On Clarkson's inequality in the real case

Lech Maligranda^{*1} and Natalia Sabourova^{**1}

¹ Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden

Received 2 February 2006, accepted 23 October 2006

Published online 7 August 2007

Key words Clarkson inequality, Lebesgue spaces, non-positive operator, algebraic inequalities, normed spaces, mixed norm spaces

MSC (2000) Primary: 26D15, 47A30, 46E30; Secondary: 46B20, 46B70

The best constant in a generalized complex Clarkson inequality is $C_{p,q}(\mathbb{C}) = \max\{2^{1-1/p}, 2^{1/q}, 2^{1/q-1/p+1/2}\}$ which differs moderately from the best constant in the real case $C_{p,q}(\mathbb{R}) = \max\{2^{1-1/p}, 2^{1/q}, B_{p,q}\}$, where $B_{p,q} = \sup_{x \in [0,1]} \frac{((1+x)^q + (1-x)^q)^{1/q}}{(1+x^p)^{1/p}}$. For $1 < q < 2 < p < \infty$ the constant $C_{p,q}(\mathbb{R})$ is equal to $B_{p,q}$ and these numbers are difficult to calculate in general. As applications of the generalized Clarkson inequalities the (p, q) -Clarkson inequalities in Lebesgue spaces, in mixed norm spaces and in normed spaces are presented.

© 2007 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 The best constant in the complex case

We will look first at a generalized complex Clarkson inequality. The result is, in fact, known but our formulation of Theorem 1.1 and its proof will help to understand better the statement and the proof of the new result in Theorem 2.1 as well as the applications of both theorems presented in the third part of the paper.

Theorem 1.1 *Let $0 < p, q \leq \infty$. Then the best constant in the generalized complex Clarkson inequality*

$$(|a + b|^q + |a - b|^q)^{1/q} \leq C (|a|^p + |b|^p)^{1/p}, \quad \text{for all } a, b \in \mathbb{C}, \quad (1.1)$$

is $C = C_{p,q}(\mathbb{C}) = \max\{2^{1-1/p}, 2^{1/q}, 2^{1/q-1/p+1/2}\}$.

Clarkson [4] proved that $C_{p,p'}(\mathbb{C}) = 2^{1/p'}$ for $1 \leq p \leq 2$, where p' is the conjugate exponent to the number p , that is, $1/p + 1/p' = 1$. Later on the best constants $C_{p,q}(\mathbb{C})$ for the remaining pairs of p and q such that $0 < p, q \leq \infty$ were found by Koskela [13] and Maligranda–Persson [17].

We note that the best constant in (1.1) is also the norm of a non-positive operator given by the matrix $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ or by the rule $T(a, b) = (a + b, a - b)$ and acting between two-dimensional complex (quasi-)Banach spaces $l_2^p(\mathbb{C})$ and $l_2^q(\mathbb{C})$, i.e., $\|T\|_{p,q} = C_{p,q}(\mathbb{C})$.

Proof. We carry out the proof in three steps.

First step. For $1 \leq p \leq 2$ the following classical complex Clarkson inequality (see [4]) holds:

$$\left(|a + b|^{p'} + |a - b|^{p'}\right)^{1/p'} \leq 2^{1/p'} (|a|^p + |b|^p)^{1/p} \quad \text{for all } a, b \in \mathbb{C}. \quad (1.2)$$

Really, we can calculate the norm of the operator T in two easy cases:

$$T : l_2^1 \longrightarrow l_2^\infty \text{ has the norm } \|T\|_{1,\infty} = \sup_{|x|+|y|=1} \max(|x+y|, |x-y|) = 1 \text{ and}$$

$$T : l_2^2 \longrightarrow l_2^2 \text{ has the norm } \|T\|_{2,2} = \sup_{|x|^2+|y|^2=1} (|x+y|^2 + |x-y|^2)^{1/2} = \sqrt{2}.$$

* Corresponding author: e-mail: lech@sm.luth.se, Phone: +46 920 491318, Fax: +46 920 491073

** e-mail: natasha@sm.luth.se, Phone: +46 768403272, Fax: +46 920 491073

Then, by the complex Riesz–Thorin interpolation theorem, $T : l_2^p(\mathbb{C}) \rightarrow l_2^q(\mathbb{C}) = l_2^{p'}(\mathbb{C})$ ($1 \leq p \leq 2$) is bounded with the norm

$$\|T\|_{p,p'} \leq \|T\|_{1,\infty}^{1-2/p'} \|T\|_{2,2}^{2/p'} = 2^{1/p'},$$

since p and q are defined by

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \quad \text{and} \quad 0 \leq \theta \leq 1.$$

Second step. For fixed $a, b \in \mathbb{C}$ let $A_p = (|a|^p + |b|^p)^{1/p}$ and $B_p = (|a+b|^p + |a-b|^p)^{1/p}$. Then A_p, B_p are decreasing and $2^{-1/p}A_p, 2^{-1/p}B_p$ are increasing in $p > 0$. The proof of these statements is standard.

Third step. Combine the first two steps by considering special four cases for p and q :

- I. $0 < q \leq 2 \leq p \Rightarrow B_q \leq 2^{1/q-1/2}B_2 = 2^{1/q}A_2 \leq 2^{1/q-1/p+1/2}A_p.$
- II. $q \geq 2, p \geq q' \Rightarrow B_q \leq 2^{1/q}A_{q'} \leq 2^{1/q+1/q'-1/p}A_p = 2^{1-1/p}A_p.$
- IIIa. $p, q \leq 2 \Rightarrow B_q \leq 2^{1/q-1/2}B_2 = 2^{1/q}A_2 \leq 2^{1/q}A_p.$
- IIIb. $q \geq 2, q' \geq p \Rightarrow B_q \leq 2^{1/q}A_{q'} \leq 2^{1/q}A_p.$

The equalities hold in (I), (II) and (III a, b) for $(a, b) = (1, i)$, $(a, b) = (1, 1)$ and $(a, b) = (1, 0)$, respectively.

As we see the classical complex Clarkson inequality (1.2) is an important estimate in the above proof. This estimate was of particular interest in a number of papers. After Clarkson paper [4] several different proofs of this inequality appeared in literature (cf. [18, pp. 534–558], [19] and [20]). All these proofs have in common that they first reduce the problem to one complex variable and then to real variables by different types of transformations.

As an example, let us assume, without loss of generality, that $0 < |b| \leq |a|$ in (1.2). Dividing the both sides of (1.2) by $|a|$ and denoting b/a by z we get

$$|1+z|^{p'} + |1-z|^{p'} \leq 2(1+|z|^p)^{p'-1}. \quad (1.3)$$

After replacing z by $re^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ the inequality (1.3) becomes

$$|1+re^{i\theta}|^{p'} + |1-re^{i\theta}|^{p'} \leq 2(1+r^p)^{p'-1}. \quad (1.4)$$

Now consider the function $g_r(\theta)$ for a fixed $r \in [0, 1]$ given by

$$g_r(\theta) = |1+re^{i\theta}|^{p'} + |1-re^{i\theta}|^{p'} = (1+r^2+2r\cos\theta)^{p'/2} + (1+r^2-2r\cos\theta)^{p'/2}.$$

Its derivative is

$$g'_r(\theta) = -p'r \sin\theta \left[(1+r^2+2r\cos\theta)^{p'/2-1} - (1+r^2-2r\cos\theta)^{p'/2-1} \right]$$

and $g'_r(\theta) = 0$ if and only if $\theta = \frac{k\pi}{2}$ for $k = 0, 1, 2, 3, 4$. Moreover, g_r is decreasing on intervals $[0, \frac{\pi}{2}]$, $[\pi, \frac{3\pi}{2}]$ and increasing on $[\frac{\pi}{2}, \pi]$, $[\frac{3\pi}{2}, 2\pi]$. Therefore,

$$\max_{\theta \in [0, 2\pi]} g_r(\theta) = g_r(0) = g_r(\pi) = (1+r)^{p'} + (1-r)^{p'}$$

and it only remains for us to prove (1.4) for $\theta = 0$, that is

$$(1+x)^{p'} + (1-x)^{p'} \leq 2(1+x^p)^{p'-1} \quad (1.5)$$

for all $0 \leq x \leq 1$ and $1 < p < 2$ (which is called real Clarkson inequality).

