\infty < 1$, then $V^+_{A_0,\boldsymbol{b},\boldsymbol{g}} + E = V^+_{A,\boldsymbol{b},\boldsymbol{g}}$ is invertible. Thus we can prove this theorem in four steps: *Step 1:* Estimate $\left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} \right\|_\infty$; *Step 2:* Estimate $\|E\|_\infty$; *Step 3:* Show that (22) implies that $\left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} E \right\|_\infty \le \left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} \right\|_\infty \|E\|_\infty < 1$, so that $V^+_{A_0,\boldsymbol{b},\boldsymbol{g}} + E = V^+_{A,\boldsymbol{b},\boldsymbol{g}}$ is invertible; *Step 4:* Prove the last statement of the theorem.

Step 1: For $k = 1, \dots, 6$ and $m = 4, 5, 6$, (20) gives that

$$E_{k,m} \overset{\text{def}}{=} (V^+_{A,\boldsymbol{b},\boldsymbol{g}} - V^+_{A_0,\boldsymbol{b},\boldsymbol{g}})_{k,m} = g_{m-3}(A_{m-3,k} - (A_0)_{m-3,k}) = g_{m-3}\varepsilon_{m-3,k}.$$

Similarly, for $k = 1, \dots, 6$ and $m = 1, 2, 3$,

$$\begin{aligned}
E_{k,m} =& (V_{A,\boldsymbol{b},\boldsymbol{g}})^2_{m,k} - (V_{A_0,\boldsymbol{b},\boldsymbol{g}})^2_{m,k} = (A_{m,k} g_m + b_m)^2 - ((A_0)_{m,k} g_m + b_m)^2 \\
=& (A^2_{m,k} - (A_0)^2_{m,k}) g^2_m + 2(A_{m,k} - (A_0)_{m,k}) g_m b_m \\
=& (A - A_0)_{m,k}(A + A_0)_{m,k} g^2_m + 2\varepsilon_{m,k} g_m b_m \\
=& \varepsilon_{m,k}(\varepsilon_{m,k} + 2(A_0)_{m,k}) g^2_m + 2\varepsilon_{m,k} g_m b_m \\
=& g^2_m \varepsilon^2_{m,k} + 2g_m(b_m + (A_0)_{m,k} g_m)\varepsilon_{m,k} \\
=& \begin{cases} g^2_m \varepsilon^2_{m,k} + 2g_m(b_m + g_m)\varepsilon_{m,k}, & \text{if } k = m, \\ g^2_m \varepsilon^2_{m,k} + 2g_m(b_m - g_m)\varepsilon_{m,k}, & \text{if } k = m + 3, \\ g^2_m \varepsilon^2_{m,k} + 2g_m b_m \varepsilon_{m,k}, & \text{otherwise.} \end{cases}
\end{aligned}$$

By (19) and the assumptions of this theorem, we now get

$$\begin{aligned}
\|E\|_\infty \le& \varepsilon^2 \left(\sum_{m=1}^{3} g^2_m \right) + 2\varepsilon \left(\sum_{m=1}^{3} |b_m| \, g_m \right) + 2\varepsilon\gamma^2 + \varepsilon \sum_{m=1}^{3} g_m \\
\le& 3\gamma^2\varepsilon^2 + (6\beta\gamma^2 + 2\gamma^2 + 3\gamma)\varepsilon = \gamma^2 \left(3\varepsilon^2 + \left(6\beta + 2 + \frac{3}{\gamma} \right) \varepsilon \right). \qquad (23)
\end{aligned}$$

Step 2: For $i = 1, 2, 3$,

$$\sum_{j=1}^{6} |M_{i,j}| = \left|\mathcal{A}^+_i\right| + \left|\mathcal{A}^-_i\right| + \sum_{j=1}^{3} \left(\left|\mathcal{B}^-_{i,j}\right| + \left|\mathcal{B}^+_{i,j}\right| \right) - \left|\mathcal{B}^-_{i,i}\right| - \left|\mathcal{B}^+_{i,i}\right|.$$

