Lech Maligranda (Luleå)

Marcinkiewicz Interpolation Theorem and Marcinkiewicz Spaces

JÓZEF MARCINKIEWICZ (1910–1940) was one of the most eminent figures in Polish mathematics and together with Stanisław Zaremba, Leon Lichtenstein, Juliusz Schauder and Antoni Zygmund, the most prominent in analysis. Zygmund considered him as his best pupil, although he had many students in Poland and the USA. We must also remember that it is because of the Master, who was Zygmund, Marcinkiewicz also became a known mathematician. Marcinkiewicz wrote 55 scientific papers in six years (1933–1939) and his name in mathematics is connected with the Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, the Marcinkiewicz integral and Marcinkiewicz function, the Marcinkiewicz multiplier theorem, the Jessen–Marcinkiewicz–Zygmund strong differentiation theorem, Marcinkiewicz–Zygmund vector-valued inequalities, Marcinkiewicz–Zygmund inequalities in probability, the Marcinkiewicz–Zygmund strong law of large numbers, Marcinkiewicz–Zygmund inequalities for trigonometric functions and polynomials, the Grünwald–Marcinkiewicz interpolation theorem, the Marcinkiewicz–Salem conjecture, the Marcinkiewicz theorem on the characteristic function, the Marcinkiewicz test for pointwise convergence of Fourier series, the Marcinkiewicz theorem on the Haar system, the Marcinkiewicz theorem on universal primitive functions and the Marcinkiewicz theorem on the Perron integral. I will present the first two parts from these achievements and the description of other parts can be found in my recent paper [31]. Finally, in the third part, the life, scientific career, work and tragic end of Józef Marcinkiewicz is presented.

1. The Marcinkiewicz interpolation theorem (1939)

The Riesz–Thorin convexity theorem (1926, 1938) and the Kolmogorov result (1925) showing that the conjugate-function operator is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$ were known for Marcinkiewicz. He tried to prove a theorem containing both these theorems. One of the simple cases of the Marcinkiewicz interpolation theorem is the following result.

Theorem. *If a linear or sublinear mapping T is of weak type $(1, 1)$ and strong type (∞, ∞) , that is, satisfies the estimates*

$$\lambda m(\{x \in I : |Tf(x)| > \lambda\}) \leq A \int_I |f(x)| dx \quad \text{for any } \lambda > 0, \quad (1)$$

$$\operatorname{ess\,sup}_{x \in I} |Tf(x)| \leq B \operatorname{ess\,sup}_{x \in I} |f(x)|, \quad (2)$$

then T is of strong type (p, p) for $1 < p \leq \infty$, i.e., we have estimate

$$\int_I |Tf(x)|^p dx \leq C_{A,B} \int_I |f(x)|^p dx, \quad C_{A,B}^{1/p} \leq \frac{2}{(p-1)^{1/p}} A^{1/p} B^{1-1/p}. \quad (3)$$

The importance of the Marcinkiewicz interpolation theorem is also connected with the so-called *weak- L^p spaces*. The Lebesgue spaces L^p on the measure space (Ω, Σ, μ) were investigated already in the thirties by F. Riesz for $\Omega = [a, b]$ and $\Omega = \mathbb{R}^n$. To prove boundedness of several operators it was needed, in the target, to have larger spaces than L^p . Marcinkiewicz observed that the so-called *weak- L^p spaces* denoted by $L^{p,\infty}$ ($1 \leq p < \infty$) are good candidates and now these spaces are also called *Marcinkiewicz spaces*. They are given by one of two quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p}.$$

Note that these spaces are really larger than L^p , since

$$\int_{\Omega} |f(x)|^p d\mu \geq \int_{\{x \in \Omega : |f(x)| > \lambda\}} |f(x)|^p d\mu \geq \lambda^p \mu(\{x \in \Omega : |f(x)| > \lambda\})$$

for any $\lambda > 0$, that is,

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \lambda \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p} \leq \left(\int_{\Omega} |f(x)|^p d\mu \right)^{1/p} = \|f\|_p.$$

The weak- L^∞ space is here by definition the L^∞ space.

Marcinkiewicz published in 1939 a two-pages paper [33] and formulated three theorems without proofs. Moreover, he sent a letter, including the proof of the main theorem for $p_0 = q_0 = 1$ and $p_1 = q_1 = 2$, to Zygmund who after the war reconstructed all the proofs and published them in 1956 (see [48]). This is the reason why *Marcinkiewicz interpolation theorem* is sometimes called *Marcinkiewicz–Zygmund interpolation theorem*. Note that in 1953 Zygmund presented all the proofs at his Chicago seminar informing that he was only developing Marcinkiewicz's ideas (cf. [36], p. 46). A proof was also given by the PhD students of Zygmund: Mischa Cotlar for $p_0 = q_0$ and $p_1 = q_1$ (PhD 1953, published in 1956 in [12]) and William J. Riordan for $1 \leq p_i \leq q_i \leq \infty, i = 0, 1$, (PhD 1955, unpublished).

Marcinkiewicz was probably the first who used the word “interpolation of operators”. Riesz and Thorin spoke on “convexity theorems” (cf. Peetre [36], p. 39 and Horwath [20], p. 618).

