ON A WEAK-TYPE THEOREM WITH APPLICATIONS

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1. Introduction

We assume that \( f \) is a measurable, complex-valued function on a measure space \((\Omega, \mu)\). The function \(|f|\) can be rearranged to become a non-increasing function, \( f^* \), on \([0, \infty[\), which is equidistributed with \(|f|\) and continuous from the right (see, for example, [5, pp. 249–51]). The Lorentz space \( \Lambda(\lambda, q) \) \((\lambda = \lambda(t) \geq 0, 0 < t < \infty, \text{and} \ q > 0)\) is the collection of all \( f \) such that \( \|f\|_{\Lambda,q} < \infty \), where

\[
\|f\|_{\Lambda,q} = \left( \int_0^{\infty} ((f^*(t))\lambda(t))^q \frac{dt}{t} \right)^{1/q}, \quad \text{for} \ q < \infty,
\]

\[
\sup_{t > 0} f^*(t)\lambda(t), \quad \text{for} \ q = \infty.
\]

(see [8, p. 37]). We note that if \( \lambda(t) = t^{1/p} \) then \( \|f\|_{\Lambda,q} \) is just the usual \( L(p,q) \)-norm \( \|f\|_{p,q}^* \).

We say that \( f \) belongs to the average space \( \Lambda_A(\lambda, q, p, r) \) \((\lambda = \lambda(t) \geq 0, 0 < q < \infty, 0 < p < \infty, \text{and} \ 0 < r < \infty)\) if

\[
\|f\|_{\Lambda_A,q,p,r} = \left( \int_0^{K} (\lambda(t))^{q-a/p} \left( \int_0^{t} (f^*(u))^{r/p} \frac{du}{u} \right)^{q/r} \frac{dt}{t} \right)^{1/q} < \infty,
\]

where \( K = \sup\{t : f^*(t) > 0\} \). (We do not exclude the case where \( K = \infty \).) For the case where \( p = r \) we write \( \|f\|_{\Lambda_A,q,p}^* = \|f\|_{\Lambda_A,q,p}^* \) and \( f \in \Lambda_A(\lambda, q, r) \) when this quantity is finite.

Moreover, we shall say that a non-negative function \( \lambda \) on \([0, \infty[\) belongs to the space \( Q(a_0, b_0) \), if, for some \( a < a_0 \), and \( b > b_0 \), \( \lambda(t)^{a_0} \) is an increasing, and \( \lambda(t)^{b_0} \) a decreasing, function of \( t \) (see [11 and 12]). Sometimes we want to say that \( \lambda \in Q(a_0, b) \) for some real number \( b \) and then we write \( \lambda \in Q(a_0, -) \).

In [11] the present author has stated

**Theorem I.** Let \( 0 < q < \infty \), let \( f \in L[0, 1] \), and let \( c_n, \text{for} \ n \in \mathbb{Z} \), be the complex Fourier coefficients of \( f \).

(a) If \( \lambda \in Q(-\frac{1}{2}, -) \), then

\[
\left( \sum_{n=1}^{\infty} (c_n^*)^q (n\lambda(n^{-1}))^{q_n-1} \right)^{1/q} \leq A \|f\|_{\Lambda_A,1}^*, \tag{1.1}
\]

where \( A \) depends on \( \lambda \) and \( q \) only.

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(b) If $\lambda \in Q\left(-\frac{1}{2}, -1\right)$, then (1.1) holds with $\|f\|_{*,1}^{\lambda}$ replaced by $\|f\|_{*,A}^{\lambda}$.

As usual, the sequence $(c_n^*)_{n=0}^{\infty}$ is the sequence $(|c_n|)_{n=0}^{\infty}$ rearranged in non-increasing order. We note that Theorem I can be seen as a generalization and a refinement of well-known results of Hardy and Littlewood, and Stein (see [14, p. 490] and [17, vol. II, p. 123]), and Zygmund and Bennett (see [1, p. 3] and [17, vol. II, p. 158]). The obvious dual version of Theorem I and complementary examples can also be found in [11].

The proof of Theorem I depends crucially upon the facts that the Fourier operator $F_d(f) = (\hat{f}(n))_{n=-\infty}^{\infty} = (c_n)_{n=-\infty}^{\infty}$ is a linear mapping of (weak-)type $(2,2)$ (Parseval’s relation) and that $|c_n| \leq \int_0^\infty |f(t)| dt$, where $n \in \mathbb{Z}$.

In this paper we shall generalize Theorem I so that the theorem can be applied to more general quasi-linear operators $T$. As usual, we shall say that an operator $T$ is quasi-linear if $T(f+g)$ is defined whenever $T(f)$ and $T(g)$ are defined and if $|T(f+g)| \leq A(\|Tf\| + \|Tg\|)$, where $A$ is a constant which does not depend on $f$ and $g$. For such operators we have the following useful estimate:

$$
(T(f+g))^*(2t) \leq A((Tf)^*(t) + (Tg)^*(t)) \quad (t > 0)
$$

(1.2) (see [5, p. 263]). It is well known that many classical operators (for example, the Hilbert transform, the Fourier transform, fractional integral operators, singular integral operators, and so on) map $L(p, q)$ boundedly into some $L(p_1, q)$ (see, for example, [5 and 16]). In particular, the results in this paper will show that the same operators even map $\Lambda(\lambda, q)$ boundedly into $\Lambda(\lambda_1, q)$ ($\lambda$ and $\lambda_1 = \lambda_1(\lambda)$ belong to certain $Q$-classes).

The main results in this paper can be found in §2. In particular, Theorem 2.1 can be seen as a refinement of a well-known interpolation theorem of Marcinkiewicz, Calderón, and Hunt (see [5, p. 264]). The proofs of the main results are carried out in §3. Some applications and relations to well-known results are pointed out in §4.

