On CLARKSON's Inequalities and Interpolation

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Abstract. We consider some elementary proofs of local versions of CLARKSON's inequalities and point out the fact that these inequalities can be generalized to hold for a much wider class of parameters. In particular it is easy to generalize our interpolation proof in various ways to higher dimensions. We point out explicitly some examples of such generalizations and we also prove some corresponding global versions. In this elementary way we obtain both new proofs of some previous results of this kind and also some new complements, unifications and further generalizations of these results.

0. Introduction

For the spaces l_p and $L_p[0,1]$, $1 < p < \infty$, CLARKSON [5] proved the following inequalities:

$$(0.1) \quad (\|x + y\|_p^p + \|x - y\|_p^p)^{1/p} \leq 2^{1/p'} (\|x\|_p^p + \|y\|_p^p)^{1/p}, \quad p \geq 2,$$

$$(0.2) \quad (\|x + y\|_p^p + \|x - y\|_p^p)^{1/p} \leq 2^{1/p} (\|x\|_p^{p'} + \|y\|_p^{p'})^{1/p'}, \quad p \geq 2,$$

$$(0.3) \quad (\|x + y\|_p^{p'} + \|x - y\|_p^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|_p^p + \|y\|_p^p)^{1/p'}, \quad 1 < p \leq 2.$$

Here, as usual, $p' = p/(p-1)$ is the conjugate number of p . We note that (0.3) only means that (0.2) holds in the opposite direction when $1 < p \leq 2$ (for $p = 2$ we have the usual parallelogram law for L_p -spaces) and (0.2) implies (0.1). The key to prove the inequalities (0.1)–(0.3) is the following well-known estimate of CLARKSON-HAUSDORFF-YOUNG type:

If $a, b \in \mathbb{C}$ and $p \geq 2$, then

$$(0.4) \quad (|a + b|^p + |a - b|^p)^{1/p} \leq 2^{1/p} (|a|^{p'} + |b|^{p'})^{1/p'}.$$

If $1 < p \leq 2$, then the reversed inequality holds.

For completeness and in order not to disturb our discussions later on we use section 1 to present three different proofs of (0.4). In section 2 we present a more general inequality of the form

$$(0.5) \quad (|a + b|^r + |a - b|^r)^{1/r} \leq C (|a|^s + |b|^s)^{1/s}, \quad s > 0, -\infty < r < \infty, r \neq 0,$$

thereby generalizing some previous results of BOAS [3] and KOSKELA [11]. In view of our interpolation and convexity proofs of (0.4) and our arguments used in section 2 it is now possible to present a strategy for generalizing (0.4)–(0.5) to inequalities for three and more complex variables. In section 3 we discuss this fact and give some examples of such generalizations thereby generalizing and complementing some previous results by PIETSCH [16] (see also KATO [10]), SHAPIRO [19] and WILLIAMS-WELLS [22]. In section 4 we use our local versions from section 3 and our arguments from section 2 to prove some global versions of some generalized forms of CLARKSON's inequalities. In particular our results generalize and unify some previous results of this kind obtained by BOAS [3], KATO [10], KOSKELA [11], PERSSON [15] and WILLIAMS-WELLS [22]. Finally, section 5 is reserved for some concluding remarks.

1. Three Proofs of the CLARKSON-HAUSDORFF-YOUNG Inequality (0.4)

In order not to disturb our discussions later on we begin to formulate the following elementary lemma:

Lemma 1.1. *Let $-\infty < \alpha < \infty$, $\beta > 0$ and let $\omega_i, b_i (i = 1, 2, \dots, n)$ be positive numbers. Then*

$$f(\alpha) = \left(\frac{\sum_{i=1}^n \omega_i b_i^\alpha}{\sum_{i=1}^n \omega_i} \right)^{1/\alpha} \quad \text{and} \quad g(\beta) = \left(\sum_{i=1}^n b_i^\beta \right)^{1/\beta}$$

are nondecreasing and nonincreasing functions of α and β , respectively.

The proof of Lemma 1.1 follows at once by using JENSEN's inequality and a subadditivity argument so we omit the details.

