Using a Natural Deconvolution for Analysis of Perturbed Integer Sampling in Shift-Invariant Spaces

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Abstract

An important cornerstone of both wavelet and sampling theory is shift-invariant spaces, that is, spaces V spanned by a Riesz basis of integer-translates of a single function. Under some mild differentiability and decay assumptions on the Fourier transform of this function, we show that V also is generated by a function with Fourier transform $\widehat{\varphi}(\xi) = \int_{\xi-\pi}^{\xi+\pi} g(\nu)\, d\nu$ for some g with $\int_{\mathbb{R}} g(\xi)\, d\xi = 1$. We explain why analysis of this particular generating function can be more likely to provide large jitter bounds ε such that any $f \in V$ can be reconstructed from perturbed integer samples $f(k+\varepsilon_k)$ whenever $\sup_{k\in\mathbb{Z}} |\varepsilon_k| \leq \varepsilon$. We use this natural deconvolution of $\widehat{\varphi}(\xi)$ to further develop analysis techniques from a previous paper. Then we demonstrate the resulting analysis method on the class of spaces for which g has compact support and bounded variation (including all spaces generated by Meyer wavelet scaling functions), on some particular choices of φ for which we know of no previously published bounds and finally, we use it to improve some previously known bounds for B-spline shift-invariant spaces.

Keywords: shift-invariant space, reproducing kernel, interpolating function, shift-invariant, deconvolution, irregular sampling, scaling function, Shannon wavelet, Franklin, B-spline, Meyer wavelet

Note: *The published paper [13] is identical to this preprint except for some proof and a few other details that were removed to make primarily Section 2 more concise.*

1. Introduction

A *shift-invariant space* is a space $V \subset L_2(\mathbb{R})$ spanned by a Riesz basis of integer-translates of a single function. One important question is under what conditions on this generating function and for what sequences of *sampling points* x_k any $f \in V$ is uniquely determined by its *samples* $(f(x_k))_{k\in\mathbb{Z}}$.

When this is the case, two additional questions arise: Is there a fast, efficient and numerically stable algorithm for computing the reconstruction and can we compute useful error estimates for any truncations or other approximations involved in such an algorithm? For such algorithms and further references, we refer, for example, to Feichtinger,

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Gröchenig and Strohmer [14] for the case of *bandlimited* V (that is, with the Fourier transform of the generating function having compact support) and to Gröchenig, Schwab and Sun in [19, 26] for a fast local reconstruction algorithm for compactly supported generating functions.

In this paper we focus on the first question, the *existence* of a numerically stable reconstruction formula that reconstructs any $f \in V$ from samples $(f(x_k))_{k\in\mathbb{Z}}$ and knowledge of the sampling points x_k. We do this for perturbed integer samples $x_k \approx k$. Under some mild differentiability and decay assumptions on the Fourier transform of the generating function, we show in Section 3.1 that there is a so-called interpolating basis $(\varphi(x-k))_{k\in\mathbb{Z}}$ for V with $\widehat{\varphi} = \chi_{[-\pi,\pi]} * g$ and $\int g(\xi)\, d\xi = 1$. In Section 3.2, using this natural deconvolution, we adapt and further develop analysis techniques proposed in [12] into new sampling theorems with g as a "design parameter". Finally, we demonstrate the resulting theorems on a few different (classes of) shift-invariant spaces in Section 4.

Notation

The notation will be mainly as in the closely related papers [11, 12] except that Fourier transforms and Fourier series coefficients have the normalizations

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dx \qquad \text{and} \qquad \widehat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(x)e^{-inx}\, dx$$

with the corresponding $L_2([0,2\pi])$ inner product $\langle f,h\rangle = \frac{1}{2\pi}\int f(x)\overline{h(x)}\, dx$. We write L_p for the spaces $L_p(\mathbb{R})$ and $\|\cdot\|$ for the l_2-, L_2- and corresponding operator norms, as well as $\|\cdot\|_p$ for l_p and L_p norms. For Banach spaces X of functions on $\mathbb{R}$, $W(X,l_p)$ denotes the *Wiener amalgam space* of complex valued functions f on $\mathbb{R}$ for which the norm

$$\|f\|_{W(X,l_p)} = \left(\sum_{k\in\mathbb{Z}} \left\|f\cdot\chi_{[k,k+1)}\right\|_X^p\right)^{1/p} < \infty.$$

Unless otherwise stated, $\sum_{k\in\mathbb{Z}}$ (with shorthand notation $\sum_k$) denotes unconditional summation. For sequences $(f_k)_{k\in\mathbb{Z}}$ we usually write $(f_k)_k$, (f_k) or simply "the sequence f_k". For functions f defined a.e., $\operatorname{supp} f$ denotes the intersection of the supports of all representatives of f. The function $\operatorname{sinc}(x) \overset{\text{def}}{=} \frac{\sin(\pi x)}{\pi x}$ for $x \neq 0$ and $\operatorname{sinc}(0) = 1$.

2. Preliminaries

A *frame* for a Hilbert space $\mathcal{H}$ with *frame bounds* $0 < A < B < \infty$ is a sequence (e_k) in $\mathcal{H}$ for which $A\|f\|^2 \leq \sum_k |\langle f, e_k\rangle|^2 \leq B\|f\|^2$ for all $f \in \mathcal{H}$. A *Riesz basis* for $\mathcal{H}$ is a frame (e_k) for $\mathcal{H}$ that ceases to be a frame whenever an element is removed, or equivalently, a basis for $\mathcal{H}$ such that for all finite-length sequences $c = (c_k)_k$, $A\|c\|_2^2 \leq \|\sum_k c_k e_k\|_{\mathcal{H}}^2 \leq B\|c\|_2^2$ with A, B now called *Riesz bounds*. A Riesz basis (but not a frame) is an orthonormal basis if and only if $A = B = 1$ [16, Section 1.6].

To every frame (e_k) for $\mathcal{H}$ corresponds a *dual frame* $(\widetilde{e_k})$ with frame bounds $\frac{1}{B}, \frac{1}{A}$, such that $f = \sum_k \langle f, \widetilde{e_k}\rangle e_k = \sum_k \langle f, e_k\rangle \widetilde{e_k}$ for all $f \in \mathcal{H}$. If (e_k) is a Riesz basis, then the dual Riesz basis and the series expansion coefficients are unique.

For a Riesz basis $(\varphi(\cdot - k))$ for a shift-invariant space V, the defining inequalities take the form [9, 30]

$$A \le \sum_k |\widehat{\varphi}(\xi + 2\pi k)|^2 \le B, \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{1}$$

The dual Riesz basis consists of integer-shifts of a dual generating function $\widetilde{\varphi} \in V$ (this follows from the fact that the so-called frame operator commutes with integer shifts, as in, for example, [17, Proposition 5.2.1] or [16, Proposition 2.17]). We show in Section 3.1 that a large class of shift-invariant spaces have a unique generating function with Fourier transform

$$\widehat{\varphi}(\xi) = \chi_{[-\pi,\pi]} * g(\xi) = \int_{\xi-\pi}^{\xi+\pi} g(\nu)\, d\nu \qquad \text{such that} \qquad \int_{\mathbb{R}} g(\xi)\, d\xi = 1. \tag{2a}$$

For such φ,

$$\|\widehat{\varphi}\|_1 = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \chi_{[-\pi,\pi]}(\xi - \nu) g(\nu)\, d\nu \right| d\xi \le \int_{\mathbb{R}} |g(\nu)| \int_{\mathbb{R}} \chi_{[-\pi,\pi]}(\xi - \nu)\, d\xi\, d\nu = 2\pi \|g\|_1,$$

so that we can choose φ to be *continuous.* In addition, we will assume that

$$\sup_x \sum_{k\in\mathbb{Z}} |\varphi_k(x)|^2 = M < \infty, \tag{2b}$$

so that $\varphi \in L_2(\mathbb{R})$ by Tonelli's theorem and the point evaluation functional is bounded since $|f(x)|^2 = |\sum_k \langle f, \widetilde{\varphi}_k\rangle \varphi_k(x)|^2 \le M \sum_k |\langle f, \widetilde{\varphi}_k\rangle|^2 \le \frac{M}{A} \|f\|_2^2$. This has two important consequences: *Firstly,*

$$\text{if } f_n \to f \text{ in } V, \text{ then } f_n \to f \text{ uniformly on } \mathbb{R}$$

(as derived from the stronger assumption $\varphi \in W(C, l_2)$ in [12, Lemma 2.1]). From this and (2a) it follows that

$$\widehat{\varphi} \in L_1(\mathbb{R}) \qquad \text{and} \qquad \text{all } f \in V \text{ are continuous.} \tag{2c}$$

Secondly, by the Riesz representation theorem, for each $x \in \mathbb{R}$ there is a unique *reproducing kernel* q_x such that $\langle f, q_x\rangle = f(x)$ for all $f \in V$. For such f,

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_k\rangle \varphi_k(x) = \left\langle f, \sum_{k\in\mathbb{Z}} \overline{\varphi_k(x)} \widetilde{\varphi}_k \right\rangle, \quad \text{so} \quad q_x = \sum_{k\in\mathbb{Z}} \overline{\varphi_k(x)} \widetilde{\varphi}_k. \tag{3}$$

Hence, if there are x_k in $\mathbb{R}$ such that (q_{x_k}) is a frame for V with frame bounds A_q, B_q and dual frame $(\widetilde{q_{x_k}})$, then

$$f = \sum_k \langle f, q_{x_k}\rangle \widetilde{q_{x_k}} = \sum_k f(x_k) \widetilde{q_{x_k}} \qquad \text{for all } f \in V. \tag{4}$$

We will consider a frame (q_{x_k}) not being a Riesz basis only in Theorem 1. For Riesz bases (q_k) and (q_{x_k}) with bounds A, B and $A_{\text{pert}}, B_{\text{pert}}$, respectively, suppose that you receive the samples $f(x_k)$ only knowing that $x_k \approx k$. By the above inequalities, the L_2 error of the approximate reconstruction $f_{\text{est}} \overset{\text{def}}{=} \sum_k f(x_k)\widetilde{q}_k$ is

$$\begin{aligned}\|f - f_{\text{est}}\|_2^2 &= \|\textstyle\sum_k (f(k) - f(x_k))\widetilde{q}_k\|_2^2 \le \frac{1}{A}\,\|\langle f, q_k\rangle - \langle f, q_{x_k}\rangle\|_{l_2}^2 \\ &= \frac{\|\langle f,q_k\rangle\|_{l_2}^2 - 2\Re\big\langle\langle f,q_k\rangle,\langle f,q_{x_k}\rangle\big\rangle_{l_2} + \big\|\langle f,q_{x_k}\rangle\big\|_{l_2}^2}{A} \le \frac{B+2\sqrt{BB_{\text{pert}}}+B_{\text{pert}}}{A}\,\|f\|_2^2 \\ &\le \frac{\big(\sqrt{B}+\sqrt{B_{\text{pert}}}\big)^2}{AA_{\text{pert}}}\,\|f(x_k)\|_{l_2}^2 \le \frac{B_{\text{pert}}\big(\sqrt{B}+\sqrt{B_{\text{pert}}}\big)^2}{AA_{\text{pert}}}\,\|f_{\text{est}}\|_2^2 .\end{aligned}$$

There is no such stability if (q_{x_k}) not is a frame. On the contrary, then for *any* $C > 0$ there is a set of sampling points with jitter error bound less than $\sup_{k\in\mathbb{Z}} |x_k|$ for which $\|f - f_{\text{est}}\|_2 \ge C\,\|f(x_k)\|_{l_2}$ and $\|f - f_{\text{est}}\|_2 \ge C\,\|f_{\text{est}}\|_2$ [10, Theorem 5.1]. Hence the frame property is of utmost importance.