Probably the shortest and the simplest among all other proofs of inequality (1.5) is one due to Friedrichs [5]. The shortness of this proof is based on the consideration of the following function

$$f(\alpha, x) = (1+\alpha^{1-p'}x)(1+\alpha x)^{p'-1} + (1-\alpha^{1-p'}x)(1-\alpha x)^{p'-1}$$

of real variables $0 \leq \alpha, x \leq 1$. Obviously,

$$f(1, x) = (1 + x)^{p'} + (1 - x)^{p'}, \quad f(x^{p-1}, x) = 2(1 + x^p)^{p'-1}$$

and

$$f'_\alpha(\alpha, x) = (p' - 1)x(1 - \alpha^{-p'})g(\alpha, x), \quad \text{where} \quad g(\alpha, x) = (1 + \alpha x)^{p'-2} - (1 - \alpha x)^{p'-2}.$$

Clearly, for $p' > 2$ we have $g(\alpha, x) \geq 0$, that in turn implies $f'_\alpha(\alpha, x) \leq 0$ since $1 - \alpha^{-p'} \leq 0$. Thus, the function $f(\alpha, x)$ is decreasing in α . Noting that $x^{p-1} \leq 1$ we have $f(1, x) \leq f(x^{p-1}, x)$ that is the same as $(1 + x)^{p'} + (1 - x)^{p'} \leq 2(1 + x^p)^{p'-1}$. $\square$

Remark 1.2 The proof using the complex Riesz–Thorin interpolation theorem gives possibility to do some generalizations of the classical complex Clarkson inequality to the higher dimensions (see [9]–[11], [17] and [24]).

2 The best constant in the real case

Now, we consider a *generalized real Clarkson inequality*:

$$(|a + b|^q + |a - b|^q)^{1/q} \leq C(|a|^p + |b|^p)^{1/p} \quad \text{for all } a, b \in \mathbb{R}. \quad (2.1)$$

The problem is how to compute the best constant $C = C_{p,q}(\mathbb{R})$ in this inequality for all $0 < p, q \leq \infty$. As in the case of generalized complex Clarkson inequality (1.1) the best constant is also the norm of the operator T but acting now between real two-dimensional spaces $l_2^p(\mathbb{R})$ and $l_2^q(\mathbb{R})$ given by the formula $T(a, b) = (a + b, a - b)$.

Theorem 2.1 If $0 < p, q \leq \infty$, then the best constant in inequality (2.1) is

$$C_{p,q}(\mathbb{R}) = \max \{2^{1-1/p}, 2^{1/q}, B_{p,q}\} \quad \text{with} \quad B_{p,q} = \sup_{x \in [0,1]} \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}. \quad (2.2)$$

Moreover, if either $p \leq 2$ or $q \geq 2$ or $p = \infty$ or $0 < q \leq 1$, then $C_{p,q}(\mathbb{R}) = \max \{2^{1-1/p}, 2^{1/q}\}$. Also, if $1 < q < 2 < p < \infty$, then $C_{p,q}(\mathbb{R}) = B_{p,q}$ and $\max \{2^{1-1/p}, 2^{1/q}\} < B_{p,q} < 2^{1/q-1/p+1/2}$.

Proof. If either $0 < p \leq 2$ or $2 \leq q \leq \infty$, then $1/q - 1/p + 1/2 \leq \max\{1 - 1/p, 1/q\}$ and thus $C_{p,q}(\mathbb{C}) = \max \{2^{1-1/p}, 2^{1/q}\}$. Since the constant $C_{p,q}(\mathbb{C})$ is attained on real numbers we obtain that $C_{p,q}(\mathbb{C}) = C_{p,q}(\mathbb{R}) = \max \{2^{1-1/p}, 2^{1/q}\}$ (see Theorem 1.1).

In the case when $p = \infty$ we have

$$C_{\infty,q}(\mathbb{R}) = \sup_{\substack{a,b \in \mathbb{R}, \\ \max\{|a|, |b|\} = 1}} (|a + b|^q + |a - b|^q)^{1/q} = \max \{2, 2^{1/q}\}.$$

Furthermore, if $0 < q < 2 < p < \infty$, then

$$C_{p,q}(\mathbb{R}) = \sup_{\substack{a,b \in \mathbb{R}, \\ |a|^p + |b|^p \neq 0}} \frac{(|a + b|^q + |a - b|^q)^{1/q}}{(|a|^p + |b|^p)^{1/p}} = \sup_{x \in [0,1]} \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}.$$

For $0 < q \leq 1$ the function

$$F(x) = \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}, \quad x \in [0, 1],$$

is decreasing on $[0, 1]$ since its derivative

$$F'(x) = \frac{[(1+x)^q + (1-x)^q]^{1/q-1}}{(1+x^p)^{1+1/p}} \{[(1+x)^{q-1} - (1-x)^{q-1}](1+x^p) - x^{p-1}[(1+x)^q + (1-x)^q]\}$$

is strictly less than zero and thus

$$C_{p,q}(\mathbb{R}) = B_{p,q} = F(0) = 2^{1/q}.$$

In the case $1 < q < 2 < p < \infty$ we will prove in the next theorem that there exists a unique $x_0 \in (0, 1)$ at which F has its maximum. Moreover, we will see that

$$\max \{2^{1-1/p}, 2^{1/q}\} < B_{p,q} < 2^{1/q-1/p+1/2}. \quad \square$$

Fig. 1 The best constants in the generalized complex Clarkson inequality

Fig. 2 The best constants in the generalized real Clarkson inequality

Remark 2.2 Let $1 \leq p, q \leq \infty$. If $\dim L^p(\mu) \geq 3$ and $\dim L^q(\nu) \geq 3$, then the norm of any bounded linear operator $T : L^p(\mu) \rightarrow L^q(\nu)$ acting between real spaces and the norm of its natural complexification $T_{\mathbb{C}} = T + iT : L^p_{\mathbb{C}}(\mu) \rightarrow L^q_{\mathbb{C}}(\nu)$ acting between complex spaces are the same $\|T\|_{p,q} = \|T_{\mathbb{C}}\|_{p,q}$ if and only if $p \leq q$ (see [6], [16] and [22]). In two-dimensional case, i.e., when $L^p(\mu) = l^p_2$ and $L^q(\nu) = l^q_2$, we have that $\|T\|_{p,q} = \|T_{\mathbb{C}}\|_{p,q}$ if and only if either $1 \leq p \leq 2$ or $2 \leq q \leq \infty$ (see [6], [16] and [23]).

