On submultiplicativity of an N -function and its conjugate

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Abstract. It is shown that if an N -function φ is submultiplicative, then its conjugate function φ^* cannot be submultiplicative. Moreover, we have the following characterization of power functions among N -functions: if an N -function φ is C_1 -submultiplicative and its conjugate function φ^* is C_2 -submultiplicative for some $C_1, C_2 > 0$, then φ is equivalent to a power function.

Mathematics Subject Classification (2000). 46E30, 46B20, 46B42.

Keywords. Orlicz functions, submultiplicative functions, power function.

1. Introduction

The present paper contains complete proofs of the results announced in the review [16] of the papers by Krnić [8] and by Kuang and Debnath [11].

Let φ be an Orlicz function (for notations see below). In [11, Theorem 2.1], the authors prove a result on an Orlicz space L_φ under the hypothesis that φ and its conjugate φ^* are 1-submultiplicative. However, our Theorem 1 below implies that the set of such functions φ is empty. To save the result somehow, one can try to consider φ and φ^* to be C_1 - and C_2 -submultiplicative, for some $C_1 > 1$ and $C_2 \geq 1$, respectively, but then our Corollary 1 says such a φ is equivalent to a power function. This means that, under the latter hypothesis on φ and φ^* , some statements of the paper [8] are proved only for weighted Lebesgue spaces (yet not for “nontrivial” Orlicz spaces). One should add that the above-cited results from [8, 11] also appeared in the recent book [9, Theorems 9.1 and 9.2].

A φ -function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous increasing unbounded function with $\varphi(0) = 0$. A convex φ -function is called an *Orlicz function*, and an Orlicz function is called an *N -function* if, moreover,

$$\lim_{u \rightarrow 0^+} \frac{\varphi(u)}{u} = 0 \text{ and } \lim_{u \rightarrow \infty} \frac{\varphi(u)}{u} = \infty. \quad (1)$$

For every φ -function φ , one can define the associated convex conjugate function φ^* by the formula

$$\varphi^*(v) = \sup_{u>0} [uv - \varphi(u)], \quad v \in [0, \infty).$$

The function φ^* may take values 0 and ∞ on some intervals; nevertheless φ^* fulfils the Young inequality:

$$uv \leq \varphi(u) + \varphi^*(v) \text{ for all } u, v \geq 0.$$

Moreover, φ^{**} is a convex minorant of φ , i.e., $\varphi^{**} \leq \varphi$ with equality if and only if φ is convex. The function φ^* is only zero at zero if and only if the first condition in (1) holds and is finite-valued if and only if the second condition in (1) holds. Moreover, if φ is an N -function, then φ^* is also an N -function and

$$u \leq \varphi^{-1}(u) \varphi^{*-1}(u) \leq 2u \text{ for all } u \geq 0, \quad (2)$$

(see [7, 14] and [17] for details).

A φ -function φ is called C -submultiplicative with $C > 0$ (see [7, 13]) if

$$\varphi(uv) \leq C \varphi(u) \varphi(v) \text{ for all } u, v \geq 0, \quad (3)$$

and if inequality (3) is reversed, then φ will be called C -supermultiplicative.

1- and C -submultiplicative functions were studied and applied in several papers (see e.g. [1, 2, 4–6, 10, 13–15, 17]).

2. Main results

Let us start with a simple remark and two examples showing the assumptions (about the simultaneous 1-submultiplicativity of φ and φ^*) in the aforementioned paper [11] give an empty assertion.

Remark 1. Note that if φ is a 1-submultiplicative function, then $\varphi(1) \geq 1$.

Example 1. The power function $\varphi_p(u) = u^p$ for $p > 1$ is a convex 1-submultiplicative N -function and $\varphi_p^*(v) = (p-1)(v/p)^{p/(p-1)}$. Since, by the Bernoulli inequality,

$$p^{\frac{p}{p-1}} = (1 + p - 1)^{\frac{p}{p-1}} \geq 1 + \frac{p}{p-1}(p-1) = p + 1$$

it follows that $\varphi_p^*(1) = \frac{p-1}{p^{\frac{p}{p-1}}} \leq \frac{p-1}{p+1} < 1$, and therefore, by Remark 1, φ_p^* is not 1-submultiplicative.

Example 2. For $1 < p < q < \infty$ let

$$\varphi_{p,q}(u) := \begin{cases} u^p & \text{if } 0 \leq u \leq 1, \\ u^q & \text{if } 1 \leq u < \infty. \end{cases}$$

The function $\varphi_{p,q}$ is a convex 1-submultiplicative N -function and

$$\varphi_{p,q}^*(v) := \begin{cases} (p-1)(v/p)^{p/(p-1)} & \text{if } 0 \leq v \leq p, \\ v-1 & \text{if } p \leq v \leq q, \\ (q-1)(v/q)^{q/(q-1)} & \text{if } q \leq v < \infty. \end{cases}$$

As in Example 1, we have $\varphi_{p,q}^*(1) = \frac{p-1}{p^{\frac{p}{p-1}}} < 1$, and $\varphi_{p,q}^*$ is not 1-submultiplicative (by Remark 1 again).