First proof: We must prove that

$$|a + b|^p + |a - b|^p \leq 2(|a|^{p'} + |b|^{p'})^{p-1}, \quad a, b \in \mathbb{C}.$$

Without loss of generality we may assume that $|a| \geq |b|$ and it is sufficient to prove that

$$(1.1) \quad |1 + z|^p + |1 - z|^p \leq 2(1 + |z|^{p'})^{p-1}, \quad |z| \leq 1.$$

We put $z = re^{i\theta}$ and consider (1.1) on the form

$$g(\theta) = |1 + re^{i\theta}|^p + |1 - re^{i\theta}|^p \leq 2(1 + r^{p'})^{p-1}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

It is easy to see that $\max g(\theta) = g(0) = g(\pi)$ and it is sufficient to prove that

$$(1 + r)^p + (1 - r)^p \leq 2(1 + r^{p'})^{p-1}, \quad 0 \leq r \leq 1.$$

or, equivalently, that

$$(1.2) \quad f(u) = ((1 + u)^p + (u - 1)^p)^{p'-1} 2^{1-p'} - 1 - u^{p'} \leq 0, \quad u \geq 1.$$

We differentiate and find that

$$\begin{aligned} f'(u) &= (p' - 1) ((1 + u)^p + (u - 1)^p)^{p'-2} 2^{1-p'} p ((1 + u)^{p-1} \\ &\quad + (u - 1)^{p-1}) - p' u^{p'-1} \\ &= p' \left(\left(\frac{(\omega_1 a^{p-2} + \omega_2 b^{p-2})^{1/(p-2)}}{(\omega_1 a^{p-1} + \omega_2 b^{p-1})^{1/(p-1)}} \right)^{p-2} - u^{p'-1} \right), \\ &\quad \text{where } \omega_1 = a = \frac{u+1}{2} \text{ and } \omega_2 = b = \frac{u-1}{2}. \end{aligned}$$

Hence, by Lemma 1.1, we find that $f'(u) \leq 0$ for $u \geq 1$. Moreover $f(1) = 0$ and we conclude that (1.2) (and thus (1.1)) holds and the proof is complete.

Remark. This proof is a modification of the proof presented first by SOBOLEV [20] (see also [13] and [21]).

Second proof. According to the M. RIESZ convexity theorem ([see 17, p. 219]) applied to the linear forms $a + b$ and $a - b$ we know that the function

$$f(\alpha, \beta) = \log \max \frac{(|a + b|^{1/\alpha} + |a - b|^{1/\alpha})^\alpha}{(|a|^{1/\beta} + |b|^{1/\beta})^\beta}$$

is convex in the triangle T : $0 \leq \alpha \leq \beta \leq 1$. Moreover, appealing to the parallelogram law and trivial estimates we find that $f(1/2, 1/2) = 2^{-1} \log 2$, $f(0, 0) = \log 2$ and $f(0, 1) = 0$. The linear function $g(\alpha, \beta)$ which coincides with $f(\alpha, \beta)$ at the points $(1/2, 1/2)$, $(0, 0)$ and $(0, 1)$ is equal to $g(\alpha, \beta) = (1 - \beta) \log 2$. Moreover, the convexity implies that $f(\alpha, \beta) \leq g(\alpha, \beta) = (1 - \beta) \log 2$ and by choosing $\beta = 1/p$, $1 \leq p \leq 2$, and $\alpha = 1/p'$ we obtain

$$\log \left(\frac{(|a + b|^{p'} + |a - b|^{p'})^{p'}}{(|a|^p + |b|^p)^{1/p}} \right) \leq \log 2^{1/p'}$$

and the proof follows.

Remark. The idea of this proof was observed by BOAS [3] and by SHAPIRO [19].

In view of the second proof it is natural that (0.4) can also be proved by using pure complex interpolation.

Third proof. We consider the operator $T = T(a, b) = (a + b, a - b)$ and note that

$$(1.3) \quad T: l_1^{(2)} \rightarrow l_\infty^{(2)} \text{ has the norm } \|T\|_{1,\infty} = \sup_{|a|+|b| \leq 1} \max(|a+b|, |a-b|) = 1$$

and, by the parallelogram law,

$$(1.4) \quad T: l_2^{(2)} \rightarrow l_2^{(2)} \text{ has the norm } \|T\|_{2,2} = \sup_{|a|^2+|b|^2 \leq 1} \sqrt{|a+b|^2 + |a-b|^2} = \sqrt{2}.$$

Using (1.3)–(1.4) and complex interpolation (see e.g. [2]) we find that for $1 \leq p \leq 2$

$$T: l_p^{(2)} \rightarrow l_{p'}^{(2)} \text{ with the norm}$$

$$\|T\|_{p,p'} \leq (\|T\|_{1,\infty})^{1-2/p'} (\|T\|_{2,2})^{2/p'} = (\sqrt{2})^{2/p'} = 2^{1/p'},$$

i.e.,

$$(|a + b|^{p'} + |a - b|^{p'})^{1/p'} \leq 2^{1/p'} (|a|^p + |b|^p)^{1/p}.$$

The proof is complete.