Theorem 1.1 (Marcinkiewicz 1939, Zygmund 1956). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and for $0 < \theta < 1$ define p, q by the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \quad (4)$$

If $p_0 \leq q_0$ and $p_1 \leq q_1$ (lower triangle) with $q_0 \neq q_1$, then from boundedness of any linear or sublinear operator $T: L^{p_0} \rightarrow L^{q_0, \infty}$ and $T: L^{p_1} \rightarrow L^{q_1, \infty}$ – i.e., T is of weak type (p_0, q_0) and of weak type (p_1, q_1) – follows boundedness of $T: L^p \rightarrow L^q$ (that is, T is of strong type (p, q)) and

$$\|T\|_{L^p \rightarrow L^q} \leq C \|T\|_{L^{p_0} \rightarrow L^{q_0, \infty}}^{1-\theta} \|T\|_{L^{p_1} \rightarrow L^{q_1, \infty}}^{\theta}, \quad (5)$$

where $C = C(\theta, p_0, p_1, q_0, q_1) \rightarrow \infty$ as $\theta \rightarrow 0^+$ or $\theta \rightarrow 1^-$.

The Marcinkiewicz interpolation theorem was proved even for pointwise quasi-additive operators, that is, there exists a constant $\gamma \geq 1$ such that for any measurable functions f, g we have the following inequality μ -almost everywhere on Ω :

$$|T(f+g)(x)| \leq \gamma (|Tf(x)| + |Tg(x)|).$$

The Marcinkiewicz interpolation theorem is true if $p \leq q$ and $q_0 \neq q_1$, that is, if one point can be in the upper triangle but the point S must appear in the lower triangle. Proof of this theorem was given by: Calderón (1963), Lions–Peetre (1964), O’Neil (1964), Hunt (1964) and Krée (1967). Berenstein–Cotlar–Kerzman–Krée (1967) proved that if for the segment W_0W_1 Marcinkiewicz

interpolation theorem is true, then it also holds for another segment obtained from the rotation of W_0W_1 around the point S (except horizontal and vertical segments).

1. Marcinkiewicz interpolation theorem is true in lower triangle:

$$W_0 = (\frac{1}{p_0}, \frac{1}{q_0}), W_1 = (\frac{1}{p_1}, \frac{1}{q_1}), S = (\frac{1}{p}, \frac{1}{q}) \text{ and } q_0 \neq q_1 \\ \text{(except horizontal segments)}$$

The Marcinkiewicz interpolation theorem is not true in the upper triangle, that is, if $p_0 > q_0$ and $p_1 > q_1$. A counter-example is due to Hunt (1964). He also observed that Theorem 1.1 is true in the extended range $0 < p_0, p_1, q_0, q_1 \leq \infty$ provided $p \leq q$ and $q_0 \neq q_1$ (cf. Hunt [21], p. 807).

2. Marcinkiewicz interpolation theorem (diagonal case):

$$W_0 = (\frac{1}{p_0}, \frac{1}{p_0}), W_1 = (\frac{1}{p_1}, \frac{1}{p_1}), S = (\frac{1}{p}, \frac{1}{p}).$$

Particular cases of Theorem 1.1 have the following form (cf. [19], p. 32).

Theorem (Marcinkiewicz interpolation theorem – diagonal case). If $1 \leq p_0 < p_1 \leq \infty$ and T is an arbitrary linear or sublinear operator of weak type (p_0, p_0) and of weak type (p_1, p_1) , that is, $T: L^{p_0} \rightarrow L^{p_0, \infty}$ and $T: L^{p_1} \rightarrow L^{p_1, \infty}$ is

bounded, then it is of strong type (p, p) , i.e., $T: L^p \rightarrow L^p$ is bounded for any $p_0 < p < p_1$. Moreover, if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for some $0 < \theta < 1$, then

$$\|T\|_{L^p \rightarrow L^p} \leq 2 \left(\frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{1-\theta} \|T\|_{L^{p_1} \rightarrow L^{p_1, \infty}}^{\theta}. \quad (6)$$

Theorem (Little Marcinkiewicz interpolation theorem). *If $1 < p \leq \infty$ and T is an arbitrary linear or sublinear operator of weak type $(1, 1)$ and strong type (∞, ∞) , that is, $T: L^1 \rightarrow L^{1, \infty}$ and $T: L^\infty \rightarrow L^\infty$ is bounded, then it is of strong type (p, p) , i.e., $T: L^p \rightarrow L^p$ is bounded and*

$$\|T\|_{L^p \rightarrow L^p} \leq 2 \left(\frac{p}{p-1} \right)^{1/p} \|T\|_{L^1 \rightarrow L^{1, \infty}}^{1/p} \|T\|_{L^\infty \rightarrow L^\infty}^{1-1/p}. \quad (7)$$

In connection to the Marcinkiewicz interpolation theorem we formulate several remarks:

1. Marcinkiewicz's proof is based on an idea of decomposition of a function which generates later on the concept of the K -functional playing a central role in modern interpolation theory.

