Symmetric cracking of a wedge by transverse displacements

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SUMMARY

Two path-independent integrals are used to determine the mode I stress intensity factors of two cases of a symmetrically cracked infinite wedge, subjected to transverse displacements of constant magnitude along the outer faces, in plane elastostatics. Results for two mode II cases are also given. The results are in closed forms. The successful use of the integrals depend firstly on the expansional invariance of the elastic fields, secondly on suitable forms of the boundary conditions, and thirdly on an inversely proportional far-field decay of the stresses.

ZUSAMMENFASSUNG

Zwei wegunabhängige Integrale werden hier benützt um die Spannungsintensitätsfaktoren von zwei symmetrisch gespaltenen Keile unter ebenem Zug in der Elastostatik zu berechnen. Die Aussenkanten der Keilen werden von Querverschiebungen konstanter Grösse deformiert. Resultate für zwei Fälle von ebenem Schub werden auch gegeben, alle in abgeschlossener Form. Drei Faktoren machen die Integrale anwendbar. Erstens ist das elastische Feld expansionsinvariabel, zweitens sind die Randbedingungen daran geeignet, und drittens nehmen die Spannungen des Fernfeldes umgekehrt proportional von dem Abstand ab.

1. Introduction

A very powerful method of linear elastic fracture mechanics (LEFM) analysis of self-similar cracks is the use of certain path-independent integrals given by Knowles and Sternberg [1], Eshelby [2], and Carlsson [3]. Their path-independence implies a conservation law. An evaluation of them along a crack tip loop for example essentially yields the local energy release rate, as was shown by Budiansky and Rice [4] for one of the two suitable integrals.

I used this approach in [5] to determine closed form stress intensity factors both for the symmetrically cracked wedge with an arbitrary force loading at the ends and for the star crack with equally spaced crack arms of the same length which is subjected to expansion forces. Later [6] it became obvious that Freund [7] had worked along the same lines. He has among other things shown that closed form stress intensity factors also result for symmetrically cracked planes and half-planes with mixed boundary conditions.

The purpose of this research note is to show that constant transverse displacements along the outer faces of a symmetrically cracked wedge of arbitrary angle also lead to closed form stress intensity factors. Path-independent integrals are used in the process and the success of the method hinges on a far field decay of the stresses, which is inversely proportional to the distance from the origin, and a suitable form of the boundary conditions. The asymptotic expressions for the stresses are derived from an analysis of uncracked wedges which are subjected to nearly identical boundary conditions on the outer faces. That analysis is mainly performed in the complex parameter plane of the Mellin transform.

2. Problem statement and Mellin transforms

Consider an infinite plane wedge of unit thickness in plane strain (PD), made of a homogeneous and isotropic linear solid which is described by its Young's modulus E and its Poisson's ratio ν . In polar coordinates (r, θ) the faces of the wedge are given by $\theta = \pm\alpha$ and $0 \leq r < \infty$ in Figure 1. Along $\theta = 0$ there is a crack of length a with stress-free faces. Along the outer faces the wedge is loaded by displacements of constant magnitude δ which are transverse to the crack plane. This results in a mode I loading. The supplementary boundary conditions along the faces are either vanishing longitudinal displacements (case *a* in Figure 1) or vanishing longitudinal stresses (case *b* in Figure 1). Thus the boundary conditions are

$$\left. \begin{aligned} \sigma_\theta = \tau_{r\theta} = 0 \quad \text{on} \quad 0 \leq r < a \quad \text{when} \quad \theta = 0, \\ v_y = \pm\delta \quad \text{and} \quad u_x = 0 \quad \text{on} \quad 0 \leq r < \infty \quad \text{when} \quad \theta = \pm\alpha \quad (\text{a}), \text{ or} \\ v_y = \pm\delta \quad \text{and} \quad T_x = 0 \quad \text{on} \quad 0 \leq r < \infty \quad \text{when} \quad \theta = \pm\alpha \quad (\text{b}). \end{aligned} \right\} \quad (2.1a+b)$$

Here σ_θ and $\tau_{r\theta}$ denote stress components in the polar coordinate system, u_x and v_y

Figure 1. Symmetrically cracked infinite wedge opened by transverse displacements v_y of constant magnitude δ on the outer faces. Supplementary boundary conditions are either vanishing longitudinal displacements u_x in case *a* or vanishing longitudinal stresses T_x in case *b*.

denote the components of the displacement vector $\mathbf{u}$ in the inscribed rectangular system, and T_x denotes the x -component of the boundary stress vector $\mathbf{T}$. The boundary conditions in equations (2.1a+b) can easily be expressed solely in the polar coordinate system.