The above two examples suggest there is a more general result. Indeed, we have

Theorem 1. *If φ is a 1-submultiplicative N -function, then its complementary function φ^* cannot be 1-submultiplicative.*

Proof. For $x > 0$, set $y = \frac{\varphi(x)}{x}$. Then, by the 1-submultiplicativity of φ ,

$$\begin{aligned} \varphi^*(1) &= \varphi^*\left(\frac{xy}{\varphi(x)}\right) = \sup_{u>0} \left[\frac{uxy}{\varphi(x)} - \varphi(u) \right] = \frac{1}{\varphi(x)} \sup_{u>0} [uxy - \varphi(x)\varphi(u)] \\ &\leq \frac{1}{\varphi(x)} \sup_{u>0} [uxy - \varphi(xu)] = \frac{1}{\varphi(x)} \varphi^*(y) = \frac{1}{\varphi(x)} \varphi^*\left(\frac{\varphi(x)}{x}\right). \end{aligned}$$

On the other hand, using the fact that φ is an N -function we obtain (cf. [14, Theorem 8.3, p. 50]), for some $0 < u_0 < x$, that

$$\begin{aligned} \varphi^*\left(\frac{\varphi(x)}{x}\right) &= \sup_{u>0} \left[\frac{\varphi(x)}{x}u - \varphi(u) \right] = \sup_{0 < u < x} \left[\frac{\varphi(x)}{x}u - \varphi(u) \right] \\ &= \frac{\varphi(x)}{x} u_0 - \varphi(u_0) < \varphi(x). \end{aligned}$$

Thus $\varphi^*(1) < 1$ whence, by Remark 1, the conjugate function φ^* cannot be 1-submultiplicative.

Although, by Theorem 1, φ and φ^* cannot be 1-submultiplicative simultaneously, the next result shows there is a relationship between the submultiplicativity of φ and the supermultiplicativity of φ^* . $\square$

Theorem 2. *If φ is a C -submultiplicative N -function ($C \geq 1$), then its conjugate function φ^* is C_1 -supermultiplicative for some $C_1 > 0$.*

Proof. We show first that φ^* satisfies the Δ_2 -condition, that is, there exists a constant $C_2 \geq 1$ such that $\varphi^*(2u) \leq C_2 \varphi^*(u)$ for all $u > 0$. By the first condition in (1) we can find $a \in (0, 1)$ such that $\frac{\varphi(a)}{a} \leq \frac{1}{2C^2}$ with the constant C from the definition of the C -submultiplicativity of φ . Moreover, by the C -submultiplicativity of φ , we obtain for all $u > 0$

$$\varphi(au) \leq C\varphi(a)\varphi(u) \leq C\frac{a}{2C^2}\varphi(u) = \frac{a}{2C}\varphi(u),$$

and so for all $v > 0$

$$\begin{aligned}\varphi^*(2Cv) &= \sup_{u>0} [2Cuv - \varphi(u)] \leq \sup_{u>0} [2Cuv - \frac{2C}{a}\varphi(au)] \\ &= \frac{2C}{a} \sup_{u>0} [uav - \varphi(au)] = \frac{2C}{a} \varphi^*(v),\end{aligned}\quad (4)$$

that is, φ^* satisfies the Δ_2 -condition.

Now, we are ready to prove the C_1 -supermultiplicativity of φ^* . For $u, v > 0$, set $s = \varphi(u)$ and $t = \varphi(v)$. From the inequality $\varphi(uv) \leq C \varphi(u)\varphi(v)$ we obtain $\varphi^{-1}(s) \varphi^{-1}(t) \leq \varphi^{-1}(Cst)$ for all $s, t > 0$. Denoting

$$f(u) = \frac{u}{\varphi^{-1}(u)}, \quad u > 0,$$

and using (2) we obtain $f(u) \leq \varphi^{*-1}(u) \leq 2f(u)$ for all $u > 0$ and

$$f(Cst) = \frac{Cst}{\varphi^{-1}(Cst)} \leq \frac{Cst}{\varphi^{-1}(s)\varphi^{-1}(t)} = Cf(s)f(t) \text{ for all } s, t > 0.$$

Hence,

$$\varphi^{*-1}(Cst) \leq 2f(Cst) \leq 2Cf(s)f(t) \leq 2C\varphi^{*-1}(s)\varphi^{*-1}(t). \quad (5)$$

Next, since $x, y > 0$, we can set $s = \varphi^*(x)$ and $t = \varphi^*(y)$. Hence, by (4) and (5), we obtain

$$C\varphi^*(x)\varphi^*(y) \leq \varphi^*(2Cxy) \leq \frac{2C}{a}\varphi^*(xy).$$

This shows that φ^* is C_1 -supermultiplicative with $C_1 = a/2$. $\square$

Corollary 1. *If φ is a C_1 -submultiplicative N -function and its conjugate function φ^* is C_2 -submultiplicative with $C_1, C_2 \geq 1$, then φ is equivalent to a power function.*

Proof. If φ is a C_1 -submultiplicative N -function ($C_1 \geq 1$), then, by Theorem 2, its conjugate function φ^* is C_3 -supermultiplicative and thus the N -function φ^* , which is C_3 -supermultiplicative and C_2 -submultiplicative, is equivalent to a power function (see [10, pp. 753–754] or [17, pp. 31–32]) and the estimates (2) give that φ is also equivalent to a power function. $\square$

Remark 2. The fact that an N -function, which is C_1 -submultiplicative and C_0 -supermultiplicative ($C_1 \geq C_0 > 0$), is equivalent to a power function was proved on several occasions. For example, a proof can be found in: [10, pp. 753–754], [17, Proposition 12, pp. 31–32], [19, Lemma 1 and Remark on pp. 386–387], and [3, Lemma 5.2, pp. 82–83] (see also [12, p. 60], [20, pp. 269–271] and [1, p. 289]). Corollary 1 in the form “for large u and v ”, with a long proof, was also given by Salekhov [18, Theorem 1].

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Received: June 24, 2013

Revised: October 22, 2013