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INTERPOLATION OF BESOV SPACES IN THE NONDIAGONAL CASE

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Abstract. In the nondiagonal case, interpolation spaces for a collection of Besov spaces are described. The results are consequences of the fact that, whenever the convex hull of points $(\bar{s}_0, \eta_0), \dots, (\bar{s}_n, \eta_n) \in \mathbb{R}^{m+1}$ includes a ball of $\mathbb{R}^{m+1}$, we have

$$(l_{q_0}^{\bar{s}_0}((X_0, X_1)_{\eta_0,p_0}), \dots, l_{q_n}^{\bar{s}_n}((X_0, X_1)_{\eta_n,p_n}))_{\bar{\theta},q} = l_q^{\bar{s}_{\bar{\theta}}}((X_0, X_1)_{\eta_{\bar{\theta}},q}),$$

where $\bar{\theta} = (\theta_0, \dots, \theta_n)$ and $(s_{\bar{\theta}}, \eta_{\bar{\theta}}) = \theta_0(\bar{s}_0, \eta_0) + \dots + \theta_n(\bar{s}_n, \eta_n)$.

§0. Introduction

An important problem in interpolation theory is the study of interpolation properties of Besov and Sobolev spaces, either isotropic or anisotropic. It is well known that, in the diagonal case ($\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$), when applied to a *couple* of isotropic Besov spaces, the real method yields again a Besov space:

$$(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_{\theta,q} = B_{p,q}^{s_\theta} \quad (s_0 \neq s_1, p = q, s_\theta = (1-\theta)s_0 + \theta s_1); \tag{0.1}$$

see [3]. However, in the nondiagonal case, (0.1) is true only under substantial restrictions on the parameters and, generally speaking, the space $(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_{\theta,q}$ falls out of the Besov scale. At the same time, switching from couples of Besov spaces to triples, we again arrive at a space within the same scale; see [2].

Difficulties presented by interpolation of Besov spaces are related to those arising in interpolation of vector-valued spaces $l_q^s(L_p)$ (see §1 for the definition).

In the diagonal case ($\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$), we have a remarkable formula (see [3]):

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_{\theta,q} = l_q^{s_\theta}((A_0, A_1)_{\theta,q}), \tag{0.2}$$

where $s_\theta = (1-\theta)s_0 + \theta s_1$. But in the nondiagonal case ($\frac{1}{q} \neq \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$) this formula cannot be valid for an arbitrary couple (A_0, A_1).

This can be shown by a very simple example. Consider the couple $(l_p^{s_0}(A_0), l_p^{s_1}(A_1))$, where $s_0 \neq s_1$, and the inner spaces A_0 and A_1 are $l_p^{r_0}$ and $l_p^{r_1}$ ($r_0 \neq r_1$), respectively. Then, by the Gilbert formula [5], the space

$$X = (l_p^{s_0}(l_p^{r_0}), l_p^{s_1}(l_p^{r_1}))_{\theta,q}, \quad q \neq p,$$

corresponds to the norm

$$\|\{a_{n_1,n_2}\}_{(n_1,n_2)\in\mathbb{Z}^2}\|_X = \left(\sum_k \|2^{s_\theta n_1} 2^{r_\theta n_2} a_{n_1,n_2} \chi_{\Omega_k}\|_{l_p}^q\right)^{1/q},$$

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where $\Omega_k = \{(n_1, n_2) : 2^{k-1} < 2^{n_1(s_0-s_1)}2^{n_2(r_0-r_1)} \leq 2^k\}$, $s_\theta = (1-\theta)s_0 + \theta s_1$, and $r_\theta = (1-\theta)r_0 + \theta r_1$. Clearly, this is not the norm of the space

$$l_q^{s_\theta}(l_q^{r_\theta}) = l_q^{s_\theta}((A_0, A_1)_{\theta,q})$$

if $q \neq p$.