2.1. Analysis methods

There are several different approaches for analyzing under what conditions (q_{x_k}) is a Riesz basis [1–4, 7, 12, 18, 21, 27, 28, 32], often based on the fact that (q_{x_k}) is a Riesz basis for V if and only if there is a bounded bijective operator $L\colon V \to V$ such that $L\widetilde{\varphi}_k = q_{x_k}$ [8, 17, 31]. For perturbed integer sampling points $x_k \approx k$, it can be particularly useful [12, 19, 25, 26] to analyze the corresponding coefficient operator $\Phi = R_{\widetilde{\varphi}}^{-1} L R_{\varphi}^{*\,-1}\colon l_2 \to l_2$ defined, as in [12], such that, with doubly infinite matrix notation,

$$\Phi\colon l_2 \to l_2, \qquad (\Phi c)_j = \sum_{k\in\mathbb{Z}} \Phi_{jk} c_k \qquad \text{and} \qquad \Phi_{j,k} = \langle q_{x_k}, \varphi_j\rangle = \overline{\varphi_j(x_k)}. \tag{5a}$$

This makes Φ a well-defined bounded bijective operator if and only if

$$\|\Phi\Lambda - \mathrm{I}\| < 1 \tag{5b}$$

for some bounded bijective operator $\Lambda\colon l_2 \to l_2$ and the identity operator I. As in [12], we will evaluate (5b) by using the Schur interpolation estimate

$$\|M\|^2 \le \left(\sup_j \sum_k |M_{jk}|\right)\left(\sup_k \sum_j |M_{jk}|\right) \tag{5c}$$

[6, 24], now for φ satisfying (2) and for perturbed integer sampling points

$$x_k{=}k + \varepsilon_k \qquad \text{with} \qquad \sup_k |\varepsilon_k| < \varepsilon < 1/2. \tag{5d}$$

Regular sampling ($x_k = k$)

For *regular sampling*, that is when $x_k = k$, Φ is a "convolution type" operator on l_2, so that for convolution operator Λ, the operator norm in (5b) can be estimated using the following lemma.

Lemma 1. *For $(c_k) \in l_2$ we can define the convolution operator $C\colon l_2 \to l_\infty$, $(Cx)_j = \sum_{k\in\mathbb{Z}} c_{j-k}x_k$. The $l_2 \to l_2$ operator norm of C is*

$$\|C\|_{l_2\to l_2} = \|f\|_\infty \le \infty \qquad \textit{for} \qquad f = \sum_{m\in\mathbb{Z}} c_m e^{im\cdot} \qquad (L_2\textit{-convergence}). \tag{6}$$

Proof. For $(c_k) \in l_2$, both C and $f \in L_2(\mathbb{R})$ are well-defined.

If $Cx \in l_2$ for all $x \in l_2$, then we can apply Parseval's formula and the fact that the Fourier coefficients of a product is the convolution of the Fourier coefficients of its factors:

$$\begin{aligned} \|Cx\|_{l_2} &= \left(\sum_{j\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} c_{j-k}x_k \right|^2 \right)^{1/2} = \left\| \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} c_{j-k}x_k \right) e^{ij\cdot} \right\|_{L_2([0,2\pi])} \\ &= \left\| \sum_{m\in\mathbb{Z}} c_m e^{im\cdot} \sum_{n\in\mathbb{Z}} x_n e^{in\cdot} \right\|_{L_2([0,2\pi])} \le \|f\|_\infty \|x\|_{l_2} \qquad \text{for all } x \in l_2. \end{aligned} \tag{7}$$

For all $c < \|f\|_\infty \le \infty$ there is a set M of positive measure where $f > c$. If x is the Fourier coefficients of χ_M, then $\|Cx\|_{l_2} > c\,\|x\|_{l_2}$. Hence the inequality in (7) is sharp and (6) follows for operators $C\colon l_2 \to l_2$.

It remains to consider the case when $Cx \notin l_2$ for some $x \in l_2$, so that $\|C\| = \infty$. Then also $\|f\|_\infty$ must be infinite, because otherwise the last line of (7) would give that $\sum_{m\in\mathbb{Z}} c_m e^{im\cdot} \sum_{n\in\mathbb{Z}} x_n e^{in\cdot} \in L_2([0,2\pi])$ and thus has l_2 Fourier coefficients Cx, which contradicts our initial assumption. This completes the proof of (6). ◻

Note from (3) that for $q \overset{\text{def}}{=} q_0$ and integers k, $q_k(x) = q(x-k)$. Note also that $\widehat{q}(\xi) = \widehat{\widehat{\varphi}}(\xi) \sum_{k\in\mathbb{Z}} \overline{\varphi_k(0)} e^{-ik\xi} = \widehat{\widehat{\varphi}}(\xi) \overline{\sum_{n\in\mathbb{Z}} \varphi(n) e^{-in\xi}}$. Hence the following Riesz basis condition (8) follows directly from the Riesz basis condition (1) (see, for example, [29, Proposition 9.1] or [12, Lemma 2.3]):

Lemma 2. *For φ satisfying* (2b)*, define $m_\varphi(\xi) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} \varphi(n) e^{-in\xi}$. If*

$$0 < C_1 \le |m_\varphi| \le C_2 < \infty \qquad a.e., \tag{8}$$

then (q_k) is a Riesz basis. If $\widehat{\varphi} \in L_1$, then $m_\varphi(\xi) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - 2\pi k)$ a.e.

The last statement follows from the Poisson summation formula, with $\sum_{n\in\mathbb{Z}} \varphi(n) e^{-in\xi}$ converging both in L_2 and almost everywhere for symmetrical partial sums, whereas $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - 2\pi k)$ converges in L_1 and unconditionally pointwise. Both $\widehat{\varphi} \in L_1$ and *uniform* convergence of $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - 2\pi k)$ follows for all spaces studied in this paper via the convolution property (2a) leading to the decay condition $\widehat{\varphi} \in W(C, l_1)$ in (11a) below.

Remark 1. For uniform convergence of $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - 2\pi k)$, it is *not* sufficient that φ is an interpolating generating function for which (2b) and (2c) holds. In fact, if $\widehat{\varphi}(2\pi\xi) = \sum_{n=0}^\infty \chi_{(n-2^{-n}, n-2^{-(n+1)}]}(\xi)$, then clearly $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - k) = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi - k)|^2 = 1$, so $\widehat{\varphi} \in L_1$ and, via the Poisson summation formula, $(\varphi_k)_{k\in\mathbb{Z}}$ is an interpolating orthonormal basis for its span V. Provided that (2b) holds, V is also a reproducing kernel Hilbert space, *but clearly* $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi - 2\pi k)$ does not converge uniformly, which thus must be blamed on $\widehat{\varphi}$ not being of convolution type (2a).

To check that (2b) holds, note for $B_n = 2^{-(n+1)}$ that

$$|\varphi(x)| \leq \sum_{n=0}^{\infty} |B_n \operatorname{sinc}(B_n x)| \leq C \sum_{n=0}^{\infty} B_n (1 + B_n |x|)^{-1},$$

Fix $\epsilon \in (0,1)$ and set $N_x = \left\lfloor \log_2 \left(\frac{|x|}{\frac{1}{\epsilon}-1} \right) - 1 \right\rfloor$ (so that for $n \leq N_x$, $(1 + B_n |x|)^{-1} \leq \epsilon$). Then $|\varphi(x)| \leq \varphi(0) = \frac{1}{2\pi}$ and it is not difficult to check that separate treatments of the sums $\sum_{n=0}^{N_x}$ and $\sum_{n=N_x+1}^{\infty}$ gives $|\varphi(x)| \leq \min\left(\frac{1}{2\pi}, C_\epsilon \left(\frac{\log_2(|x|)}{|x|} + \frac{1}{|x|}\right)\right)$ so that (2b) holds.

Perturbed integer sampling ($x_k \approx k$)

As long as $(q_k)_{k\in\mathbb{Z}}$ is a Riesz basis, thus providing the stable reconstruction (4) from integer samples, the same holds true also for *some* $\varepsilon > 0$ in (5d) [12, Theorem 3.2]. It always holds that $\varepsilon < 1/2$, as shown for the Franklin scaling function φ in [21] and for arbitrary $\varphi \in L_2$ here:

Theorem 1 (No stability if $\varepsilon \geq 1/2$)**.** *If $\varphi \in W(L_\infty, l_1)$ generates a shift-invariant space V with reproducing kernels q_x, then there are ε_k such that* $\sup_k |\varepsilon_k| = 1/2$ *and* $(q_{k+\varepsilon_k})_{k\in\mathbb{Z}}$ not *is a frame for V.*

Proof. Set $\varepsilon_k = -1/2$ if $k \leq 0$, $\varepsilon_k = 1/2$ for $k > 0$ and assume that $(q_{k+\varepsilon_k})$ is a frame for V. This would imply that the sequence $\left(q_{k+1/2}\right)$ (obtained by adding the element $q_{\frac{1}{2}}$ to $(q_{k+\varepsilon_k})$) is a frame, but not a Riesz basis, for V. However, it follows from [12, Proposition 3.1] that if $\left(q_{k+1/2}\right)$ is frame, then it is, in fact, a Riesz basis. Thus our initial assumption is wrong and the theorem follows. □

2.2. One space, different generators

It follows from our construction (2a) that for integers n, $\varphi(n) = \delta_{0,n}$. This prevents shifts of the generating function, just like in wavelet multiresolution analysis (MRA), where a noninteger shift X_0 of the scaling function would require dilations around X_0 instead of from dilations around 0. However, a natural and equivalent generalization of (5d) is to reconstruct f from samples $f(X_0 + k + \varepsilon_k)$. Then, a sufficient condition for the reconstruction (4) to be possible with jitter error bound ε [12, Theorem 3.1] is that

$$\sum_{k\neq 0} \sup_{|x|\leq\varepsilon} |\varphi(X_0 + k + x)| < \inf_{|x-X_0|<\varepsilon} \varphi(x). \tag{9}$$

This together with the continuity of φ suggests choosing X_0 so that φ is large near X_0 and small near $X_0 + k$ for nonzero integers k. The importance of this choice was investigated in [11], where for 95 different Daubechies, Symmlet and B-spline wavelets, we computed the value of X_0 that gives the Hilbert-adjoint Φ^* of Φ the “simplest possible” structure in the sense that it minimizes the operator norm error of 7 different low complexity (near diagonal) approximations of Φ^{*-1}. The computed optimal X_0 was consistently, with small or (for the symmetric B-spline scaling functions) unrecognizable deviation, coinciding with the location of the maximum of $|\varphi(x)|$.