The analysis of this problem follows closely that of my paper [8]. A normalized radial coordinate $\rho = r/a$ is introduced as is the Airy's stress function $\Phi(\rho, \theta)$ which satisfies the biharmonic differential equation in polar coordinates. Then Mellin transforms, defined according to Davies [9] for example, are introduced. One finds that the transformed stress function $\tilde{\Phi}(s, \theta)$ has the general solution

$$\tilde{\Phi}(s, \theta) = A \sin s\theta + B \cos s\theta + C \sin (s+2)\theta + D \cos (s+2)\theta. \quad (2.2)$$

Here A , B , C , and D are arbitrary functions of the parameter s . An application of the transformed stress and displacement equations to transformed boundary conditions would determine A , B , C , and D . For a cracked body that would lead to the solution of an equation of Wiener-Hopf type as in [8].

Once the particular solution $\tilde{\Phi}$ is obtained, an inversion of the equations mentioned above would yield the stresses and the displacements in the cracked wedge. In this LEFM context a complete inversion will be unnecessary since the stress intensity factors are crack tip quantities and as such obtainable from the initial-value theorems of the finite Mellin transform. Similarly the asymptotic expressions of the near- and far-fields may be obtained from the corresponding final-value theorems. When applying these theorems to obtain far-field stresses which decay $\sim 1/r$, one has to make sure that the near-field stresses behave like $r\sigma \rightarrow 0$ at the origin. Here and in the following σ denotes any one of the three polar stress components. In most relevant cases England [10] establishes this.

3. Analysis of uncracked wedges

3.1. Displacement-displacement boundary conditions

According to de Saint-Venant's principle, the far-field stresses in the cracked wedge can not differ significantly from those in an uncracked wedge subjected to the same boundary conditions along the outer faces. Hence to obtain the far-field, consider a wedge defined as in Section 2 except that it is uncracked. As general displacement-displacement boundary conditions along the faces we choose the appropriate line of equation (2.1a) with a linearly increasing part on $0 \leq r \leq a$ to obtain a finite strip of convergence of the corresponding transform integral. As above, the latter modification won't change the far-field stresses.

Because symmetry requires only B and D to be determined it is sufficient to consider the conditions on $\theta = \alpha$. With the normalized radial coordinate $\rho = r/a$, the boundary conditions and the transformed displacement equations yield the expressions for B and D . With a generalization to include plane stress (PS) there results

that

$$\left. \begin{aligned} B &= -\frac{2Ga\delta}{s(s+1)(s+2)} \cdot \frac{\kappa \cos(s+1)\alpha - (s+1) \cos(s+3)\alpha}{\kappa \sin 2(s+1)\alpha - (s+1) \sin 2\alpha} \\ D &= \frac{2Ga\delta}{(s+1)(s+2)} \cdot \frac{\cos(s+1)\alpha}{\kappa \sin 2(s+1)\alpha - (s+1) \sin 2\alpha} \end{aligned} \right\} \quad (3.1)$$

Here $\kappa = 3 - 4\nu$ in *PD*, $\kappa = (3 - \nu)/(1 + \nu)$ in *PS*, and $G = E/2(1 + \nu)$. The characteristic equation which determines the power of the leading term in the near-field is essentially $\kappa \sin 2(s+1)\alpha - (s+1) \sin 2\alpha = 0$. This equation is nearly identical to equation (4.7) of case 2 in England's paper [10]. His conclusions imply that the dominant power will always be real and that the leading term will be at most $O(\rho^{-1/2})$ when $0 < \alpha \leq \pi$.

Since both B and $D \sim 1/(s+1)^2$ when $s \rightarrow -1$ the far-field stress decay of $1/r$ is ensured. There result stresses when $r \rightarrow \infty$ according to

$$\{\sigma_\theta, \sigma_r, \tau_{r\theta}\} = \frac{2G\delta}{(2\kappa\alpha - \sin 2\alpha)r} \{(\kappa+1) \cos \theta, (3-\kappa) \cos \theta, (\kappa+1) \sin \theta\}. \quad (3.2)$$

If one considers the case when the displacements are parallel to $\theta = 0$ then one ends up with far-field stresses which decay faster than $1/r$.