2. A Generalization of the CLARKSON-HAUSDORFF-YOUNG Inequality

We present the following generalization of some results of KOSKELA [11] and BOAS [3] and our elementary proof follows easily from the basic estimate (0.4) and Lemma 1.1.

Proposition 2.1. *Let $a, b \in \mathbb{C}$, $r \in \mathbb{R}$, $r \neq 0$, and $s \in \mathbb{R}_+$. Then*

$$(|a + b|^r + |a - b|^r)^{1/r} \leq 2^\gamma (|a|^s + |b|^s)^{1/s}$$

where $\gamma = \frac{1}{r} - \frac{1}{s} + \frac{1}{q}$, $q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2$.

Proof. Put $A_r = (|a + b|^r + |a - b|^r)^{1/r}$ and $B_s = (|a|^s + |b|^s)^{1/s}$. According to (0.4) we have $A_{p'} \leq 2^{1/p'} B_p$, $1 < p \leq 2$. Using this fact together with Lemma 1.1 we obtain the following estimates:

$$r \leq 2, s \leq 2: A_r \leq 2^{1/r} B_2 \leq 2^{1/r} B_s,$$

$$r \leq 2, s \geq 2: A_r \leq 2^{1/r} B_2 \leq 2^{1/r - 1/s + 1/2} B_s,$$

$$r > 2, s \leq r': A_r \leq 2^{1/r} B_{r'} \leq 2^{1/r} B_s,$$

$$r > 2, s \geq r': A_r \leq 2^{1/r} B_{r'} \leq 2^{1/r - 1/s + 1/r'} B_s.$$

The proof follows by combining these inequalities.

Remark. For the case $r > 0$ another proof has been presented by KOSKELA [11].

By applying Proposition 2.1 with the r and s interchanged and by making the substitution $a + b = a_1$ and $a - b = b_1$ we obtain the following estimates in the opposite direction:

Proposition 2.2. *Let $a, b \in \mathbb{C}$, $r \in \mathbb{R}_+$ and $s \in \mathbb{R}$, $s \neq 0$. Then*

$$(|a + b|^r + |a - b|^r)^{1/r} \geq 2^\gamma (|a|^s + |b|^s)^{1/s}$$

where $\gamma = \frac{1}{r} - \frac{1}{s} + \frac{1}{q}$, $q = \min(2, r)$ if $s \leq 2$ and $q = \min(s', r)$ if $s > 2$.

Here as usual $1/p' = 1 - 1/p$. In particular, by applying Propositions 2.1–2.2 with $r = s = p$ we obtain the following well-known estimates: If $0 < p \leq \infty$, then,

$$\begin{aligned} \min(2^{1/p}, 2^{1/p'}) (|a|^p + |b|^p)^{1/p} &\leq (|a - b|^p + |a + b|^p)^{1/p} \\ &\leq \max(2^{1/p}, 2^{1/p'}) (|a|^p + |b|^p)^{1/p}. \end{aligned}$$

3. Some Generalizations to Higher Dimensions

Let $a_1, a_2, \dots, a_n$ be complex numbers. In view of our convexity and interpolation proofs of the basic estimate (0.4) we now consider any linear operator T such that $T: l_1^{(n)} \rightarrow l_\infty^{(m)}$ with the norm $\|T\|_{1,\infty} = M_1$ and $T: l_2^{(n)} \rightarrow l_2^{(m)}$ with the norm $\|T\|_{2,2} = M_2$. By using complex interpolation we obtain the (HAUSSDORFF-YOUNG type) estimate

$$(3.1) \quad \|T(\bar{a})\|_{l_p^{(m)}} \leq M_1^{1-2/p'} M_2^{2/p'} \|\bar{a}\|_{l_p^{(n)}},$$

which may be seen as a genuine generalization of (0.4). By also using Lemma 1.1 we obtain in this way generalized forms of our Proposition 2.1. We point out some examples of results which can be obtained in this way, thereby generalizing some previous results of this kind (see e.g. [10], [16], [19] and [22]).