For asymmetric φ, such as any continuous compactly supported (anti)symmetric MRA wavelet scaling function ([5, p. 47] or [22, pp. 312–313]), these two observations would suggest choosing a nonzero X_0.

In this paper, however, we exploit that for a large class of shift-invariant spaces V, instead of fine-tuning X_0, we can *choose* a generating function φ for V such that $\varphi(n) = \delta_{0,n}$. Then (9) suggests that for $X_0 = 0$ (at least unless φ is highly asymmetric), analysis of this particular φ is likely to provide larger ε then analysis of other generators for the same space. In Section 4.5 we show for some B-spline spaces that this choice of φ actually does result in larger bounds and that one particular trick in our main theorem further improves this bound.

Remark 2. Points X_0 for which all $f \in V$ satisfy $f(x) = \sum_k f(X_0 + k)S(x - k)$ for some frame $(S(x-k))_k$ are called *regular points* and investigated closer in [27]. One further generalization of (5d) is to reconstruct from samples $(\mathcal{L}f)(k + \varepsilon_k)$ for a linear time-invariant filter $\mathcal{L}$ [15], with sampling points $x_k{=}X_0{+}k{+}\varepsilon_k$ corresponding to a filter with impulse response $\delta(\cdot + X_0)$.

3. Main results

Our main theorems follow in Section 3.2, where we use the setup and estimates in (2), (5), Lemma 1 and Lemma 2 to compute sufficient conditions for the Riesz basis reconstructions (4) to hold. This setup and these estimates are the same as in [12], except that we make use of the deconvolution $\widehat{\varphi} = \chi_{[-\pi,\pi]} * g$ in (2a). This is much less restrictive than it might seem, since we show in Theorem 2 that under some differentiability and decay restrictions on $\widehat{\varphi}$, a characteristic property of the basis $(\widetilde{q}_k)$ is exactly that it allows a deconvolution $\widehat{\widetilde{q}} = \chi_{[-\pi,\pi]} * g$ with $\int_{\mathbb{R}} g(\xi)\, d\xi = 1$.

3.1. Properties of the interpolating basis $(\widetilde{q}_k)$

For φ satisfying (2a) and (2b), the integer shifts φ_k generates a shift-invariant space V with reproducing kernels $q_x = \sum_k \overline{\varphi_k(x)} \widetilde{\varphi}_k$, for which we know from Lemma 2 that if $0 < C_1 \le |\sum_k \widehat{\varphi}(\cdot + 2\pi k)| = |m_\varphi| \le C_2 < \infty$, then $(q_k)_{k\in\mathbb{Z}}$ is a Riesz basis for V with dual basis $(\widetilde{q}_k)_{k\in\mathbb{Z}}$.

Now set $\widehat{s} \stackrel{\text{def}}{=} \widehat{\varphi}/m_\varphi$. It is not difficult to check that $s \in V$, (s_k) is a Riesz basis for V and $\sum_{k\in\mathbb{Z}} \widehat{s}(\cdot + 2\pi k) = 1$, so that by the Poisson summation formula, $s(k) = \delta_{0,k}$ or equivalently, $s = \widetilde{q}$ (due to the uniqueness of coefficients in $s = \sum_k \langle s, q_k\rangle \widetilde{q}_k = \sum_k s(k)\widetilde{q}_k$). Hence

$$\widehat{\widetilde{q}} = \frac{\widehat{\varphi}}{\sum_k \widehat{\varphi}(\cdot + 2\pi k)} = \frac{\widehat{\varphi}}{m_\varphi} \tag{10}$$

is the unique element in V with the characterizing property $\widetilde{q}(k) = \delta_{0,k}$ for integers k. The function $\widetilde{q}$ and basis $(\widetilde{q}_k)$ are usually referred to as *interpolating.* Instead of computing $\widetilde{q}$ from (10), the following theorem shows that the construction (2a) actually gives $m_\varphi = 1$ and $\varphi = \widetilde{q}$:

Theorem 2. *The following are equivalent:*

i. $\widehat{s}(\xi) = \chi_{[-\pi,\pi]} * g(\xi) = \int_{\xi-\pi}^{\xi+\pi} g(\nu)\, d\nu$ *with* $\int_{\mathbb{R}} g(\nu)\, d\nu = 1$.

ii. *s is interpolating,*

$$\widehat{s} \in W(C, l_1), \quad \widehat{s} \text{ is absolutely continuous,} \quad \widehat{s}\,' \in L_1(\mathbb{R}) \text{ and} \tag{11a}$$

$$g(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \widehat{s}\,'(\xi - (2k+1)\pi) \in L_1(\mathbb{R}) \qquad \text{(convergence a.e.).} \tag{11b}$$

In both these cases, it follows that $s \in W(C, l_2)$ *and that* (s_k) *is a Riesz basis for the closure* V *of its span with reproducing kernel dual basis* (q_k). *Moreover, if* $\operatorname{supp} g \subseteq [-\frac{\pi}{3}, \frac{\pi}{3}]$, *then* s *is the scaling function of a multiresolution analysis (MRA).*

It will be clear from the proof of Theorem 2 that condition (11a) is sufficient for showing that $\widehat{s} = \chi_{[-\pi,\pi]} * g$ with $g \in L_{1,\text{loc}}$, so for the implication $ii \Rightarrow i$, (11b) and the interpolation property are needed only for obtaining that $\int_{\mathbb{R}} g(\xi)\, d\xi = 1$.

It may seem difficult to check if the condition (11) holds, for example, in situations when only a non-interpolating basis (φ_k) for V is known. Therefore we present some simplified and sufficient conditions for (11) to hold in Theorem 3 before continuing with some examples and finally the proofs of Theorems 2 and 3.

Theorem 3. *Suppose that* (φ_k) *is a Riesz basis for a shift-invariant space* V *and that* m_φ *satisfies the boundedness condition* (8). *Set* $\widehat{s} = \widehat{\varphi}/m_\varphi$. *A sufficient condition for* (11a) *to hold is that*

$$\widehat{\varphi} \in W(C, l_1), \quad \widehat{\varphi} \text{ is differentiable on } \mathbb{R} \quad \text{and} \quad \widehat{\varphi}\,' \in W(L_\infty, l_1). \tag{11a$'$}$$

If (11a′) *holds, then a sufficient condition for* (11b) *to hold is that*

$$\int_{\mathbb{R}} \left(1 + \frac{|\xi|}{2\pi}\right) |\widehat{s}\,'(\xi)|\, d\xi < \infty. \tag{11b$'$}$$

If

$$\operatorname{supp} \widehat{\varphi} \subseteq [a, b] \qquad \text{and} \qquad \widehat{\varphi} \text{ is absolutely continuous,} \tag{11a$''$}$$

then (11a) *and* (11b) *hold with* g *becoming a finite sum*

$$g(\xi) = \chi_{[a+\pi, b-\pi]}(\xi) \sum_{k=0}^{\lfloor \frac{b-a-2\pi}{2\pi} \rfloor} \widehat{s}\,'(\xi - (2k+1)\pi) \tag{11b$''$}$$

with notation $\lfloor x \rfloor$ *for the largest integer* $n \leq x$.

Example 1 (Meyer). Meyer [23, pp. 22–23] introduced orthonormal MRA scaling functions whose Fourier transforms are even C^∞ functions $\widehat{\varphi} \colon \mathbb{R} \to [0, 1]$ such that

$$\widehat{\varphi}(\xi) = 1 \text{ for } \xi \in [-2\pi/3, 2\pi/3] \qquad \text{and} \qquad \operatorname{supp} \widehat{\varphi} \subseteq [-4\pi/3, 4\pi/3].$$

Hence (11a″) holds and m_φ has at most two nonzero terms in $[0, 2\pi]$, so the orthonormality condition $\sum_k \widehat{\varphi}(\xi + 2\pi k)^2 = 1$ holds if and only if $\widehat{\varphi}(\xi)^2 + \widehat{\varphi}(\xi - 2\pi)^2 = 1$ for $0 \leq \xi \leq 2\pi$, which implies that $1 \leq m_\varphi \leq \sqrt{2}$, and, by (11b″), $\widehat{\widehat{q}} = \chi_{[-\pi,\pi]} * g$ with $g(\xi) = \chi_{[-\pi/3, \pi/3]}(\xi) \widehat{\widehat{q}}\,'(\xi - \pi)$.

Example 2 (Haar and Shannon). Haar and Shannon wavelet scaling functions are interpolating, so $\varphi = \widetilde{q}$. However, they are clearly not of the convolution type (2a), since not both φ and $\widehat{\varphi}$ are continuous.

Remark 3. Some results in Theorem 2 are related to but should not be confused with results in [29, Section 10.1], which examines similar properties and deconvolution of so-called Meyer type scaling functions ϕ with the properties $\phi(x) = O((1+|x|)^{1+\epsilon})$, $\widehat{\phi}(\xi) = O((1+|\xi|)^{1+\epsilon})$, $\epsilon > 0$ and $\widehat{\phi}(\xi) = \left(\int_{\xi-\pi}^{\xi+\pi} g(\xi)\, d\xi\right)^{1/2} = \widehat{\widetilde{q}}(\xi)^{1/2}$ for distributions g such that $\int_{\xi-\pi}^{\xi+\pi} g(\xi)\, d\xi \geq 0$. Then the computation (13) with $\widehat{s} = \widehat{\phi}^2$ shows that contrary to $\widetilde{q}$, ϕ is orthogonal, but in general not interpolating. However, from its construction follows an oversampled reconstruction formula

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2}\right)\phi(2x-n), \qquad \text{for all } f \in V.$$

Proof of Theorem 2.

$ii \Rightarrow i$: Since $\sum_{k=0}^{\infty} \int_{\xi-\pi}^{\xi+\pi} |\widehat{s}\,'(\nu - (2k+1)\pi)|\, d\nu = \int_{-\infty}^{\xi} |\widehat{s}\,'(\nu)|\, d\nu < \infty$, the series $g(\cdot) = \sum_{k=0}^{\infty} \widehat{s}\,'(\cdot - (2k+1)\pi) \in L_1([\xi-\pi, \xi+\pi])$ with convergence almost everywhere. Hence for absolutely continuous $\widehat{s} \in W(C, l_1)$,

$$\begin{aligned}\chi_{[-\pi,\pi]} * g(\xi) = \int_{\xi-\pi}^{\xi+\pi} g(\nu)\, d\nu &= \sum_{k=0}^{\infty} \int_{\xi-\pi}^{\xi+\pi} \widehat{s}\,'(\nu - (2k+1)\pi)\, d\nu \\ &= \sum_{k=0}^{\infty} (\widehat{s}(\xi - 2k\pi) - \widehat{s}(\xi - 2(k+1)\pi)) = \widehat{s}(\xi).\end{aligned}$$

If, in addition, $g \in L_1(\mathbb{R})$, then

$$\int_{\mathbb{R}} g(\xi)\, d\xi = \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} g(\xi + 2\pi n)\, d\xi = \sum_{n\in\mathbb{Z}} (\chi_{[-\pi,\pi]} * g)(2\pi n) = \sum_{n\in\mathbb{Z}} \widehat{s}(2\pi n) = 1.$$

$i \Rightarrow ii$ and $s \in W(C, l_2)$: With notation $\check{g}$ for the inverse Fourier transform of g,

$$s(t) = \operatorname{sinc}(t) 2\pi \check{g}(t). \tag{12}$$

so that $s \in W(C, l_2)$ and s is interpolating.