3.2. Displacement-stress boundary conditions

Again consider an uncracked wedge, this time with displacement-stress boundary conditions along the faces according to the appropriate line of equation (2.1b) and with a linearly increasing part on $0 \leq r \leq a$. Transforming into polar coordinates and taking similar steps as in Section 3.1 there now results

$$\left. \begin{aligned} B &= -\frac{2Ga\delta}{(\kappa+1)s(s+1)(s+2)} \cdot \frac{\sin(s+1)\alpha + (s+1) \sin(s+3)\alpha}{\sin^2(s+1)\alpha} \\ D &= \frac{2Ga\delta}{(\kappa+1)(s+1)(s+2)} \cdot \frac{1}{\sin(s+1)\alpha} \end{aligned} \right\} \quad (3.3)$$

Both of them are $\sim 1/(s+1)^2$ when $s \rightarrow -1$ for $0 < \alpha \leq \pi$.

The characteristic equation which determines the leading term of the near-field stresses is now $\sin(s+1)\alpha = 0$. The leading term is thus $O(\rho^{-1+\pi/\alpha})$ and not even singular since $\alpha \leq \pi$. Hence the condition $r\sigma \rightarrow 0$ when $r \rightarrow 0$ is once again met. Converting B and D to far-field stresses then yields as $r \rightarrow \infty$

$$\{\sigma_\theta, \sigma_r, \tau_{r\theta}\} = \frac{2G\delta(2\alpha + \sin 2\alpha)}{(\kappa+1)\alpha^2 r} \left\{ \cos \theta, \frac{2\alpha - \sin 2\alpha}{2\alpha + \sin 2\alpha} \cos \theta, \sin \theta \right\}. \quad (3.4)$$

If one considers other combinations of displacement-stress boundary conditions of identical form in terms of either v_θ and $\tau_{r\theta}$ or u_r and σ_θ , the only two which yield a far-field stress decay of $1/r$ are $\alpha = \pi$ in the first case and $\alpha = \pi/2$ in the second case. Both of these are included in the boundary conditions used above. All other wedge angles give a faster decay. Congruent boundary conditions in terms of u_x and T_y also yield a faster far-field stress decay than $1/r$.

4. Analysis of cracked wedges

4.1. Preliminaries on path-independent integrals

The basic formulas for the path-independent integrals follow from Carlsson's [3] and Eshelby's [2] papers. Here Carlsson's notation is used and hence $M \equiv J_x$. The basic formulas in a polar coordinate system using physical vector components are found in my paper [5]. Thus

$$\left. \begin{aligned} J_x &= \oint_C [W(\mathbf{n} \cdot \mathbf{r}) - r \mathbf{T} \cdot \mathbf{u}_r] ds \quad \text{and} \\ \bar{J}_x &= \oint_C [\bar{W}(\mathbf{n} \cdot \mathbf{r}) - \mathbf{T} \cdot \mathbf{u} - r \sigma_{\alpha\beta,r} u_\alpha n_\beta] ds, \end{aligned} \right\} \quad (4.1)$$

along a closed curve C . $W = \bar{W}$ is the energy density. $\mathbf{r}$ is the position vector at a point on C , $\mathbf{n} = (n_r, n_\theta)$ is the unit outward normal vector there and ds is the length of an incremental line element along C . The indices α and β takes the "values" r, θ and the summation convention is valid.

In evaluating cracked bodies it is convenient to let C consist of segments C_i which are either radial lines, centered circular arcs or crack tip loops as in Figure 2. See the comments by Ouchterlony [5] and Freund [7]. The conservation law inherent in the

Figure 2. Cracked wedge with integration contour suitable for J_x -evaluation of the stress intensity factor K_I .

integrals implies that both

$$J_x = \sum_i J_x^i = 0 \quad \text{and} \quad \bar{J}_x = \sum_i \bar{J}_x^i = 0 \quad (4.2)$$

since C encompasses homogeneous material without singularities. The contributions to J_x and $\bar{J}_x$ from radial segments and centered circular arcs assume simple forms. They are given by equations (3.2) and (3.3) in [5]. Especially when the dominating stress term is $\sim 1/r$, equations (3.5) and (3.6) in [5] permit closed form integrations along centered circular arcs in the near- and far-fields.

The contributions J_x^k and $\bar{J}_x^k$ from the crack tip loops are essentially the energy release rate multiplied by the crack tip distance, see equation (3.4) in [5]. Together with equations (2.2) and (2.6), equations (3.2) to (3.6) in [5] contain all the information necessary to apply the path-independent integrals to the wedge problems stated in Section 2.