Thus, for arbitrary A_0 and A_1, formula (0.2) fails in the nondiagonal case. However, as was shown in [1], it can be extended to the nondiagonal case if we consider a collection of more than two vector-valued spaces in place of a couple. Specifically, if $A_n = A_{n-1}$, $s_n \neq s_{n-1}$, then

$$(0.3) \qquad (l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1), \dots, l_{q_n}^{s_n}(A_n))_{\bar{\theta},q} = l_q^{s_{\bar{\theta}}}((A_0, \dots, A_n)_{\bar{\theta},q}),$$

where $\bar{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$, $s_{\bar{\theta}} = \sum_{i=0}^n \theta_i s_i$. It is worthy of notice that there are no restrictions on $A_0, A_1, \dots, A_{n-2}$ and $s_0, s_1, \dots, s_{n-2}$ in (0.3).

Let $\bar{s}_k \in \mathbb{R}^m$, $k = 0, 1, \dots, n$, let $\vec{X} = (X_0, X_1)$ be a couple of Banach or quasi-Banach spaces, and let $A_k = (X_0, X_1)_{\lambda_k, p_k}$, $k = 0, 1, \dots, n$. We shall show that formula (0.3) extends to the case of the collection

$$(l_{q_0}^{\bar{s}_0}(A_0), l_{q_1}^{\bar{s}_1}(A_1), \dots, l_{q_n}^{\bar{s}_n}(A_n));$$

as a consequence, we shall be able to interpolate collections of Besov and Sobolev spaces in the nondiagonal case.

§1. Basic definitions

As usual, we denote by $\mathbb{R}_+^n$ the set of n-vectors with positive coordinates.

Let $A_0, A_1, \dots, A_n$ be Banach or quasi-Banach spaces. We say that $A_0, A_1, \dots, A_n$ constitute an $(n+1)$-tuple if they are embedded linearly and continuously in a certain Hausdorff linear topological space X.

Much as for the case of couples, we can define the K-functional of an element $a \in A_0 + A_1 + \cdots + A_n$:

$$(1.1) \qquad K(\bar{t}, a, \vec{A}) = \inf(\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + \cdots + t_n\|a_n\|_{A_n}),$$

where $\bar{t} = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, and the infimum is taken over all representations $a = a_0 + a_1 + \cdots + a_n$ with $a_i \in A_i$, $i = 0, 1, \dots, n$.

Clearly, the K-functional is a concave function on $\mathbb{R}_+^n$; for fixed t, this is a norm on $A_0 + \cdots + A_n$.

Next, for every parameter $\bar{\theta} = (\theta_0, \dots, \theta_n)$, $\theta_i > 0$, $i = 0, 1, \dots, n$, with $\sum_{i=0}^n \theta_i = 1$, we define the interpolation space $\vec{A}_{\bar{\theta},q}$, $q > 0$, as the collection of all elements $a \in A_0 + A_1 + \cdots + A_n$ for which the quasinorm

$$(1.2) \qquad \|a\|_{\bar{\theta},q} = \left(\int_{\mathbb{R}_+^n} (t_1^{-\theta_1} \cdots t_n^{-\theta_n} K(\bar{t}, a; \bar{A}))^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/q}$$

is finite (for $q \geq 1$ this is a norm). As usual, the integral is replaced with the supremum if $q = \infty$.

Let A be a Banach or quasi-Banach space. Suppose $\bar{s} = (s_1, \dots, s_m) \in \mathbb{R}^m$, $q > 0$, and $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$. We denote by $l_q^{\bar{s}}(A)$ the space of A-valued functions determined by the following norm (quasinorm if $0 < q < 1$):

$$(1.3) \qquad \|a\|_{l_q^{\bar{s}}(A)} = \|\{a_k\}_{k \in \mathbb{Z}^m}\|_{l_q^{\bar{s}}(A)} = \left(\sum_{\bar{k} \in \mathbb{Z}^m} (2^{\bar{k} \cdot \bar{s}} \|a_{\bar{k}}\|_A)^q \right)^{1/q};$$

here $\bar{k} \cdot \bar{s} = \sum_{i=1}^m k_i s_i$, and the sum in (1.3) is replaced with the supremum if $q = \infty$.

