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SOME ENGINEERING AND MATHEMATICAL ASPECTS ON THE HOMOGENIZATION METHOD

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Abstract—Effective properties and local stress variations of elastic fiber composites are studied by means of the homogenization method. Two different methods, “the displacement method” and “the force method”, for solving the so called cell problem are derived, applied and compared. Numerical results for three different types of representative volume elements are presented. We also calculate the effective moduli of a unidirectional fiber composite for different volume fractions of fibers. The numerical results are discussed and compared with the Hashin-Shtrikman bounds.

1. INTRODUCTION

Composite materials play an important role in many branches of engineering. A common feature for this type of material is that the locally heterogeneous material behaves as a homogeneous medium when the characteristic size of the inclusion is much smaller than the size of the whole sample and the characteristic wavelength of external fields. In such a situation the properties of the composite can be described by the effective moduli which are obtained by some type of averaging. One way to proceed is to use the homogenization theory, developed in the studies of partial differential equations with rapidly varying coefficients.

In this paper we use the homogenization method to analyze some stress problems in elastic fiber composites. For the sake of simplicity we have assumed that the temperature and the moisture are constant. This paper may be seen as a follow-up and generalization of the previous article Holmbom *et al.* (1992) published in this journal and of Persson and Svanstedt (1993).

We begin by presenting our model problem (see Section 2) and giving a short description of the homogenization theory for this problem and make physical interpretations when it is possible (see Section 3). We emphasize that this handling of the homogenization theory gives us the following:

- (a) A concrete *homogenization algorithm* for computing both effective moduli and microstress variations in composite materials (even with more than two phases and for nonlinear cases).
- (b) Due to some fairly new theoretical results we have good control of the actual error estimates, stability and convergence questions.

The main problem in the homogenization algorithm is to identify and solve a characteristic *cell problem* (see (4)). Here we derive, present and compare two different methods, “the displacement method” and “the force method”, for solving this crucial problem (see Sections 4 and 5, respectively).

In Section 6 we illustrate the homogenization algorithm above by computing the effective stiffness tensor and the local stress variations for a unidirectional elastic fiber composite (glass-epoxy, volume fraction of fibers is 45%). Three different types of representative volume elements (RVE) have been considered.

In Section 7 we use the homogenization method to calculate the effective moduli of

a unidirectional fiber composite for six different volume fractions of fiber. The numerical results are compared with the Hashin–Shtrikman bounds (1963) and we observe that these results are close to the lower bounds. This indicates that the lower bounds may be used as first approximations of the effective properties.

Section 8 is reserved for concluding remarks and a final discussion of the homogenization method, our modeling and the numerical experiments.

2. THE MODEL PROBLEM

We consider a linear elastic body which occupies a region Ω in R^3 . We introduce a Cartesian coordinate system (x_i) . Moreover, let us introduce $\sigma = (\sigma_{ij})$, $\mathbf{f} = (f_i)$, $\mathbf{t} = (t_i)$, $\mathbf{u} = (u_i)$ and $\mathbf{n} = (n_i)$ as the stress tensor, the internal force field, the surface force field, the displacement field and the outer unit normal to the boundary $\partial\Omega$ of Ω , respectively.

The governing force balance equation is

$$\begin{cases} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \Gamma_1 \quad \text{and} \quad \sigma_{ij} n_j = t_i & \text{on } \Gamma_2, \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The most general anisotropic form of linear stress–strain relations is given by Hooke’s generalized law

$$\sigma_{ij} = C_{ijkl} e_{kl},$$

where e_{kl} is the linear strain tensor and C_{ijkl} is the fourth order tensor of elastic moduli, called the stiffness tensor. It is also known that the stiffness tensor has the symmetries

$$C_{ijkl} = C_{jikl} = C_{jilk} = C_{lkji}.$$

The strain tensor is defined in terms of displacement components, u_i , as

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Concerning these basic facts we refer to Fung (1965) and Christensen (1979). Now by using this information we obtain the following equilibrium equations

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl} e_{kl}(\mathbf{u})) = f_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \Gamma_1 \quad \text{and} \quad C_{ijkl} e_{kl}(\mathbf{u}) n_j = t_i & \text{on } \Gamma_2. \end{cases}$$

Let us now assume that the body consists of two or more different linear elastic materials which are periodically distributed in the sense that we can define a unit cell which is periodically repeated. We introduce the local variable $\mathbf{y} = \mathbf{x}/\varepsilon$. Assume that $C_{ijkl}^\varepsilon = C_{ijkl}(\mathbf{x}/\varepsilon) = C_{ijkl}(\mathbf{y})$ is Y -periodic. By Y -periodicity we mean that $C_{ijkl}(\mathbf{y}_1) = C_{ijkl}(\mathbf{y}_2)$ whenever $\mathbf{y}_1$ and $\mathbf{y}_2$ have the same positions in the corresponding cells. This means that ε is a parameter for varying the fineness of the cell structure.