4.2. Displacement-displacement boundary conditions

The cracked wedge with an integration contour is shown in Figure 2. The boundary conditions are given by equations (2.1a). First of all $J_x^i = 0$ along the stress-free crack faces. Likewise $J_x^i = 0$ along the outer faces since $|v_y| = \delta$ along $\theta = \pm\alpha$ implies $v_{\theta,r} = u_{r,r} = 0$ there. Referring to England [10] case 4 and equation (4.10), his conclusion is that the leading term of the near-field stresses at most is singular of order $-\frac{1}{2} + (i/2\pi) \ln \kappa$ since $\alpha \leq \pi$ in Figure 1. This implies that $r\sigma \rightarrow 0$ at the origin and consequently that $J_x^i = 0$ along the small arcs around it.

The only contributions to J_x which remain are thus J_x^k from the crack-tip loop and J_x^∞ from the centered circular arc in the far-field. The latter is preferably evaluated as $\bar{J}_x^\infty$ from equations (3.5) and (3.6) in [5] and converted to J_x^∞ with the aid of equations (2.2) and (2.6) in [5]. Denoting the end-points of C_∞ by A and B , equations (2.1a) and (3.2) yield the stress and displacement magnitudes there. This determines $\bar{J}_x^\infty$ and hence

$$J_x^\infty = \frac{2(\kappa+1)G\delta^2}{2\kappa\alpha - \sin 2\alpha}. \quad (4.3)$$

The conservation law for J_x in equations (4.2) reduces to $J_x^k + J_x^\infty = 0$ from which K_I is

Figure 3. Transformation of wedge configuration into infinite strip of constant height $2h$ through $\alpha a \approx h$ as wedge-angle $\alpha \rightarrow 0$.

determined to be

$$(4.2) \quad K_I(\alpha) = \frac{4G\delta}{\sqrt{\pi a}} \cdot \sqrt{\pi/(2\kappa\alpha - \sin 2\alpha)} \quad \text{when } 0 < \alpha \leq \pi. \quad (4.4)$$

Here the expression $2\kappa\alpha - \sin 2\alpha > 0$ when $\alpha > 0$ since $\kappa \geq 1$.

Two cases with extremely simple K_I -expressions are $\alpha = \pi$ and $\alpha = \pi/2$. The first one corresponds to the wedging open of a crack in front of a semi-infinite rough tool of constant thickness 2δ . The other one corresponds to the pulling apart of a symmetrically cracked half-plane with constant transverse displacements $\pm\delta$ along stiff boundaries. The limiting case $\alpha \rightarrow 0$ is also of interest, see Figure 3. Then the configuration gradually transforms into that of the symmetrically cracked semi-infinite strip subjected to constant transverse displacements. The height of the strip is $2h$ and thus $\alpha a \approx h$ at the crack tip is the link between the two. Equation (4.4) yields

$$K_I(\alpha \rightarrow 0) \rightarrow \frac{2G\delta}{\sqrt{h}} \cdot \sqrt{\frac{2}{\kappa-1}} = \begin{cases} \frac{E\delta}{\sqrt{h}} \cdot \frac{1}{\sqrt{1-\nu^2}} & \text{in PS and} \\ \frac{E\delta}{\sqrt{h}} \cdot \frac{1}{\sqrt{(1+\nu)^2(1-2\nu)}} & \text{in PD.} \end{cases} \quad (4.5)$$

This is in complete accordance with Rice's [11] results. The comments following equations (3.2) imply that a horizontal displacement component $u_x \neq 0$ doesn't contribute to $K_I(\alpha)$ of equation (4.4).

4.3. Displacement-stress boundary conditions

The boundary conditions in this case are given by equations (2.1b). Just like before $J_x^i = 0$ along the stress-free crack faces. Along the outer faces, J_x^i vanishes too since a component of traction vanishes and its complementary component of displacement is uniform, see Freund [7].

For the small arcs in the near-field, the leading stress term for the most severe wedge half-angle $\alpha = \pi$ is given by England [10] case 3 to be $O(r^{-1/2})$. Hence $r\sigma \rightarrow 0$ at the origin and it follows that $J_x^i = 0$ along the small arcs. It is believed that this also is true for $\alpha < \pi$. Consequently the only contributions to J_x which remain are again J_x^k and J_x^∞ . Proceeding in the same steps as before there results

$$J_x^\infty = \frac{2G\delta^2}{\kappa+1} \cdot \frac{2\alpha + \sin 2\alpha}{\alpha^2}. \quad (4.6)$$