Theorem 2.3 For all $1 < q < 2 < p < \infty$ there exists a unique $x_0 = x_0(p, q) \in (0, 1)$ such that

$$B_{p,q} = \sup_{x \in [0,1]} F(x, p, q) = F(x_0, p, q),$$

where

$$F(x) = F(x, p, q) = \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}, \quad x \in [0, 1]. \quad (2.3)$$

Moreover, $\max \{2^{1-1/p}, 2^{1/q}\} < B_{p,q} < 2^{1/q-1/p+1/2}$.

Proof. Using standard calculus we find the maximum of the function F which is also the best constant in the generalized real Clarkson inequality (2.1). Since

$$F'(x) = \frac{[(1+x)^q + (1-x)^q]^{1/q-1}}{(1+x^p)^{1+1/p}} [(1+x)^{q-1}(1-x^{p-1}) - (1-x)^{q-1}(1+x^{p-1})], \quad (2.4)$$

it follows that the derivative $F'(x)$ is equal to zero on $(0, 1)$ if and only if

$$\frac{1-x^{p-1}}{1+x^{p-1}} = \frac{(1-x)^{q-1}}{(1+x)^{q-1}}. \quad (2.5)$$

Denoting $f(x) = (1-x)/(1+x)$ we can rewrite (2.5) as $f(x^{p-1}) = f(x)^{q-1}$. Let $g(x) = \ln f(x)$ and define $h(x) = g(x^{p-1})/g(x)$. Then, for x which satisfies (2.5) we have

$$h(x) = \frac{g(x^{p-1})}{g(x)} = \frac{\ln f(x^{p-1})}{\ln f(x)} = \frac{\ln f(x)^{q-1}}{\ln f(x)} = q-1. \quad (2.6)$$

With help of the following auxiliary lemma we find that there exists a unique solution $x_0(p, q) \in (0, 1)$ of (2.6) and therefore (2.5). Note that clearly 0 and 1 satisfies (2.5).

Lemma 2.4 For $g(x) = \ln(1-x)/(1+x)$ and $2 < p < \infty$ the function $h(x) = g(x^{p-1})/g(x)$ is strictly increasing on the interval $(0, 1)$ and $h((0, 1)) = (0, 1)$.

Proof. First we show that the function $h(x)$ has positive derivative on $(0, 1)$. Indeed,

$$h'(x) = \frac{[g(x^{p-1})]'g(x) - g(x^{p-1})g'(x)}{g^2(x)},$$

where $g'(x) = -2/(1-x^2)$ and

$$[g(x^{p-1})]' = (p-1)x^{p-2}g'(x^{p-1}) = -\frac{2(p-1)x^{p-2}}{1-x^{2(p-1)}}.$$

Thus,

$$\begin{aligned} h'(x) &= \left(-\frac{2(p-1)x^{p-2}}{1-x^{2(p-1)}} \ln \frac{1-x}{1+x} + \frac{2}{1-x^2} \ln \frac{1-x^{p-1}}{1+x^{p-1}} \right) / g^2(x) \\ &= \frac{2 \left(-(p-1)x^{p-2}(1-x^2) \ln \frac{1-x}{1+x} + (1-x^{2(p-1)}) \ln \frac{1-x^{p-1}}{1+x^{p-1}} \right)}{(1-x^2)(1-x^{2(p-1)})g^2(x)}. \end{aligned} \quad (2.7)$$

Using the Taylor expansion we get

$$\ln \frac{1-x}{1+x} = \ln(1-x) - \ln(1+x) = -2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

and

$$\ln \frac{1-x^{p-1}}{1+x^{p-1}} = -2 \sum_{k=0}^{\infty} \frac{x^{(2k+1)(p-1)}}{2k+1}.$$

Now, in order to prove that $h'(x) > 0$ for all $x \in (0, 1)$ and $2 < p < \infty$ it is sufficient to show that the numerator in (2.7) is positive for all such x and p . To do this we consider the function

$$\begin{aligned} \phi(x) &= (p-1)(1-x^2)x^{p-2} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} - (1-x^{2(p-1)}) \sum_{k=0}^{\infty} \frac{x^{(2k+1)(p-1)}}{2k+1} \\ &= (p-1) \sum_{k=0}^{\infty} \frac{x^{p+2k-1} - x^{p+2k+1}}{2k+1} + \sum_{k=0}^{\infty} \frac{-x^{(2k+1)(p-1)} + x^{(2k+3)(p-1)}}{2k+1} \\ &= (p-2)x^{p-1} + (p-1) \sum_{k=1}^{\infty} x^{p+2k-1} \left(\frac{1}{2k+1} - \frac{1}{2k-1} \right) \\ &\quad + \sum_{k=1}^{\infty} x^{(2k+1)(p-1)} \left(-\frac{1}{2k+1} + \frac{1}{2k-1} \right) \\ &= \sum_{k=1}^{\infty} \frac{2x^{p-1}}{(2k+1)(2k-1)} \left(p-2 + x^{2k(p-1)} - (p-1)x^{2k} \right). \end{aligned}$$

We note that $\sum_{k=1}^{\infty} 2(2k+1)^{-1}(2k-1)^{-1} = 1$. For each $k = 1, 2, \dots$ the function

$$\phi_k(x) = p-2 + x^{2k(p-1)} - (p-1)x^{2k}$$

is strictly decreasing on $(0, 1)$ since for $p \in (2, \infty)$

$$\phi'_k(x) = 2k(p-1)x^{2k-1} (x^{2k(p-2)} - 1) < 0.$$

Hence, $\phi_k(x) > \phi_k(1) = 0$. Thus, all functions $\phi_k(x)$ are positive which clearly implies that $\phi(x) > 0$ and therefore $h'(x) > 0$ for all required x and p , i.e., the function $h(x)$ is strictly increasing on $(0, 1)$.

It remains to show that $h((0, 1)) = (0, 1)$. For $p \in (2, \infty)$ and $x \in (0, 1)$ we have that $x^{p-1} < x$ and since $g(x)$ is decreasing and negative on $(0, 1)$, then $0 < h(x) = g(x^{p-1})/g(x) < 1$. Moreover, using de l'Hospital rule we calculate the following limits

$$\lim_{x \rightarrow 0^+} \frac{\ln(1 - x^{p-1})(1 + x^{p-1})}{\ln(1 - x)(1 + x)} = \lim_{x \rightarrow 0^+} \frac{-2(p-1)x^{p-2} \cdot \frac{1 - x^2}{-2}}{1 - x^{2(p-1)}} = 0,$$

$$\lim_{x \rightarrow 1^-} \frac{\ln(1 - x^{p-1})(1 + x^{p-1})}{\ln(1 - x)(1 + x)} = \lim_{x \rightarrow 1^-} \frac{(p-1)x^{p-2}(1 - x^2)}{1 - x^{2(p-1)}} = \lim_{x \rightarrow 1^-} \frac{-2(p-1)x}{-2(p-1)x^{2p-3}} = 1.$$

Thus, $h(x) \rightarrow 0$ as $x \rightarrow 0^+$ and $h(x) \rightarrow 1$ as $x \rightarrow 1^-$. All together implies that the image of $(0, 1)$ by h is $(0, 1)$, i.e., $h((0, 1)) = (0, 1)$. $\square$

Now we continue our proof of Theorem 2.3. Since $h(x)$ is strictly increasing on $(0, 1)$ and $h((0, 1)) = (0, 1)$ as was established in Lemma 2.4, then for all $p \in (2, \infty)$ and $q \in (1, 2)$ there exists a unique $x_0 = x_0(p, q) \in (0, 1)$ such that $h(x_0) = q - 1$ and thus $F(x_0)$ is the extreme value of $F(x)$. Moreover, x_0 is the point of maximum of $F(x)$.