For $\frac{|b_i|}{g_i} \le \beta \le \frac{1}{2}$ it follows that $1 - S \pm \frac{b_i}{g_i} + \frac{b^2_i}{g^2_i} \ge 1 - S + \frac{b^2_i}{g^2_i} - \frac{|b_i|}{g_i} \ge 1 - \frac{2}{4} - \frac{1}{2} = 0$, so that

$$\begin{aligned}
\sum_{j=1}^{6} |M_{i,j}| =& \frac{1 - S - \frac{b_i}{g_i} + \frac{b^2_i}{g^2_i}}{2g^2_i} + \frac{1 - S + \frac{b_i}{g_i} + \frac{b^2_i}{g^2_i}}{2g^2_i} \\
&+ \sum_{j=1}^{3} \left(\frac{|b_j|(g_j - b_j)}{2g^2_i g^2_j} + \frac{|b_j|(b_j + g_j)}{2g^2_i g^2_j} \right) - \frac{|b_i|(g_i - b_i)}{2g^2_i g^2_i} - \frac{|b_i|(g_i + b_i)}{2g^2_i g^2_i} \\
=& \frac{1}{g^2_i} \left(1 - S + \frac{b^2_i}{g^2_i} \right) + \sum_{j=1}^{3} \left(\frac{|b_j|}{g^2_i g_j} \right) - \frac{|b_i|}{g^2_i g_i} = \frac{1}{g^2_i} \left(1 - S + \frac{b^2_i}{g^2_i} - \frac{|b_i|}{g_i} + \sum_{j=1}^{3} \frac{|b_j|}{g_j} \right)
\end{aligned}$$

$$\leq \frac{1}{g_i^2}\left(1 - S + \sum_{j=1}^{3} \frac{|b_j|}{g_j}\right).$$

For $\sigma \stackrel{\text{def}}{=} \min g_m$, $|b_m|/g_m \leq \beta \leq 1/2$ and $i = 1, 2, 3$ we obtain

$$\begin{aligned}\sum_{j=1}^{6} |M_{i,j}| \leq & \frac{1}{\sigma^2}\left(1 - S + \sum_{j=1}^{3} \frac{|b_j|}{g_j}\right) = \frac{1}{\sigma^2}\left(\frac{7}{4} - \sum_{j=1}^{3}\left(\frac{1}{2} - \frac{|b_j|}{g_j}\right)^2\right) \\ \leq & \frac{1}{\sigma^2}\left(\frac{7}{4} - 3\left(\frac{1}{2} - \beta\right)^2\right) = \frac{1}{\sigma^2}\left(1 + 3\beta - 3\beta^2\right).\end{aligned}$$

Similarly, for $i = 4, 5, 6$ we have

$$\sum_{j=1}^{6} |M_{i,j}| = \left|\mathcal{C}_{i-3}^-\right| + \left|\mathcal{C}_{i-3}^-\right| + \sum_{j=1}^{3}\left(\left|\mathcal{D}_{i-3,j}^-\right| + \left|\mathcal{D}_{i-3,j}^+\right|\right) - \left|\mathcal{D}_{i-3,i-3}^-\right| - \left|\mathcal{D}_{i-3,i-3}^+\right|.$$

For $|b_m|/g_m \leq \beta \leq 1/2$ we get $g_m/2 \pm b_m \geq |b_m| \pm b_m \geq 0$ and $b_m^2/g_m \pm b_m^3/g_m^2 = b_m^2/g_m(1 \pm b_m/g_m) \geq 0$. Consequently,

$$\left|(S-1)\left(b_{i-3} \mp \frac{g_{i-3}}{2}\right) \pm \frac{b_{i-3}^2}{g_{i-3}} - \frac{b_{i-3}^3}{g_{i-3}^2}\right| = \pm\left((S-1)\left(b_{i-3} \mp \frac{g_{i-3}}{2}\right) \pm \frac{b_{i-3}^2}{g_{i-3}} - \frac{b_{i-3}^3}{g_{i-3}^2}\right) \qquad \text{and}$$