The conservation law again reduces to $J_x^k + J_x^\infty = 0$ whereby

$$K_I(\alpha) = \frac{4G\delta}{\sqrt{\pi a}} \cdot \frac{1}{\kappa+1} \cdot \sqrt{\pi(2\alpha + \sin 2\alpha)/\alpha^2} \quad \text{when } 0 < \alpha \leq \pi. \quad (4.7)$$

The case $\alpha = \pi$, when a semi-infinite friction free tool of constant thickness 2δ

ht $2h$

wedges open a crack, has been considered by Freund [7]. Letting $2\delta \rightarrow B$, equation (4.7) in *PD* reduces to equation (14) in [7]. The case $\alpha = \pi/2$ which corresponds to the prying open of a symmetrically cracked half-plane by traction free transverse displacements $\pm\delta$ along the outer faces has also been considered by Freund. In *PD* with $2\delta \rightarrow B$ my result is identical to the mode I version of equation (16) in his paper [7].

As before the limiting case $\alpha \rightarrow 0$ is of interest, namely

$$K_I(\alpha \rightarrow 0) \rightarrow \frac{2G\delta}{\sqrt{h}} \cdot \frac{4}{\kappa+1} = \begin{cases} \frac{E\delta}{\sqrt{h}} & \text{in PS and} \\ \frac{E\delta}{\sqrt{h}} \cdot \frac{1}{1-\nu^2} & \text{in PD.} \end{cases} \quad (4.8)$$

Again this is in complete accordance with Rice's [11] results. The discussion following equations (3.4) shows that boundary conditions on the outer faces $\theta = \pm\alpha$, like those in equations (2.1b) but expressed in terms of $|v_\theta| = \delta$ and $\tau_{r\theta} = 0$, or $u_r = \delta$ and $\sigma_\theta = 0$, or $u_x = \delta$ and $T_y = 0$, usually yield $K_I(\alpha) = 0$. The only and trivial exceptions are the wedge angles where they coincide with the formulation of equations (2.1b).

4.4. Mode II versions of cracked wedges

Using the method of this note it is quite easy to derive the closed form stress intensity factors of the mode II counterparts of equations (2.1a+b). For longitudinal displacement-displacement boundary conditions according to

$$\left. \begin{aligned} \sigma_\theta = \tau_{r\theta} = 0 & \quad \text{on } 0 \leq r < a \quad \text{when } \theta = 0 \\ u_x = \pm\delta & \quad \text{and } v_y = 0 \quad \text{on } r \geq 0 \quad \text{when } \theta = \pm\alpha \end{aligned} \right\} \quad (4.9)$$

there results

$$K_{II}(\alpha) = \frac{4G\delta}{\sqrt{\pi a}} \cdot \sqrt{\pi/(2\kappa\alpha + \sin 2\alpha)} \quad \text{when } 0 < \alpha \leq \pi. \quad (4.10)$$

The limiting case $\alpha \rightarrow 0$ corresponds to pure shearing of a symmetrically cracked semi-infinite strip of height $2h$. It yields

$$K_{II}(\alpha \rightarrow 0) \rightarrow \frac{2G\delta}{\sqrt{h}} \cdot \sqrt{\frac{2}{\kappa+1}} = \begin{cases} \frac{E\delta}{\sqrt{h}} \cdot \sqrt{\frac{2}{1+\nu}} & \text{in PS and} \\ \frac{E\delta}{\sqrt{h}} \cdot \sqrt{\frac{1}{2(1+\nu)^2(1-\nu)}} & \text{in PD.} \end{cases} \quad (4.11)$$

Naturally, a set of rigid body displacements added to either of equations (2.1a) and

(4.9) contributes nothing to either K_I or K_{II} . For displacement-stress boundary conditions according to

$$\left. \begin{aligned} \sigma_\theta = \tau_{r\theta} = 0 \quad \text{on} \quad 0 \leq r < a \quad \text{when} \quad \theta = 0 \\ u_x = \pm \delta \quad \text{and} \quad T_y = 0 \quad \text{on} \quad r \geq 0 \quad \text{when} \quad \theta = \pm \alpha, \end{aligned} \right\} \quad (4.12)$$

there results

$$K_{II}(\alpha) = \frac{4G\delta}{\sqrt{\pi a}} \cdot \frac{1}{\kappa + 1} \cdot \sqrt{\pi(2\alpha - \sin \alpha)/\alpha^2} \quad \text{when} \quad 0 < \alpha \leq \pi. \quad (4.13)$$

This equation vanishes in the limit $\alpha \rightarrow 0$. However, for $\alpha = \pi/2$ in *PD* and with $2\delta \rightarrow B$ it reduces to equation (16) of Freund's paper [7].

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