Namely, by (2.5) the derivative $F'(x)$ defined by (2.4) has the same sign as the expression $f(x^{p-1}) - f(x)^{q-1}$. Consider the functions $u(x) = f(x^{p-1})$ and $v(x) = f(x)^{q-1}$. Then,

$$u'(x) = -2(p-1) \frac{x^{p-2}}{(1 + x^{p-1})^2} \quad \text{and} \quad v'(x) = -2(q-1) \frac{(1-x)^{q-2}}{(1+x)^q}.$$

Therefore, both $u(x)$ and $v(x)$ are decreasing on $(0, 1)$, $u'(0) = 0$ and $v'(0) = -2(q-1)$. Moreover, $u(0) = v(0) = 1$. Hence, $u(x) - v(x) > 0$ which means that $F(x)$ is increasing at this neighborhood. Thus, x_0 is the point of maximum of $F(x)$ for $p \in (2, \infty)$ and $q \in (1, 2)$.

It remains for us to show that $\max\{2^{1-1/p}, 2^{1/q}\} < B_{p,q} < 2^{1/q-1/p+1/2}$. Since $F(x)$ is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, 1]$ for $1 < q < 2 < p < \infty$, then

$$\max\{2^{1-1/p}, 2^{1/q}\} = \max\{F(0), F(1)\} < F(x_0) = B_{p,q}.$$

On the other hand, using strict convexity of the functions $u^{2/q}$ and $u^{p/2}$ we obtain for $x \in (0, 1)$

$$\begin{aligned} [(1+x)^q + (1-x)^q]^{1/q} &< 2^{1/q-1/2} [(1+x)^2 + (1-x)^2]^{1/2} \\ &= 2^{1/q} (1+x^2)^{1/2} \\ &< 2^{1/q-1/p+1/2} (1+x^p)^{1/p} \end{aligned}$$

and thus $B_{p,q} = F(x_0) < 2^{1/q-1/p+1/2}$. $\square$

Theorem 2.3 only shows the existence of a point inside the interval $(0, 1)$ at which the function $F(x)$ given by (2.3) has its maximum. But it is even more interesting to find this point and as a consequence to obtain the best constant in the generalized real Clarkson inequality for $1 < q < 2 < p < \infty$. It turns out that it is difficult, if even possible, to solve the problem for all such p and q . We consider only the case when $q = p'$.

As it was established in Theorem 2.3, the point of maximum of $F(x)$ is then defined by

$$\frac{1 - x^{p-1}}{1 + x^{p-1}} = \frac{(1-x)^{p'-1}}{(1+x)^{p'-1}}.$$

Denoting $f(x) = (1-x)/(1+x)$, we rewrite this equality in the form $u(x) = f(x^{p-1}) = f^{p'-1}(x) = v(x)$. Observe that $u \circ v = f \circ f^{(p'-1)(p-1)} = f \circ f = id$ and thus the point of maximum of $F(x)$ is also defined by $f(x^{p-1}) = x$ which is equivalent to the equation

$$x^p + x^{p-1} + x - 1 = 0. \quad (2.8)$$

If we denote the root of (2.8) by x_0 , then $\max \{F(x) : x \in (0, 1)\} = F(x_0)$ and thus the best constant in the generalized real Clarkson inequality for $2 < p < \infty$ will be

$$C_{p,p'}(\mathbb{R}) = (1 + x_0)(1 + x_0^p)^{1/p' - 1/p}. \quad (2.9)$$

We could find the exact expression for this constant when $p = 3, 4, 8$ and compare it with the constant $C_{p,p'}(\mathbb{C})$.

Thus, in the case when $p = 3$ Equation (2.8) becomes cubic with the real root

$$x_0 = 1/3 \left(\sqrt[3]{17 + 3\sqrt{33}} + \sqrt[3]{17 - 3\sqrt{33}} - 1 \right) \approx 0.544.$$

After substitution of this expression into (2.9) we get $C_{3, \frac{3}{2}}(\mathbb{R}) = (1 + x_0)(1 + x_0^3)^{1/3} \approx 1.622$. At the same time we have $C_{3, \frac{3}{2}}(\mathbb{C}) = 2^{5/6} \approx 1.781$ and so $C_{3, \frac{3}{2}}(\mathbb{C})/C_{3, \frac{3}{2}}(\mathbb{R}) \approx 1.098$.

The constant $C_{4, \frac{4}{3}}(\mathbb{R})$ ($p = 4$) is connected to the real root of the equation

$$x^4 + x^3 + x - 1 = (x^2 + 1)(x^2 + x - 1) = 0,$$

which is $x_0 = (\sqrt{5} - 1)/2 \in (0, 1)$ and therefore $C_{4, \frac{4}{3}}(\mathbb{R}) = \sqrt{3} \approx 1.732$. We also have that $C_{4, \frac{4}{3}}(\mathbb{C}) = 2$ and thus $C_{4, \frac{4}{3}}(\mathbb{C})/C_{4, \frac{4}{3}}(\mathbb{R}) = 2/\sqrt{3} \approx 1.154$.

When $p = 8$ the real root of the equation

$$x^8 + x^7 + x - 1 = (x^2 + 1)(x^3 - x + 1)(x^3 + x^2 - 1) = 0.$$

can be found from $x^3 + x^2 - 1 = 0$ and is equal to

$$x_0 = 1/3 \left(\sqrt[3]{(25 + 3\sqrt{69})/2} + \sqrt[3]{(25 - 3\sqrt{69})/2} - 1 \right) \approx 0.755.$$

Therefore, $C_{8, 8/7}(\mathbb{R}) \approx 1.892$ and since $C_{8, 8/7}(\mathbb{C}) = 2^{5/4} \approx 2.378$ it follows that $C_{8, 8/7}(\mathbb{C})/C_{8, 8/7}(\mathbb{R}) \approx 1.256$.