We also assume that the functions C_{ijkl}^ε are real-valued, measurable and satisfy the following coercivity and boundedness conditions

$$\lambda \xi_{ij} \xi_{ij} \leq C_{ijkl}^\varepsilon \xi_{kl} \xi_{ij} \leq \beta \xi_{ij} \xi_{ij}$$

for every symmetric real-valued tensors ξ_{ij} where $0 < \lambda \leq \beta < \infty$. Physically this means that the strain energy is positive and bounded.

We now study the following class of problems, one problem for each choice of ε ,

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon e_{kl}(\mathbf{u}^\varepsilon)) = f_i^\varepsilon & \text{in } \Omega, \\ u_i^\varepsilon = 0 & \text{on } \Gamma_1 \quad \text{and} \quad C_{ijkl}^\varepsilon e_{kl}(\mathbf{u}^\varepsilon) n_j = t_i & \text{on } \Gamma_2. \end{cases} \quad (1)$$

We emphasize that if ε is small, i.e. if we have a fine microstructure, then the functions C_{ijkl}^ε will oscillate very rapidly. Therefore a direct numerical treatment (e.g. by using some FEM code) is impossible and we must attack the problem by using an averaging or homogenization procedure. In the next section we will briefly discuss the homogenization method for treating such problems.

3. THE HOMOGENIZATION METHOD

The main idea in the homogenization theory is to approximate the solutions $\mathbf{u}^\varepsilon$ of our model problem (1) by means of a function $\mathbf{u}$ which solves the problem corresponding to a homogenized material

$$\begin{cases} -\bar{C}_{ijkl} \frac{\partial}{\partial x_j} e_{kl}(\mathbf{u}) = g_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \Gamma_1 \quad \text{and} \quad \bar{C}_{ijkl} e_{kl}(\mathbf{u}) n_j = t_i & \text{on } \Gamma_2, \end{cases} \quad (2)$$

where $\bar{C}_{ijkl}$ is a constant tensor and g_i is defined by (5). The homogenized tensor $\bar{C}_{ijkl}$ may be interpreted as the physical parameters of a homogeneous material, whose overall response is "close" to that of the heterogeneous periodic material, when the size of the cell tends to zero. The main problem is to find the homogenized tensor $\bar{C}_{ijkl}$. In our case it can be proved, by using some results from functional analysis and the theory of partial differential equations and assuming two scales expansion of $u_i^\varepsilon(\mathbf{x})$ and $f_i^\varepsilon(\mathbf{x})$ on the forms

$$u_i^\varepsilon(\mathbf{x}) = u_i^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon u_i^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 u_i^{(2)}(\mathbf{x}, \mathbf{y}) + \dots,$$

and

$$f_i^\varepsilon(\mathbf{x}) = f_i^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon f_i^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 f_i^{(2)}(\mathbf{x}, \mathbf{y}) + \dots,$$

where $u_i^{(n)}(\mathbf{x}, \mathbf{y})$ and $f_i^{(n)}(\mathbf{x}, \mathbf{y})$, $n = 0, 1, 2, \dots$, are Y -periodic in the variable $\mathbf{y}$, that the homogenized tensor $\bar{C}_{ijkl}$, in fact, can be given explicitly by the formula

$$\bar{C}_{ijrs} = \frac{1}{|Y|} \int_Y (C_{ijrs} + C_{ijkl} e_{kl}(\mathbf{W}^{rs})) dy, \quad (3)$$

where $\mathbf{W}^{rs}$ is the solution of the cell problem

$$\begin{cases} -\frac{\partial}{\partial y_j} C_{ijkl} e_{kl}(\mathbf{W}^{rs}) = \frac{\partial}{\partial y_j} C_{ijrs} & \text{on } Y, \\ \mathbf{W}^{rs} & \text{is } Y\text{-periodic.} \end{cases} \quad (4)$$

From this proof we also obtain the following results

$$g_i = \langle f_i^{(0)} \rangle, \quad (5)$$

where $\langle \cdot \rangle$ denotes the arithmetic mean over the Y -cell, i.e.

$$\langle f_i^{(0)} \rangle = \frac{1}{|Y|} \int_Y f_i^{(0)} dy,$$

where $|Y|$ is the volume of Y and $u_i^{(0)}(\mathbf{x}, \mathbf{y})$ depends only on $\mathbf{x}$ and is the solution of the homogenized problem (2).