Since $\widehat{s}(\xi) = (\chi_{[-\pi,\pi]} * g)(\nu) = \int_0^{\xi} g(\nu+\pi) - g(\nu-\pi)\, d\nu + C$, $\widehat{s}$ is absolutely continuous on $\mathbb{R}$ and for almost all $\xi \in \mathbb{R}$, $\widehat{s}\,' = g(\nu+\pi) - g(\nu-\pi)$. Hence $\widehat{s}\,' \in L_1(\mathbb{R})$ and

$$\begin{aligned}\|\widehat{s}\|_{W(C,l_1)} &= \sum_{k\in\mathbb{Z}} \sup_{\xi\in[0,1]} |\widehat{s}(\xi+k)| \leq \sum_{k\in\mathbb{Z}} \sup_{\xi\in[0,1]} \int_{\xi+k-\pi}^{\xi+k+\pi} |g(\nu)|\, d\nu \\ &\leq \sum_{k\in\mathbb{Z}} \int_{k-\pi}^{1+k+\pi} |g(\nu)|\, d\nu \leq 8 \int_{\mathbb{R}} |g(\nu)|\, d\nu < \infty.\end{aligned}$$

(We leave the second last inequality unproven since it just is an artefact of choosing a non-unitary normalization of the Fourier transform.) Finally, as in the first lines of this proof, the fact that $\widehat{s}\,' \in L_1(\mathbb{R})$ implies almost everywhere convergence of the series

$$\sum_{k=0}^{\infty} \widehat{s}\,'(\xi - (2k+1)\pi) = \sum_{k=0}^{\infty} (g(\xi - 2k\pi) - g(\xi - 2(k-1)\pi)) = g(\xi).$$

Now suppose that i and ii hold. Then we claim that $(s_k)_k$ is a Riesz basis for the closure V of its span, or equivalently, that the function $\phi \stackrel{\text{def}}{=} \sum_k |\widehat{s}(\xi + 2\pi k)|^2$ satisfies the double inequality (1). The right-hand inequalities in (1) as well as uniform convergence to a continuous function ϕ follows from the facts that $\widehat{s}$ is continuous, $\|\widehat{s}\|_\infty \leq \|g\|_1 < \infty$ and

$$\begin{aligned}\sum_{|k|\geq N} |\widehat{s}(\xi + 2\pi k)|^2 &\leq \|\widehat{s}\|_\infty \sum_{|k|\geq N} |\widehat{s}(\xi + 2\pi k)| \leq \|\widehat{s}\|_\infty \sum_{|k|\geq N} \int_{\xi-\pi+2\pi k}^{\xi+\pi+2\pi k} |g(\nu)| \ d\nu \\ &\leq \|\widehat{s}\|_\infty \int_{|\nu|\geq 2\pi(N-1)} |g(\nu)| \ d\nu, \qquad \text{for all } \xi \in [-\pi, \pi].\end{aligned}$$

Moreover, the left-hand inequalities in (1) hold, because ϕ is continuous and

$$\sum_k \widehat{s}(\xi + 2\pi k) = \sum_k \int_{\xi-\pi+2\pi k}^{\xi+\pi+2\pi k} g(\nu) \ d\nu = \int_{\mathbb{R}} g(\nu) \ d\nu = 1. \tag{13}$$

Thus every $f \in V$ has the series expansion $f = \sum_{k\in\mathbb{Z}} c_k s_k$, for which the interpolation property of s gives that $c_k = f(k)$. Since $s \in W(C, l_2)$, (2b) holds and implies (3), that is, that V is equipped with reproducing kernels q_k with the characterizing property $\langle f, q_k \rangle = f(k) = c_k$. Thus (q_k) is the dual Riesz basis of (s_k).

Finally, if $\operatorname{supp} g \subseteq [-\pi/3, \pi/3]$ and $\int g(\nu)\, d\nu = 1$, then $\widehat{s} = 1$ on $[-2\pi/3, 2\pi/3]$ and $\widehat{s}(\xi) = 0$ for $|\xi| \geq 4\pi/3$. Hence, if m is the 4π-periodic function $m(\xi) = \sum_{k\in\mathbb{Z}} \widehat{s}(\xi + 4\pi k)$, then clearly $m \in L_2([0, 4\pi])$ and

$$\widehat{s}(\xi) = \widehat{s}(\xi/2) m(\xi) \qquad \text{so that} \qquad s = \sum_k c_k s(2 \cdot - k) \qquad \text{for some} \qquad (c_k) \in l_2.$$

Thus $V \subset \{f(2\cdot) \mid f \in V\}$, so that (for example, by theorems 1.6 and 1.7 of Chapter 2 in [20]), $(s_k)_k$ generates an MRA if and only if

$$0 \neq \widehat{s}(0) = \int_{-\pi}^{\pi} g(\nu)\, d\nu = \int_{\mathbb{R}} g(\nu)\, d\nu,$$

which holds, since $\int_{\mathbb{R}} g(\nu)\, d\nu = 1$. □

Proof of Theorem 3. Suppose first that (8) and (11a′) hold. Then by uniform convergence of $\sum_k \widehat{\varphi}'(\xi + 2\pi k)$ on compact sets, $\widehat{s} = \widehat{\varphi}/m_\varphi$ is differentiable on $\mathbb{R}$ and

$$|\widehat{s}\,'(\xi)| = \left| \frac{\widehat{\varphi}'(\xi) m_\varphi(\xi) - \widehat{\varphi}(\xi) m_\varphi'(\xi)}{m_\varphi(\xi)^2} \right| \leq \frac{|\widehat{\varphi}'(\xi)|}{C_1} + \frac{|\widehat{\varphi}(\xi)| \cdot 2\, \|\widehat{\varphi}'\|_{W(L_\infty, l_1)}}{C_1^2},$$

where we used (8) and the fact that $\sum_{k\in\mathbb{Z}} \sup_{\xi\in[0,2\pi)} |\widehat{\varphi}'(\xi+2\pi k)| \leq 2\,\|\widehat{\varphi}'\|_{W(L_\infty,l_1)}$. Hence $\widehat{s}\,' \in W(L_\infty, l_1) \subseteq L_1 \cap L_\infty$. Thus (11a) follows. By differentiability everywhere and uniform convergence on compact sets, it follows also that

$$\sum_{k\in\mathbb{Z}} \widehat{s}\,'(\xi-(2k+1)\pi) = \frac{\partial}{\partial\xi}\sum_{k\in\mathbb{Z}} \widehat{s}(\xi-(2k+1)\pi) = \frac{\partial}{\partial\xi}1 = 0. \tag{14}$$

Hence, if $\int_{\mathbb{R}} \left(1+\frac{|\xi|}{2\pi}\right) |\widehat{s}\,'(\xi)|\,d\xi < \infty$, then

$$\begin{aligned}
\int_0^\infty |g(\xi)|\,d\xi &= \int_0^\infty \left|\sum_{k=0}^\infty \widehat{s}\,'(\xi-(2k+1)\pi)\right| d\xi \qquad \text{(apply (14))}\\
&= \int_0^\infty \left|\sum_{k=-\infty}^{-1} \widehat{s}\,'(\xi-(2k+1)\pi)\right| d\xi \leq \int_{\mathbb{R}} |\widehat{s}\,'(\xi)| \sum_{k=1}^\infty \chi_{[(2k-1)\pi,\infty)}(\xi)\,d\xi\\
&\leq \int_\pi^\infty \left(1+\frac{|\xi|}{2\pi}\right)|\widehat{s}\,'(\xi)|\,d\xi < \infty
\end{aligned}$$

and similarly but simpler,

$$\begin{aligned}
\int_{-\infty}^0 |g(\xi)|\,d\xi &\leq \int_{-\infty}^0 \sum_{k=0}^\infty |\widehat{s}\,'(\xi-(2k+1)\pi)|\,d\xi = \int_{\mathbb{R}} |\widehat{s}\,'(\xi)| \sum_{k=0}^\infty \chi_{(-\infty,-(2k+1)\pi]}(\xi)\,d\xi\\
&\leq \int_{-\infty}^{-\pi} \left(1+\frac{|\xi|}{2\pi}\right)|\widehat{s}\,'(\xi)|\,d\xi < \infty.
\end{aligned}$$

Hence (8), (11a′) and (11b′) imply (11b).

Next, suppose that $\operatorname{supp}\widehat{\varphi} \subseteq [a,b]$. Then (14) holds again, since the series reduces to a finite sum on every finite interval. Thus $\operatorname{supp}\widehat{s}\,' \subseteq \operatorname{supp}\widehat{\varphi} \subseteq [a,b]$ and

$$g(\xi) \overset{\text{def}}{=} \sum_{k=0}^\infty \widehat{s}\,'(\xi-(2k+1)\pi) = -\sum_{k=-\infty}^{-1} \widehat{s}\,'(\xi-(2k+1)\pi),$$

so that $\operatorname{supp} g \subseteq [a+\pi, b-\pi]$ and the series in (11b) reduces to the sum in (11b″). Hence, if $\widehat{\varphi}$ is absolutely continuous, then so is m_φ and thus also $\widehat{s}$, because

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\widehat{s}(b_k)-\widehat{s}(a_k)| &\leq \sum_{k\in\mathbb{Z}} \left(\frac{|\widehat{\varphi}(b_k)-\widehat{\varphi}(a_k)|}{|m_\varphi(b_k)|} + |\widehat{\varphi}(a_k)|\,\frac{|m_\varphi(a_k)-m_\varphi(b_k)|}{|m_\varphi(a_k)m_\varphi(b_k)|}\right)\\
&\leq \sum_{k\in\mathbb{Z}} \left(\frac{|\widehat{\varphi}(b_k)-\widehat{\varphi}(a_k)|}{C_1} + \|\widehat{\varphi}\|_\infty\,\frac{|m_\varphi(a_k)-m_\varphi(b_k)|}{C_1^2}\right).
\end{aligned}$$

Consequently, $\widehat{s} \in W(C, l_1)$ and $\widehat{s}\,' \in L_1(\mathbb{R})$ so that also $g \in L_1(\mathbb{R})$. Hence (8) and (11a″) imply (11a), (11b) and (11b″). □

3.2. The sampling theorem

Theorem 2 shows that convolutions $\widehat{\widehat{q}} = \chi_{[-\pi,\pi]} * g$ with $\int g(x)\,dx = 1$ give the interpolating Riesz bases of a large class of shift-invariant spaces and a *regular* sampling reconstruction $f = \sum_k f(k)\widetilde{q}_k$. From this and [12, Theorem 3.2] follows that for some *jitter bound* $\varepsilon > 0$, an *irregular* sampling reconstruction $f = \sum_k f(k+\varepsilon_k)\widetilde{q_{k+\varepsilon_k}}$ holds whenever $\sup|\varepsilon_k| \le \varepsilon$.