2. It is not true, as some authors write, that Marcinkiewicz obtained in his short paper the result related only to the diagonal case. Marcinkiewicz had the theorem in the general case and equations (4) were written by formulas $p_0 < p < p_1$ and $\frac{q-q_0}{q_1-q} = \frac{q_0 p_1}{p_0 q_1} \cdot \frac{p-p_0}{p_1-p}$. Moreover, Marcinkiewicz's second theorem was formulated even for Orlicz spaces in the thesis (in this case it was indeed the diagonal case): if a linear or sublinear operator is bounded $T: L^{p_0} \rightarrow L^{p_0, \infty}$ and $T: L^{p_1} \rightarrow L^{p_1, \infty}$, and a function $\varphi: [0, \infty) \rightarrow [0, \infty)$ is continuous, increasing, vanishing at zero and satisfying three conditions $\varphi(2u) = O(\varphi(u))$, $\int_1^u t^{-p_0-1} \varphi(t) dt = O(u^{-p_0} \varphi(u))$ and $\int_u^\infty t^{p_1-1} \varphi(t) dt = O(u^{-p_1} \varphi(u))$ as $u \rightarrow \infty$, then for f such that $\varphi(|f|) \in L^1[0, 1]$ we obtain $\int_0^1 \varphi(|Tf(x)|) dx \leq C \int_0^1 \varphi(|f(x)|) dx + C$, where C is independent of f (see Zygmund [48, Thm 2] and [49, XII. Thm 4.22]).

3. E. Stein and G. Weiss (1959) generalized the Marcinkiewicz interpolation theorem replacing the spaces L^{p_i} in the domain of an operator by the smaller Lorentz spaces $L^{p_i, 1}$, $i = 0, 1$ (in fact, they have in the assumption of the Marcinkiewicz theorem only estimates for characteristic functions of measurable sets). Hence, if $p_i \leq q_i$, $p_i \neq \infty$, $i = 0, 1$ and $q_0 \neq q_1$, then the boundedness of $T: L^{p_0, 1} \rightarrow L^{q_0, \infty}$ and $T: L^{p_1, 1} \rightarrow L^{q_1, \infty}$ implies the boundedness $T: L^p \rightarrow L^q$.

4. Using reiteration theorems for the real method of interpolation we are getting a *generalized Marcinkiewicz interpolation theorem*: Suppose $1 \leq p_0, p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$. If a quasi-linear operator $T: L^{p_0,1} \rightarrow L^{q_0,\infty}$ and $T: L^{p_1,1} \rightarrow L^{q_1,\infty}$ is bounded, then $T: L^{p,r} \rightarrow L^{q,r}$ is bounded for any $1 \leq r \leq \infty$. In particular, $T: L^p \rightarrow L^{q,p}$ is bounded.

Note that if we have, as in Marcinkiewicz interpolation theorem, $p_0 \leq q_0$ and $p_1 \leq q_1$, then $L^{q,p} \hookrightarrow L^q$.

5. A very important progression to a generalization of the Marcinkiewicz interpolation theorem was done by Calderón (1966), who found the maximal operator in the sense that if an operator $T: L^{p_0,1} \rightarrow L^{q_0,\infty}$ and $T: L^{p_1,1} \rightarrow L^{q_1,\infty}$ is bounded, then $(Tf)^*(t) \leq CS_\sigma(f^*)(t)$ for all $t > 0$, where S_σ is the maximal Calderón operator:

$$\begin{aligned} S_\sigma f(t) &= \int_0^\infty f(s) \min\left\{\frac{s^{1/p_0}}{t^{1/q_0}}, \frac{s^{1/p_1}}{t^{1/q_1}}\right\} \frac{ds}{s} \\ &= t^{-1/q_0} \int_0^{t^m} s^{1/p_0-1} f(s) ds + t^{-1/q_1} \int_{t^m}^\infty s^{1/p_1-1} f(s) ds, \end{aligned}$$

with $m = (1/q_0 - 1/q_1)/(1/p_0 - 1/p_1)$. To get Marcinkiewicz interpolation theorem it is enough to investigate boundedness of the last two operators of Hardy type.

6. The Marcinkiewicz interpolation theorem was proved for symmetric spaces by Boyd (1967 $\Rightarrow$, 1969 $\Leftarrow$): if $1 \leq p_0 < p_1 < \infty$, E is a symmetric space with the Fatou property of the norm on either $I = (0, 1)$ or $I = (0, \infty)$ and every linear operator $T: L^{p_0,1} \rightarrow L^{p_0,\infty}$ and $T: L^{p_1,1} \rightarrow L^{p_1,\infty}$ is bounded, then it yields that $T: E \rightarrow E$ is bounded if and only if $1/p_1 < \alpha_E \leq \beta_E < 1/p_0$, where the numbers α_E, β_E are the so-called Boyd indices of the space E defined by

$$\alpha_E = \lim_{a \rightarrow 0^+} \frac{\ln \|\sigma_a\|_{E \rightarrow E}}{\ln a}, \quad \beta_E = \lim_{a \rightarrow \infty} \frac{\ln \|\sigma_a\|_{E \rightarrow E}}{\ln a}$$

and $\sigma_a f(x) = f(x/a) \chi_I(x/a)$. Krein–Petunin–Semenov (1977) proved that Boyd's theorem is true for arbitrary symmetric spaces (even without the Fatou property of the norm). In particular, E is an interpolation space between L^{p_0} and L^{p_1} . If $p_1 = \infty$, a one-sided estimate for a symmetric space E , $\beta_E < 1/p_0$, $1 \leq p_0 < \infty$, implies that E is an interpolation space between L^{p_0} and L^∞ (see [28, Theorem 4.6], where it is proved even for

Lipschitz operators). Moreover, Astashkin and Maligranda [1] proved the following one-sided Boyd theorem: if a symmetric space E has either the Fatou property or it is separable and $\alpha_E > 1/p_1$, $1 < p_1 < \infty$, then E is an interpolation space between L^1 and L^{p_1} .