From the asymptotic expansion it also follows that the lowest order approximation of the stress field can be expressed as

$$\sigma_{ij}^{(0)}(\mathbf{x}, \mathbf{y}) = (C_{ijrs}(\mathbf{y}) + C_{ijkl}(\mathbf{y}) e_{kl}(\mathbf{W}^{rs}(\mathbf{y}))) e_{rs}(\mathbf{u}(\mathbf{x})), \quad (6)$$

where the operators e_{klx} and e_{kly} are defined by

$$e_{klx}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \quad \text{and} \quad e_{kly}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_k}{\partial y_l} + \frac{\partial u_l}{\partial y_k} \right).$$

This fact gives us very good information about the local stress variations.

Early results concerning these well-known formulas can be found in Spagnolo (1968) and Tartar (1977). For more information we also refer to Bakhvalov and Panasenko (1989), Sanchez-Palencia (1980) and Persson *et al.* (1993).

The following important information is available:

- The model problem (1) can be solved by using the following homogenization algorithm.
 - (i) Solve the cell problem (4) numerically.
 - (ii) Insert the solution of the cell problem in the expression (3) for the homogenized coefficients.
 - (iii) Insert the homogenized coefficients into the homogenized equation (2) and solve it numerically.
 - (iv) Use the solution of the homogenized equation to compute the lowest order approximation of the stress field, see (6).
- Due to known mathematical results we have good control of the algorithm above. For example the following hold true:
 - (a) We have the following error estimate (see e.g. Bakhvalov and Panasenko (1989))

$$\|u_i^{(0)} - u_i^\varepsilon\|_{L^2} \leq K\varepsilon,$$

where the L^2 norm (energy norm) is defined by

$$\|f\|_{L^2} = \left(\int_{\Omega} |f|^2 dx \right)^{1/2}.$$

Physically this means that we have fairly rapid convergence measured in the energy norm.

- (b) The homogenization method is stable in the sense that it gives the same homogenized equation for any $\varepsilon > 0$. Here we remark that, unlike homogenization theory, the accuracy in predicting effective stiffness by standard mechanics depends on the representative volume element (RVE) size compared to the global analysis region, see Hill (1963) and Hollister and Kikuchi (1992).
- (c) The class of differential operators in the model problem (1) *G-converges* to the homogenized differential operator. In fact, it yields (in a special weak sense) that

$$u_i^\varepsilon \rightarrow u_i \quad \text{and} \quad \frac{\partial}{\partial x_j} (C_{ijkl} e_{kl}(\mathbf{u}^\varepsilon)) \rightarrow \bar{C}_{ijkl} \frac{\partial}{\partial x_j} (e_{kl}(\mathbf{u})),$$

where $u_i = u_i^{(0)}$ is the solution of the homogenized equation (2). A precise description of such convergence results can be found, for example, in Tartar (1977), Persson *et al.* (1993) and Dal Maso (1993).

Summing up, we find that the homogenization method gives us the following information about the model problem (1): For a fixed material we have a fixed ε , say $\varepsilon = \varepsilon_0$. The average stresses and local stress variations can be obtained by the homogenization algorithm. According to the *G-convergence* result we can think of this procedure as a continuous deformation of the heterogeneous material to a homogeneous material. We can stop this deformation at any value of ε , $0 < \varepsilon \leq \varepsilon_0$, and study the corresponding problem. In each step we can get the lowest order approximation of the stress field (the last step in the homogenization algorithm) and the corresponding error estimate (see (a) above) and the stability (see the *G-convergence* result) guarantees that we end up in the same homogenized problem. We close this section by pointing out that the crucial step in the homogenization algorithm is to solve the cell problem (4) and in the next two sections we will present two useful methods for solving this problem.