By Theorem 1, $\varepsilon < 1/2$. It can be a very difficult task to find good estimates of ε from below. Our main result in this section is Theorem 5, where we under an additional mild decay assumption on the inverse Fourier transform $\check{g}$ of g derive the invertibility condition (19), which we thereafter use in Section 4 for computing jitter bounds ε. We present our main results first and then end this section with the proofs.

As outlined in Section 2, our approach is to study the coefficient mapping with doubly infinite matrix representation

$$\Phi_{jk} = \overline{\varphi(x_k - j)}, \qquad x_k = k + \varepsilon_k, \qquad \sup|\varepsilon_k| = \varepsilon < \frac{1}{2}. \tag{15a}$$

More precisely, for different interpolating $\varphi(x) = 2\pi\check{g}(x)\,\mathrm{sinc}(x)$, we aim to find ε such that the invertibility condition $\|\Phi\,\Lambda - \mathrm{I}\| < 1$ holds with Λ chosen in a way that seems likely to make $\|\Phi\Lambda - I\|$ smaller than $\|\Phi - I\|$. We choose Λ to be the $l_2 \to l_2$ operator with diagonal matrix-representation

$$(\Lambda)_{j,k} = \delta_{jk}/\overline{\varphi(\varepsilon_k)} \qquad \text{and} \qquad 0 < \inf_k |\varphi(\varepsilon_k)| \le \sup_k |\varphi(\varepsilon_k)| < \infty, \tag{15b}$$

where the last inequalities make Λ bounded and bijective. Hence $(\Phi\Lambda)_{k,k} = 1$ and

$$\begin{aligned}(\Phi\Lambda)_{j,k} =& \frac{2\pi\overline{\check{g}(k-j+\varepsilon_k)}}{2\pi\overline{\check{g}(\varepsilon_k)}}\,\frac{\sin(\pi(k-j+\varepsilon_k))}{\pi(k-j+\varepsilon_k)}\,\frac{\pi\varepsilon_k}{\sin(\pi\varepsilon_k)}\\ =& \frac{(-1)^{k-j}}{k-j+\varepsilon_k}\overline{\check{g}(k-j+\varepsilon_k)\varepsilon_k/\check{g}(\varepsilon_k)}, \qquad\qquad \text{for } k \ne j.\end{aligned}$$

Thus there is a diagonal matrix Λ_d, a convolution matrix A and a perturbation operator B such that

$$\begin{aligned}\Phi\Lambda - I =& (\mathrm{A}+\mathrm{B})\Lambda_\mathrm{d} \qquad \text{with} \qquad (\Lambda_\mathrm{d})_{j,k} = \overline{\varepsilon_k/\check{g}(\varepsilon_k)}\delta_{j,k}, \qquad \mathrm{A}_{k,k} = \mathrm{B}_{k,k} = 0,\\ \mathrm{A}_{j,k} =& \frac{(-1)^{k-j}}{k-j}\overline{\check{g}(k-j)} \qquad \text{and} \qquad \mathrm{B}_{j,k} = \frac{(-1)^{k-j}}{k-j+\varepsilon_k}\overline{\check{g}(k-j+\varepsilon_k)} - \mathrm{A}_{j,k}.\end{aligned} \tag{16}$$

Separate estimates of Λ_d, A and B give the following theorem:

Theorem 4. *Let g, Φ, A and B be defined by (2a), (15) and (16) with $\int_\mathbb{R} g(\xi)\,d\xi = 1$. Then Φ is a bounded bijective mapping of l_2 onto l_2 (thus with bounded inverse) if*

$$\sup_{|\alpha|\le\varepsilon}\left|\frac{\alpha}{\check{g}(\alpha)}\right|(\|\mathrm{A}\| + \|\mathrm{B}\|) < 1. \tag{17a}$$

Moreover, for the 2π-periodization $Pg = \sum_{k\in\mathbb{Z}} g(\cdot + 2\pi k)$,

$$\|\mathrm{A}\| = \left\|\sum_{n\ne0}\frac{(-1)^n}{n}\check{g}(n)e^{in\cdot}\right\|_\infty = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi Pg(\cdot-\xi)\,d\xi\right\|_\infty \le \frac{\|g\|_1}{2} \tag{17b}$$

and

$$\|\mathrm{B}\| \le \left(\sum_{k\neq 0} \sup_{|\alpha_k|\le\varepsilon} \frac{\gamma(k,\alpha_k)}{|(k+\alpha_k)k|} \right)^{1/2} \left(\sup_{|\alpha|\le\varepsilon} \sum_{k\neq 0} \frac{\gamma(k,\alpha)}{|(k+\alpha)k|} \right)^{1/2}, \tag{17c}$$

with

$$\gamma(k,\alpha) = |k\check{g}(k+\alpha) - (k+\alpha)\check{g}(k)| = \frac{1}{2\pi}\left| \int_{\mathbb{R}} \left(k\left(e^{i\alpha\xi}-1\right)-\alpha\right) g(\xi)e^{ik\xi}\,d\xi \right|. \tag{17d}$$

Next we use a simplified version of (17c) to obtain more easily computed, albeit possibly smaller jitter bounds ε:

Corollary 1. *For* A, Φ *and* γ *as in Theorem 4, define*

$$G(k,\varepsilon) \stackrel{\text{def}}{=} \sup\{\gamma(l,\alpha):\ l=\pm k,\ |\alpha|\le\varepsilon\} \qquad \text{for } 0\le\varepsilon\le\tfrac{1}{2}. \tag{18}$$

Then, with $\|\mathrm{A}\|$ *given in* (17b), Φ *is a bounded bijective mapping of* l_2 *onto* l_2 *if*

$$S(\varepsilon) \stackrel{\text{def}}{=} \sup_{|\alpha|\le\varepsilon} \left|\frac{\alpha}{\check{g}(\alpha)}\right| \left(\|\mathrm{A}\| + 2\left(\sum_{k=1}^{\infty} \frac{G(k,\varepsilon)}{(k-\varepsilon)k} \sum_{k=1}^{\infty} \frac{G(k,\varepsilon)}{k^2-\varepsilon^2} \right)^{1/2} \right) < 1. \tag{19}$$

Finally, under a mild decay assumption on $\check{g}$, we get our main sampling theorem:

Theorem 5. *Suppose that for some* $\nu > 0$, $\check{g}(x) = O(|x|^{-\nu})$ *as* $|x|\to\infty$. *Then the function* S *in* (19) *is continuous and increasing with* $S(0)=0$ *and* $S(\frac{1}{2})\ge 1$. *Hence, the equation* $S(\varepsilon)=1$ *has either one solution or a solution set* $[a,b]$ *in* $(0,1/2]$. *For* φ *satisfying* (2b), *if* $S(\varepsilon)<1$ *and* $\sup|\varepsilon_k|\le\varepsilon$, *then* $(q_{k+\varepsilon_k})$ *is a Riesz basis for* V.

Proof of Theorem 4. By (15a) and (16), $\|\Phi\Lambda - I\| \le \sup_{|\alpha|\le\varepsilon}\left|\frac{\alpha}{\check{g}(\alpha)}\right|(\|\mathrm{A}\|+\|\mathrm{B}\|)$, so that (17a) implies bounded invertible Φ. By (16),

$$(\mathrm{A}x)_j = \sum_{k\in\mathbb{Z}} a_{j-k}x_k \qquad \text{with} \qquad a_n = -\frac{(-1)^n}{n}\overline{\check{g}(-n)}(1-\delta_{0,n}).$$

Hence by Lemma 1, $\|\mathrm{A}\| = \|f\|_\infty \le \infty$ for $f=\sum_{n\in\mathbb{Z}} a_n e^{in\cdot} \in L_2$. Moreover,

$$a_n = b_n c_n \qquad \text{with} \qquad b_n = -\frac{(-1)^n}{n}(1-\delta_{0,n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} i\xi e^{-in\xi}\,d\xi \tag{20}$$

and, via the Fubini–Tonelli theorem,

$$c_n = \overline{\frac{1}{2\pi}\int_{\mathbb{R}} g(\xi)e^{-in\xi}\,d\xi} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}\overline{g(\xi+2\pi k)}e^{in\xi}\,d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Pg(-\xi)}e^{-in\xi}\,d\xi.$$

Consequently f is the cyclic convolution

$$f(\nu) = \frac{i}{2\pi}\int_{-\pi}^{\pi}\xi\overline{Pg(-(\nu-\xi))}\,d\xi = -\frac{i}{2\pi}\int_{-\pi}^{\pi}\xi\overline{Pg(-\nu-\xi)}\,d\xi,$$

so that (17b) follows from the fact $\|f\|_\infty = \|f(-\cdot)\|_\infty$ and the Hölder inequality.

Next, for the operator B defined in (16), the first and second factors in the Schur interpolation theorem (5c) are

$$\begin{aligned}\sup_{j\in\mathbb{Z}}\sum_{k\neq j}\left|\frac{\overline{\check g(k-j+\varepsilon_k)}}{k-j+\varepsilon_k}-\frac{\overline{\check g(k-j)}}{k-j}\right| &\leq \sum_{k\neq 0}\sup_{|\alpha_k|\leq\varepsilon}\left|\frac{\check g(k+\alpha_k)}{k+\alpha_k}-\frac{\check g(k)}{k}\right| \\ &= \sum_{k\neq 0}\sup_{|\alpha_k|\leq\varepsilon}\frac{\gamma(k,\alpha_k)}{|(k+\alpha_k)k|}\end{aligned}$$

and

$$\begin{aligned}\sup_{k\in\mathbb{Z}}\sum_{j\neq k}\left|\frac{\overline{\check g(k-j+\varepsilon_k)}}{k-j+\varepsilon_k}-\frac{\overline{\check g(k-j)}}{k-j}\right| &\leq \sup_{|\alpha|\leq\varepsilon}\sum_{j\neq 0}\left|\frac{\check g(j+\alpha)}{j+\alpha}-\frac{\check g(j)}{j}\right| \\ &= \sup_{|\alpha|\leq\varepsilon}\sum_{j\neq 0}\frac{\gamma(j,\alpha)}{|(j+\alpha)j|},\end{aligned}$$

respectively, from which (17c) follows. Finally, a direct evaluation of the inverse Fourier transform gives the last equality in (17d). □

Proof of Corollary 1. Since $\varepsilon < 1/2$, we get upper bounds for the factors in (17c) from

$$\sum_{k\neq 0}\sup_{|\alpha_k|\leq\varepsilon}\frac{\gamma(k,\alpha_k)}{|(k+\alpha_k)k|}\leq\sum_{k\neq 0}\sup_{|\alpha_k|\leq\varepsilon}\frac{G(k,\varepsilon)}{|(k+\alpha_k)k|}=2\sum_{k=1}^{\infty}\frac{G(k,\varepsilon)}{(k-\varepsilon)k},$$

and, since $1/(\alpha+k) - 1/(\alpha-k) = 2k/(k^2-\alpha^2)$,

$$\sup_{|\alpha|\leq\varepsilon}\sum_{k\neq 0}\frac{\gamma(k,\alpha)}{|(k+\alpha)k|}\leq\sup_{|\alpha|\leq\varepsilon}\sum_{k\neq 0}\frac{G(k,\varepsilon)}{|(k+\alpha)k|}=2\sum_{k=1}^{\infty}\frac{G(k,\varepsilon)}{k^2-\varepsilon^2}.$$