For the remaining values of $p \in (2, \infty)$ we could not calculate exactly the constants $C_{p,p'}(\mathbb{R})$ but we noticed that (2.8) can be solved for p :

$$p(x) = 1 + \log_x \frac{1-x}{1+x}. \quad (2.10)$$

Now, taking arbitrary value of $x \in [\sqrt{2} - 1, 1)$ we find the corresponding value p and therefore by (2.9) we can calculate the constant $C_{p,p'}(\mathbb{R})$ for this particular p . The left end of the interval is determined by putting $p = 2$ in (2.10) and solving this for x (limit case). It is also worth to note that the function $p(x)$ is increasing on $[\sqrt{2} - 1, 1)$. Indeed, its derivative

$$p'(x) = \left(\frac{\ln \frac{1-x}{1+x}}{\ln x} \right)' = \frac{-\frac{2}{1-x^2} \ln x - \frac{1}{x} \ln \frac{1-x}{1+x}}{(\ln x)^2} \geq 0.$$

For example, if we take $x = 1/2$, then $p = 1 + \log_2 3 \approx 2.584$, $C_{p,p'}(\mathbb{R}) = 7^{\log_6 3/2} \approx 1.553$, $C_{p,p'}(\mathbb{C}) = 2^{1/2 + \log_6 3/2} \approx 1.654$ and thus $C_{p,p'}(\mathbb{C})/C_{p,p'}(\mathbb{R}) \approx 1.065$. If $x_0 = 9/10$, then $p = 1 + \log_{9/10} 1/19 \approx 28.9463$, $C_{p,p'}(\mathbb{R}) \approx 1.9836$, $C_{p,p'}(\mathbb{C}) \approx 2.6962$ and $C_{p,p'}(\mathbb{C})/C_{p,p'}(\mathbb{R}) \approx 1.3592$. For $x_0 = 99/100$ we have $p = 1 + \log_{99/100} 1/199 \approx 527.6794$, $C_{p,p'}(\mathbb{R}) \approx 1.9999$, $C_{p,p'}(\mathbb{C}) \approx 2.8210$ and $C_{p,p'}(\mathbb{C})/C_{p,p'}(\mathbb{R}) \approx 1.4106$. Observe that $C_{p,p'}(\mathbb{C})/C_{p,p'}(\mathbb{R}) \rightarrow C_{\infty,1}(\mathbb{C})/C_{\infty,1}(\mathbb{R}) = \sqrt{2}$ as p approaches ∞ (see Krivine [14]).

3 Some applications of the generalized Clarkson inequalities

We show how the generalized Clarkson inequalities can be used in calculation of the norm of a certain operator as well as in the proof of the (p, q) -Clarkson inequalities in Lebesgue spaces, mixed norm spaces and in normed spaces.

For $1 \leq p, q \leq \infty$ consider a linear operator $T : L^p[a, b] \rightarrow L^q[a, b]$ given by

$$Tx(t) = \begin{cases} x(t) + x\left(t + \frac{b-a}{2}\right), & t \in \left[a, \frac{a+b}{2}\right), \\ x\left(t - \frac{b-a}{2}\right) - x(t), & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

As an application of Theorems 1.1 and 2.1 we calculate the norm of this operator acting between complex and real spaces.

Theorem 3.1 *Let $1 \leq p, q \leq \infty$. Then the operator $T : L^p[a, b] \rightarrow L^q[a, b]$ is bounded if and only if $p \geq q$. Moreover,*

$$\|T\|_{p,q} = \left(\frac{b-a}{2}\right)^{\frac{p-q}{pq}} C_{p,q}, \quad \text{where } C_{p,q} = C_{p,q}(\mathbb{R}) \text{ or } C_{p,q}(\mathbb{C}) \quad (3.1)$$

depending if the spaces are real or complex.

Proof. First, by Theorems 1.1, 2.1 (substituting $a = x + y$ and $b = x - y$ for $x, y \in \mathbb{R}$ or $x, y \in \mathbb{C}$)

$$\begin{aligned} \|Tx\|_q &= \left(\int_a^{(a+b)/2} \left| x(t) + x\left(t + \frac{b-a}{2}\right) \right|^q dt + \int_{(a+b)/2}^b \left| x\left(t - \frac{b-a}{2}\right) - x(t) \right|^q dt \right)^{1/q} \\ &= \left(\int_a^{(a+b)/2} \left| x(t) + x\left(t + \frac{b-a}{2}\right) \right|^q dt + \int_a^{(a+b)/2} \left| x(s) - x\left(s + \frac{b-a}{2}\right) \right|^q ds \right)^{1/q} \\ &= \left[\int_a^{(a+b)/2} \left(\left| x(t) + x\left(t + \frac{b-a}{2}\right) \right|^q + \left| x(t) - x\left(t + \frac{b-a}{2}\right) \right|^q \right) dt \right]^{1/q} \\ &\geq \min \{2^{1/q}, 2^{1/q'}\} \left[\int_a^{(a+b)/2} \left(|x(t)|^q + \left| x\left(t + \frac{b-a}{2}\right) \right|^q \right) dt \right]^{1/q} \\ &= \min \{2^{1/q}, 2^{1/q'}\} \left(\int_a^{(a+b)/2} |x(t)|^q dt + \int_{(a+b)/2}^b \left| x\left(t - \frac{b-a}{2}\right) \right|^q dt \right)^{1/q} \\ &= \min \{2^{1/q}, 2^{1/q'}\} \|x\|_q. \end{aligned}$$

Thus, in the case $p < q$ the operator T is unbounded, since otherwise we obtain that

$$\min \{2^{1/q}, 2^{1/q'}\} \|x\|_q \leq \|Tx\|_q \leq C \|x\|_p$$

and therefore $L^p[a, b] \subset L^q[a, b]$ that is obviously not true.

If $p \geq q$, then by Theorems 1.1, 2.1 and the Hölder inequality we have

$$\begin{aligned} \|Tx\|_q &= \left(\int_a^{(a+b)/2} \left| x(t) + x\left(t + \frac{b-a}{2}\right) \right|^q dt + \int_{(a+b)/2}^b \left| x\left(t - \frac{b-a}{2}\right) - x(t) \right|^q dt \right)^{1/q} \\ &\leq C_{p,q} \left[\int_a^{(a+b)/2} \left(|x(t)|^p + \left| x\left(t + \frac{b-a}{2}\right) \right|^p \right)^{q/p} dt \right]^{1/q} \leq \end{aligned}$$

$$\begin{aligned} &\leq C_{p,q} \left(\int_a^{(a+b)/2} \left(|x(t)|^p + \left| x\left(t + \frac{b-a}{2}\right) \right|^p \right) dt \right)^{1/p} \left(\int_a^{(a+b)/2} dt \right)^{\frac{p-q}{pq}} \\ &= C_{p,q} \left(\frac{b-a}{2} \right)^{\frac{p-q}{pq}} \|x\|_p, \end{aligned}$$

where $C_{p,q} = C_{p,q}(\mathbb{R})$ in the real case and $C_{p,q} = C_{p,q}(\mathbb{C})$ in the complex case.

To verify that equality may occur in (3.1) we can take a pair (u, v) of numbers from Theorem 1.1 and Theorem 2.3 (in the case $1 < q < 2 < p < \infty$ we take $x_0 = v/u$) at which the generalized Clarkson inequalities become equalities and consider the following function

$$x(t) = u\chi_{[a, (a+b)/2)}(t) + v\chi_{[(a+b)/2, b]}(t). \quad \square$$

The main application of the classical complex Clarkson inequality (1.2) was for Clarkson [4] to prove that for $1 < p < \infty$ L^p -spaces are uniformly convex and to find the lower estimates of the modulus of uniform convexity defined by him in [4]. A normed space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $0 < \varepsilon \leq 2$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ it follows that $\|(x + y)/2\| < 1 - \delta$. The function $\delta_X(\varepsilon) : [0, 2] \rightarrow [0, 1]$ defined by the formula

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| \geq \varepsilon \right\}$$

is called the *modulus of uniform convexity of X* . Clearly, a normed space X is uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Using inequality (1.2) Clarkson proved two inequalities, one for $1 < p \leq 2$ and the other for $2 \leq p < \infty$, which can be written in one formula (see e.g. [3, p. 21]): if $1 < p < \infty$ and $r = \max\{p, p'\}$, then

$$\left(\|x + y\|_p^r + \|x - y\|_p^r \right)^{1/r} \leq 2^{1/r'} \left(\|x\|_p^{r'} + \|y\|_p^{r'} \right)^{1/r'} \quad \text{for all } x, y \in L^p.$$

Thus, if $\|x\|_p = \|y\|_p = 1$ and $\|x - y\| \geq \varepsilon$, $\varepsilon \in [0, 2]$, then $\|x + y\|_p \leq 2[1 - (\varepsilon/2)^r]^{1/r}$ and so

$$\delta_{L^p}(\varepsilon) \geq 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^r \right]^{1/r} \geq \frac{1}{r} \left(\frac{\varepsilon}{2} \right)^r, \quad \text{where } r = \max\{p, p'\}. \quad (3.2)$$

For $2 \leq p < \infty$ estimate (3.2) is sharp. For $1 < p < 2$ the sharp estimate $\delta_{L^p}(\varepsilon) > \frac{p-1}{8}\varepsilon^2$ was found by Hanner [7] and Kadec [8].

