Lech Maligranda, Department of Mathematics, Luleå University of
Technology, S-971 Luleå, Sweden, e-mail: lech@sm.luth.se

THE RIESZ-WIENER AND THE STEIN-HERZ ESTIMATES OF THE MAXIMAL FUNCTION

The Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

plays an important role in the study of differentiation, singular integrals, and almost everywhere convergence. A great interest in estimates of rearrangement and distribution of the maximal function started after the classical paper of Hardy and Littlewood (1930). They defined the maximal operator (in the one-dimensional case) and proved boundedness of this operator in $L_p(\mathbb{R}^1)$ for $p > 1$. F. Riesz (1932) proved the rearrangement inequality

$$(Mf)^*(t) \leq A F^{**}(t) \quad \forall t > 0, \quad f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds. \quad (1)$$

N. Wiener (1939) defined the maximal function Mf in the n -dimensional case and proved the weak type $(1,1)$ of maximal operator and also the stronger inequality for the distribution

$$\lambda |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq B \int_{\{x: |f(x)| > \lambda/2\}} |f(x)| dx \quad \forall \lambda > 0. \quad (2)$$

The inverse inequalities to those of Riesz and Wiener

$$\frac{1}{\lambda} \int_{\{x \in \mathbb{R}^n : |f(x)| > \lambda/2\}} |f(x)| dx \leq C |\{x \in \mathbb{R}^n : |f(x)| > \lambda/2\}| \quad \forall \lambda > 0, \quad (3)$$

$$f^{**}(t) \leq D (Mf)^*(t) \quad \forall t > 0, \quad (4)$$

were found by Stein (1968) and Herz (1968).

We discuss the conditions on the measure $d\mu(x) = w(x)dx$ under which similar estimates for the distribution and rearrangement of the weighted maximal function

$$M_w f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)| w(y) dy$$

are valid. It is proved that the Riesz inequality (1_w) is equivalent to the Wiener inequality (2_w) and that they are true if and only if the maximal operator M_w is of the weak type $(1,1)$. The Stein inequality (3_w) and the Herz inequality (4_w) are valid without any restriction on the measure w . In particular,

$$(M_w f)_w^*(t) \approx \frac{1}{t} \int_0^t f_w^*(s) ds \text{ iff the operator } M_w \text{ is of weak type } (1,1).$$

These are my joint results with I. U. Asekritova, N. Ya. Krugljak, and L. E. Persson.

References

- [1] I. U. Asekritova, N. Ya. Krugljak, L. Maligranda, and L. E. Persson, *Distribution and rearrangement estimates of the maximal function and interpolation*, Studia Math. 124 (1997), 107 – 132.
- [2] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985.
- [3] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press 1993.