7. The Marcinkiewicz interpolation theorem is not true for bilinear operators without additional assumptions. In fact, Strichartz (1969) proved the following: the operator $S(f, g)(x) = \int_0^\infty f(xt)g(t)dt$ is bounded $S: L^1 \times L^\infty \rightarrow L^{1,\infty}$ and $S: L^2 \times L^2 \rightarrow L^{2,\infty}$, but it is not bounded $S: L^p \times L^{p'} \rightarrow L^p$, and Maligranda (1989) proved that the operator $T(f, g)(x) = \int_0^1 \int_0^1 f(s)g(t) \min(\frac{1}{st}, \frac{1}{x}) ds dt$ is bounded $T: L^1 \times L^1 \rightarrow L^{1,\infty}$ and $T: L^2 \times L^2 \rightarrow L^{2,\infty}$, but it is not bounded $T: L^p \times L^p \rightarrow L^p$ for $1 < p \leq 2$.

J. L. Lions and J. Peetre (1964) proved that for the real method of interpolation we have the following interpolation theorem for bilinear operators: if a bilinear operator $T: L^{p_0} \times L^{q_0} \rightarrow L^{r_0,\infty}$ and $T: L^{p_1} \times L^{q_1} \rightarrow L^{r_1,\infty}$ is bounded, then $T: L^p \times L^q \rightarrow L^r$ is bounded, if besides the natural interpolation equality

$$\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right) = (1 - \theta)\left(\frac{1}{p_0}, \frac{1}{q_0}, \frac{1}{r_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}\right)$$

we have also $1/r \leq 1/p + 1/q - 1$. More theorems of this type can be found in papers by Sharpley (1977), Zafran (1978), Janson (1988) and Grafakos-Kalton (2001).

8. The Marcinkiewicz interpolation theorem for spaces of sequences (and some of its analogues) was given by Sargent in 1961.

9. In 1972 Y. Sagher introduced the notion of a *Marcinkiewicz quasi-cone*. If (A_0, A_1) is a pair of quasi-normed spaces, then a subset Q of $A_0 + A_1$ is called a quasi-cone if $Q + Q \subset Q$. Q is a cone if we also have $\lambda Q \subset Q$ for all $\lambda > 0$. A quasi-cone Q is called *Marcinkiewicz quasi-cone* in (A_0, A_1) if

$$(A_0 \cap Q, A_1 \cap Q)_{\theta,p} = (A_0, A_1)_{\theta,p} \cap Q \quad \text{for all } 0 < \theta < 1, 0 < p \leq \infty,$$

where $(\cdot, \cdot)_{\theta,p}$ means the real K -method of interpolation of Lions-Peetre. For example, $Q = \{(x_k)_{k=1}^\infty : x_k \downarrow 0\}$ is a Marcinkiewicz quasi-cone in (l^p, l^∞) .

10. In 1978 Dmitriev and Krein [13] extended Marcinkiewicz interpolation theorem to operators mapping a couple of Banach spaces (A_0, A_1) into a couple of Marcinkiewicz spaces $(M_{\varphi_0}^*, M_{\varphi_1}^*)$.

The Marcinkiewicz interpolation theorem is cited in several classical books on analysis, harmonic analysis and interpolation theory as, for example, in 55 books mentioned in chronological order in [31].

2. Marcinkiewicz spaces

Marcinkiewicz investigated three types of spaces: two symmetric spaces and one space of another type, which will be considered in the next part. All are called now *Marcinkiewicz spaces*.

2.1. *Marcinkiewicz function and sequence spaces (1939)*. Earlier we informed about the weak- L^p space $L^{p,\infty}$ ($1 \leq p < \infty$) or *Marcinkiewicz space* given by one of the quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p},$$

where f^* denotes the decreasing rearrangement of $|f|$. The above type of space can be easily generalized. Let $I = (0, 1)$ or $I = (0, \infty)$ and let $\varphi: I \cup \{0\} \rightarrow [0, \infty)$ be an arbitrary concave function on I such that $\varphi(0) = 0$ (it is also possible to take as φ only a quasi-concave function, that is, a function for which the inequality $\varphi(s) \leq \max(1, s/t) \varphi(t)$ is true for all $s, t \in I$).

The *Marcinkiewicz function space* M_φ^* on I consists of classes of measurable functions given by the quasi-norm

$$\|f\|_\varphi^* = \sup_{t \in I} \varphi(t) f^*(t) < \infty.$$

Another (smaller) *Marcinkiewicz function space* M_φ on I is generated by the norm

$$\|f\|_\varphi = \sup_{t \in I} \varphi(t) f^{**}(t), \quad \text{where } f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.$$

In the case where $\varphi(t) = t^{1/p}$, $1 < p < \infty$ we have $M_\varphi^* = M_\varphi = L^{p,\infty}$, but for $\varphi(t) = t$ we get $M_\varphi^* = L^{1,\infty}$ (weak- L^1 space is a quasi-Banach space but not a Banach space, since the triangle inequality holds only with constant 2) and $M_\varphi = L^1$.