2. The main results

We shall use the following notation:

$$
K = \sup\{t \mid f^*(t) > 0\};
$$

$$
\gamma = \left(\frac{1}{p_1} - \frac{1}{p_0}\right) / \left(\frac{1}{p'_1} - \frac{1}{p'_0}\right);
$$

$$
I = I(K, \gamma) = \begin{cases} 
[0, 2K\gamma], & \text{for } \gamma > 0, \\
[2K\gamma, \infty[, & \text{for } \gamma < 0;
\end{cases}
$$

and

$$
\lambda_1 = \lambda_1(t, \lambda, \gamma) = t^{(1/p_1p'_0) - (1/p'_1p_0)}/((1/p_1) - (1/p_0)) \lambda(t^{1/\gamma}).
$$
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THEOREM 2.1. Let \( q, p_i, q_i, p'_i, q'_i \) where \( i = 0, 1 \), be real numbers such that
\( 0 < q \leq \infty, \ 0 < p_0 < p_1 \leq \infty, \ 0 < q_0 < \infty, \ 0 < q_1 \leq \infty \) \((p_1 = \infty \text{ when } q_1 = \infty)\), and \( p'_0 \neq p'_1 \). Moreover, let \( \lambda \in \mathcal{Q}(-1/p_1, -1/p_0) \) and let \( T \) be a quasi-linear operator such that
\[
\| Tf \|_{p_i, q_i}^* \leq A_i \| f \|_{p_i, q_i}^* \quad (i = 0, 1). \tag{2.1}
\]
Then, for every \( s \geq q \),
\[
\| Tf \|_{p, q}^* \leq A \| f \|_{p, q}^*
\]
The constant \( A \) depends only on \( \lambda, q, A_i, p_i, \) and \( q_i \).

We note that if
\[
\frac{1}{p_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{p'_0} = \frac{1-\theta}{p'_0} + \frac{\theta}{p'_1} \quad (0 < \theta < 1),
\]
and if \( \lambda(t) = t^{1/p_0} \), then \( \lambda(t) \in \mathcal{Q}(-1/p_1, -1/p_0) \) and \( \lambda_1(t) = t^{1/p'}. \) Therefore, for every \( s \geq q \),
\[
\| Tf \|_{p, q}^* \leq A \| f \|_{p, q}^* \tag{2.2}
\]
and this is just a well-known interpolation estimate of Marcinkiewicz, Calderón, and Hunt [5, p. 264]. We can even state the following corollary.

COROLLARY 2.2. Let \( a \in \mathbb{R}, \) let \( T, q, p_i, q_i, p'_i, p'_0, \) and \( p'_0 \) be defined as above, and let \( f \in \Lambda(t^{1/p_0}(\log(2 + t))^{a}, q) \). Then
\[
Tf \in \Lambda(t^{1/p'_0}(\log(2 + t^{\text{sgn}(p'_1-p'_0)}))^{a}, q)
\]
(where \( \text{sgn} b = b/|b| \)).

The proof of the corollary depends on the fact that, for every \( b \neq 0 \), there exist positive constants \( A_i (i = 0, 1) \), such that

\[
A_0 \log(2 + t^{\text{sgn} b}) \leq \log(2 + b^t) \leq A_1 \log(2 + t^{\text{sgn} b}).
\]

Therefore we find that if \( \lambda(t) = t^{1/p_0}(\log(2 + t))^{a}, \) then

\[
\lambda_1(t) \simeq t^{1/p'_0}(\log(2 + t^{\text{sgn}(p'_1-p'_0)}))^{a}.
\]

(We can, for example, choose \( A_0 = \frac{1}{2} \min(|b|, 1) \) and \( A_1 = 2 \log(2 + 2|b|) \).)

It is easy to find examples to show that (2.2) does not hold, in general, for the case where \( \theta = 0 \). However, it is possible to make a similar estimate which does not exclude this and other exceptional cases.

THEOREM 2.3. Assume that \( p_i, q_i, p'_i, q'_i \) \((i = 0, 1)\) are positive numbers such that \( q_i < \infty, \ p_0 < p_1 < \infty, \) and \( p'_0 \neq p'_1 \). If \( T \) is a quasi-linear operator satisfying (2.1) and if \( \lambda \in \mathcal{Q}(-1/p_1, -1/p_0) \), then, for every \( q \), with \( 0 < q < \infty \),
\[
\int_f \langle (Tf)^*(t)(\lambda_1(t))^q \rangle^\frac{dt}{t} \leq A(f)^{\| f \|_{p, q}}
\]
where the constant \( A \) depends at most on \( \lambda, q, A_i, p_i, \) and \( q_i \).
The proofs of Theorems 2.1 and 2.3 depend crucially upon some estimates, which we shall state separately as lemmas. Our first lemma is used to estimate the functionals

\[ L(t, f) = t^{-1/p} \left( \int_0^{t^{1/r}} (f^*(u))^{q_0} u^{q_0 - 1} \frac{du}{u} \right)^{1/q_0} + t^{-1/p'} \left( \int_{t^{1/r}}^{\infty} (f^*(u))^{q_1} u^{q_1 - 1} \frac{du}{u} \right)^{1/q_1}. \]

**Lemma 2.4.** Assume that \( p_i, q_i, p'_i, q'_i \) \((i = 0, 1)\) are positive numbers such that \( q_0 < \infty, p_0 < p_1 < \infty, \) and \( p'_0 \neq p'_1 \). If \( T \) is a quasi-linear operator satisfying (2.1), then, for \( t > 0, \)

\[ (Tf)^*(2t) \leq AL(t, f), \]

where \( A \) depends only on \( A, p_i, \) and \( q_i \).

The next lemma can be seen as a generalization of a well-known estimate of Hardy (see, for example, [17, vol. I, p. 20]).