$$\begin{aligned}\sum_{j=1}^{6} |M_{i,j}| =& \frac{(S-1)(b_{i-3} - \frac{g_{i-3}}{2}) + \frac{b_{i-3}^2}{g_{i-3}} - \frac{b_{i-3}^3}{g_{i-3}^2}}{g_{i-3}^2} - \frac{(S-1)(b_{i-3} + \frac{g_{i-3}}{2}) - \frac{b_{i-3}^2}{g_{i-3}} - \frac{b_{i-3}^3}{g_{i-3}^2}}{g_{i-3}^2} \\ &+ \sum_{j=1}^{3}\left(\frac{|b_j b_{i-3}|(g_j - b_j)}{g_j^2 g_{i-3}^2} + \frac{|b_j b_{i-3}|(b_j + g_j)}{g_j^2 g_{i-3}^2}\right) \\ &- \frac{|b_{i-3} b_{i-3}|(g_{i-3} - b_{i-3})}{g_{i-3}^2 g_{i-3}^2} - \frac{|b_{i-3} b_{i-3}|(g_{i-3} + b_j)}{g_{i-3}^2 g_{i-3}^2} \\ =& \frac{1}{g_{i-3}^2}\left(g_{i-3}(1-S) + \frac{2|b_{i-3}|^2}{g_{i-3}}\right) + \sum_{j=1}^{3}\left(\frac{2|b_j b_{i-3}|}{g_{i-3}^2 g_j}\right) - \frac{2b_{i-3}^2}{g_{i-3}^2 g_{i-3}} \\ =& \frac{1}{g_{i-3}^2}\left(g_{i-3}(1-S) + 2|b_{i-3}|\sum_{j=1}^{3}\left(\frac{|b_j|}{g_j}\right)\right) \\ \leq& \frac{1}{\sigma^2}\left(\gamma(1-S) + 2\beta\gamma\sum_{j=1}^{3}\left(\frac{|b_j|}{g_j}\right)\right) = \frac{\gamma}{\sigma^2}\left(1 - \left(S - 2\beta\sum_{j=1}^{3}\left(\frac{|b_j|}{g_j}\right) + 3\beta^2\right) + 3\beta^2\right) \\ =& \frac{\gamma}{\sigma^2}\left(1 + 3\beta^2 - \sum_{j=1}^{3}\left(\beta - \frac{|b_j|}{g_j}\right)^2\right) \leq \frac{1}{\sigma^2}\gamma\left(1 + 3\beta^2\right).\end{aligned}$$

Thus, $\|M\|_\infty \leq \frac{1}{\sigma^2}\max\left\{1 + 3\beta - 3\beta^2, \gamma\left(1 + 3\beta^2\right)\right\} \stackrel{\text{def}}{=} \frac{\Omega}{\sigma^2}$ and

$$\left\|{V_{A_0,\boldsymbol{b},\boldsymbol{g}}^{+}}^{-1}\right\|_\infty = \frac{\|M\|_\infty}{1-S} \leq \frac{\Omega}{\sigma^2(1-S)} \leq \frac{\Omega}{\sigma^2(1-3\beta^2)}. \tag{24}$$

Step 3: By the inequalities (23) and (24) we now get $\|E\|_\infty \left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} \right\|_\infty < 1$ for ε satisfying

$$\|E\|_\infty \leq \gamma^2 \left(3\varepsilon^2 + \left(6\beta + 2 + \frac{3}{\gamma}\right)\varepsilon\right) < \frac{\sigma^2(1-3\beta^2)}{\Omega} \leq \frac{1}{\left\| {V^+_{A_0,\boldsymbol{b},\boldsymbol{g}}}^{-1} \right\|_\infty}$$

$$3\varepsilon^2 + (6\beta + 2 + \frac{3}{\gamma})\varepsilon < \frac{\sigma^2(1-3\beta^2)}{\gamma^2\Omega}, \qquad \varepsilon > 0.$$

with solution (22).