Of course, $M_\varphi \subset M_\varphi^*$ and $\|f\|_\varphi^* \leq \|f\|_\varphi$ for $f \in M_\varphi$. The Marcinkiewicz function space M_φ on I is a symmetric Banach space and for an arbitrary symmetric space X on I with the fundamental function $\varphi(t) := \|\chi_{(0,t)}\|_X$, M_φ is the largest symmetric space containing X with the same fundamental function, i.e., $\|\chi_{(0,t)}\|_X = \|\chi_{(0,t)}\|_{M_\varphi} = \varphi(t)$ for any $t \in I$. This follows from the fact that $\int_0^t f^*(s) \, ds \leq \frac{t}{\varphi(t)} \|f^* \chi_{(0,t)}\|_X$ for $f \in X$.

Let us recall that a symmetric space X on I is an ideal Banach space on I (the assumption $|f(t)| \leq |g(t)|$ almost everywhere on I , $g \in X$ and f is

measurable on I implies that $f \in X$ and $\|f\|_X \leq \|g\|_X$ with the additional property that two arbitrary equimeasurable functions f and g , i.e. satisfying $m(\{x \in I : |f(x)| > \lambda\}) = m(\{x \in I : |g(x)| > \lambda\})$ for any $\lambda > 0$, with $f \in X$ and g measurable on I implies that $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$.

Note that when we investigate the Marcinkiewicz spaces $L^{p,\infty}$ or operators with values in this space, then the so-called Kolmogorov–Cotlar equivalence is important (cf. [17, pp. 485–486]): if $0 < q < p < \infty$ and $f \in L^{p,\infty}$, then

$$\|f\|_{p,\infty} \approx \sup_{\substack{A \subset I \\ 0 < \mu(A) < \infty}} \mu(A)^{1/p-1/q} \left(\int_A |f(x)|^q d\mu \right)^{1/q}. \quad (8)$$

Marcinkiewicz sequence spaces m_φ^* and m_φ are defined analogously by quasi-norms and norms

$$\|x\|_\varphi^* = \sup_{n \in \mathbb{N}} \varphi(n) x_n^* < \infty, \quad \|x\|_\varphi = \sup_{n \in \mathbb{N}} \varphi(n) \frac{1}{n} \sum_{k=1}^n x_k^*,$$

where $\varphi: \mathbb{N} \rightarrow [0, \infty)$ satisfies $\varphi(k) \leq \max(1, k/n) \varphi(n)$ for any $k, n \in \mathbb{N}$ and (x_n^*) is a rearrangement of the sequence $(|x_n|)$ in decreasing order. These spaces are also symmetric sequence spaces.

Marcinkiewicz symmetric function and sequence spaces are now classical spaces and they are objects of investigation of their structure and geometry. They also appear naturally in the interpolation theory. The name *Marcinkiewicz space* appeared already in Encyclopedia of Mathematics, Vol. 10, Kluwer, Dordrecht 1990: S. G. Krein, *Marcinkiewicz space*, pp. 93–94.

2.2. Marcinkiewicz $M^p(\mathbb{R})$ spaces (1939). The third type of Marcinkiewicz function spaces (sometimes also called Besicovitch–Marcinkiewicz spaces) consists of functions defined on $\mathbb{R}$ satisfying certain condition, more precisely, if $1 \leq p < \infty$, then the *Marcinkiewicz space* $M^p = M^p(\mathbb{R})$ contains all measurable real (or complex) functions $f: \mathbb{R} \rightarrow \mathbb{R}$ (or $\mathbb{C}$) locally p -integrable on $\mathbb{R}$ and such that the semi-norm

$$\|f\|_{M^p} = \limsup_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |f(t)|^p dt \right)^{1/p} < \infty. \quad (9)$$

The closure in the semi-norm $\|\cdot\|_{M^p}$ of the set of trigonometric polynomials $\sum a_k e^{i\lambda_k t}$ is the Besicovitch space of almost periodic functions (B^p a.p.).

Marcinkiewicz himself was calling this space the Besicovitch space and he proved the completeness of this space in the paper [32] from 1939.

Theorem 2.1 (Marcinkiewicz 1939). *The space $M^p(\mathbb{R})$ with semi-norm (9) is complete.*

Marcinkiewicz and Orlicz were trying to prove completeness in Lwów, when Marcinkiewicz was there in the academic year 1935/1936, but they were not able to do it. Two years later, on 24 November 1938, Marcinkiewicz wrote in a letter from Paris to Orlicz the proof of completeness (cf. [31, p. 35]).

Note here that the completeness of the space, where “lim sup” is replaced by “sup”, in the terms of a new convergence, was given already in 1914 by the Italian mathematician Pia Nalli (1886–1964). Marcinkiewicz gave a proof of completeness for (9) in [32] (see also [26, pp. 249–252]), and other proofs were also found by Bohr and Følner (1945), Hartman and Wintner (1947), Luxemburg and Zaanen (1963), and Corduneanu (2009).