**Lemma 2.5.** Let \( \lambda \) and \( \psi \) be non-negative functions on \([0, \infty]\) such that \( \psi(u) \leq A_\lambda \psi(t) \) for some \( A_\lambda \in \mathbb{R}^+ \) and for \( u \in [t, 2t] \) \((t > 0)\). Moreover, let \( p(u) = (\lambda(t))^q(\psi(t))^{-q-1} \).

(a) If \( p_q \in Q(\lambda', 1) \), then
\[
\left( \int_0^\infty \varphi_q(t) \left( \int_t^{2t} ((f^*(u))(\psi(u)))^{q_1} u^{q_1 - 1} \frac{du}{u} \right)^{1/q_1} dt \right)^{1/q} \leq \| f \|_{\mathcal{L}^q},
\]

(b) If \( p_q \in Q(1, \lambda) \), then
\[
\left( \int_0^\infty \varphi_q(t) \left( \int_t^{2t} ((f^*(u))(\psi(u)))^{q_1} u^{q_1 - 1} \frac{du}{u} \right)^{1/q_1} dt \right)^{1/q} \leq \| f \|_{\mathcal{L}^q}.
\]

The constant \( A \) depends only on \( \lambda, \psi, \) and \( q \).

### 3. Proofs

**Proof of Lemma 2.4.** We put

\[
f_v(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq y, \\ y \mathrm{sgn} f(x) & \text{if } |f(x)| > y, \end{cases}
\]

\[ f_v(x) = f(x) - f_v(x), \]

\[ g_1 = Tf_v \text{ and } g_2 = Tf_v. \]

For each \( t \) we choose \( y = y(t) = f^*(t^{1/r}) \) and note that

\[ (f_v)^*(u) = \begin{cases} y & \text{if } 0 < u \leq t^{1/r}, \\ f^*(u) & \text{if } u > t^{1/r}, \end{cases} \]

\[ (f^v)^*(u) = 0 \text{ for } u > t^{1/r} \text{ and } (f^v)^*(u) \leq f^*(u) \text{ for } 0 < u \leq t^{1/r}. \]

Since \( \| f \|_{\mathcal{L}^p, r_1} \leq \| f \|_{p, r_2} \) for \( r_2 \leq r_1 \) (see [5, p. 253]), we can, without loss of generality, assume that \( r_0 = r'_1 = \infty \) in (2.1). Therefore when we apply
(2.1) (with $i = 0$) to the function $f^v(x)$ we find that, for each $t > 0,$
\[
g^*_v(t)^{1/p^*} \leq A_0 \|f^v\|_{p_0,q_0}
\]
\[
\leq A_0 \left( \int_0^{1/y} \left( f^*(u) \right)^{p_0/q_0} \frac{du}{u} \right)^{1/q_0} = A_0 I^1_{1/q_0}.
\]
(3.1)

Analogously, by (2.1) (with $i = 1$),
\[
g^*_v(t)^{1/p^*} \leq A_1 \|f_v\|_{p_1,q_1}
\]
\[
= A_1 \left( \int_0^{1/y} \left( f^*(u) \right)^{p_1/q_1} \frac{du}{u} \right)^{1/q_1} = A_1 (I_1 + I_2)^{1/q_1}.
\]
Therefore, by the inequality
\[
(a + b)^\beta \leq \max(2^{\beta - 1}, 1)(a^\beta + b^\beta) \quad (a > 0, b > 0, \beta > 0),
\]
(3.2)
we find that, for each $t > 0,$
\[
g^*_v(t)^{1/p^*} \leq A_2 (I_2^{1/q_1} + I_3^{1/q_1}).
\]
(3.3)

Also
\[
I_2^{1/q_1} = \left( \int_0^{1/y} \left( f^*(u) \right)^{1/y} \frac{du}{u} \right)^{1/q_1}
\]
\[
= A_2 \left( \int_0^{1/y} \left( f^*(u) \right)^{p_1/q_1} \frac{du}{u} \right)^{1/q_1} = A_2 (I_2^{1/q_1} + I_3^{1/q_1}).
\]
(3.4)

Thus, according to (1.2), the definition of $\gamma,$ and (3.1)-(3.4),
\[
(Tf)^*_v(2t) \leq A_6 (t^{1/\gamma p_1} - (1/\gamma p_0) + t^{-1/p_1}) I_1^{1/q_0} + A_6 t^{-1/p^*} I_3^{1/q_1}
\]
\[
\leq A \left( t^{-1/p^*} I_1^{1/q_0} + t^{-1/p_1} I_3^{1/q_1} \right) = AL(t, f).
\]

The proof is complete.

The proof of Lemma 2.5 is based upon the following corresponding result for sequences:

**Lemma.** Let $(a_k)_{n_0}^{n_1}$ and $(b_k)_{n_0}^{n_1}$ be sequences where $a_k > 0$ and $b_k \geq 0$ for every $k,$ and $|n_0| + |n_1| = \infty.$ Then, for $d \geq 1,$
\[
\sum_{k=n_0}^{n_1} a_k \left( \sum_{k=n_0}^{n_1} b_k \right)^d \leq d^d \sum_{k=n_0}^{n_1} (a_k)^{1-d} \left( \sum_{k=n_0}^{n_1} a_n \right) (b_k)^d
\]
(3.5)

and
\[
\sum_{k=n_0}^{n_1} a_k \left( \sum_{k=n_0}^{n_1} b_k \right)^d \leq d^d \sum_{k=n_0}^{n_1} (a_k)^{1-d} \left( \sum_{k=n_0}^{n_1} a_n \right) (b_k)^d.
\]
(3.6)

The constant $d^d$ is the best possible.
Proof. For the case where \( n_0 = 1 \) and \( n_1 = \infty \), the lemma has been proved by Leindler [7, p. 279]. The proof of this more general case can be carried out in a similar way, so we omit the details.