4. "THE DISPLACEMENT METHOD" FOR SOLVING THE CELL PROBLEM

Let us consider the cell problem (4)

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{ijkl} e_{kl}(\mathbf{W}^{rs})) = \frac{\partial}{\partial y_j} (C_{ijkl}), \\ \mathbf{W}_i^{rs} \text{ is } Y\text{-periodic.} \end{cases}$$

First we note that this equation can be written as

$$\frac{\partial}{\partial y_j} (C_{ijkl} e_{kl}(\mathbf{W}^{rs} + \mathbf{V}^{rs})) = 0, \quad (7)$$

where $V_i^{rs} = \delta_{ir} y_s$ or, explicitly,

$$\begin{aligned} \mathbf{V}^{11} &= \begin{bmatrix} y_1 \\ 0 \\ 0 \end{bmatrix} & \mathbf{V}^{12} &= \begin{bmatrix} y_2 \\ 0 \\ 0 \end{bmatrix} & \mathbf{V}^{13} &= \begin{bmatrix} y_3 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{V}^{21} &= \begin{bmatrix} 0 \\ y_1 \\ 0 \end{bmatrix} & \mathbf{V}^{22} &= \begin{bmatrix} 0 \\ y_2 \\ 0 \end{bmatrix} & \mathbf{V}^{23} &= \begin{bmatrix} 0 \\ y_3 \\ 0 \end{bmatrix} \\ \mathbf{V}^{31} &= \begin{bmatrix} 0 \\ 0 \\ y_1 \end{bmatrix} & \mathbf{V}^{32} &= \begin{bmatrix} 0 \\ 0 \\ y_2 \end{bmatrix} & \mathbf{V}^{33} &= \begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix}. \end{aligned} \quad (8)$$

The corresponding components in the strain tensors e_{ij}^{rs} are given by

$$e_{ij}^{rs} = \frac{1}{2} \left(\frac{\partial V_i^{rs}}{\partial y_j} + \frac{\partial V_j^{rs}}{\partial y_i} \right)$$

and we get the following strain vectors

$$\begin{aligned} \mathbf{e}^{11} &= (1, 0, 0, 0, 0, 0)^T & \mathbf{e}^{22} &= (0, 1, 0, 0, 0, 0)^T & \mathbf{e}^{33} &= (0, 0, 1, 0, 0, 0)^T \\ \mathbf{e}^{23} &= (0, 0, 0, 1, 0, 0)^T & \mathbf{e}^{13} &= (0, 0, 0, 0, 1, 0)^T & \mathbf{e}^{12} &= (0, 0, 0, 0, 0, 1)^T, \end{aligned} \quad (9)$$

where $\mathbf{e}^{rs} = (e_{11}^{rs}, e_{22}^{rs}, e_{33}^{rs}, \gamma_{23}^{rs}, \gamma_{13}^{rs}, \gamma_{12}^{rs})^T$ and $\gamma_{ij}^{rs} = 2e_{ij}^{rs}$.

We put $\mathbf{U}^{rs} = \mathbf{W}^{rs} + \mathbf{V}^{rs}$ and our equation (7) becomes

$$\frac{\partial}{\partial y_j} (C_{ijkl} e_{kl}(\mathbf{U}^{rs})) = 0.$$

According to (8) we see that our new boundary conditions are that points on opposite faces of the cell are coupled to each other but with one exception, namely that the points on sides with the normal vector parallel to the y_s -direction move identically except for Δy_r , where Δy_r is the length of the cell in the y_r -direction.

We emphasize that the last observation together with (9) is important because it explicitly tells us how to prescribe the displacements so that the cell is subject to a unit average strain field with only one nonzero component.

We can now use a FEM-code and compute $\mathbf{U}^{rs}$. We recall the formula (3) for the homogenized coefficients $\bar{C}_{ijrs}$ and find that

$$\begin{aligned} \bar{C}_{ijrs} &= \frac{1}{|Y|} \int_Y C_{ijrs} + C_{ijkl} e_{kl}(\mathbf{W}^{rs}) \, dy = \frac{1}{|Y|} \int_Y C_{ijrs} + C_{ijkl} e_{kl}(\mathbf{U}^{rs} - \mathbf{V}^{rs}) \, dy \\ &= \frac{1}{|Y|} \int_Y C_{ijkl} e_{kl}(\mathbf{U}^{rs}) \, dy = \frac{1}{|Y|} \int_Y \sigma_{ij} \, dy, \end{aligned} \quad (10)$$

where σ_{ij} is the stress corresponding to $\mathbf{U}^{rs}$. Moreover, the expression (6) for the lowest order approximation of the stress field can be rewritten as

$$\begin{aligned}\sigma_{ij}^{(0)}(\mathbf{x}, \mathbf{y}) &= (C_{ijkl}(\mathbf{y}) + C_{ijkl}(\mathbf{y})e_{kl}(\mathbf{W}^{rs}(\mathbf{y})))e_{rsx}(\mathbf{u}^{(0)}) \\ &= (C_{ijkl}(\mathbf{y}) + C_{ijkl}(\mathbf{y})e_{kl}(\mathbf{U}^{rs}(\mathbf{y}) - \mathbf{V}^{rs}(\mathbf{y})))e_{rsx}(\mathbf{u}^{(0)}) \\ &= C_{ijkl}(\mathbf{y})e_{kl}(\mathbf{U}^{rs}(\mathbf{y}))e_{rsx}(\mathbf{u}^{(0)}) = \sigma_{ij}e_{rsx}(\mathbf{u}^{(0)}).\end{aligned}\quad (11)$$