Observe that if $1 \leq p \leq q < \infty$ then

$$L^\infty(\mathbb{R}) \xhookrightarrow{1} M^q(\mathbb{R}) \xhookrightarrow{1} M^p(\mathbb{R}),$$

that is, $L^\infty(\mathbb{R}) \subset M^q(\mathbb{R}) \subset M^p(\mathbb{R})$ with $\|f\|_{M^p} \leq \|f\|_{M^q}$ for any $f \in M^q(\mathbb{R})$ and $\|f\|_{M^q} \leq \|f\|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R})$.

Sometimes the name *Marcinkiewicz space* is used on the quotient space $\mathcal{M}^p = M^p(\mathbb{R})$, i.e. the space $M^p(\mathbb{R})$ modulo the kernel of $\|\cdot\|_{M^p}$ (that is, $\|\cdot\|_{M^p} = 0$). This space is then a Banach ideal space on $\mathbb{R}$.

Lau [24] investigated geometry of the ball and geometry of the Marcinkiewicz space. He proved, among other things, the following theorem.

Theorem. *The Marcinkiewicz space $\mathcal{M}_p(\mathbb{R})$ contains an isomorphic copy of l^∞ . Therefore, $\mathcal{M}_p(\mathbb{R})$ is a non-separable and non-reflexive space.*

The convolution operator in Marcinkiewicz spaces M^p was investigated by Bertrandias (1966) and Lau (1981). Harmonic analysis in M^p spaces was done by Urbanik [47], Bertrandias [5] and in [6].

Note that there are generalizations of these spaces, the so-called *Marcinkiewicz–Orlicz spaces* generated by modulars

$$\rho_\varphi(f) = \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \varphi(|f(t)|) dt,$$

where $\varphi: [0, \infty) \rightarrow [0, \infty)$ are convex Orlicz functions. These spaces were developed by Albrycht (1956, 1959, 1962), Wang (1959) and Hillmann (1986).

In 1995 Kucher and Plichko investigated also the *Marcinkiewicz symmetric spaces* M_E defined by semi-norms $\|f\|_{M_E} = \limsup_{T \rightarrow \infty} \|f_T\|_E$, where $f_T(t) = f(tT)$ and E is an arbitrary symmetric Banach function space on $[-\frac{1}{2}, \frac{1}{2}]$. Moreover, Vo-Khac in [6] and Cohen-Losert (2006) defined *generalized Marcinkiewicz spaces* based upon arbitrary measure spaces and limits of averages over more general families of sets.

3. Józef Marcinkiewicz (1910–1940)

Józef Marcinkiewicz was born on 12th April 1910 (30th March 1910 in old style) in Cimoszka near Białystok (Poland). His parents were Klemens Marcinkiewicz (1866–1941) and Aleksandra Marcinkiewicz née Chodakiewicz (1878–1941). Józef was the fourth of five children: Stanisława (1903–1988), Mieczysław (1904–1976), Edward (1908–1985) and Kazimierz (1913–1946).

In the period 1924–1930 he studied at the State Gymnasium of the King Zygmunt August in Białystok and then three years at the Department of Mathematics and Natural Science, University of Stefan Batory (USB) in Wilno (then in Poland, now in Lithuania). Here he met Antoni Zygmund (1900–1992) who was not only a master for him, but also an advocate for his achievements. In 1940 Zygmund emigrated to United States, and from 1947 he worked in Chicago, where he created the famous school of mathematics. It is thanks to Zygmund, who survived the war and became an important mathematician in the world, the name of Marcinkiewicz also became widely known among mathematicians.

Marcinkiewicz graduated in 1933, after only three years of study and in June 1933 he obtained a MSc degree (in mathematics) at USB. The title of his master thesis was *Convergence of the Fourier–Lebesgue series* and the supervisor was Zygmund. From September 1933 he was assistant to the Zygmund chair of mathematics at USB. In June 1935 he defended his PhD thesis entitled *Interpolation polynomials of absolutely continuous functions* at the University of Stefan Batory in Wilno, under supervision of Zygmund.

He spent the academic year 1935/1936 at the University of Jan Kazimierz in Lwów at the chair of Stefan Banach (Dec. 1935 – Aug. 1936). Marcinkiewicz visited the *Scottish Café* and solved some problems from the *Scottish Book* (83, 106, and 131). He also posed his own problem number 124. In Lwów he cooperated with Juliusz Paweł Schauder (1899–1943), Stefan Kaczmarz (1895–1939) and Władysław Orlicz (1903–1990).

In June 1937 he wrote his habilitation thesis *On summability of orthogonal series*, and was nominated docent by USB. His habilitation lecture was entitled *Arithmetization of notion of eventual variable*. At the age of 27 Marcinkiewicz was the youngest PhD who had ever habilitation at USB.

In 1938 he received of one-year scholarship for journeys to Paris, London, and Stockholm to complement his knowledge of probability theory and mathematical statistics. He stayed in Paris six months (October 1938 – March 1939), where he collaborated with Stefan Bergman (1895–1977), Raphaël Salem (1898–1963) and also had contact with the famous mathematician Paul Lévy (1886–1971). After Paris, Marcinkiewicz continued on to London on his scientific travels. He remained five months (April–August 1939) at University College, London. A planned trip to Stockholm was never realized, since Marcinkiewicz returned from London to Wilno at the end of August 1939.