Proof of Lemma 2.5. We can make some straightforward calculations to see that the condition \( \varphi_q \in Q(-, 1) \) implies that, for every \( x > 0 \),

\[
0 < \int_x^\infty \varphi_q(t) dt < B_1 \int_x^{2x} \varphi_q(t) dt \leq B_2 \int_{x/2}^x \varphi_q(t) dt.
\]

Therefore, by applying the estimate (3.5) in the lemma above with \( n_0 = -\infty \) and \( n_1 = \infty \),

\[
a_k = \int_{2^k}^{2^{k+1}} \varphi_q(t) dt \quad \text{and} \quad b_k = \int_{2^k}^{2^{k+1}} (f^*(u))(\psi(u)) r \frac{du}{u} ,
\]

we obtain

\[
\int_0^\infty \varphi_q(t) \left( \int_0^t (f^*(u))(\psi(u)) r \frac{du}{u} \right)^{q/r} dt \leq \sum_{-\infty}^\infty \int_{2^k}^{2^{k+1}} \varphi_q(t) dt \left( \sum_{-\infty}^\infty \int_{2^k}^{2^{k+1}} (f^*(u))(\psi(u)) r \frac{du}{u} \right)^{q/r} = \sum_{-\infty}^\infty a_k \left( \sum b_n \right)^{q/r} \leq B_3 \sum_{-\infty}^\infty a_k b_k^{q/r} .
\]

(For the case where \( q < r \) we must make the trivial estimate

\[
\left( \sum b_n \right)^{q/r} \leq \sum b_n^{q/r}
\]

before we apply (3.5) with \( d = 1 \).) Furthermore, by the growth conditions on \( \psi \) and (3.7),

\[
\sum_{-\infty}^\infty a_k b_k^{q/r} = \sum_{-\infty}^\infty \int_{2^k}^{2^{k+1}} \varphi_q(t) dt \left( \int_{2^k}^{2^{k+1}} (f^*(u))(\psi(u)) r \frac{du}{u} \right)^{q/r} \leq B_4 \sum_{-\infty}^\infty \int_{2^k}^{2^{k+1}} \varphi_q(t)(f^*(2^k))^q(\psi(2^k))^q dt \leq B_5 \| f \|_{\psi^q_q}^q . \]

The proof of (a) follows when we combine this estimate with (3.8).

Analogously, it is easily seen that if \( \varphi_q \in Q(1, -) \), then, for every \( x > 0 \),

\[
0 < \int_0^{2x} \varphi_q(t) dt \leq B_6 \int_x^{2x} \varphi_q(t) dt \leq B_7 \int_{x/2}^x \varphi_q(t) dt .
\]
Therefore we can use (3.6) and make calculations similar to those above to prove part (b) of the lemma.

**Proof of Theorem 2.3.** According to Lemma 2.4 and (3.2) we see that for every $L \geq 0$,

$$
\int_0^L ((Tf)^*(2t\gamma))^{q_0/(q_0-p_0)}(\lambda(t))^{q_0} \frac{dt}{t} \leq B_0 \left( \int_0^L (\lambda(t))^{q_0-q_0-\mu_0} \left( \int_0^\infty (f^*(u))^{\alpha_0 u^{\alpha_0-p_0}} \frac{du}{u} \right)^{q_0} \frac{dt}{t} \right.
$$

$$
+ \left. \int_0^L (\lambda(t))^{q_0-q_0-\mu_1} \left( \int_0^\infty (f^*(u))^{\alpha_1 u^{\alpha_1-p_1}} \frac{du}{u} \right)^{q_0} \frac{dt}{t} \right). \tag{3.9}
$$

The hypothesis $\lambda \in Q(-1/p_1, -)$ implies that

$$
\varphi_q(t) = (\lambda(t)t^{-1/p_1})^{q_0-1} \in Q(1, -)
$$

and thus, by Lemma 2.5(b) and a trivial estimate,

$$
\int_0^\infty (\lambda(t))^{q_0-q_0-\mu_1} \left( \int_0^\infty (f^*(u))^{\alpha_0 u^{\alpha_0-p_0}} \frac{du}{u} \right)^{q_0} \frac{dt}{t} \leq (A \|f\|_{L^2}^q)^q \leq B_1(\|f\|_{L^2(\mu_0, p_0, \gamma_0, \gamma_1)}). \tag{3.10}
$$

In view of the definitions of $I, \gamma,$ and $\lambda_1$ the proof follows by making a change of variables to the integral on the left-hand side of (3.9) and by combining (3.9) ($L = K$) with (3.10).

**Proof of Theorem 2.1.** In particular, the hypothesis $\lambda \in Q(-1/p_1, -1/p_0)$ implies that, for every $k \in \mathbb{Z}$ and $2^{k-1} \leq t \leq 2^k,$

$$
0 < B_0 \lambda(2^k) \leq \lambda(t) \leq B_1 \lambda(2^{k-1})
$$

and thus, for $s \geq q$,

$$
(\|f\|_{L^2}^*)^q = \left( \int_{-\infty}^\infty \left( \int_{2^{k-1}}^{2^k} ((f^*(t))(\lambda(t)))^{q/s} \frac{dt}{t} \right)^q \right)^{q/s} \leq B_2 \left( \sum_{-\infty}^\infty \left( \int_{2^{k-1}}^{2^k} ((f^*(2^{k-1})))(\lambda(2^{k-1}))^{q/s} \right)^q \right)^{q/s} \leq B_2 \sum_{-\infty}^\infty \left( (f^*(2^{k-1})))(\lambda(2^{k-1}))^{q/s} \right)^q \leq B_3 \sum_{-\infty}^\infty \int_{2^{k-1}}^{2^k} (f^*(t))^q(\lambda(t))^{q/s} \frac{dt}{t} \leq B_3(\|f\|_{L^2}^*)^{q/s}.
$$