Theorem 3.1. *Let $a_1, a_2, \dots, a_n$ be complex numbers. Let $r \in \mathbb{R}$, $r \neq 0$ and $s \in \mathbb{R}_+$. Then*

$$\left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_1^n \varepsilon_i a_i \right|^r \right)^{1/r} \leq n^{1/q-1/s} \left(\sum_1^n |a_i|^s \right)^{1/s},$$

where $q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2$.

We note that if $r > 0$, then

$$\left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_1^n \varepsilon_i a_i \right|^r \right)^{1/r} = \left(\int_0^1 \left| \sum_1^n \varphi_i(t) a_i \right|^r dt \right)^{1/r},$$

where $\varphi_i(t) = \text{sign}(\sin(2^i \pi t))$ are the usual RADEMACHER functions, and we obtain the following corollary:

Corollary 3.2. *Let $a_1, a_2, \dots, a_n$ be complex numbers and let $r, s \in \mathbb{R}_+$. Then*

$$\left(\int_0^1 \left| \sum_1^n \varphi_i(t) a_i \right|^r dt \right)^{1/r} \leq n^{1/q-1/s} \left(\sum_1^n |a_i|^s \right)^{1/s},$$

where $q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2$.

Remark. For the case $n = 1$ Corollary 3.2 coincides with a result of KOSKELA [11, Th. 1]. For the case $s = r'$, $r > 2$, another proof of Corollary 3.2 is due to WILLIAMS-WELLS [22, Th. 5: formula (27)]. For the case $s = r$ the inequality in Corollary 3.2 reads:

$$\left(\int_0^1 \left| \sum_1^n \varphi_i(t) a_i \right|^r dt \right)^{1/r} \leq \begin{cases} \left(\sum_1^n |a_i|^r \right)^{1/r}, & 0 < r \leq 2, \\ n^{(r-2)/r} \left(\sum_1^n |a_i|^r \right)^{1/r}, & r > 2. \end{cases}$$

For the case $r \leq 2$ this variant of KHINTCHINE's type inequality coincides with formula (26) in Th. 5 in [22]. For the case $r > 2$ the constant $n^{(r-2)/r}$ is not the best possible if $n \geq 3$, in fact, in this case the best constant is equal to (see [9])

$$n^{-1/r} \left(\int_0^1 \left| \sum_1^n \varphi_i(t) a_i \right|^r dt \right)^{1/r}.$$

Proof of Theorem 3.1. Consider the operator

$$T: \vec{a} \rightarrow \left(\sum_1^n a_i, \dots, \sum_1^n \varepsilon_i a_i, \dots, \sum_1^n -a_i \right),$$

where $\varepsilon_i = \pm 1$, $1 \leq i \leq n$ (each coordinate of the vector $T(\vec{a}) \in R^m$, $m = 2^n$, is equal to the sums of the type $\sum_1^n \varepsilon_i a_i$). We note that

$$T: l_1^{(n)} \rightarrow l_\infty^{(m)} \text{ has the norm } \|T\|_{1,\infty} = M_1 = 1, \text{ and}$$

$$T: l_2^{(n)} \rightarrow l_2^{(m)} \text{ has the norm } \|T\|_{2,2} = M_2 = 2^{n/2}.$$

Therefore, by using complex interpolation (see (3.1)) we find that

$$T: l_p^{(n)} \rightarrow l_p^{(m)}, \quad 1 \leq p \leq 2, \text{ with the norm } M_p \leq (2^{n/2})^{2/p'} = 2^{n/p'}, \text{ i.e.}$$

$$(3.2) \quad \left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_1^n \varepsilon_i a_i \right|^{p'} \right)^{1/p'} \leq \left(\sum_1^n |a_i|^p \right)^{1/p}, \quad 1 \leq p \leq 2.$$

We let

$$A_r = \left(\sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_1^n \varepsilon_i a_i \right|^r \right)^{1/r} \quad \text{and} \quad B_s = \left(\frac{1}{n} \sum_1^n |a_i|^s \right)^{1/s}.$$