In June 1939 he was appointed Extraordinary Professor at the University of Poznań. Beginning with the academic year 1939/40 Marcinkiewicz was to take up the chair of mathematics at the University of Poznań.

As a reserve officer he took part in the defence of Lwów (12–21 September 1939). After the capitulation of Lwów (22 September 1939) he, together with other officers, was taken prisoner by the Red Army. He was kept in the Starobielsk camp from September 1939 until April or May 1940. Marcinkiewicz was then murdered in Kharkov, where thousands of Polish officers were executed. He is probably buried in the village Piatykhvatky (Piatichatki). The exact date of Józef Marcinkiewicz's death remains unknown because some official Soviet documents are inaccessible or have been destroyed. The only known information is that this was between 5 April and 12 May 1940.

Józef Marcinkiewicz should be remembered as a true Polish patriot, and especially as an excellent mathematician. Mathematics was his passion. He possessed an outstanding ability to focus on problems of mathematical thinking and had extraordinary insights into mathematics. Marcinkiewicz's premature death was a huge blow to Polish and to world mathematics.

Zygmund, in his article about Marcinkiewicz [50, p. 1], wrote:

His first mathematical paper appeared in 1933; the last one he sent for publication in the Summer of 1939. This short period of mathematical activity left, however, a definite imprint on Mathematics, and but for his premature death he would probably have been one of the most outstanding contemporary mathematicians. Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.

References

- [1] S. V. Astashkin, L. Maligranda, *Interpolation between L_1 and L_p , $1 < p < \infty$* , Proc. Amer. Math. Soc. 132 (2004), no. 10, 2929–2938.
- [2] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Boston 1988. [Marcinkiewicz interpolation theorem, pp. 216–230].
- [3] C. A. Berenstein, M. Cotlar, N. Kerzman, P. Krée, *Some remarks on the Marcinkiewicz convexity theorem in the upper triangle*, Studia Math. 29 (1967), 79–95.
- [4] J. Bergh, J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin–New York 1976. [Marcinkiewicz theorem, pp. 6–11 and application, pp. 11–12].
- [5] J.-P. Bertrandias, *Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p* , Bull. Soc. Math. France Mém. 5 (1966), 1–106. [Besikovich–Marcinkiewicz space M^p , pp. 12–51; Marcinkiewicz theorem on completeness, p. 12].
- [6] J. P. Bertrandias, J. Couot, J. Dhombres, M. Mendès France, V.-K. Pham Phu Hien, Kh. Vo-Khac, *Espaces de Marcinkiewicz: Corrélations, Mesures, Systèmes Dynamiques*, Masson, Paris 1987. [Marcinkiewicz spaces, pp. 2–3, 11, 13, 59, 78].
- [7] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. 21 (1969), 1245–1254.
- [8] Yu. A. Brudnyi, N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*, North-Holland, Amsterdam 1991. [Marcinkiewicz theorem, pp. 66–83; Marcinkiewicz space $M(\varphi)$, p. 472].
- [9] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), 273–299.
- [10] A. Cianchi, *An optimal interpolation theorem of Marcinkiewicz type in Orlicz spaces*, J. Funct. Anal. 153 (1998), no. 2, 357–381.
- [11] C. Corduneanu, *Almost Periodic Oscillations and Waves*, Springer, New York 2009. [Marcinkiewicz space, pp. 41–46].
- [12] M. Cotlar, *A general interpolation theorem for linear operations*, Rev. Mat. Cuyana 1 (1955), 57–84 (1956).
- [13] V. I. Dmitriev, S. G. Krein, *Interpolation of operators of weak type*, Anal. Math. 4 (1978), no. 2, 83–99.
- [14] N. Dunford, J. T. Schwartz, *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*, John Wiley & Sons, New York–London 1963. [interpolation theorem of Marcinkiewicz, pp. 1166–1168].
- [15] R. E. Edwards, *Fourier Series. Vol. 2. A Modern Introduction*, 2nd ed., Springer, New York–Berlin 1982. [13.8. Marcinkiewicz interpolation theorem].
- [16] G. B. Folland, *Real Analysis. Modern Techniques and their Applications*, 2nd ed., Wiley, New York 1999. [Marcinkiewicz interpolation theorem, pp. 203–208].
- [17] J. García-Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam 1985. [Marcinkiewicz interpolation theorem, pp. 148–150].
- [18] D. J. H. Garling, *Inequalities: A Journey into Linear Analysis*, Cambridge Univ. Press, Cambridge 2007. [Marcinkiewicz interpolation theorem, pp. 154–156 and 162–165].