Hence (19) implies (17a) so that Theorem 4 completes our proof. □

Proof of Theorem 5. By definition, $G(k,\cdot)$ is continuous, increasing and $G(k,0) = 0$. If $\check g(x) = O(|x|^{-\nu})$ as $|x| \to \infty$, then $G(k,\varepsilon) = O(k^{1-\nu})$ so that both series in (19) converge uniformly to a continuous function. Hence also S is continuous, increasing and $S(0) = 0$ because $\check g(0) = \int g(x)\,dx/(2\pi) = 1/(2\pi)$ and $\|\mathrm{A}\| < \infty$. We also know from Theorem 1 that $1 \leq S(1/2) \leq \infty$. Hence $S(\varepsilon) = 1$ for exactly one $\varepsilon \in (0, 1/2]$ or for all ε in some interval $[a,b] \subseteq (0,1/2]$. For φ satisfying (2b), (3) follows and the last statement of the theorem follows exactly as explained in the beginning of this section. □

Remark 4. It is clear from (17d)–(19) that $S(\varepsilon)$ is constant in an interval $[a,b]$ only if for $|x| \leq b$ and integers $k \neq 0$,

$$\check g(k+x) = \left(1+\tfrac{x}{k}\right)\check g(k) \qquad\text{and}\qquad \sup_{|x|\leq b}\left|\tfrac{x}{\check g(x)}\right| = \sup_{|x|\leq a}\left|\tfrac{x}{\check g(x)}\right|.$$

It is not very difficult to construct a g that satisfies this condition as well as the conditions $\int g(\xi)\,d\xi = 1$, (2b) and $|\check{g}(x)| = O(|x|^{-\nu})$ of Theorem 5. One example is $\check{g}(x) = \left(\sin^2(2\pi x) + \frac{1}{2\pi}\cos^2(\pi x)\right)\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x)$ with $a = 0.1$ and $b = 0.2$.

4. Examples

We will now demonstrate a few different kinds of applications of our main theorems to different spaces, in all cases by computing $\varepsilon \in (0, 1/2)$ satisfying equation (19), using different estimates of $\sup_{|\alpha|\leq\varepsilon} |\alpha/\check{g}(\alpha)|$, $\|\mathrm{B}\|$ or $G(k,\varepsilon)$ when necessary.

We begin with the classical example $\varphi(x) = \mathrm{sinc}(x)$ in Section 4.1. In Section 4.2 we demonstrate how to use our results for computing joint jitter error bounds for whole classes of spaces with g having bounded variation and compact support, including all Meyer scaling functions. In Section 4.3 and 4.4 we compute bounds for two particular choices of φ for which we know of no previously published bounds. Finally, in Section 4.5, we improve some previously known bounds for B-spline wavelets.

4.1. Shannon (g = Dirac measure)

Until now we have always assumed g to be a function but the results that are crucial for computing jitter bounds hold also if g is the Dirac measure, that is, if $\widehat{\varphi} = \chi_{[-\pi,\pi]}$ and $\varphi(x) = \frac{\sin(\pi x)}{\pi x} = \mathrm{sinc}(x)$, for which (2b) guarantees the existence of reproducing kernels q_x. As in Theorem 2, φ is interpolating and (1) holds, so (φ_k) is a Riesz basis for the closure V of its span and $\widetilde{\varphi} = q$. Moreover, $\check{g}(t) = 1/(2\pi)$, so that (17b) via (20) gives that

$$\|\mathrm{A}\| = \left\| \sum_{n\neq 0} \frac{(-1)^n}{2\pi n} e^{in\cdot} \right\|_\infty = \frac{1}{2}.$$

Further, $G(k,\varepsilon) = \frac{\varepsilon}{2\pi}$, reducing the invertibility condition (19) to

$$S(\varepsilon) = \varepsilon\left(\pi + 2\varepsilon\sqrt{\sum_{k=1}^{\infty}\frac{1}{(k-\varepsilon)k}}\sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2-\varepsilon^2}}\right) < 1, \qquad 0 \leq \varepsilon < \frac{1}{2}.$$

A numerical solution of $S(\varepsilon) = 1$ and rounding down gives $\varepsilon \approx 0.2463$, which is smaller than the well-known largest possible upper bound $\varepsilon = 1/4$ (see Example 1.1 and further references in [12]).

Under the additional restriction that all $\varepsilon_k = \varepsilon_{-k}$ we get the bound $\varepsilon = 1/4$ by noting in the proof of Corollary 1 that the sum $\sum \frac{1}{(k-\varepsilon)k}$ then should be replaced with $\sum \frac{1}{k^2-\varepsilon^2}$, giving the equation $S(\varepsilon) = \varepsilon\left(\sum_{k=1}^{\infty}\frac{2\varepsilon}{k^2-\varepsilon^2} + \pi\right) = 1$, which we rewrite as follows and identify the partial fraction expansion of cot:

$$\pi = \frac{1}{\varepsilon} + \sum_{k=1}^{\infty}\frac{2\varepsilon}{\varepsilon^2 - k^2} = \pi\cot(\pi\varepsilon), \qquad \text{hence } \varepsilon = 1/4.$$

4.2. Compactly supported g with bounded variation

Let $(\widehat{\widehat{q}} =)\widehat{\varphi} = \chi_{[-\pi,\pi]} * g$ with g having total variation V, $\int g(\xi)\, d\xi = 1$ and $\operatorname{supp} g \subset (-M, M)$. Via Example 1, this includes all Meyer scaling functions as defined in [23, pp. 22–23]. It follows that $\check{g}$ is differentiable, $|\check{g}'(x)| = \left| \frac{1}{2\pi} \int_{-M}^{M} i\xi g(\xi) e^{ix\xi}\, d\xi \right| \le \frac{M\|g\|_1}{2\pi}$ and

$$\begin{aligned} |\check{g}(x)| \ge& |\check{g}(0) - |\check{g}(x) - \check{g}(0)|| \\ =& \tfrac{1}{2\pi} \left| 1 - \left| \int_{-M}^{M} g(\xi)(e^{i\xi x} - 1)\, d\xi \right| \right| \ge \tfrac{1 - M\|g\|_1 |x|}{2\pi} \quad \text{for all} \quad |x| \le \tfrac{1}{M\|g\|_1}. \end{aligned}$$

Hence the first factor in (19) satisfies

$$\sup_{|\alpha| \le \varepsilon} \left| \tfrac{\alpha}{\check{g}(\alpha)} \right| \le \tfrac{2\pi\varepsilon}{1 - M\|g\|_1 \varepsilon} \qquad \text{for} \qquad 0 \le \varepsilon < \min\left(\tfrac{1}{M\|g\|_1}, \tfrac{1}{2} \right). \tag{21}$$

For such ε, insertion of the estimates (17b) and (21) in (19) gives

$$S(\varepsilon) \le \tfrac{2\pi\varepsilon}{1 - M\|g\|_1 \varepsilon} \left(2 \left(\sum_{k=1}^{\infty} \tfrac{G(k,\varepsilon)}{(k-\varepsilon)k} \sum_{k=1}^{\infty} \tfrac{G(k,\varepsilon)}{k^2 - \varepsilon^2} \right)^{1/2} + \tfrac{\|g\|_1}{2} \right), \qquad \varepsilon \le \tfrac{1}{M\|g\|_1}. \tag{22}$$

(Hence the right-hand side is larger than 1 unless $\frac{2\pi\varepsilon}{1 - M\|g\|_1 \varepsilon} \cdot \frac{\|g\|_1}{2} \le 1$, that is, unless $\varepsilon \le \frac{1}{M + \pi\|g\|_1}$.) We will estimate $G(k, \varepsilon)$ using the three different estimates (23) below for $\gamma(k, \alpha)$. First, from (17d) we get for $|\alpha| \le \varepsilon$ that

$$\begin{aligned} \gamma(k, \alpha) =& \tfrac{1}{2\pi} \left| \int_{-M}^{M} \left(k \left(e^{i\alpha\xi} - 1 \right) - \alpha \right) g(\xi) e^{ik\xi}\, d\xi \right| \\ \le& \tfrac{1}{2\pi} \int_{-M}^{M} \left(|k\xi\alpha| + |\alpha| \right) |g(\xi)|\, d\xi \le \tfrac{\varepsilon}{2\pi} (|k|\, M + 1) \|g\|_1 . \end{aligned} \tag{23a}$$

This estimate does not satisfy the bound $G(k, \varepsilon) = O(k^{1-\nu})$ that we used in the proof of Theorem 5. Thus we will also use the estimate that we get from integration by parts and use of the bounded variation in the first integral:

$$\begin{aligned} \gamma(k, \alpha) =& \tfrac{1}{2\pi} \left| \int_{-M}^{M} \left(\tfrac{k}{-i(\alpha+k)} e^{-i(\alpha+k)\xi} - \tfrac{k}{-ik} e^{-ix\xi} - \tfrac{\alpha}{-ik} e^{-ik\xi} \right) dg \right| \\ \le& \tfrac{1}{2\pi} \left(\left| \tfrac{k}{\alpha+k} \right| + 1 + \left| \tfrac{\alpha}{k} \right| \right) V \le \tfrac{V}{2\pi} \left(\tfrac{|k|}{|k| - \varepsilon} + 1 + \tfrac{\varepsilon}{|k|} \right) \end{aligned} \tag{23b}$$

for all $|\alpha| \le \varepsilon$. In the integration by parts, if we instead integrate only $e^{-ik\xi}$ then we do instead have to calculate the total variation of $(k(e^{-i\alpha\xi} - 1) - \alpha) g(\xi)$, which, again for $|\alpha| \le \varepsilon$, equals $(|k|\, M\varepsilon + \varepsilon) V + 2 \|g\|_\infty |k|\, M\varepsilon$ and gives

$$\gamma(k, \alpha) \le \tfrac{\varepsilon}{2\pi} \left(\left(M + \tfrac{1}{|k|} \right) V + 2 \|g\|_\infty M \right). \tag{23c}$$

We will use the estimates (22)–(23) in the following two ways:

1. The estimates (23a) and (23b) combined give

$$G(k, \varepsilon) \le \tfrac{1}{2\pi} \min \left(\varepsilon (Mk + 1) \|g\|_1 , V \left(\tfrac{2k - \varepsilon}{k - \varepsilon} + \tfrac{\varepsilon}{k} \right) \right). \tag{24a}$$

   Hence partial sums with $O(1/\varepsilon)$ terms give error $O(\varepsilon)$ in (22). For nonnegative g, $\|g\|_1 = \int g(\xi)\, d\xi = 1$, so that (22) and (24a) depend only on ε, M and V. Note also