§2. Interpolation of spaces of vector-valued functions

Suppose $\bar{s}_k = (s_1^k, \dots, s_m^k) \in \mathbb{R}^m$, $\eta_k \in (0,1)$, $k = 0, 1, \dots, n$. We denote by $(\bar{s}_k, \eta_k)$ the point of $\mathbb{R}^{m+1}$ with the coordinates $(s_1^k, \dots, s_m^k, \eta_k)$.

Let $\vec{X} = (X_0, X_1)$ be a couple of Banach or quasi-Banach spaces.

Theorem 1. *If the convex hull of the $(n+1)$-tuple $(\bar{s}_0, \eta_0)$, $(\bar{s}_1, \eta_1), \dots, (\bar{s}_n, \eta_n)$ contains some ball of $\mathbb{R}^{m+1}$, then*

$$(l_{q_0}^{\bar{s}_0}(\vec{X}_{\eta_0,p_0}), \dots, l_{q_n}^{\bar{s}_n}(\vec{X}_{\eta_n,p_n}))_{\bar{\theta},q} = l_q^{\bar{s}_{\bar{\theta}}}(\vec{X}_{\eta_{\bar{\theta}},q}), \tag{2.1}$$

where $(\bar{s}_{\bar{\theta}}, \eta_{\bar{\theta}}) = \theta_0(\bar{s}_0, \eta_0) + \dots + \theta_n(\bar{s}_n, \eta_n)$.

For the proof, we need the following lemma.

Lemma 1. *Let $\bar{e}_i$ $(i = 1, \dots, m)$ be the standard basis of $\mathbb{R}^m$. Put $\bar{e}_0 = \bar{e}_{m+1} = (0, \dots, 0) \in \mathbb{R}^m$. Then for every $a \neq 0$ we have*

$$(l_p^{a\bar{e}_0}(X_0), l_p^{a\bar{e}_1}(X_1), \dots, l_p^{a\bar{e}_{m+1}}(X_1))_{\bar{\theta},q} = l_q^{a(\theta_1,\dots,\theta_m)}(\vec{X}_{\theta_1+\dots+\theta_{m+1},q}), \tag{2.2}$$

where $\bar{\theta} = (\theta_0, \theta_1, \dots, \theta_{m+1})$.

Proof. We proceed by induction on m. If $m = 1$, then $\bar{e}_0 = \bar{e}_2 = 0$, $a\bar{e}_1 = a \neq 0$. Therefore, formula (2.2) takes the form

$$(l_p(X_0), l_p^a(X_1), l_p(X_1))_{\bar{\theta},q} = l_q^{a\theta_1}(\vec{X}_{\theta_1+\theta_2,q}), \tag{2.3}$$

where $\bar{\theta} = (\theta_0, \theta_1, \theta_2)$. This is true indeed by (0.3) with $n = 2$.

Suppose (2.2) is true for $m \le N$. We show it is true for $m = N+1$; namely, we prove the formula

$$(l_p^{a\bar{e}_0}(X_0), l_p^{a\bar{e}_1}(X_1), \dots, l_p^{a\bar{e}_{N+2}}(X_1))_{\bar{\theta},q} = l_q^{a(\theta_1,\dots,\theta_{N+1})}(\vec{X}_{\theta_1+\dots+\theta_{N+2},q}), \tag{2.4}$$

where $\bar{\theta} = (\theta_0, \dots, \theta_{N+1}, \theta_{N+2})$.

We denote by $\bar{e}_i^N$, $i = 0, 1, \dots, N+2$, the vectors in $\mathbb{R}^N$ with coordinates equal to the first N coordinates of the vectors $\bar{e}_i$, and consider the spaces

$$Y_0 = l_p^{a\bar{e}_0^N}(X_0), \quad Y_i = l_p^{a\bar{e}_i^N}(X_1), \quad i = 1, \dots, N+2. \tag{2.5}$$

It is easily seen that the space on the left in (2.4) can be rewritten in the form

$$\begin{aligned}(l_p^{a\bar{e}_0}(X_0), l_p^{a\bar{e}_1}(X_1), \dots, l_p^{a\bar{e}_{N+2}}(X_1))_{\bar{\theta},q} \\ = (l_p(Y_0), l_p(Y_1), \dots, l_p^a(Y_{N+1}), l_p(Y_{N+2}))_{\bar{\theta},q}.\end{aligned} \tag{2.6}$$