Step 4: Now it only remains to prove the last statement. For each column vector $\boldsymbol{a_0}$ in A_0 and corresponding column vector $\boldsymbol{a}$ in A, it is not difficult to check that $\|\boldsymbol{a} - \boldsymbol{a_0}\|_\infty \stackrel{\text{def}}{=} \max_k |\boldsymbol{a} - \boldsymbol{a_0}|_k \leq \|\boldsymbol{a} - \boldsymbol{a_0}\|_2$. Hence, (21) holds if, for all column vectors, $\|\boldsymbol{a} - \boldsymbol{a_0}\|_2 \leq \varepsilon$. Since the lengths $\|\boldsymbol{a}\|_2 = \|\boldsymbol{a_0}\|_2 = 1$, the angle α between $\boldsymbol{a_0}$ and $\boldsymbol{a}$ then satisfies $\sin(\alpha/2) = \|\boldsymbol{a} - \boldsymbol{a_0}\|_2/2 \leq \varepsilon/2$ with $0 \leq \alpha \leq \pi$. Thus $\alpha = 2\sin^{-1}(\varepsilon/2)$. □

2.3. **Comparison with a related iterative calibration method.** Our calibration method is closely related to a remarkably fast converging iterative calibration method proposed by Won and Golnaraghi in [6]. Their algorithm starts from initial guesses $\hat{b}_m^{(0)}$ and $\hat{g}_m^{(0)}$. At the kth iteration, the estimated bias and gain is $\hat{b}_m^{(k)}$ and $\hat{g}_m^{(k)}$, which gives estimated accelerations $\hat{A}_{m,n}^{(k)} \stackrel{\text{def}}{=} \frac{V_{m,n} - \hat{b}_m^{(k)}}{\hat{g}_m^{(k)}}$.

A crucial idea in [6] is for each iteration to do the calibration parameter updating

$$\hat{b}_m^{(k)} = \hat{b}_m^{(k-1)} + \tilde{b}_m^{(k)}, \qquad \hat{g}_m^{(k)} = \hat{g}_m^{(k-1)}\tilde{g}_m^{(k)} \tag{25}$$

and investigate how to choose the updating parameters $\tilde{g}_m^{(k)}$ and $\tilde{b}_m^{(k)}$ for making the algorithm to converge in the kth step. This would mean that the kth estimates $\hat{b}_m^{(k)}$ and $\hat{g}_m^{(k)}$ coincide with the true biases and gains, and consequently, the *error* after the $(k-1)$th iteration is the vector $\boldsymbol{E}^{(k-1)}$ with nth entry

$$E_n^{(k-1)} \stackrel{\text{def}}{=} (\hat{A}_{1,n}^{(k-1)})^2 + (\hat{A}_{2,n}^{(k-1)})^2 + (\hat{A}_{3,n}^{(k-1)})^2 - 1 \tag{26}$$

$$= \sum_{m=1}^{3} (\hat{A}_{m,n}^{(k-1)})^2 - \sum_{m=1}^{3} (\hat{A}_{m,n}^{(k)})^2, \qquad n = 1, \ldots, 6. \tag{27}$$

Insertion of $\hat{A}_{m,n}^{(k)} = \frac{V_{m,n} - \hat{b}_m^{(k-1)} + \hat{b}_m^{(k-1)} - \hat{b}_m^{(k)}}{\hat{g}_m^{(k-1)}\hat{g}_m^{(k)}/\hat{g}_m^{(k-1)}} = \frac{\hat{A}_{m,n}^{(k-1)}}{\tilde{g}_m^{(k)}} - \frac{\tilde{b}_m^{(k)}}{\hat{g}_m^{(k-1)}\tilde{g}_m^{(k)}}$ in (27) gives

$$E_n^{(k-1)} = \sum_{m=1}^{3} \left(1 - \frac{1}{(\tilde{g}_m^{(k)})^2}\right)(\hat{A}_{m,n}^{(k-1)})^2 + \frac{2\tilde{b}_m^{(k)}}{\hat{g}_m^{(k-1)}(\tilde{g}_m^{(k)})^2}\hat{A}_{m,n}^{(k-1)} - \frac{(\tilde{b}_m^{(k)})^2}{(\hat{g}_m^{(k-1)}\tilde{g}_m^{(k)})^2}. \tag{28}$$