We conclude that $\|f\|_{L^2}^* \leq B_4\|f\|_{L^2}^{q_i}$ for $s \geq q$ and therefore it is sufficient to prove the theorem with $s = q$ and $q_i = \infty$ ($i = 0, 1$).
The case $p_1 < \infty$, $q < \infty$. The hypothesis $\lambda \in Q(-1/p_1, -1/p_0)$ ensures that $(\lambda(t)t^{-1/p_0})^{q-1} \in Q(-1, 1)$ and $(\lambda(t)t^{-1/p_1})^{q-1} \in Q(1, -1)$ and therefore, by Lemma 2.5, the estimate (3.10) and the inequality

$$\int_0^\infty (\lambda(t))^{q-1/p_0} \left( \int_0^1 (f^*(u)) u^{q_0/p_0} \frac{du}{u} \right)^{q_0} \frac{dt}{t} \leq (A \| f \|_{\lambda, \infty}^*)^q$$

are satisfied. Hence the proof follows by making a suitable change of variables and by applying (3.9) with $L = \infty$.

The case $p_1 < \infty$, $q = \infty$. According to Lemma 2.4

$$(Tf)^*(t) t^{(1/p_1) - (1/p_1r)} \leq B_\delta t^{1-1/p_0} \left( \int_0^{(u/2)^{1/r}} (f^*(u)) u^{q_0/p_0} \frac{du}{u} \right)^{1/q_0} + t^{1-1/p_1} \left( \int_0^\infty (f^*(u)) u^{q_1} \frac{du}{u} \right)^{1/q_1}.$$  (3.11)

For fixed $t > 0$ we choose $k_0 \in Z$ such that $2^{k_0-1} \leq (\frac{\lambda t}{t})^{1/\gamma} \leq 2^{k_0}$. Then, by the hypothesis $\lambda \in Q(-1/p_1, -1)$, we find that

$$\left( \int_0^{(u/2)^{1/r}} (f^*(u)) u^{q_0/p_0} \frac{du}{u} \right)^{1/q_0} \leq \| f \|_{\lambda, \infty}^* \left( \int_0^{2^{k_0}} u^{q_0/p_0} u^{1/q_0} \frac{du}{u} \right)^{1/q_0} \leq B_\delta \| f \|_{\lambda, \infty}^* \left( \sum_{k=-\infty}^{k_0} (2^{k_0}/\lambda(2^{k_0}))^{q_0} \right)^{1/q_0} \leq B_\delta \| f \|_{\lambda, \infty}^* 2^{k_0/p_0} (\lambda(2^{k_0}))^{-1} \leq B_\delta \| f \|_{\lambda, \infty}^* 2^{k_0/p_0} (\lambda(2^{k_0}))^{-1}.$$  (3.12)

Since $\lambda \in Q(-1/p_1, -1)$ we find that

$$\left( \int_0^\infty (f^*(u)) u^{q_1} \frac{du}{u} \right)^{1/q_1} \leq B_\delta \| f \|_{\lambda, \infty}^* \left( \sum_{k_0-1}^{k_0} (2^{k_0}/\lambda(2^{k}))^{q_1} \right)^{1/q_1} \leq B_{10} \| f \|_{\lambda, \infty}^* 2^{k_0/p_1} (\lambda(2^{k_0}))^{-1} \leq B_{11} \| f \|_{\lambda, \infty}^* 2^{k_0/p_1} (\lambda(2^{k_0}))^{-1}.$$  (3.13)

We combine (3.11)-(3.13) and find that, for every $t > 0$,

$$(Tf)^*(t) t^{(1/p_1) - (1/p_1r)} \lambda(2^{1/\gamma}) \leq B_{12} \| f \|_{\lambda, \infty}^*.$$ 

Therefore, according to the definitions of $\gamma$ and $\lambda$, the theorem is proved for this case too.

The case $p_1 = q_1 = q = \infty$. We use the notation from the proof of Lemma 2.4 and write $f(x) = f^u(x) + f^v(x)$. Then by the hypothesis (2.1)
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Since \((f^v)^*(u) \leq f^*(u)\) for \(0 < u < t \) and \((f^v)^*(u) = 0\) for \(u > t\) it follows from (3.13) and (2.1) \((i = 0)\) that
\[
g^*_1(t) t^{1/p^*_1 - 1/p^*_T} \lambda(t^{1/\gamma})^q \frac{dt}{t} \leq B_{14} \|f\|_{\lambda, \infty}^*.
\]

According to the definition of \(\lambda_1\) the proof of this case follows when we combine (1.2), (3.14), and (3.15).

The case \(p_x = q_x = \infty, q < \infty\). We apply (2.1) \((i = 0)\) to the function \(f^v(x)\) and use Lemma 2.5(a) to obtain that
\[
\int_0^\infty \left( g^*_1(t) t^{1/p^*_1 - 1/p^*_T} \lambda(t^{1/\gamma})^q \frac{dt}{t} \right) dt = B_{15} \|f\|_{\lambda, q}^*.
\]
The desired conclusion follows easily from (1.2), (3.14), and (3.16). The proof is complete.

4. Some applications

I. A more general version of Theorem I

We shall state a theorem which, for example, can be applied to the Fourier operators \(F_d\) (see p. 296) and \(F_c\) (see this page).

**THEOREM 4.1.** Let \(T\) be a quasi-linear operator of weak-type \((2r, 2r)\), where \(r > 0\), and let \(\|Tf\|_{\lambda, \infty}^* \leq A_0 \|f\|_{\gamma, r}^*\).