Since $\bar{e}_{N+1}^N = \bar{e}_{N+2}^N = \bar{0} \in \mathbb{R}^N$ and $a \neq 0$, we see that $Y_{N+1} = Y_{N+2}$, and (0.3) can be applied to the right-hand side of (2.6). This yields

$$\begin{aligned}(l_p^{a\bar{e}_0}(X_0), l_p^{a\bar{e}_1}(X_1), \dots, l_p^{a\bar{e}_{N+2}}(X_1))_{\bar{\theta},q} \\ = l_q^{a\theta_{N+1}}((Y_0, \dots, Y_{N+1}, Y_{N+2})_{\bar{\theta},q}) \\ = l_q^{a\theta_{N+1}}((Y_0, \dots, Y_{N+1})_{\theta_0,\dots,\theta_{N+1}+\theta_{N+2},q}).\end{aligned} \tag{2.7}$$

It remains to observe that the collection $(Y_0, \dots, Y_{N+1})$ satisfies the inductive hypothesis. Therefore,

$$\begin{aligned}(l_p^{a\bar{e}_0}(X_0), l_p^{a\bar{e}_1}(X_1), \dots, l_p^{a\bar{e}_{N+2}}(X_1))_{\bar{\theta},q} \\ = l_q^{a\theta_{N+1}}(l_q^{a(\theta_1,\dots,\theta_N)}((X_0, X_1)_{\theta_1+\dots+\theta_{N+2},q})) \\ = l_q^{a(\theta_1,\dots,\theta_{N+1})}(\vec{X}_{\theta_1+\dots+\theta_{N+2},q}).\end{aligned} \tag{2.8}$$

The lemma is proved. □

Proof of Theorem 1. The right-hand side of (2.1) does not depend on the p_i and q_i ($i = 0, 1, \dots, n$). Therefore, the continuous embeddings

$$l_{r_0}^{\bar{s}_i}(\vec{X}_{\eta_i, r_0}) \hookrightarrow l_{q_i}^{\bar{s}_i}(\vec{X}_{\eta_i, p_i}) \hookrightarrow l_{r_1}^{\bar{s}_i}(\vec{X}_{\eta_i, r_1}),$$

where

$$r_0 = \min_{0 \le i,j \le n} \{p_i, q_j\}, \quad r_1 = \max_{0 \le i,j \le n} \{p_i, q_j\},$$

show that it suffices to prove Theorem 1 only in the case where all p_i and q_i, $i = 0, \dots, n$, are equal to one and the same number r. Namely, it suffices to prove that

$$(l_r^{\bar{s}_0}(\vec{X}_{\eta_0, r}), \dots, l_r^{\bar{s}_n}(\vec{X}_{\eta_n, r}))_{\bar{\theta}, q} = l_q^{\bar{s}_\theta}(\vec{X}_{\eta_{\bar{\theta}}, q}). \tag{2.9}$$

Next, without loss of generality we may assume that $\bar{s}_i \in \mathbb{R}_+^m$, $i = 0, 1, \dots, n$. Indeed, since

$$\begin{aligned} &\|\{a_{\bar{k}}\}_{\bar{k} \in \mathbb{Z}^m}\|_{l_r^{\bar{s}_i + \bar{s}}(\vec{X}_{\eta_i, r})} \\ &= \left(\sum_{k \in \mathbb{Z}^m} (2^{(\bar{s}_i + \bar{s}) \cdot \bar{k}} \|a_{\bar{k}}\|_{\bar{X}_{\eta_i, r}})^r \right)^{1/r} = \|\{2^{\bar{s} \cdot \bar{k}} a_{\bar{k}}\}_{\bar{k} \in \mathbb{Z}^m}\|_{l_r^{\bar{s}_i}(\vec{X}_{\eta_i, r})}, \end{aligned} \tag{2.10}$$