Then, according to the basic estimate (3.2) and Lemma 1.1, we have the following estimates:

$$r \leq 2, s \leq 2: A_r \leq \left(\sum_1^n |a_i|^2 \right)^{1/2} \leq \left(\sum_1^n |a_i|^s \right)^{1/s} = n^{1/s} B_s,$$

$$r \leq 2, s \geq 2: A_r \leq \left(\sum_1^n |a_i|^2 \right)^{1/2} \leq n^{1/2-1/s} \left(\sum_1^n |a_i|^s \right)^{1/s} = n^{1/2} B_s,$$

$$r \geq 2, s \leq r': A_r \leq \left(\sum_1^n |a_i|^{r'} \right)^{1/r'} \leq \left(\sum_1^n |a_i|^s \right)^{1/s} = n^{1/s} B_s,$$

$$r \geq 2, s \geq r': A_r \leq \left(\sum_1^n |a_i|^{r'} \right)^{1/r'} \leq n^{1/r'-1/s} \left(\sum_1^n |a_i|^s \right)^{1/s} = n^{1/r'} B_s.$$

The proof is complete.

Next we consider the LITTLEWOOD-matrices $A_2^n = (\varepsilon_{ij})$, $1 \leq i, j \leq 2^n$, defined recursively in the following way:

$$A_2^1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_2^n = \begin{pmatrix} A_2^{n-1} & A_2^{n-1} \\ A_2^{n-1} & -A_2^{n-1} \end{pmatrix}, \quad n = 2, 3, \dots$$

Theorem 3.3. Let $a_1, a_2, \dots, a_n$ be complex numbers and let $r \in R$, $r \neq 0$ and $s \in R_+$. Then

$$\left(\sum_1^{2^n} \left| \sum_{j=1}^{2^n} \varepsilon_{ij} a_j \right|^r \right)^{1/r} \leq 2^{n(1/r-1/s+1/q)} \left(\sum_1^{2^n} |a_j|^s \right)^{1/s},$$

where $q = \min(2, s)$ if $r \leq 2$ and $q = \min(r', s)$ if $r > 2$.

Remark. For some cases Theorem 3.3 was also proved by PIETSCH [16, p. 15] (see also KATO [10, p. 164]).

Proof. We consider the operator

$$T: \vec{a} \rightarrow \left(\sum_1^{2^n} \varepsilon_{1j} a_j, \sum_1^{2^n} \varepsilon_{2j} a_j, \dots, \sum_1^{2^n} \varepsilon_{2^n j} a_j \right).$$

and note that

$$T: l_1^{(2^n)} \rightarrow l_\infty^{(2^n)} \text{ has the norm } M_1 = 1, \text{ and} \\ T: l_2^{(2^n)} \rightarrow l_2^{(2^n)} \text{ has the norm } M_2 = 2^{n/2}.$$

As before we interpolate and obtain the following estimate (see (3.1)):

$$\left(\sum_{i=1}^{2^n} \left| \sum_{j=1}^{2^n} \varepsilon_{ij} a_j \right|^{p'} \right)^{1/p'} \leq 2^{n/p'} \left(\sum_1^{2^n} |a_j|^p \right)^{1/p}, \quad 1 \leq p \leq 2.$$

Now the proof follows by using this estimate, Lemma 1.1 and arguing as in the proof of Theorem 3.1.

Theorem 3.4. Let $a_1, a_2, \dots, a_n$ be complex numbers. Let $r \in \mathbb{R}$, $r \neq 0$ and $s \in \mathbb{R}_+$. Then

$$\left(\left| \sum_1^n a_i \right|^r + \sum_{1 \leq i < j \leq n} |a_i - a_j|^r \right)^{1/r} \leq C \left(\sum_1^n |a_i|^s \right)^{1/s},$$

where $C = m^{1/r-1/2} n^{1/2+1/p-1/s}$, $q = \min(2, s)$, $m = n(n-1)/2$, if $r \leq 2$ and $C = n^{1/r+1/q-1/s}$, $q = \min(s, r)$, if $r > 2$.

Remark. For the case $n = 2$ Theorem 3.4 coincides with Proposition 2.1. Moreover, for the case $r = s = p \geq 2$ we have the estimate

$$(3.3) \quad \left| \sum_1^n a_i \right|^p + \sum_{1 \leq i < j \leq n} |a_i - a_j|^p \leq n^{p-1} \sum_1^n |a_i|^p.$$

For the case $n = 3$ this and some similar estimates were presented by SHAPIRO in a talk at the meeting of the Swedish Mathematical Society in Stockholm in 1973 (see [19]).