(a) If \(\lambda \in Q(-1/2r, -1)\) then, for \(0 < q < \infty\),
\[
\left( \int_{2/r}^\infty \left( \frac{1}{(Tf)^*(t)} \lambda(t^{1/\gamma})^q \frac{dt}{t} \right)^{1/q} \right) \leq A \|f\|_{\lambda, q}^*.
\]

(b) If \(\lambda \in Q(-1/2r, -1/r)\) then, for \(q > 0\) and \(s \geq q\),
\[
\|Tf\|_{\lambda, \gamma, 1/q, s}^* \leq A \|f\|_{\lambda, q}^*.
\]
The constant \(A\) depends only on \(\lambda, q, s, r, T\).

**Proof.** The hypothesis asserts that the operator \(T\) satisfies (2.1) with \(p_0 = q_0 = r, p_1 = q_1 = 2r, p_0' = q_0' = q_1' = \infty\). Hence \(\gamma = -1\) and \(\lambda_1(t) = t^{1/r} \lambda(t^{-1})\) and the proof follows when we apply Theorems 2.1 and 2.3.

We note that Theorem I and the analogous estimates for multiple Fourier series are special cases of Theorem 4.1 with \(r = 1\).

The Fourier transform, \(F_c\), of a function \(f \in L(R^n)\) is defined by
\[
F_c(f) = \hat{f}(\xi) = \int_{R^n} f(\xi) e^{-2\pi i \xi \cdot \tilde{d}} d\tilde{d}.
\]
The Fourier transform can also be defined for all \( f \in L^2(\mathbb{R}^n) \) and \( \| f \|_{L^2}^* = \| \hat{f} \|_{L^2}^* \) (Plancherel’s formula). It is easily seen that if \( \lambda \in Q(-\frac{1}{2}, -1) \) and if \( f \in \Lambda(\lambda, q) \) then \( f \in L^4 + L^2 \). Thus \( \hat{f} \) is defined for every \( f \in \Lambda(\lambda, q) \) when \( \lambda \in Q(-\frac{1}{2}, -1) \). We conclude that by applying Theorem 4.1(b) with \( r = 1 \) we find that, for every \( s \geq q, q > 0 \), and \( \lambda \in Q(-\frac{1}{2}, -1) \),

\[
\| \hat{f} \|_{\Lambda(1/10, s)}^* \leq B \| f \|_{\Lambda q}^*.
\]  

(4.1)

For the case where \( \lambda(t) = t^{1/p}, 1 < p < 2 \) \( (\lambda(t^{-1}) = t^{(1/p)}), \) see also Hunt [5, p. 270].

II. Necessary and sufficient integrability conditions of functions and their Fourier transforms

We shall now state a more general version of a theorem of Hunt [5, p. 270]. For the proof of this theorem we need the following extension of a lemma of Zygmund [17, vol. II, p. 130].

**Lemma 4.2.** Assume that \( f \) is an even function, which is non-negative, locally integrable, non-increasing on \([0, \infty[ \) and \( f(x) \to 0 \) as \( x \to \infty \). Furthermore, suppose that \( q > 0 \), \( g(x) = \int_0^\infty f(s) \cos xs \, ds, \lambda \in Q(0, -1) \), and \( \lambda_1(t) = t\lambda(1/t) \). Then \( f \in \Lambda(\lambda, q) \) if and only if \( g \in \Lambda(\lambda_1, q) \).

**Proof.** Assume that \( f \in \Lambda(\lambda, q) \). We have, for every \( x \in \mathbb{R} \),

\[
|g(x)| \leq B_1 \int_0^{1/|x|} f(s) \, ds \leq B_1 \int_0^{1/|x|} f^*(u) \, du
\]

(see [5, p. 271]). We conclude that \( g^*(t^{-1}) \leq B_1 \int_0^\infty f^*(u) \, du \) and thus

\[
\int_0^\infty (g^*(t^{-1}))^{q-\lambda(\lambda(t))} \frac{dt}{t} \leq B_1^q \int_0^\infty (\lambda(t))^{q-\lambda} \left( \int_0^\infty f^*(u) \, du \right)^q \frac{dt}{t}.
\]  

(4.2)

We use the hypothesis \( \lambda \in Q(-1, -1) \) and find that

\[
\varphi_q(t) = \lambda((t))^{q-\lambda-1} \in Q(-, 1).
\]

Therefore, according to Lemma 2.5(a), the integral on the right-hand side of (4.2) is majorized by \( (A \| f \|_{\Lambda q}^*)^2 \). Thus we can make a change of variables in (4.2) and obtain that \( \| g \|_{\Lambda q}^* \leq AB_1 \| f \|_{\Lambda q}^* \).

On the other hand, suppose that \( g \in \Lambda(\lambda_1, q) \). We have that

\[
f(t) \leq B_2 \left| \int_0^{1/t} g(x) \, dx \right|
\]

(see [5, p. 270] and compare with [17, vol. II, p. 129]). Therefore \( f^*(t) \leq B_2 \int_0^{1/t} g^*(u) \, du \) so that

\[
\int_0^\infty (f^*(t))^2 \lambda(t) \frac{dt}{t} \leq B_2^2 \int_0^\infty \lambda(t) \left( \int_0^{1/t} g^*(u) \, du \right)^2 \frac{dt}{t}.
\]  

(4.3)
By hypothesis \( \lambda(t^{-1}) \in Q(-1,0) \) and thus we can make a change of variables in the integral on the right-hand side in (4.3) and apply Lemma 2.5(a) with \( \varphi_0(t) = (\lambda(t))^{2p-1} \) to obtain that \( \|f\|_{A_q}^q \leq AB \|g\|_{A_{aq}}^q \). The proof is complete.