the mapping

$$A_{\bar{s}} : \{a_{\bar{k}}\}_{\bar{k} \in \mathbb{Z}^m} \mapsto \{2^{\bar{s} \cdot \bar{k}} a_{\bar{k}}\}_{\bar{k} \in \mathbb{Z}^m}$$

is an isometry between $l_r^{\bar{s}_i + \bar{s}}(\vec{X}_{\eta_i, r})$ and $l_r^{\bar{s}_i}(\vec{X}_{\eta_i, r})$ and also between $l_q^{\bar{s}_{\bar{\theta}} + \bar{s}}(\vec{X}_{\eta_{\bar{\theta}}, q})$ and $l_q^{\bar{s}_{\bar{\theta}}}(\vec{X}_{\eta_{\bar{\theta}}, q})$. Moreover,

$$\theta_0(\bar{s}_0 + \bar{s}) + \theta_1(\bar{s}_1 + \bar{s}) + \dots + \theta_n(\bar{s}_n + \bar{s}) = \bar{s}_{\bar{\theta}} + \bar{s}.$$

Consequently, it suffices to prove (2.9) for $\bar{s}_i \in \mathbb{R}_+^m$, $i = 0, \dots, n$.

Now, we show that if $\bar{s}_i \in \mathbb{R}_+^m$, then all spaces $l_r^{\bar{s}_i}(\vec{X}_{\eta_i, r})$ can be obtained by the real method of interpolation from the "basic" collection

$$\vec{Z} = (l_r^{a\bar{e}_0}(X_0), l_r^{a\bar{e}_1}(X_1), \dots, l_r^{a\bar{e}_{m+1}}(X_1)), \tag{2.11}$$

treated in Lemma 1.

For this, we choose a sufficiently large number a $\left(a > \max_{i=0,n} \frac{\sum_{j=1}^m s_j^i}{\eta_i}\right)$ and define vectors $\bar{\lambda}_i = (\lambda_0^i, \dots, \lambda_{m+1}^i)$, $i = 0, 1, \dots, n$, in the following way:

$$\lambda_j^i = \frac{1}{a} s_j^i, \quad j = 1, \dots, m, \quad \lambda_{m+1}^i = \eta_i - \sum_{j=1}^m \frac{s_j^i}{a}. \tag{2.12}$$

Also, we put $\lambda_0^i = 1 - \sum_{j=1}^{m+1} \lambda_j^i$. Clearly, if a is sufficiently large, then all λ_j^i ($j = 0, \dots, m+1$) are strictly positive and $\sum_{j=0}^{m+1} \lambda_j^i = 1$. Therefore, from Lemma 1 and formula (2.12) we deduce that

$$(l_r^{a\bar{e}_0}(X_0), l_r^{a\bar{e}_1}(X_1), \dots, l_r^{a\bar{e}_{m+1}}(X_1))_{\bar{\lambda}_i, r} = l_r^{a(\lambda_1^i, \dots, \lambda_m^i)}(\vec{X}_{\lambda_1^i + \dots + \lambda_{m+1}^i, r}) = l_r^{\bar{s}_i}(\vec{X}_{\eta_i, r}) \tag{2.13}$$

for all $i = 0, 1, \dots, n$.

Thus, the left-hand side of (2.9) has the form

$$(l_r^{\bar{s}_0}(\vec{X}_{\eta_0, r}), l_r^{\bar{s}_1}(\vec{X}_{\eta_1, r}), \dots, l_r^{\bar{s}_n}(\vec{X}_{\eta_n, r}))_{\bar{\theta}, q} = (\bar{Z}_{\bar{\lambda}_0, r}, \bar{Z}_{\bar{\lambda}_1, r}, \dots, \bar{Z}_{\bar{\lambda}_n, r})_{\bar{\theta}, q}, \tag{2.14}$$

where $\vec{Z}$ is defined by (2.11). Next, from [6, Propositions 6.4, 10.1, and Theorem 1], the reiteration theorem applies to the collection $\bar{Z}$:

