The AM-GM Inequality is Equivalent to the Bernoulli Inequality

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The AM-GM (= arithmetic-mean/geometric-mean) inequality is sometimes called the *Cauchy inequality* (1821):

$$A_n = \frac{x_1 + \dots + x_n}{n} \geq (x_1 \cdot \dots \cdot x_n)^{1/n} = G_n \quad (\text{all } x_k > 0). \quad (\text{AG})$$

In the books [1] and [2] there are given 52 and 74 proofs of this inequality, respectively, and mathematicians still find new proofs (!).

Much earlier, Isaac Barrow (1670) and Jacob Bernoulli (1689) proved the inequality, which now has the name *Bernoulli inequality*:

$$x^n \geq 1 + n(x - 1) \quad \text{for any } x > 0 \text{ and } n \in \mathbb{N}. \quad (\text{B})$$

In the paper [5] we can read about the history, generalizations, and different proofs of the Bernoulli inequality.

In this note I show that these inequalities are in fact equivalent, that is, each follows one from the other. This was observed in [4, Thm 3] (see also [2], p. 213, and [5, Thm 10]), but I present here even simpler arguments.

THEOREM Inequalities (AG) and (B) are equivalent.

PROOF. (B) $\implies$ (AG). Since $\frac{A_n}{A_{n-1}} > 0$ it follows from the Bernoulli inequality (B) that

$$\begin{aligned} \left(\frac{A_n}{A_{n-1}} \right)^n &\geq 1 + n \left(\frac{A_n}{A_{n-1}} - 1 \right) \\ &= \frac{A_{n-1} + nA_n - nA_{n-1}}{A_{n-1}} \\ &= \frac{nA_n - (n-1)A_{n-1}}{A_{n-1}} = \frac{x_n}{A_{n-1}} \end{aligned}$$

or

$$A_n^n \geq x_n \cdot A_{n-1}^{n-1}. \quad (\text{C})$$

Now, using inequality (C) successively, we obtain

$$\begin{aligned} A_n^n &\geq x_n \cdot A_{n-1}^{n-1} \geq x_n \cdot x_{n-1} \cdot A_{n-2}^{n-2} \\ &\geq \dots \geq x_n \cdot x_{n-1} \cdot \dots \cdot x_2 \cdot A_1^1 \\ &= x_n \cdot x_{n-1} \cdot \dots \cdot x_2 \cdot x_1 = G_n^n, \end{aligned}$$

and thus $A_n \geq G_n$.

(AG) $\implies$ (B). For $n = 1$ we have even equality in (B). If $n \geq 2$ and $0 < x \leq 1 - \frac{1}{n}$, then $x^n > 0 \geq 1 + n(x - 1)$, that is, (B) holds. Therefore, we can assume that $n \geq 2$ and $x > 1 - \frac{1}{n}$. Then $1 + n(x - 1) > 0$, and now applying assumption (AG) to the n positive numbers

$$1 + n(x - 1), \overbrace{1, 1, \dots, 1}^{(n-1) \text{ times}}$$

we obtain

$$\begin{aligned} x^n &= \left\{ \frac{[1 + n(x - 1)] + 1 + \dots + 1}{n} \right\}^n \\ &\geq [1 + n(x - 1)] \cdot 1 \cdot \dots \cdot 1 = 1 + n(x - 1), \end{aligned}$$

and the inequality (B) is proved.

From the theorem, it follows that one way to get a simple proof of inequality (AG) is to give a simple proof of inequality (B). Here is one:

$$\begin{aligned} x^n - 1 - n(x - 1) &= (x - 1)(x^{n-1} + x^{n-2} + \dots + 1) - n(x - 1) \\ &= (x - 1)(x^{n-1} + x^{n-2} + \dots + 1 - n) \geq 0. \end{aligned}$$

The last inequality holds because the expression $x^{n-1} + x^{n-2} + \dots + 1$ is bigger than n for $x \geq 1$ and smaller than n for $0 < x \leq 1$. Thus (B) is proved, and consequently also (AG). Other proofs of (AG) via inequality (B) were given in [6] and [3].

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