**Theorem 4.3.** Let \( q > 0 \) and \( \lambda_1(t) = t\lambda(t^{-1}) \).

(a) Assume that \( \lambda \in Q(-\frac{1}{2}, -1) \). Then \( f \in A(\lambda, q) \) if and only if for every function \( h \), such that \( h^* = f^* \), the Fourier transform, \( \hat{h} \), exists and \( \hat{h} \in A(\lambda_1, q) \). Moreover, \( \|\hat{h}\|_{A_{aq}} \leq B\|f\|_{A_q}^q \) for every such function \( h \).

(b) Suppose that \( \lambda \in Q(0, -\frac{1}{2}) \). Then \( f \in A(\lambda, q) \) if and only if for some function \( h \), such that \( h^* = f^* \), the Fourier transform, \( \hat{h} \), exists and \( \hat{h} \in A(\lambda_1, q) \). Furthermore, \( \|f\|_{A_q}^q \leq B\|\hat{h}\|_{A_{aq}}^q \) for every such function \( h \).

**Proof.** Assume that \( n = 1 \).

(a) Let \( f \in A(\lambda, q) \). Then \( h \in A(\lambda, q) \) and the existence of \( \hat{h} \) is guaranteed (see p. 304). Also, by (4.1), \( \|\hat{h}\|_{A_{aq}} \leq B\|f\|_{A_q}^q = B\|f\|_{A_{aq}}^q \). Conversely we choose \( h = f^* \) and note that, by assumption, \( \hat{h} \in A(\lambda_1, q) \). Therefore we can use Lemma 4.2 to conclude that \( \hat{h} \in A(\lambda, q) \). Since \( h^* = f^* \) we find that \( f \in A(\lambda, q) \) and the proof of (a) is complete.

(b) Suppose that \( f \in A(\lambda, q) \). Put \( h = f^* \in A(\lambda, q) \) and use Lemma 4.2 to see that \( \hat{h} \in A(\lambda_1, q) \). On the other hand, since \( \hat{h}(x) = -h(-x) \) and \( \lambda(t) \in Q(0, -\frac{1}{2}) \) if and only if \( \lambda_1(t) \in Q(-\frac{1}{2}, -1) \), we can use (4.1) to obtain that \( \|f\|_{A_q}^q = \|\hat{h}\|_{A_{aq}}^q \leq B\|f\|_{A_q}^q \). For the case where \( n > 1 \) we study the function \( h(x_1, x_2, \ldots, x_n) = f(x_1)x_{[0,1]}(x_2)\cdots x_{[0,1]}(x_n) \), where \( f(x_1) \) is defined as in Lemma 4.2 and \( x_{[0,1]} \) is the characteristic function of the interval \([0,1]\). Then the proof follows just as above.

**Remark.** For the case where \( \lambda(t) = t^{1/p} \) \( (\lambda_1(t) = t^{1-(1/p)} = t^{1/p'}) \), \( p > 1 \), \( q \geq 1 \), see Hunt [5, p. 270].

**Remark.** If we study Fourier analysis when the underlying dual groups are \( G = \mathbb{R} \) (mod 1) and \( \hat{G} = \mathbb{Z} \), then the corresponding theorem has been stated by the present author in [13].

**III. On operators simultaneously of weak-type \((r,r)\) and weak-type \((p,p)\): a suggestion.**

We shall state a theorem, which can be seen as a refined version of the usual \( L^p \)-estimates (see, for example, [16, p. 21]). The theorem also improves a result of O'Neil (see [10, p. 464] and [2, p. 235]).

**Theorem 4.4.** Let \( T \) be a quasi-linear operator simultaneously of weak-type \((r,r)\) and weak-type \((p,p)\), where \( 0 < r < p < \infty \).

(a) If \( \lambda \in Q(-1/p, -) \) then, for \( q > 0 \),

\[
\left( \int_0^{(t)} ((Tf)^*(t))^q(\lambda(t))q\left(\int_0^t \frac{d\lambda(t)}{t}\right)^{1/q} \right)^{1/q} \leq B\|f\|_{A_{aq}}^q.
\]

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(b) If $\lambda \in Q(-1/p, -1/r)$, then, for every $s \geq q$,
\[
\|Tf\|_{L^{s,r}} \leq B \|f\|_{L^{s,q}}. \tag{4.4}
\]

*Proof.* The hypotheses on the operator $T$ are equivalent to saying that the condition (2.1) is satisfied with $p_0 = q_0 = p' = r$, $p_1 = q_1 = p'^1 = p$, and $q_0' = q_1' = \infty$. Therefore $\gamma = 1$ and $\lambda(t) = \lambda(t)$, so the proof follows when we apply Theorems 2.1 and 2.3 to the considered operator $T$.