The algorithm converges if each $\tilde{g}_m^{(k)}$ converges to 1 and each $\tilde{b}_m^{(k)}$ converges to 0 when $k \to \infty$. Then also the nonlinear last term in (28) will converge to 0, and thus removing it gives a reasonable linearization of (28) to the equation

$$\boldsymbol{E}^{(k-1)} = \Lambda^{(k-1)}\boldsymbol{c}^{(k)} \tag{29}$$

with $\quad \Lambda^{(k)} \stackrel{\text{def}}{=} [P_2((\hat{A}^{(k)})^T)\ (\hat{A}^{(k)})^T], \qquad (P_2(M))_{m,n} \stackrel{\text{def}}{=} M_{m,n}^2,$

$$\text{and} \qquad \boldsymbol{c}^{(k)} \stackrel{\text{def}}{=} \left(1 - \frac{1}{(\tilde{g}_1^{(k)})^2}\ \ 1 - \frac{1}{(\tilde{g}_2^{(k)})^2}\ \ 1 - \frac{1}{(\tilde{g}_3^{(k)})^2}\ \ \frac{2\tilde{b}_1^{(k)}}{\hat{g}_1^{(k-1)}(\tilde{g}_1^{(k)})^2}\ \ \frac{2\tilde{b}_2^{(k)}}{\hat{g}_2^{(k-1)}(\tilde{g}_2^{(k)})^2}\ \ \frac{2\tilde{b}_3^{(k)}}{\hat{g}_3^{(k-1)}(\tilde{g}_3^{(k)})^2}\right)^{\mathrm{T}}. \tag{30}$$

The linear equation (29) is solved in each iteration of the algorithm proposed in [6]. To simplify a direct comparison with Theorem 2.3 and Corollary 2.2, we observe from the definition (26) that $\boldsymbol{E}^{(k-1)} = \Lambda^{(k-1)}(1\ 1\ 1\ 0\ 0\ 0)^T - (1\ 1\ 1\ 1\ 1\ 1)^T$, which on insertion in (29) gives

$$\begin{aligned}\Lambda^{(k-1)}(1\ 1\ 1\ 0\ 0\ 0)^T - (1\ 1\ 1\ 1\ 1\ 1)^T =& \Lambda^{(k-1)}\boldsymbol{c}^{(k)} \\ \Lambda^{(k-1)}\boldsymbol{u}^{(k)} =& \mathbf{1} \end{aligned} \tag{31}$$

with $\boldsymbol{u}^{(k)} = \left(\frac{1}{(\tilde{g}_1^{(k)})^2}\ \frac{1}{(\tilde{g}_2^{(k)})^2}\ \frac{1}{(\tilde{g}_3^{(k)})^2}\ -\frac{2\tilde{b}_1^{(k)}}{\hat{g}_1^{(k-1)}(\tilde{g}_1^{(k)})^2}\ -\frac{2\tilde{b}_2^{(k)}}{\hat{g}_2^{(k-1)}(\tilde{g}_2^{(k)})^2}\ -\frac{2\tilde{b}_3^{(k)}}{\hat{g}_3^{(k-1)}(\tilde{g}_3^{(k)})^2}\right)^T$. We can thus rewrite (25) in terms of $\boldsymbol{u}^{(k)}$:

$$\begin{gathered}\hat{b}_m^{(k)} = \hat{b}_m^{(k-1)} - \frac{w_m^{(k)}}{2u_m^{(k)}}\hat{g}_m^{(k-1)} \qquad \text{and} \qquad \hat{g}_m^{(k)} = \frac{\hat{g}_m^{(k-1)}}{\sqrt{u_m^{(k)}}} \\ \text{with} \qquad \boldsymbol{u}^{(k)} \stackrel{\text{def}}{=} (u_1^{(k)}\ u_2^{(k)}\ u_3^{(k)}\ w_1^{(k)}\ w_2^{(k)}\ w_3^{(k)}).\end{gathered} \tag{32}$$