Proof. We consider the operator

$$T: \vec{a} \rightarrow \left(\sum_1^n a_i, a_1 - a_2, a_1 - a_3, a_2 - a_3, a_1 - a_4, \dots, a_1 - a_n, \dots, a_{n-1} - a_n \right)$$

and note that

$$T: l_1^{(n)} \rightarrow l_\infty^{(m)}, \quad m = n(n-1)/2 + 1, \text{ has the norm } \|T\|_{1,\infty} = M_1 = 1, \text{ and} \\ T: l_2^{(n)} \rightarrow l_2^{(m)}, \text{ has the norm } \|T\|_{2,2} = M_2 = \sqrt{n}.$$

Hence, by interpolating as before, we find that

$$\left(\left| \sum_1^n a_i \right|^{p'} + \sum_{1 \leq i < j \leq n} |a_i - a_j|^{p'} \right)^{1/p'} \leq n^{1/p'} \left(\sum_1^n |a_i|^p \right)^{1/p}, \quad 1 \leq p \leq 2$$

and the proof follows as usual by also invoking Lemma 1.1 in an appropriate way.

4. Some Global Version of CLARKSON's Inequalities

Let T denote a linear class of functions $f = f(t)$ defined on a non-empty set E . Moreover, in the sequel we let $B: \Gamma \rightarrow \mathbb{R}$, denote an isotone sublinear functional ("isotone" means that for every $f, g \in \Gamma$ such that $|f(t)| \geq |g(t)|$ on E it holds that $B(f) \geq B(g)$). We also say that $f \in B_p$ if $B_p(f) = (B(|f|^p))^{1/p} < \infty$, $0 < p < \infty$. It is well-known that local versions of CLARKSON's inequalities always imply some corresponding global versions yielding e.g. for L_p -spaces or more general function spaces. As one example of this fact we state the following global version of Theorem 3.3:

Theorem 4.1. *Let $0 < p, s < \infty$ and $-\infty < r < \infty$, $r \neq 0$. Then, for any $n \in \mathbb{Z}_+$ and any $f_1, f_2, \dots, f_{2^n} \in B_p$,*

$$\left(\sum_1^{2^n} B_p^r \left(\sum_1^{2^n} \varepsilon_{ij} f_j \right) \right)^{1/r} \leq 2^{n(1/r - 1/s + 1/q)} \left(\sum_1^{2^n} B_p^s(f_j) \right)^{1/s},$$

where $q = \min(p, p', s, r')$ with the convention that p' is omitted if $0 < p \leq 1$ and r' is omitted if $r \leq 1$.

For the case $n = 1$ and when $B_p(f) = \|f\|_{X^p}$ (X is any BANACH function space) see [15, Th. 5.1]. For the case $B(f) = \int |f(t)| d\mu(t)$ (f is a measurable function on a measure space (Ω, Σ, μ)) we have $B_p(f) = \|f\|_p$ and, thus, we obtain the following generalization of some previous results by KATO [10] and KOSKELA [11]:

Corollary 4.2. *Let $0 < p, s < \infty$ and $-\infty < r < \infty$, $r \neq 0$. Then, for any $n \in \mathbb{Z}_+$ and $f_1, f_2, \dots, f_{2^n} \in L_p(\Omega)$,*

$$\left(\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^r \right)^{1/r} \leq 2^{n(1/r - 1/s + 1/q)} \left(\sum_{j=1}^{2^n} \|f_j\|_p^s \right)^{1/s},$$

where $q = \min(p, p', r', s)$ with the convention that p' is omitted if $0 < p \leq 1$ and r' is omitted if $r \leq 1$.

Remark. For the case $1 < p < \infty$, $1 \leq r, s < \infty$ and with some further restrictions on the parameters another proof of Corollary 4.2 has been presented by KATO [10, Th.1]. Moreover, for the case $n = 1$, $r > 0$, Corollary 4.2 is also proved by KOSKELA [11, Th. 2].

Proof of Theorem 4.1. 1°. Let $0 < s \leq p \leq r < \infty$. It is well-known that $B_{p/r}$ is superadditive and $B_{p/s}$ is subadditive (see e.g. [14, Lemma 1.2]). Hence

$$(4.1) \quad \sum_1^{2^n} B_p^r \left(\sum_1^{2^n} \varepsilon_{ij} f_j \right) = \sum_1^{2^n} B_{p/r} \left(\left| \sum_1^{2^n} \varepsilon_{ij} f_j \right|^r \right) \leq B_{p/r} \left(\sum_1^{2^n} \left| \sum_1^{2^n} \varepsilon_{ij} f_j \right|^r \right)$$

and

$$(4.2) \quad B_{p/r} \left(\left(\sum_1^{2^n} |f_j|^s \right)^{r/s} \right) = \left(B_{p/s} \left(\sum_1^{2^n} |f_j|^s \right) \right)^{r/s} \leq \left(\sum_1^{2^n} B_{p/s}(|f_j|^s) \right)^{r/s} = \left(\sum_1^{2^n} B_p^s(f_j) \right)^{r/s}.$$