$$(\bar{Z}_{\bar{\lambda}_0, r}, \bar{Z}_{\bar{\lambda}_1, r}, \dots, \bar{Z}_{\bar{\lambda}_n, r})_{\bar{\theta}, q} = \bar{Z}_{\theta_0 \bar{\lambda}_0 + \dots + \theta_n \bar{\lambda}_n, q}, \tag{2.15}$$

provided that the linear hull of the vectors $\bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_n$ coincides with $\mathbb{R}^{m+2}$. Since the convex hull of the $(\bar{s}_i, \eta_i) \in \mathbb{R}^{m+1}$, $i = 0, 1, \dots, n$, includes some ball of $\mathbb{R}^{m+1}$, and the points $p_i = (\lambda_1^i, \lambda_2^i, \dots, \lambda_{m+1}^i)$, $i = 0, 1, \dots, n$, are obtained from the $(\bar{s}_i, \eta_i)$ with the help of a nondegenerate linear mapping (see (2.12)), we see that the convex hull of the p_i also includes some ball of $\mathbb{R}^{m+1}$. Consequently, the convex hull of the points $(\lambda_0^i, \lambda_1^i, \dots, \lambda_{m+1}^i) \in \mathbb{R}^{m+2}$, $i = 0, 1, \dots, n$, which lie on the hyperplane $t_0 + t_1 + \dots + t_{m+1} = 1$, also includes a ball of dimension $m + 1$. This implies immediately that the linear hull of the vectors $\bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_n$ is the entire $\mathbb{R}^{m+2}$.

Now, from formulas (2.14) and (2.15), Lemma 1, and identity (2.12) we obtain

$$
\begin{aligned}
&(l_r^{\bar{s}_0}(\vec{X}_{\eta_0,r}), l_r^{\bar{s}_1}(\vec{X}_{\eta_1,r}), \dots, l_r^{\bar{s}_n}(\vec{X}_{\eta_n,r}))_{\bar{\theta},q} \\
&\qquad = (l_r^{a\bar{e}_0}(X_0), l_r^{a\bar{e}_1}(X_1), \dots, l_r^{a\bar{e}_{m+1}}(X_1))_{\bar{\lambda}_{\bar{\theta}},q} = l_q^{\bar{s}_\theta}(\vec{X}_{\bar{\eta}_{\bar{\theta}}}, q),
\end{aligned}
$$

which completes the proof of the theorem. □

Theorem 1 treats the collection of spaces

$$l_{q_i}^{\bar{s}_i}(\vec{X}_{\eta_i,p_i}) = l_{q_i}^{\bar{s}_i}((X_0, X_1)_{\eta_i,p_i});$$

here, as usual, $0 < \eta_i < 1$, $i = 0, \dots, n$. However, the case where the "extreme" space X_1 is involved in place of some spaces $(X_0, X_1)_{\eta_i,p_i}$ is of interest. Then we put $X_1 = (X_0, X_1)_{1,p_i}$, i.e., the parameter η_i may belong to the semi-interval $(0, 1]$.

Remark. Theorem 1 remains true in the case where $\eta_i \in (0, 1]$.

This follows immediately from the fact that Lemma 1 is true also for $\theta_0 = 0$ and the reiteration theorem (see [6]) is true also in the case where some coordinates of $\bar{\theta}$ are equal to 0.

§3. Applications

Consider the collection of vector-valued spaces

$$(l_{q_0}^{\bar{s}_0}(L_{p_0}), \dots, l_{q_n}^{\bar{s}_n}(L_{p_n})),$$

where $\bar{s}_i \in \mathbb{R}^m$, $m \geq 1$, $p_i, q_i > 0$, $i = 0, 1, \dots, n$.

Since for $r < \min_{0 \leq i \leq n} p_i$ we have

$$L_{p_i} = (L_r, L_\infty)_{\eta_i,p_i}, \quad \eta_i = 1 - \frac{r}{p_i}, \quad i = 0, \dots, n,$$

Theorem 1 readily implies the following statement.