The resulting algorithm goes as follows

Choose real-valued initial values $\hat{b}_m^{(0)}$ and $\hat{g}_m^{(0)} > 0$ such that $\Lambda^{(0)}\boldsymbol{u}^{(1)} = \mathbf{1}$ has a unique solution with $u_m^{(1)} > 0$ for $m = 1, 2, 3$.

The kth iteration ($k \geq 1$): Compose the matrix $\Lambda^{(k-1)}$ from (30).
Solve Equation (31) to find $\boldsymbol{u}^{(k)}$.
Compute $\hat{b}_m^{(k)}$ and $\hat{g}_m^{(k)}$ from (32).
Stop iterating if $\boldsymbol{u}^{(k)} = (1\ 1\ 1\ 0\ 0\ 0)^T$ so that $\tilde{b}_m^{(k)} = 0$ and $\tilde{g}_m^{(k)} = 1$.

Next, we prove that this algorithm actually always converges in at most two iterations. In the first iteration, the nth row of (31) is

$$\sum_{m=1}^{3}\left(\frac{V_{m,n} - \hat{b}_m^{(0)}}{\hat{g}_m^{(0)}}\right)^2 u_m^{(1)} + \frac{V_{m,n} - \hat{b}_m^{(0)}}{\hat{g}_m^{(0)}} w_m^{(1)} = 1.$$

We insert $\hat{g}_m^{(0)} = \hat{g}_m^{(1)}\sqrt{u_m^{(1)}}$ and $\hat{b}_m^{(0)} = \hat{b}_m^{(1)} + \frac{w_m^{(1)}}{2u_m^{(1)}}\hat{g}_m^{(0)}$ from (32) into this equation, expand the first square and simplify to

$$1 = \sum_{m=1}^{3}\left(\frac{V_{m,n} - \hat{b}_m^{(1)}}{\hat{g}_m^{(1)}}\right)^2 \frac{\operatorname{sgn}(u_m^{(1)})}{1 + \sum_{k=1}^{3}\frac{(w_k^{(1)})^2}{4u_k^{(1)}}}. \tag{33}$$

For invertible $\Lambda^{(1)}$, comparison with (31) for $k = 2$ gives that

$$\boldsymbol{u}^{(2)} = (u_1^{(2)}\ u_2^{(2)}\ u_3^{(2)}\ 0\ 0\ 0)^T \qquad \text{with} \qquad u_m^{(2)} = \frac{\operatorname{sgn}(u_m^{(1)})}{1 + \sum_{k=1}^{3}\frac{(w_k^{(1)})^2}{4u_k^{(1)}}}. \tag{34}$$

Insertion of (34) in (32) gives

$$\begin{gathered}\hat{b}_m^{(2)} = \hat{b}_m^{(1)} = \hat{b}_m^{(0)} - \frac{w_m^{(1)}}{2u_m^{(1)}}\hat{g}_m^{(0)} \qquad \text{and} \\ \hat{g}_m^{(2)} = \frac{\hat{g}_m^{(1)}}{\sqrt{u_m^{(2)}}} = \frac{\hat{g}_m^{(0)}}{\sqrt{u_m^{(1)}}}\sqrt{\frac{1 + \sum_{k=1}^{3}\frac{(w_k^{(1)})^2}{4u_k^{(1)}}}{\operatorname{sgn}(u_m^{(1)})}}.\end{gathered} \tag{35}$$

Consequently, insertion of (35) in (33) gives

$$1 = \sum_{m=1}^{3} \left(\frac{V_{m,n} - \hat{b}_m^{(1)}}{\hat{g}_m^{(1)}} \right)^2 u_m^{(2)} = \sum_{m=1}^{3} \left(\frac{V_{m,n} - \hat{b}_m^{(2)}}{\hat{g}_m^{(2)}} \right)^2 .$$

By (31), this means that $\Lambda^{(2)}\boldsymbol{u}^{(3)} = \mathbf{1}$ with $\boldsymbol{u}^{(3)} = (1\ 1\ 1\ 0\ 0\ 0)^T$ so the algorithm has converged to the solution (35), which is identical to (12) in general and identical to (6) in the special case with all $b_m = 0$ and $g_m = 1$.