Now by combining (4.1) – (4.2) and using Theorem 3.3 together with the fact that B is an isotone functional we find that

$$(4.3) \quad \left(\sum_1^{2^n} B_p^r \left(\sum_1^{2^n} \varepsilon_{ij} f_j \right) \right)^{1/r} \leq 2^{n(1/r - 1/s + 1/q)} \left(\sum_1^{2^n} B_p^s(f_j) \right)^{1/s},$$

where $q = \min(s, r')$ if $r \geq 2$ and $q = s$ if $0 < r \leq 2$.

2°. Let $p \leq s, r$. First we use (4.3) for the case 1° with $s = p$. After that we use Lemma 1.1 to obtain that in this case (4.3) holds with $q = \min(p, r')$ if $r \geq 2$ and $q = p$ if $0 < r \leq 2$.

3° Let $s, r \leq p$. We use (4.3) for the case 1° with $r = p$ and Lemma 1.1 to find that in this case (4.3) holds with $q = \min(s, p')$ if $p \geq 2$ and $q = s$ if $0 < p \leq 2$.

The proof follows by combining 1°–3°.

By arguing exactly as in the proof of Theorem 4.1 we can obviously also state the corresponding global versions of our Theorem 3.1, Corollary 3.2 and Theorem 3.4. Here we only present the following global B_p -version of Theorem 3.1:

Theorem 4.3. Let $0 < p, s, < \infty$. $-\infty < r < \infty, r \neq 0$. Then, for any $n \in \mathbb{Z}_+$ and any $f_1, f_2, \dots, f_{2^n} \in B_p$,

$$\left(\sum_{\varepsilon_i = \pm 1} 2^{-n} B_p^r \left(\sum_1^n \varepsilon_i f_i \right) \right)^{1/r} \leq n^{1/p - 1/s} \left(\sum_1^n B_p^s(f_i) \right)^{1/s},$$

where $q = \min(p, p', s, r')$ with the convention that p' is omitted if $0 < p \leq 1$ and r' is omitted if $r \leq 1$.

5. Concluding Remarks

1. Best constants. By choosing a and b as 1, i or $1 + i$ we see that we can have equality in the inequalities in Proposition 2.1 in all cases. This fact, in its turn, implies that the constants in the global versions in section 4 can not in general be replaced by any smaller constants.

2. Limiting cases. All of our inequalities hold also in the limiting cases $r = \infty$ or $s = \infty$ and $s = -\infty$ with the usual supremum and infimum interpretations, respectively. By considering the scale of power means (or more generally the scale of Gini means) we also obtain the limiting inequalities corresponding to $r = 0$ or $s = 0$. For example by letting $r \rightarrow 0$ in the inequality in Corollary 4.2 we obtain the following (somewhat curious) limiting inequality:

$$\left(\prod_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p \right)^{1/2^n} \leq n^{1/q - 1/s} \left(\sum_1^{2^n} \|f_j\|_p^s \right)^{1/s}$$

where $n \in \mathbb{Z}_+$, $0 < p, s < \infty$ and $q = \min(p, p', s)$ if $p \geq 1$ and $q = \min(p, s)$ if $0 < p \leq 1$.

3. Further generalizations. Our procedure to obtain global versions of CLARKSON's inequalities from the corresponding local versions can be repeated once more thereby obtaining some versions of CLARKSON's inequalities for mixed-norm spaces e.g. for spaces of the type $L_p(L_q)$. Another type of generalization is obtained by using, instead of Lemma 1, the fact that even the general scale of GINI means $G(\alpha, \beta; f)$ is increasing in

both α and β (see [14]). For example the inequality in Corollary 4.2 can be generalized in the following way: Let p, r, s and q be defined as in Corollary 4.2, $b \leq 0$ and $c \geq 0$. Then

$$\left(\frac{\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^r}{\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^b} \right)^{\frac{1}{r-b}} \leq 2^{n/q} \left(\frac{\sum_{j=1}^{2^n} \|f_j\|_p^s}{\sum_{j=1}^{2^n} \|f_j\|_p^c} \right)^{\frac{1}{s-c}}.$$

The inequality does not hold in general if $b > 0$ or $c < 0$.