Later on Boas [2] obtained, by the Riesz convexity theorem, the following result: if $1 < p \leq r \leq q$, then $(\|x + y\|_r^q + \|x - y\|_r^q)^{1/q} \leq 2^{1/p'} (\|x\|_r^p + \|y\|_r^p)^{1/p}$ for all $x, y \in L^r(\mu)$. Koskela [13] removed the restrictions on p, q and r and proved for $0 < p, q, r < \infty$ the inequality

$$(\|x + y\|_r^q + \|x - y\|_r^q)^{1/q} \leq 2^{1/q-1/p+1/t} (\|x\|_r^p + \|y\|_r^p)^{1/p} \quad \text{for all } x, y \in L^r(\mu), \quad (3.3)$$

where $t = \min\{p, q', r, r'\}$ with the convention that the conjugate exponent to a number is omitted if the number is not bigger than one. Moreover, if $\dim L^r(\mu) \geq 2$ then there exist functions $x, y \in L^r(\mu)$ for which the equality in (3.3) holds. For convenience we call inequality (3.3) the (p, q) -Clarkson inequality in $L^r(\mu)$.

Let us note that the generalized complex Clarkson inequality (1.1) can also be written in the form

$$(|a + b|^q + |a - b|^q)^{1/q} \leq 2^{1/q-1/p+1/t} (|a|^p + |b|^p)^{1/p} \quad \text{for all } a, b \in \mathbb{C}, \quad (3.4)$$

where $t = \min\{p, q', 2\}$, i.e., the sharp constants in (1.1) and (3.4) are the same. Obviously, the constant in (3.4) is not bigger than that in (3.3). At the same time this constant is the best constant in the generalized Clarkson inequality in real or complex Hilbert space $(H, \|\cdot\|)$ (put $\|\cdot\|$ instead of $|\cdot|$ in (3.4) and take $a, b \in H$). The latter can be proved by using arguments similar to those we used in Theorem 1.1.

Some results on a generalization of (3.3) to the mixed norm spaces were obtained by Boas [2], Koskela [13] and Sobolev [21] and this work was continued by Kato–Miyazaki [9] and Kato–Miyazaki–Takahashi [10], [11]. In contrast with the proofs presented in [9], [10] and [12], where the interpolation techniques were used, our proof of the (p, q) -Clarkson inequality in the mixed norm spaces will only rely on the Jensen, Minkowski and (p, q) -Clarkson inequalities in $L^r(\mu)$ spaces. We will also complete the chain of generalizations of the (p, q) -Clarkson inequality to the mixed norm spaces.

Let us recall that the mixed norm spaces $L^s[L^r]$ (see Benedek–Panzone [1]) are classes of measurable functions on $\Omega_1 \times \Omega_2$ generated by the quasi-norms (norms when $r, s \geq 1$)

$$\|x\|_{r,s} = \left\| \|x\|_r \right\|_s = \left(\int_{\Omega_2} \left| \int_{\Omega_1} |x(t_1, t_2)|^r d\mu_1 \right|^{s/r} d\mu_2 \right)^{1/s}, \quad \text{where } r, s > 0.$$

Theorem 3.2 *If $0 < p, q, r, s \leq \infty$, then*

$$\left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p} \quad (3.5)$$

for all $x, y \in L^s[L^r]$, where $t = \min \{p, q', r, r', s, s'\}$ and the conjugate exponent is omitted if the number is not bigger than one. Moreover, if $\dim L^r(\mu_1) \geq 2$ and $\dim L^s(\mu_2) \geq 2$, then the constant $2^{1/q-1/p+1/t}$ is sharp.

Proof. We prove the inequality for real positive numbers p, q, r and s . Simple modification of this proof establishes that inequality (3.5) also holds when some of these numbers are equal to infinity.

Case 1: If $s \leq q$, then by applying the reverse Minkowski inequality we get

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &= \left[\left(\int_{\Omega_2} \|x + y\|_r^{q \cdot s/q} d\mu_2 \right)^{q/s} + \left(\int_{\Omega_2} \|x - y\|_r^{q \cdot s/q} d\mu_2 \right)^{q/s} \right]^{1/q} \\ &\leq \left(\int_{\Omega_2} (\|x + y\|_r^q + \|x - y\|_r^q)^{s/q} d\mu_2 \right)^{1/s}. \end{aligned} \quad (3.6)$$

Further, if $p \leq s$, then by sequential use of the (p, q) -Clarkson and Minkowski inequalities it follows

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &\leq 2^{1/q-1/p+1/t} \left(\int_{\Omega_2} (\|x\|_r^p + \|y\|_r^p)^{s/p} d\mu_2 \right)^{1/s} \\ &\leq 2^{1/q-1/p+1/t} \left[\left(\int_{\Omega_2} \|x\|_r^s d\mu \right)^{p/s} + \left(\int_{\Omega_2} \|y\|_r^s d\mu \right)^{p/s} \right]^{1/p} \\ &= 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p}, \end{aligned}$$

where $t = \min \{p, q', r, r'\}$. Since $s \leq q$ (or $q' \leq s'$) and $p \leq s$ we can write $t = \min \{p, q', r, r', s, s'\}$.

If we instead have $p \geq s$ and apply the (s, q) -Clarkson inequality to (3.6) we get

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &\leq 2^{1/q-1/s+1/t} \left(\int_{\Omega_2} (\|x\|_r^s + \|y\|_r^s) d\mu_2 \right)^{1/s} \\ &= 2^{1/q-1/s+1/t} \left(\|x\|_{r,s}^s + \|y\|_{r,s}^s \right)^{1/s} \\ &\leq 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p}, \end{aligned}$$

where $t = \min \{s, q', r, r'\} = \min \{p, q', r, r', s, s'\}$, since $s \leq q$ ($q' \leq s'$) and $s \leq p$.

Case 2: If $s \geq q$, then

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &\leq 2^{1/q-1/s} \left(\|x + y\|_{r,s}^s + \|x - y\|_{r,s}^s \right)^{1/s} \\ &= 2^{1/q-1/s} \left(\int_{\Omega_2} (\|x + y\|_r^s + \|x - y\|_r^s) d\mu_2 \right)^{1/s}. \end{aligned} \quad (3.7)$$

Moreover, if $s \leq p$ then by the (s, s) -Clarkson inequality we have

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &\leq 2^{1/q-1/s+1/t} \left(\int_{\Omega_2} (\|x\|_r^s + \|y\|_r^s) d\mu_2 \right)^{1/s} \\ &= 2^{1/q-1/s+1/t} \left(\|x\|_{r,s}^s + \|y\|_{r,s}^s \right)^{1/s} \\ &\leq 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p}, \end{aligned}$$

where $t = \min \{s, s', r, r'\} = \min \{p, q', r, r', s, s'\}$, since $s \geq q$ ($s' \leq q'$) and $p \geq s$.