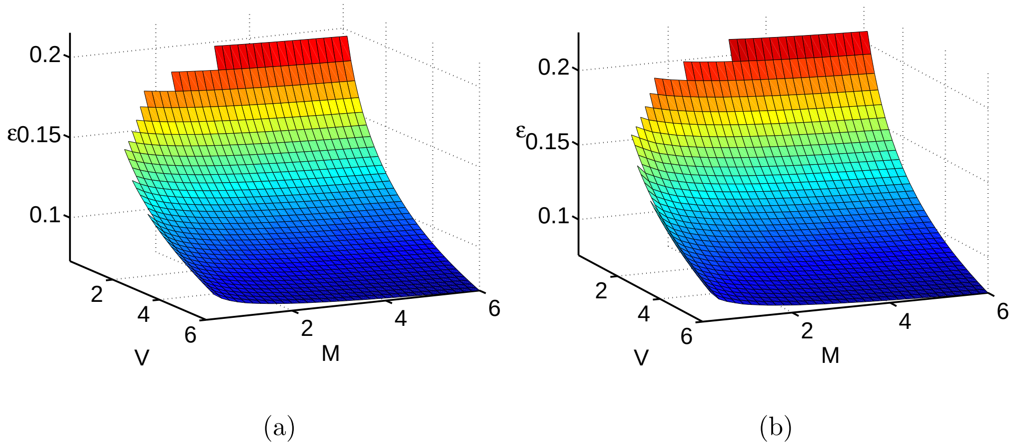


Figure 1: Jitter error bounds for g with total variation V and support in $(-M, M)$ for (a) nonnegative g and (b) g increasing in one interval $(-M, a)$ and (b) decreasing in (a, M). Computed from the estimates (22) and (24).

that $MV \geq \int g(\xi)\, d\xi = 1$. For some such M and V, we have plotted the resulting jitter error bounds ε in Figure 1 (a)
Since the $S(\varepsilon)$ obtained from (22) and (24a) is an increasing[2] function of $\|g\|_1$, the plotted bounds are better than the corresponding bounds obtained from g not being nonnegative.

2. In addition to the above, if g is increasing on $(-M, a]$ and decreasing on $[a, M)$, then its variation will be $V = 2g(a) = 2\|g\|_\infty$, so that (23) gives

$$G(k, \varepsilon) \leq \tfrac{1}{2\pi} \min\left(\varepsilon(Mk+1)\|g\|_1, V\left(\tfrac{2k-\varepsilon}{k-\varepsilon} + \tfrac{\varepsilon}{k}\right), \varepsilon V\left(2M + \tfrac{1}{k}\right)\right). \qquad (24b)$$

Insertion in (22), again with $\|g\|_1 = 1$ gives an equation with results plotted in Figure 1 (b).

Remark 5. We got our basic estimate (22) from the inequalities (17b) and (21), which both can be expected to be good estimates for well localized g. In fact, for g being ("close to a Dirac"), (17b) should be a good estimate and $\check{g}$ should be slowly varying, so we can expect to have $\check{g}(\alpha) \approx 1/(2\pi)$ for small α, as in (21) for small M.

4.3. $\varphi(x) = \operatorname{sinc}(x)a^{|x|}$ *with* $0 < a < 1$

For $g_0(\xi) = 1/(\pi(1+\xi^2))$, $\int g_0(\xi)d\xi = 1$, so the same also holds for any dilation

$$g(\xi) = -\frac{1}{\ln(a)} g_0\left(-\frac{\xi}{\ln(a)}\right) = -\frac{\ln a}{\pi((\ln(a))^2 + \xi^2)}, \qquad 0 < a < 1,$$

[2]A jitter bound should of course not depend on a normalization of φ or g. This is the case in Theorem 4: Replacing g with cg for some $c \in \mathbb{R}$ would not change the left-hand side of (17a) and is therefore uninteresting. The same holds for the other estimates here, which is clear if you note that the first right-hand side denominator in (21) actually is $\int g(\xi)\, dx - M\|g\|_1 \varepsilon$.

which also satisfies the conditions (2) since

$$\check{g}(x) = \frac{1}{2\pi} a^{|x|}, \qquad \varphi(x) = \operatorname{sinc}(x) a^{|x|} \qquad \text{and}$$

$$\widehat{\varphi}(\xi) = \frac{1}{\pi}\left(\arctan\left(\frac{\xi-\pi}{\ln a}\right) - \arctan\left(\frac{\xi+\pi}{\ln a}\right)\right).$$

Thus Theorem 5 applies. As in (17b), $\|\mathrm{A}\| = \|f\|_\infty$ with

$$\begin{aligned}
f(x) &= \sum_{n\neq 0} \frac{(-1)^n}{n}\frac{1}{2\pi} a^{|n|} e^{inx} = \frac{i}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} a^n \sin(nx) \\
&= \frac{i}{\pi}\operatorname{Im}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} a^n e^{inx} = \frac{i}{\pi}\operatorname{Im}\sum_{n=1}^{\infty}\frac{z^n}{n}, \qquad z = -ae^{ix}, \qquad 0<a<1, \\
f'(x) &= \frac{i}{\pi}\operatorname{Im}\sum_{n=0}^{\infty} z^n = \frac{i}{\pi}\operatorname{Im}\frac{1}{1-z} \qquad \text{and} \qquad f(0)=0, \qquad \text{so that} \\
f(x) &= -\tfrac{i}{\pi}\operatorname{Im}\log(1-z) = -\tfrac{i}{\pi}\arctan\left(\tfrac{\operatorname{Im}(1-z)}{\operatorname{Re}(1-z)}\right) = -\tfrac{i}{\pi}\arctan\left(\tfrac{a\sin(x)}{1+a\cos(x)}\right).
\end{aligned}$$

Hence for A and the first factor in (17a) we get

$$\|\mathrm{A}\| = \|f\|_\infty = f(\arccos a - \pi) = \frac{1}{\pi}\arctan\left(\frac{a}{\sqrt{1-a^2}}\right) \qquad \text{and}$$

$$\sup_{|\alpha|\le\varepsilon}\left|\frac{\alpha}{\check{g}(\alpha)}\right| = 2\pi\varepsilon a^{-\varepsilon}.$$

From (17d) with $\check{g}(x) = \frac{1}{2\pi}a^{|x|}$,

$$\begin{aligned}
\gamma(k,\alpha) &= \tfrac{1}{2\pi}\left|k\check{g}(k+\alpha) - (k+\alpha)\check{g}(k)\right| = \tfrac{1}{2\pi}\left|ka^{|k+\alpha|} - (k+\alpha)a^{|k|}\right| \\
&= \tfrac{1}{2\pi}\left|ka^{|k|+\alpha\operatorname{sgn}(k)} - (k+\alpha)a^{|k|}\right| = \tfrac{1}{2\pi}a^{|k|}\left|ka^{\alpha\operatorname{sgn}(k)} - (k+\alpha)\right|
\end{aligned}$$

Hence if $\operatorname{sgn}(\alpha) = \operatorname{sgn}(k)$ and $|\alpha|\le\varepsilon$, then

$$\begin{aligned}
\gamma(k,\alpha) &= \tfrac{1}{2\pi}a^{|k|}\left(|k| + |\alpha| - |k|\,a^{|\alpha|}\right) \le \tfrac{1}{2\pi}a^{|k|}\left(|k| + \varepsilon - |k|\,a^{\varepsilon}\right) \\
&\le \tfrac{1}{2\pi}a^{|k|}\left(|k|\,(1-a^{\varepsilon}) + \varepsilon\right) \overset{\text{def}}{=} \gamma_0.
\end{aligned}$$

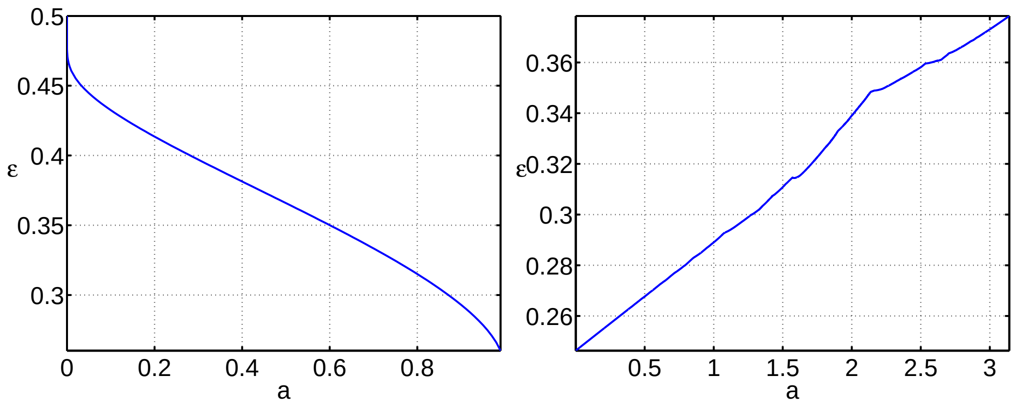
Similarly, if $\operatorname{sgn}(\alpha) = -\operatorname{sgn}(k)$ and $|\alpha|\le\varepsilon$, then

$$\begin{aligned}
\gamma(k,\alpha) &= \tfrac{1}{2\pi}a^{|k|}\left(|k|\,a^{-|\alpha|} - (|k| - |\alpha|)\right) \le \tfrac{1}{2\pi}a^{|k|}\left(|k|\,a^{-\varepsilon} - (|k|+\varepsilon)\right) \\
&\le \tfrac{1}{2\pi}a^{|k|}\left(|k|\,(a^{-\varepsilon}-1) + \varepsilon\right) = \tfrac{1}{2\pi}a^{|k|}\left(|k|\,(1-a^{\varepsilon})a^{-\varepsilon} + \varepsilon\right) \ge \gamma_0
\end{aligned}$$

Insertion in (18) and then (19) gives

$$G(k,\varepsilon) \le \tfrac{1}{2\pi}a^k(k(a^{-\varepsilon}-1)+\varepsilon) \qquad \text{and} \tag{25a}$$

$$S(\varepsilon) \le 2\varepsilon a^{-\varepsilon}\left(\arctan\left(\frac{a}{\sqrt{1-a^2}}\right) + 2\pi\left(\sum_{k=1}^{\infty}\frac{G(k,\varepsilon)}{k(k-\varepsilon)}\sum_{k=1}^{\infty}\frac{G(k,\varepsilon)}{k^2-\varepsilon^2}\right)^{1/2}\right). \tag{25b}$$



(a) (b)

Figure 2: Jitter error bounds for (a) $\check{g}(x) = \frac{1}{2\pi} a^{|x|}$ and (b) $\check{g}(x) = \frac{1}{2\pi}\,\mathrm{sinc}\left(\frac{a}{\pi}x\right)$. Computed from the estimates (25) and (26), respectively.

A numerical solution is shown in Figure 2 (a). Note in (25b) that if we fix $\varepsilon < 1/2$, then $S(\varepsilon) \to 0$ when $a \to 0+$. Hence for any $\varepsilon \in (0, 1/2)$, a can be chosen so that $\varphi(x) = \frac{\sin \pi x}{\pi x} a^{|x|}$ spans a shift-invariant space that allows for reconstruction from samples with jitter error bound $\sup |\varepsilon_k| \le \varepsilon$. This is a bit more restrictive than the Franklin scaling function, which alone allows for $\sup |\varepsilon_k| < 1/2$ (see, for example, [12, Theorem 3.3]), whereas we know from Theorem 1 that the same is not possible for $\sup |\varepsilon_k| = 1/2$.