We note that if $\lambda(t) = t^{1/a}$, where $r < a < p$, then $\lambda \in Q(-1/p, -1/r)$ and (4.4) is turned into the estimate $\|Tf\|_{L^{s,q}} \leq B \|f\|_{L^{s,q}}$. For the case where $s = q = a$ and $r = 1$ see [16, p. 21]. Moreover, Theorem 4.4(a) contains the fact that if $f \in \Lambda(t^{1/a}(\log(2 + t))^b, q)$, for $r < a < p$, $b \in \mathbb{R}$, then, for $s \geq q$, $Tf \in \Lambda(t^{1/a}(\log(2 + t))^b, s)$ for every operator of the type considered. We also observe that, by applying Theorem 4.4(a), with $K = 1$ and $\lambda(t) = t^{1/r}(\log t)^{b-1/(1/a)}$, where $1 \leq q < \infty$, we obtain the inequality
\[
\int_0^1 (Tf)^s(t)dt \leq B \int_0^1 (\log t)^{b-1/q} \left( \int_0^t f^s(u)du \right)^{1/q} dt
\]
for every operator $T$ which is of weak-type $(r, r)$ and weak-type $(p, p)$. In particular, for the case where $q = r$ we can make a partial summation and use a standard argument (see p. 307) to see that
\[
\int_0^1 |Tf(x)|^r(\log^+ |Tf(x)|)^{b-1} dx \leq B \int_0^1 |f(x)|^r(\log^+ |f(x)|)^{b} dx + B.
\]
Therefore it is tempting to suggest that if the quasi-linear operator $T$ is of weak-type $(r, r)$ and if
\[
T : Lr(\log^+ L)^{b+r} \rightarrow Lr(\log L)^{b+r-1},
\]
where $b_0 \geq 0$, then
\[
T : \Lambda(t^{1/r}(\log t)^{b-1/(1/a)}, q, r) \rightarrow \Lambda(t^{1/r}(\log t)^{b-1/(1/a)}, q)
\]
where $b_0 - 1 < b < b_0$, $q > 0$. According to a recent result of Bennett (see [2, p. 934] and [3, p. 215]) the suggestion is correct at least when $r = 1$, $b_0 = 1$, and $q \geq 1$. Bennett's proof depends crucially upon the fact that $\Lambda(X) = X$ for the spaces $L$ and $L\log^+ L$ ($\Lambda(X)$ denotes the $\Lambda$-space associated with $X$, see, for example, [3, p. 220]). Unfortunately, the spaces $Lr\log^+ L$, where $r > 1$, for example, are not $\Lambda$-spaces (see [9, p. 127]) so the technique developed by Bennett cannot be extended to the case where $r > 1$.

IV. Summability relations between functions and their maximal and conjugate functions

For the sake of convenience we shall assume that $f \in L[0, 1]$. As usual, $Mf$ stands for the Hardy–Littlewood maximal function of $f$ (see [17,
The conjugate function of \( f \) is denoted \( \tilde{f} \) (see [17, vol. I, p. 131]). It is easy to verify that
\[
\tilde{f}(2x) \leq M(\tilde{f}^*)(x) \leq \tilde{f}^*(x) \quad (0 < x \leq 1),
\]
where \( \tilde{f}^*(x) = x^{-1} \int_0^x f^*(u) \, du \) (see [3, p. 218]). The following nice estimate is due to Herz (see [4]):
\[
\tilde{f}^*(2x) \leq 6(Mf)^*(2x) \leq 24\tilde{f}^*(x) \quad (0 < x \leq \frac{1}{2}).
\]
It is well known that the mapping \( f \to \tilde{f} \) is of weak-type \((1,1)\) and of (weak-) type \((p,p)\) for all \( p > 1 \) (see, for example, [6, pp. 66–68]). Therefore, by Theorem 4.2,
\[
\|\tilde{f}\|_{L_1}^* \leq B \|f\|_{L_1}^*
\]
for every \( \lambda \in Q(0, -1) \) and \( q > 0 \). Since \( \tilde{f}(x) = -f(x) \) a.e. (see, for example, [6, p. 63]) we can combine (4.5)–(4.7) and use Lemma 2.5(a) to obtain the following theorem:

**Theorem 4.5.** Let \( q > 0 \), let \( \lambda \in Q(0, -1) \), and let \( h \) be any of the functions \( f \) or \( \tilde{f} \). Then the following conditions are equivalent:
\[
\begin{align*}
& h \in \Lambda(\lambda, q), \\
& h \in \Lambda_+(\lambda, q, 1), \\
& Mh \in \Lambda(\lambda, q), \\
& Mh \in \Lambda_+(\lambda, q, 1), \\
& Mh^* \in \Lambda(\lambda, q), \\
& Mh^* \in \Lambda_+(\lambda, q, 1).
\end{align*}
\]

Theorem 4.5 is false, for example, when \( \lambda(t) = t(\log t)^{\theta} \), where \( \theta > -1 \) (see [17, vol. I, Chapter 5]). This observation depends on the fact that the estimate (4.7) does not hold for this case. (However, the estimate \( \|\tilde{f}\|_{L_1}^* \leq B \|f\|_{L_1}^* \) holds for every \( \lambda \in Q(0, -1) \).

Finally we shall only state the following generalization of a recent result of Bennett (see [3, p. 218]).

**Theorem 4.6.** If \( \varphi(2^t) \in Q(-, -) \), if \( \varphi_{\Lambda}(2^t) = \int_0^t \varphi(2^x) \, dx \), and if \( f \in L[0, 1] \), then the following conditions are equivalent:
\[
\begin{align*}
& f \in \Lambda(t\varphi(t), 1), \\
& f \in \Lambda_+(t\varphi(t), 1, 1), \\
& f \in L\varphi_1(L), \\
& Mf \in \Lambda(t\varphi(t), 1), \\
& Mf \in L\varphi(L), \\
& Mf^* \in \Lambda(t\varphi(t), 1), \\
& Mf^* \in L\varphi(L).
\end{align*}
\]

**Proof.** The proof follows easily by applying (4.5) and (4.6), by making partial summations, and by using the fact that we may, without loss of generality, assume that \( u^{-1} \leq f^*(u) \leq u^{-1} \) and thus \( \varphi(t) \simeq \varphi(f^*) \) and \( \varphi_{\Lambda}(t) \simeq \varphi_{\Lambda}(f^*) \) (see [11]).

For example, if \( \varphi(t) = (\log^+ t)^{\theta} \), where \( \theta \geq -1 \), then \( \varphi_{\Lambda}(t) = \log^+ t^{-1} \theta + 1 \) when \( \theta > -1 \) and \( \varphi_{\Lambda}(t) = \log^+ \log^+ t^{-1} \) when \( \theta = -1 \). Compare with the work of Bennett [3, p. 218] and Stein [15, p. 305].
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