If we instead have $s \geq p$, then by (p, s) -Clarkson and Minkowski inequalities applied to (3.7) we get

$$\begin{aligned} \left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} &\leq 2^{1/q-1/s+1/s-1/p+1/t} \left(\int_{\Omega_2} (\|x\|_r^p + \|y\|_r^p)^{s/p} d\mu_2 \right)^{1/s} \\ &\leq 2^{1/q-1/p+1/t} \left(\left(\int_{\Omega_2} \|x\|_r^s d\mu_2 \right)^{p/s} + \left(\int_{\Omega_2} \|y\|_r^s d\mu_2 \right)^{p/s} \right)^{1/p} \\ &= 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p}, \end{aligned}$$

where $t = \min \{p, s', r, r'\} = \min \{p, q', r, r', s, s'\}$, since $s \geq q$ ($s' \leq q'$) and $s \geq p$.

Summarizing we can write

$$\left(\|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left(\|x\|_{r,s}^p + \|y\|_{r,s}^p \right)^{1/p},$$

where $t = \min \{p, q', r, r', s, s'\}$.

In order to check that the equality may occur in (3.5) let us take disjoint sets A_1, A_2 in Ω_1 and disjoint sets B_1, B_2 in Ω_2 with corresponding positive finite measures and define the functions

$$\begin{aligned} \alpha(u, v) &= \chi_{A_1}(u) \chi_{B_1}(v) \mu_1^{-1/s}(A_1) \mu_2^{-1/r}(B_1), \\ \beta(u, v) &= \chi_{A_1}(u) \chi_{B_2}(v) \mu_1^{-1/s}(A_1) \mu_2^{-1/r}(B_2), \\ \gamma(u, v) &= \chi_{A_2}(u) \chi_{B_1}(v) \mu_1^{-1/s}(A_2) \mu_2^{-1/r}(B_1). \end{aligned}$$

Then the required equality occurs on the following pairs of functions

$$\begin{aligned} 1. \quad &x(u, v) = \alpha(u, v), \quad y(u, v) = 0; & 5. \quad &x(u, v) = \alpha(u, v) + \gamma(u, v), \\ &x(u, v) = y(u, v) = \alpha(u, v); & &y(u, v) = \alpha(u, v) - \gamma(u, v); \\ 3. \quad &x(u, v) = \alpha(u, v), \quad y(u, v) = \beta(u, v); & 6. \quad &x(u, v) = \alpha(u, v) + \beta(u, v), \\ 4. \quad &x(u, v) = \alpha(u, v), \quad y(u, v) = \gamma(u, v); & &y(u, v) = \alpha(u, v) - \beta(u, v). \end{aligned}$$

We note, the functions defined with help of the sets A_1, A_2, B_1 and B_2 are real-valued, which means that the best constants in (3.5) are the same in both the complex and the real case. $\square$

Observe that by proving inequality (3.5) in $L^s[L^r]$ space we referred each time to the (p, q) -Clarkson inequality in $L^r(\mu)$ space. Thus, by induction, the inequality similar to (3.5) is also true in the settings of the mixed norm spaces $L^{r_n}[\dots[L^{r_1}]\dots]$ equipped with the quasi-norms (norms when $r_1, \dots, r_n \geq 1$)

$$\|x\|_{r_1, \dots, r_n} = \left\| \dots \| \|x\|_{r_1} \|_{r_2} \dots \right\|_{r_n}, \quad \text{where } r_1, \dots, r_n > 0.$$

Using inequality (3.5), it is easy to give the following estimation of the modulus of convexity of $L^s[L^r]$ spaces similar to (3.2): if $\|x\|_{r,s} = \|y\|_{r,s} = 1$ and $\|x - y\|_{r,s} \geq \varepsilon$, $\varepsilon \in [0, 2]$, then $\|x + y\|_{r,s} \leq 2 \left[1 - (\varepsilon/2)^{\bar{p}} \right]^{1/\bar{p}}$ and

$$\delta_{L^s[L^r]}(\varepsilon) \geq 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^{\bar{p}} \right]^{1/\bar{p}} \geq \frac{1}{\bar{p}} \left(\frac{\varepsilon}{2} \right)^{\bar{p}}, \quad \text{where } \bar{p} = \max \{r, r', s, s'\}.$$

This estimation is sharp when $2 \leq r, s < \infty$. The sharp estimate $\delta_{L^s[L^r]}(\varepsilon) \geq k_{r,s} \varepsilon^{\underline{p}}$, where $\underline{p} = \max\{r, s, 2\}$ was found by Maleev–Troyanski [15].

Theorem 3.3 *If $0 < p, q, r_1, \dots, r_n \leq \infty$, then*

$$\left(\|x + y\|_{r_1, \dots, r_n}^q + \|x - y\|_{r_1, \dots, r_n}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left(\|x\|_{r_1, \dots, r_n}^p + \|y\|_{r_1, \dots, r_n}^p \right)^{1/p}$$

for all $x, y \in L^{r_n}[\dots[L^{r_1}]\dots]$, where $t = \min\{p, q', r_1, r_1', r_2, r_2', \dots, r_n, r_n'\}$ and the conjugate exponent is omitted if the number is not bigger than one. Moreover, if $\dim L^{r_i}(\mu_i) \geq 2$ for all $1 \leq i \leq n$, then the constant $2^{1/q-1/p+1/t}$ is sharp.

Again, if $\|x\|_{r_1, \dots, r_n} = \|y\|_{r_1, \dots, r_n} = 1$ and $\|x - y\|_{r_1, \dots, r_n} \geq \varepsilon$, $\varepsilon \in [0, 2]$, then $\|x + y\|_{r_1, \dots, r_n} \leq 2[1 - (\varepsilon/2)^{\bar{p}}]^{1/\bar{p}}$ and so

$$\delta_{L^{r_n}[\dots[L^{r_1}]\dots]}(\varepsilon) \geq 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^{\bar{p}}\right]^{1/\bar{p}} \geq \frac{1}{\bar{p}} \left(\frac{\varepsilon}{2}\right)^{\bar{p}}, \quad \text{where } \bar{p} = \max\{r_1, r_1', \dots, r_n, r_n'\}. \quad (3.8)$$

By induction the result of Maleev–Troyanski, mentioned above, can be extended to $L^{r_n}[\dots[L^{r_1}]\dots]$ spaces. The sharp estimate of the modulus of convexity of these spaces will be $\delta_{L^{r_n}[\dots[L^{r_1}]\dots]}(\varepsilon) \geq k_{r_1, \dots, r_n} \varepsilon^{\underline{p}}$, where $\underline{p} = \max\{r_1, \dots, r_n, 2\}$. Thus, the estimation (3.8) is sharp for $2 \leq r_1, \dots, r_n < \infty$.