- [19] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Springer, New York 2008. [Marcinkiewicz interpolation theorem, pp. 31–34 and off-diagonal Marcinkiewicz interpolation theorem, pp. 55–63].
- [20] J. Horváth, *Encounters with Mischa Cotlar*, Notices Amer. Math. Soc. 56 (2009), no. 5, 616–620.
- [21] R. A. Hunt, *An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces*, Bull. Amer. Math. Soc. 70 (1964), 803–807.
- [22] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984. (Russian) [Appendix 1.2. Marcinkiewicz interpolation theorem, pp. 442–443]; English transl. AMS, Providence 1989 [Appendix 2.1. Marcinkiewicz interpolation theorem, pp. 390–392].
- [23] S. G. Krein, Yu. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*, AMS, Providence 1982, [extensions of Marcinkiewicz’ theorem, pp. 129–130, 132–133 and 350; Marcinkiewicz spaces, pp. 112–118 and 127–130].
- [24] K.-S. Lau, *On the Banach spaces of functions with bounded upper means*, Pacific J. Math. 91 (1980), no. 1, 153–172.
- [25] K.-S. Lau, *Extension of Wiener’s Tauberian identity and multipliers on the Marcinkiewicz space*, Trans. Amer. Math. Soc. 277 (1983), no. 2, 489–506.
- [26] B. M. Levitan, *Almost-Periodic Functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow 1953, [Marcinkiewicz theorem, pp. 249–252].
- [27] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*, Springer-Verlag, Berlin–New York 1979, [Marcinkiewicz interpolation theorem, pp. 149–150].
- [28] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. 71 (1981), no. 1, 69–83.
- [29] L. Maligranda, *Indices and interpolation* 234 (1985), 1–54, [Chapter 6, Part B, Marcinkiewicz interpolation theorem in Orlicz spaces].
- [30] L. Maligranda, *On interpolation of nonlinear operators*, Comment. Math. Prace Mat. 28 (1989), no. 2, 253–275.
- [31] L. Maligranda, *Józef Marcinkiewicz (1910–1940) – on the centenary of his birth*, Banach Center Publ. 95 (2011), 133–234.
- [32] J. Marcinkiewicz, *Une remarque sur les espaces de M. Besicovitch*, C. R. Acad. Sci. Paris 208 (1939), 157–159, reprinted in: Józef Marcinkiewicz, *Collected Papers* (A. Zygmund, ed.), PWN, Warsaw 1964, 470–472.
- [33] J. Marcinkiewicz, *Sur l’interpolation d’opérations*, C. R. Acad. Sci. Paris 208 (1939), 1272–1273, reprinted in: Józef Marcinkiewicz, *Collected Papers* (A. Zygmund, ed.), PWN, Warsaw 1964, 539–540.
- [34] P. Nalli, *Sopra una nuova specie di convergenza in media*, Rend. Circ. Mat. Palermo (1) 38 (1914), 305–319 and 320–323.
- [35] E. T. Oklander, *Interpolación, espacios de Lorentz y teorema de Marcinkiewicz*, Cursos y Seminarios de Matemática, vol. Fasc. 20, Universidad de Buenos Aires, Buenos Aires 1965.
- [36] J. Peetre, *On the development of interpolation—instead of a history three letters*, in: *Function Spaces, Interpolation Theory and Related Topics. Proc. of the International Conference in honour of Jaak Peetre on his 65th birthday* (held at

- Lund University, Lund, Aug. 17–22, 2000) (M. Cwikel, M. Englis, A. Kufner, L.-E. Persson, G. Sparr, eds.), Walter de Gruyter, Berlin 2002, 39–48.
- [37] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston 2007, [The Marcinkiewicz interpolation theorem (1939), p. 433].
- [38] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York 1991, [Marcinkiewicz's interpolation theorem for certain Orlicz spaces, Theorem 13, pp. 247–252].
- [39] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York–London 1975, [IX.18 (Marcinkiewicz interpolation theorem)].
- [40] N. M. Rivière, *Interpolation à la Marcinkiewicz*, Rev. Un. Mat. Argentina 25 (1970/71), 363–377.
- [41] W. L. C. Sargent, *Some analogues and extensions of Marcinkiewicz's interpolation theorem*, Proc. London Math. Soc. 1 (1961), 457–468.
- [42] E. M. Stein, G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. 8 (1959), 263–284.
- [43] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton 1971, [Marcinkiewicz interpolation theorem, pp. 183–200].
- [44] R. S. Strichartz, *A multilinear version of the Marcinkiewicz interpolation theorem*, Proc. Amer. Math. Soc. 21 (1969), 441–444.
- [45] A. Torchinsky, *Interpolation of operations and Orlicz classes*, Studia Math. 59 (1976), no. 2, 177–207.
- [46] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, Orlando 1986, [Marcinkiewicz interpolation theorem, pp. 86–91].
- [47] K. Urbanik, *Fourier analysis in Marcinkiewicz spaces*, Studia Math. 21 (1961), 93–102.
- [48] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operations*, J. Math. Pures Appl. 35 (1956), 223–248.
- [49] A. Zygmund, *Trigonometric Series, Vol. I, II*, Cambridge Univ. Press, Cambridge 1959, [Marcinkiewicz's theorem on the interpolation of operators, pp. 111–120].
- [50] A. Zygmund, *Józef Marcinkiewicz*, in: Józef Marcinkiewicz, Collected Papers (A. Zygmund, ed.), PWN, Warszawa 1964, 1–33.

Lech Maligranda
 Department of Engineering Sciences
 and Mathematics
 Luleå University of Technology
 SE-971 87 Luleå
 Sweden
 lech.maligranda@ltu.se