4. Applications. A classical application of CLARKSON's inequalities is to prove that the corresponding space is uniformly convex and (at least for the case $p \geq 2$) calculate its modulus of rotundity. Another application is to find the NEUMANN-JORDAN constant for the space at hand (see [4]). By using three or more dimensional versions of the inequality we can get some more information of the geometry. For example by using (3.3) we obtain in the usual way that if $p \geq 2$, then

$$\left(\left\| \frac{1}{n} \sum_{i=1}^n f_i \right\|_p \right)^p + \sum_{1 \leq i < j \leq n} \left(\left\| \frac{f_i - f_j}{n} \right\|_p \right)^p \leq \frac{1}{n} \sum_{i=1}^n (\|f_i\|_p)^p$$

with an obvious geometric interpretation. For the case $n = 3$ see also SHAPIRO [19]. Another type of applications to approximation theory can be found in [18].

5. Further references. Some historical remarks, further references and a useful quasi-linearization proof can be found in [1]. For further contributions and references see also [6], [7], [8], [12], and [13].

References

- [1] E. E. BECKENBACH and R. BELLMANN, *Inequalities*, Springer Verlag, 1983
- [2] J. BERGH and J. LÖFSTRÖM, *Interpolation Spaces. An introduction*, Springer Verlag, 1976
- [3] R. P. BOAS, JR., Some uniformly convex spaces, *Bull. Amer. Math. Soc.* **46** (1940) 304–311
- [4] J. A. CLARKSON, The von Neumann-Jordan constant for the Lebesgue spaces, *Ann. Math.* **38** (1937) 114–115
- [5] —, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936) 396–414
- [6] C. E. CLEAVER, Interpolation and extension of Lipschitz-Hölder maps on C_p -spaces, *Colloq. Math.* **27** (1973) 83–87
- [7] F. COBOS, Clarkson's inequalities for Sobolev spaces, *Math. Japonica* **31** (1986) 17–22
- [8] F. COBOS and D. E. EDMONDS, Clarkson's inequalities, Besov spaces and Triebel-Sobolev spaces, *Z. Anal. Anwend.* **7** (1988) 229–232
- [9] T. FIGIEL, T. IWANIEC and A. PELCZYŃSKI, Computing norms and critical exponents of some operators in L^p -spaces, *Studia Math.* **79** (1984) 227–274
- [10] M. KATO, Generalized Clarkson's inequalities and the norms of the Littlewood matrices, *Math. Nachr.* **114** (1983) 163–170
- [11] M. KOSKELA, Some generalizations of Clarkson's inequalities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. **634–677** (1979) 89–93
- [12] M. MILMAN, Complex interpolation and geometry of Banach spaces, *Ann. Mat. Pura. Appl.* **136** (1984) 317–328
- [13] D. S. MITRINOVIC and P. M. VASIĆ, Addenda to the monograph "Analytic Inequalities" I, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. **577–598** (1977) 3–10

- [14] J. PEETRE and L. E. PERSSON, A general Beckenbach's inequality with applications, In: Function Spaces, Differential Operators and Nonlinear Analysis, Pitman Research Notes in Math., Ser. 211, 1989, 125–139
- [15] L. E. PERSSON, Some elementary inequalities in connection with X^p -spaces, Publishing House of the Bulgarian Academy of Sciences, 1988, 367–376
- [16] A. PIETSCH, Absolutely- p -summing operators in Ω -spaces 2, Sem. Goulaouic-Schwartz, Paris 1970/71
- [17] M. RIESZ, Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires, Acta Math. **49** (1926) 465–497
- [18] H. S. SHAPIRO, Regularity properties of the element of closest approximation, Trans. Amer. Math. Soc. **181** (1973) 127–142
- [19] –, The use of soft analysis, Research report 2, Dept. of Math., Royal Institute of Technology, Stockholm 1974
- [20] S. L. SOBOLEV, On an inequality, Uspekhi Mat. Nauk **1** (1946) 197
- [21] –, Some Applications of Functional Analysis in Mathematical Physics, Nauka, Moscow 1988
- [22] L. R. WILLIAMS and J. H. WELLS, L^p -inequalities, J. Math. Anal. Appl. **64** (1978) 518–529

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