We finish this part of the paper with the (p, q) -Clarkson inequality in a normed space $(X, \|\cdot\|)$ for $0 < p, q \leq \infty$:

$$(\|x + y\|^q + \|x - y\|^q)^{1/q} \leq C(\|x\|^p + \|y\|^p)^{1/p} \quad \text{for all } x, y \in X. \quad (3.9)$$

This inequality is always true but to find the best constant in (3.9), which we denote by $C_{p,q}(X)$, is of interest.

Theorem 3.4 *Let $0 < p, q \leq \infty$. Then*

$$C_{p,q} \leq C_{p,q}(X) \leq 2^{1/q} \max\{1, 2^{1-1/p}\},$$

where $C_{p,q} = C_{p,q}(\mathbb{R})$ when X is a real normed space and $C_{p,q} = C_{p,q}(\mathbb{C})$ when X is a complex normed space.

Proof. For $x \in X$, $a \neq 0$ and $y = (b/a)x$ the estimate (3.9) means

$$\left(\left|1 + \frac{b}{a}\right|^q \|x\|^q + \left|1 - \frac{b}{a}\right|^q \|x\|^q \right)^{1/q} \leq C \left(\|x\|^p + \left|\frac{b}{a}\right|^p \|x\|^p \right)^{1/p}$$

or

$$(|a + b|^q + |a - b|^q)^{1/q} \leq C(|a|^p + |b|^p)^{1/p}.$$

Thus, $C_{p,q}(X) \geq C_{p,q}(\mathbb{R})$ when X is a real space (then we take $a, b \in \mathbb{R}$) and $C_{p,q}(X) \geq C_{p,q}(\mathbb{C})$ when X is a complex space (then we take $a, b \in \mathbb{C}$). Estimation from above follows by applying twice the triangle inequality for the norm and the property that $f(p) = (\|x\|^p + \|y\|^p)^{1/p}$ is decreasing in p

$$(\|x + y\|^q + \|x - y\|^q)^{1/q} \leq 2^{1/q} (\|x\| + \|y\|) \leq 2^{1/q} \max\{1, 2^{1-1/p}\} (\|x\|^p + \|y\|^p)^{1/p}. \quad \square$$

Note that the upper estimate in the above inequality is attained. For $0 < p \leq 1$ take $y = 0$ and $x \neq 0$. For $p \geq 1$ consider $X = L^1(\mu)$ with $\dim L^1(\Omega, \mu) \geq 2$ and take $x = \frac{1}{\mu(A)}\chi_A$, $y = \frac{1}{\mu(B)}\chi_B$ with disjoint $A, B \subset \Omega$; we can also consider $L^\infty(\Omega, \mu)$ with $\dim L^\infty(\Omega, \mu) \geq 2$ and take $x = \chi_A + \chi_B$ and $y = \chi_A - \chi_B$ with disjoint $A, B \subset \Omega$.

Of course, there exist stronger versions of the (p, q) -Clarkson inequality in X , i.e., with a constant smaller than $2^{1/q} \max\{1, 2^{1-1/p}\}$, but then the normed space X will have additional properties. Take, for $p, q > 1$ the space $L^r(\mu)$ with $1 < r < \min\{p, q'\}$, then by applying (3.3) we get $C_{p,q}(L^r) = 2^{1/q-1/p+1/r} < 2^{1/q-1/p+1}$.

The Clarkson type inequality (3.9) for a Banach space with the constant $C = 2^{1/q}$ for $1 < p \leq 2$ and $p \leq q \leq p'$, and $C = 2^{1-1/p}$ for $2 \leq q \leq \infty$ and $q' \leq p \leq q$ was considered by Kato and Takahashi in [12].

References

- [1] A. Benedek and R. Panzone, The spaces L^p , with mixed norm, *Duke Math. J.* **28**, 301–324 (1961).
- [2] R. P. Boas, Some uniformly convex spaces, *Bull. Amer. Math. Soc. (N.S.)* **46**, 304–311 (1940).
- [3] Yu. A. Brudnyi and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces Vol. 1* (North-Holland, Amsterdam 1991).
- [4] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40**, 396–414 (1936).
- [5] K. O. Friedrichs, On Clarkson's inequalities, *Comm. Pure Appl. Math.* **23** 603–607 (1970).
- [6] J. Gasch and L. Maligranda, On vector-valued inequalities of the Marcinkiewicz–Zygmund, Hertz and Krivine type, *Math. Nachr.* **155**, 187–197 (1992).
- [7] O. Hanner, On the uniform convexity of L^p and l^p , *Ark. Mat.* **3**, 239–244 (1958).
- [8] M. I. Kadec, Unconditionally convergent series in a uniformly convex space, *Uspekhi Mat. Nauk* **11**, 185–190 (1956) (in Russian).
- [9] M. Kato and K. Miyazaki, On generalized Clarkson's inequalities for $L_p(\mu, L_q(\nu))$ and Sobolev spaces, *Math. Japon.* **43**, 505–515 (1996).
- [10] M. Kato, K. Miyazaki, and Y. Takahashi, Clarkson-type inequalities, type constants and interpolation, Manuscript of the lecture at ICM94 in Zürich (1994), 10 pages.
- [11] M. Kato, K. Miyazaki, and Y. Takahashi, Type, cotype constants for $L_p(L_q)$, norms of the Rademacher matrices and interpolation, *Nihonkai Math. J.* **6**, 81–95 (1995).
- [12] M. Kato and Y. Takahashi, Type, cotype constants and Clarkson's inequalities for Banach spaces, *Math. Nachr.* **186**, 187–196 (1997).
- [13] M. Koskela, Some generalizations of Clarkson's inequalities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **634–677**, 89–93 (1979).
- [14] J. L. Krivine, Constantes de Grothendieck et fonctions de type positif sur les sphères, *Adv. Math.* **31**, 16–30 (1979).
- [15] R. P. Maleev and S. I. Troyanski, On the moduli of convexity and smoothness in L_{pq} spaces, *Univ. Sofia, Fac. Math.* **66** (1971/72), 331–339 (1974) (in Russian).
- [16] L. Maligranda, On the norms of operators in the real and the complex case, in: *Proceedings of the Second Seminar on Banach Spaces and Related Topics, Kitakyushu 1996, Japan* (Kyushu Institute of Technology, Kitakyushu 1997), pp. 67–71.
- [17] L. Maligranda and L. E. Persson, On Clarkson's inequalities and interpolation, *Math. Nachr.* **155**, 187–197 (1992).
- [18] D. S. Mitrinović, J. E. Pecarić, and A. M. Fink, *Classical and New Inequalities in Analysis* (Kluwer Academic Publ., Dordrecht, 1993).
- [19] D. S. Mitrinović and P. M. Vasić, Addenda to the monograph “Analytic Inequalities” I, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **577–598**, 3–10 (1977).
- [20] S. L. Sobolev, On an inequality, *Uspekhi Mat. Nauk* **1**, 197 (1946) (in Russian).
- [21] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics* (Nauka, Moscow, 1988) (in Russian).
- [22] A. E. Taylor, The norm of a real linear transformation in Minkowski space, *Enseign. Math. (2)* **4**, 101–107 (1958).
- [23] I. E. Verbickiĭ and P. P. Sereda, The norm of the complexification of an operator, *Mat. Issled.* **10**, No. 3 (37), 201–206 (1975) (in Russian).
- [24] L. R. Williams and J. H. Wells, L^p inequalities, *J. Math. Anal. Appl.* **64**, 518–529 (1978).