4.4. $\varphi(x) = \mathrm{sinc}(x)\,\mathrm{sinc}\left(\frac{a}{\pi}x\right)$

For $g = \frac{1}{2a}\chi_{[-a,a]}$, $\check{g}(x) = \frac{1}{2\pi}\,\mathrm{sinc}\left(\frac{a}{\pi}x\right)$ and Theorem 5 applies. For simplicity, we will assume that $0 < a \le \pi$, so that Pg equals g on $[-\pi, \pi)$, which on insertion in (17b) gives that $\|\mathrm{A}\| = \frac{\pi - a}{2\pi}$. For $\varepsilon \le 1/2$ and $0 < a \le \pi$ the first factor in (19) is $\sup_{|\alpha| \le \varepsilon} \left| \frac{\alpha}{\check{g}(\alpha)} \right| \le \frac{2\pi a \varepsilon^2}{\sin a\varepsilon}$. To estimate $\gamma(k, \alpha)$ we Taylor expand $\check{g}(k + \alpha)$ around $\alpha = 0$ i.e.,

$$\begin{aligned} \gamma(k,\alpha) &= |k\check{g}(k+\alpha) - (k+\alpha)\check{g}(k)| \\ &= \left| (k\check{g}'(k) - \check{g}(k))\,\alpha + k\check{g}''(k)\frac{\alpha^2}{2} + k\check{g}'''(k)\frac{\alpha^3}{3!} + \ldots \right|. \end{aligned}$$

A first order expansion gives that for some $x \in [0, \alpha]$,

$$\gamma(k,\alpha) = |(k\check{g}'(k+x) - \check{g}(k))\alpha| \le \varepsilon\left(|k\check{g}'(k+x)| + |\check{g}(k)|\right).$$

For small $|ak|$ and $|\alpha| \le \varepsilon$ it is reasonable to estimate $\check{g}'$ by its global maximum:

$$\gamma(k,\alpha) \le \tfrac{\varepsilon}{2\pi}\left(\left|\tfrac{\sin ak}{ak}\right| + |k|\,aM_1\right),$$

with $M_1 = \sup_{x\in\mathbb{R}} \left| \frac{d}{dx}\frac{\sin(x)}{x} \right| = \sup_{x\in\mathbb{R}} \left| \frac{1}{2\pi x}\left(\cos(x) - \frac{\sin(x)}{x}\right) \right|$. For larger $|ak|$, a more promising estimate is

$$\gamma(k,\alpha) \le \tfrac{\varepsilon}{2\pi}\left(\left|\tfrac{\sin(ak)}{ak}\right| + \tfrac{|k|}{|k|-\varepsilon}M_2\right) \qquad \text{with} \qquad M_2 = \sup_{x\in\mathbb{R}} \left|\cos(x) - \tfrac{\sin(x)}{x}\right|.$$

Similarly, the second order expansion gives the bound

$$\gamma(k,\alpha) \leq \tfrac{1}{2\pi}\left(\varepsilon\left|\cos(ak) - 2\tfrac{\sin(ak)}{ak}\right| + \tfrac{1}{2}\left|k\right| a^2 M_3 \varepsilon^2\right),$$

with $M_3 \sup\left|\frac{d^2}{dx^2}\frac{\sin(x)}{x}\right| = \sup\left|\frac{1}{2\pi x}\cdot\frac{(2-x^2)\sin(x)-2x\cos(x)}{x^2}\right|$ or for large $|ak|$ rather

$$\gamma(k,\alpha) \leq \tfrac{1}{2\pi}\left(\varepsilon\left|\cos(ak) - 2\tfrac{\sin(ak)}{ak}\right| + \tfrac{1}{2}\tfrac{|k|a}{|k|-\varepsilon} M_4 \varepsilon^2\right)$$

with $M_4 = \sup\left|\frac{(2-x^2)\sin(x)-2x\cos(x)}{x^2}\right|$. Higher degree approximations can also be used, but we settle for these. Insertion in (19) gives

$$S(\varepsilon) \leq \frac{2\pi a\varepsilon^2}{\sin(a\varepsilon)}\left(2\left(\sum_{k=1}^{\infty}\frac{G_{\text{est}}(k,\varepsilon)}{k(k-\varepsilon)}\sum_{k=1}^{\infty}\frac{G_{\text{est}}(k,\varepsilon)}{k^2-\varepsilon^2}\right)^{1/2} + \frac{\pi - a}{2\pi}\right), \tag{26a}$$

where the estimate $G_{\text{est}}(k,\varepsilon)$ is the minimum of

$$G_1(k,\varepsilon) = \frac{\varepsilon}{2\pi}\left(\left|\frac{\sin(ak)}{ak}\right| + \min\left(kaM_1, \frac{k}{k-\varepsilon}M_2\right)\right), \tag{26b}$$

and

$$G_2(k,\varepsilon) = \frac{\varepsilon}{2\pi}\left(\left|\cos ak - 2\frac{\sin(ak)}{ak}\right| + \min\left(\frac{ka^2M_3}{2}, \frac{ka}{2(k-\varepsilon)}M_4\right)\varepsilon\right). \tag{26c}$$

The solution as a function of a is plotted in Figure 2 (b).

4.5. B-splines

There are two primary reasons why Theorem 4 improves previously known bounds. *First*, recall that (9) with $X_0 = 0$ suggests that the generating function should be large near 0 and small near other integers. From this point of view, B-spline scaling functions B_n gets worse with increasing n by their definition

$$B_0 \stackrel{\text{def}}{=} \chi_{[-1/2,1/2]}, \qquad B_n \stackrel{\text{def}}{=} B_{n-1} * B_0 \qquad \text{for positive integers } n.$$

For $\varphi = B_n$, analysis of $\widetilde{q}$, on the other hand, is more likely to provide large jitter error bounds, due to (9) and the interpolation property $\widetilde{q}(k) = \delta_{0,k}$. *One other* reason is the splitting up into matrices A and B in (16). We can apply Theorem 4 without this splitting, simply by setting $A = 0$ and replacing (17c) with

$$\|\mathrm{B}\| \leq \left(\sum_{k\neq 0}\sup_{|\alpha_k|\leq\varepsilon}\frac{\check{g}(k+\alpha_k)}{k+\alpha_k}\right)^{1/2}\left(\sup_{|\alpha|\leq\varepsilon}\sum_{k\neq 0}\frac{\check{g}(k+\alpha)}{k+\alpha}\right)^{1/2}. \tag{27}$$

Table 1 shows that the resulting jitter bounds are better than those in [12, 27] for the B-spline spaces generated by B_4–B_8, even though the bounds in [12] were computed by more carefully using a more precise knowledge of the exact shape of $\varphi = B_n$ than we have about $\widetilde{q}$, so the interpolating property must be the reason for obtaining better bounds from analysis of $\widetilde{q}$. The rightmost column in Table 1 shows that for B_2 and B_3 the split into operators $\mathrm{A}+\mathrm{B}$ does not give a larger ε, but for the spaces generated by B_4–B_8 it gives a clear improvement. This splitting was also necessary for the Shannon example in Section 4.1, for which the series in (27) does not converge, since the terms decay as $|k|^{-1}$, whereas after the splitting into operators $\mathrm{A}+\mathrm{B}$, the corresponding terms in (17c) are proportional to $|k|^{-2}$. Our understanding of Table 1 is that similar faster decay in (17c) is the reason why Theorem 4 gives the best bounds for $\varphi = B_4$–B_8.

For B_1, the full theory in this paper does not apply, since $\breve{g}$ is discontinuous in ± 1, but it is easy to check that in this case and with $\breve{g}(\pm 1) \overset{\text{def}}{=} 0$, it follows that $\mathrm{A} = 0$ and the method of Theorem 4 coincides with the one in [12], thus giving the same bounds.

4.5.1. Computing the bounds in Table 1

The B-spline examples $\varphi = B_n$ are different from the previous ones in the sense that for $n > 1$, we can only compute $\widetilde{q}$ numerically.

For $n > 1$ and $\varphi = B_n$, m_φ is a positive trigonometric polynomial, so $1/m_\varphi \in C^\infty$ and $1/m_\varphi(\xi) = \sum_k a_k e^{-ik\xi}$ with $|a_k| \le c_m |k|^{-m}$ for all positive integers m, so that, by the Fubini–Tonelli theorem,

$$\widetilde{q}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \left(\sum_{k\in\mathbb{Z}} a_k e^{-ik\xi} \right) e^{ix\xi}\, d\xi = \sum_{k\in\mathbb{Z}} a_k \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) e^{i(x-k)\xi}\, d\xi = \sum_{k\in\mathbb{Z}} a_k \varphi_k(x).$$

Recall from (8) that $0 < C_1 \le |m_\varphi| \le C_2 < \infty$. For $c \overset{\text{def}}{=} \frac{C_1+C_2}{2}$, $\left|1 - \frac{1}{c} m_\varphi(\xi)\right| \le \frac{C_2 - C_1}{C_2 + C_1} < 1$ and hence $\frac{1}{m_\varphi(\xi)} = \frac{1}{c} \frac{1}{1 - \left(1 - \frac{1}{c} m_\varphi(\xi)\right)} = \frac{1}{c} \sum_{m=0}^{\infty} \left(1 - \frac{1}{c} m_\varphi(\xi)\right)^m$. Consequently, a_k can be computed with exponential rate of convergence as an iterated convolution of the Fourier series coefficients of $1 - \frac{1}{c} m_\varphi$. Thus we can easily compute $\widetilde{q}$ with high enough precision for correct positioning of its zero-crossings, which is important for avoiding problems with singularities at the integers in $\breve{g} = \frac{\widetilde{q}}{2\pi\, \mathrm{sinc}}$. Moreover, m_φ is in C^∞, just like $\alpha/\breve{g}(\alpha)$ and $\gamma(k,\alpha)/|(k+\alpha)k|$ for $0 < |\alpha| < 1/2$. From this and the and the fact that we also easily can compute the derivatives $\widetilde{q}'$, $\widetilde{q}''$ and of course sinc', sinc'', there are no numerical problems involved in computing all the suprema in Theorem 4, thus retrieving the bounds in Table 1.

φ	*From [27]*	*From [12]*	(17b) *with* $A = 0$ *and* (27)	*Computed from* (17)
B_1	0.3535	**0.4142**	**0.4142**	**0.4142**
B_2	0.1767	**0.4068**	0.3745	0.3649
B_3	0.2222	**0.3389**	0.3244	0.3242
B_4	0.1563	0.2661	0.2982	**0.3169**
B_5	0.1123	0.1693	0.2752	**0.3051**
B_6	0.0794	0.0472	0.2584	**0.2987**
B_7	0.0563	—	0.2446	**0.2929**
B_8	0.0398	—	0.2334	**0.2886**

Table 1: B-spline jitter error bounds rounded down to four digits. **Boldface** print indicates the largest bound of those obtained from φ in [12, 27] and those obtained from $\widetilde{q}$ via Theorem 4 with or without the splitting into two operators A and B in (16).

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