For this algorithm to work, we need the same requirements on invertibility of $\Lambda^{(k)}$ and on the signs of $\hat{g}_m^{(k)}, u_m^{(k)}$ that were needed and treated more carefully in the previous sections of this paper.

3. Conclusions

We have derived a simple non-iterative calibration method in Theorem 2.3. Its main advantages are that only from the accelerometer output voltages it gives a complete knowledge of whether it is possible, with any method, to recover the accelerometer parameters b_m^{true} and g_m^{true} from the output voltages, and when this is possible, we have a simple explicit formula (6) for computing them with a smaller number of arithmetic operations than previous iterative approaches.

Next, we proved in Theorem 2.7 that b_m^{true} and g_m^{true} can be safely recovered for any deviations of the accelerometer orientations from a certain natural calibration measurement setup with angles smaller than some upper bound. We provided an estimate α from below of this upper bound, for example, $\alpha = 5°$ for the Colibrys SF3000L accelerometers in our lab.

Finally, we explained how our method is related to the iterative method proposed in [6] and showed that the latter method always converges in at most two iterations.

References

[1] M El-Diasty, A El-Rabbany, and S Pagiatakis. Temperature variation effects on stochastic characteristics for low-cost MEMS-based inertial sensor error. *Meas. Sci. Technol.*, 18(11):3321–3328, June 2007. DOI: 10.1088/0957-0233/18/11/009.

[2] Alan Lai, Daniel A. James, Jason P. Hayes, and Erol C. Harvey. Semi-automatic calibration technique using six inertial frames of reference. In *Proc. SPIE*, volume 5274, pages 531–542, 2004. DOI: 10.1117/12.530199.

[3] J.C. Lötters, J. Schipper, P.H. Veltink, W. Olthuis, and P. Bergveld. Procedure for in-use calibration of triaxial accelerometers in medical applications. *Sensor Actuat A: Phys*, 68(1–3):221–228, June 1998. DOI: 10.1016/S0924-4247(98)00049-1.

[4] Paul Lukowicz, Holger Junker, and Gerhard Tröster. Automatic calibration of body worn acceleration sensors. *Lect. Notes Comput. Sc.*, 3001:176–181, April 2004. DOI: 10.1007/978-3-540-24646-6_11.

[5] Albert Krohn, Michael Beigl, Christian Decker, Uwe Kochendörfer, Philip Robinson, and Tobias Zimmer. Inexpensive and automatic calibration for acceleration sensors. *Lect. Notes Comput. Sc.*, 3598:245–258, April 2005. DOI: 10.1007/11526858_19.

[6] Seong-hoon Peter Won and Farid Golnaraghi. A triaxial accelerometer calibration method using a mathematical model. *IEEE T Instrum Meas*, 59(8):2144–2153, August 2010. DOI: 10.1109/TIM.2009.2031849.

[7] Z.C. Wu, Z.F. Wang, and Y. Ge. Gravity based online calibration for monolithic triaxial accelerometers' gain and offset drift. In *Proceedings of the 4th World Congress on Intelligent Control and Automation*, volume 3, pages 2171–2175, Shanghai, P.R.China, 2002. DOI: 10.1109/WCICA.2002.1021471.

[8] Walter Rudin. *Real and Complex Analysis.* McGraw-Hill, third edition, 1987.

[9] Gene H. Golub and Chales F. van Loan. *Matrix Computations.* The Johns Hopkins University Press, third edition, 1996.