Theorem 2. *If vectors $\bar{s}_0, \dots, \bar{s}_n \in \mathbb{R}^m$ have the property that the convex hull of the vectors $(\bar{s}_0, \frac{1}{p_0}), \dots, (\bar{s}_n, \frac{1}{p_n})$ includes a ball of $\mathbb{R}^{m+1}$, then*

$$(l_{q_0}^{\bar{s}_0}(L_{p_0}), \dots, l_{q_n}^{\bar{s}_n}(L_{p_n}))_{\bar{\theta},q} = l_q^{\bar{s}_\theta}(L_{p_\theta,q}),$$

where $L_{p_\theta,q}$ is the Lorentz space and $(\bar{s}_{\bar{\theta}}, \frac{1}{p_\theta}) = \theta_0(\bar{s}_0, \frac{1}{p_0}) + \dots + \theta_n(\bar{s}_n, \frac{1}{p_n})$.

Theorem 2 and the fact that Besov spaces are retracts of $l_q^{\bar{s}}(L_p)$ imply interpolation formulas for Besov and Sobolev spaces.

We start with the isotropic case. It is known (see [3] for $q, p \geq 1$ and [4] for $q, p < 1$) that $B_q^s(L_p)$ (i.e., the Besov spaces of smoothness s constructed on the basis of L_p) are obtained from $l_q^s(L_p)$ with the help of some retraction independent of p, q, and s. Therefore, Theorem 2 implies the following statement.

Corollary 1. *If the convex hull of the points $(s_0, \frac{1}{p_0}), \dots, (s_n, \frac{1}{p_n})$ includes a ball of $\mathbb{R}^2$ (i.e., the points do not lie on one straight line), then*

$$(B_{q_0}^{s_0}(L_{p_0}), \dots, B_{q_n}^{s_n}(L_{p_n}))_{\bar{\theta},q} = B_q^{s_{\bar{\theta}}}(L_{p_{\bar{\theta}},q}),$$

where $(s_{\bar{\theta}}, \frac{1}{p_{\bar{\theta}}}) = \theta_0(s_0, \frac{1}{p_0}) + \dots + \theta_n(s_n, \frac{1}{p_n})$.

Since for Sobolev spaces we have

$$B_1^s(L_p) \subset W_p^s \subset B_\infty^s(L_p),$$

Corollary 1 implies yet another statement.

Corollary 2. *Under the assumptions of Corollary* 1*, we have*

$$(W_{p_0}^{s_0}, \dots, W_{p_n}^{s_n})_{\bar\theta,q} = B_q^{s_{\bar\theta}}(L_{p_{\bar\theta},q}),$$

where $(s_{\bar\theta}, \frac{1}{p_\theta}) = \theta_0(s_0, \frac{1}{p_0}) + \dots + \theta_n(s_n, \frac{1}{p_n})$.

Similar results are valid also for anisotropic Besov and Sobolev spaces with dominating smoothness.

It is well known (see, e.g., [6]) that a Besov space with dominating smoothness $\bar s \in \mathbb{R}^m$ constructed on the basis of L_p, $p \geq 1$ (it will be denoted by $\tilde B_q^{\bar s}(L_p)$) is a retract of $l_q^{\bar s}(L_p)$. We denote by $\tilde B_q^{\bar s}(L_{p,r})$ the space obtained from $l_q^{\bar s}(L_{p,r})$ under the same retraction. Then Theorem 2 implies immediately the following statement.

Corollary 3. *If the convex hull of the points* $(\bar s_0, \frac{1}{p_0}), \dots, (\bar s_n, \frac{1}{p_n})$ *contains a ball of* $\mathbb{R}^{m+1}$, *then*

$$(\tilde B^{\bar s_0}(L_{p_0}), \dots, \tilde B^{\bar s_n}(L_{p_n}))_{\bar\theta,q} = \tilde B_q^{\bar s_{\bar\theta}}(L_{p_{\bar\theta},q}),$$

where $\bar s_{\bar\theta}$ *and* $p_{\bar\theta}$ *are defined by*

$$(\bar s_{\bar\theta}) = \theta_0\Big(\bar s_0, \frac{1}{p_0}\Big) + \dots + \theta_n\Big(\bar s_n, \frac{1}{p_n}\Big).$$

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