4.1. An engineering interpretation of “the displacement method”

For our homogenized material Hooke’s law yields

$$\begin{bmatrix} \bar{C}_{1111} & \bar{C}_{1122} & \bar{C}_{1133} & 0 & 0 & 0 \\ \bar{C}_{2211} & \bar{C}_{2222} & \bar{C}_{2233} & 0 & 0 & 0 \\ \bar{C}_{3311} & \bar{C}_{3322} & \bar{C}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{1212} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}. \quad (12)$$

Our purpose is now to compute the homogenized coefficients $\bar{C}_{ijrs}$. Let us for the sake of simplicity assume that we want to calculate $\bar{C}_{1111}$, ($rs = 11$). Then, it follows from (9) that $e_{11}^{11} = 1$ and all other components in e^{11} are equal to 0. According to (12) we have

$$\bar{C}_{1111} = \sigma_{11}.$$

We would have been done if σ_{11} was constant all over the cell. However, since σ_{11} varies over the cell it seems reasonable, but not obvious, to assume that one can replace σ_{11} by the arithmetic mean of σ_{11} over the cell, i.e. that

$$\bar{C}_{1111} = \frac{1}{|Y|} \int_Y \sigma_{11} \, d\mathbf{y}.$$

Fortunately, this natural conjecture is in agreement with (10) which was derived by the homogenization method.

5. “THE FORCE METHOD” FOR SOLVING THE CELL PROBLEM

Again we consider the cell problem (4)

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{ijkl}(\mathbf{W}^{rs})) = \frac{\partial}{\partial y_j} C_{ijrs}, \\ \mathbf{W}_i^{rs} \text{ is } y\text{-periodic.} \end{cases}$$

First we notice that this equation is of the same type as the original equation (1). Thus it seems natural to interpret the right hand side as an internal force per unit volume. The right hand side consists of derivatives of C_{ijrs} which are constant over the cell, except at the interface between the two different materials where it is infinite. This means that we will have a distributed force per unit length of magnitude equal to the difference in C_{ijrs} acting on the interface. For example, if we want to compute $\bar{C}_{1111}$ ($rs = 11$) explicitly our cell problem takes the form

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{1jkl}e_{kl}(\mathbf{W}^{11})) = \frac{\partial}{\partial y_1} \bar{C}_{1111}, \\ -\frac{\partial}{\partial y_j} (C_{2jkl}e_{kl}(\mathbf{W}^{11})) = \frac{\partial}{\partial y_2} \bar{C}_{2211}, \\ -\frac{\partial}{\partial y_j} (C_{3jkl}e_{kl}(\mathbf{W}^{11})) = 0. \end{cases} \quad (13)$$

Fig. 1. The forces that act on the interface between the different materials when $rs = 11$.

Hence we have one distributed force acting in the y_1 -direction which is equal to the jump in C_{1111} per unit length in the y_1 -direction, one distributed force acting in the y_2 -direction which is equal to the jump in C_{2211} per unit length in the y_2 -direction and the force in the y_3 -direction is equal to zero. The different forces are illustrated in Fig. 1. A physical interpretation of this method is that the given displacement in the foregoing method is replaced by an "equivalent" force.

6. NUMERICAL RESULTS

To illustrate the methods given above we present some numerical results. We have considered three different cell geometries for unidirectional fiber composites (see Fig. 2).

For the sake of simplicity we just compute the transversal components in the stiffness matrix in (12). According to the Y -periodic boundary condition, $e_{33} = e_{23} = e_{13} = 0$ for the corresponding applied displacements. This shows that it is sufficient to use a 2-D FEM-program and use plane strain. In all cases the composite consisted of glass as fiber material, epoxy as matrix material and the volume fraction of fibers was 45%. We have used the following data on Young's modulus (E) and Poisson's ratio (ν):

Epoxy: $E = 3.5$ GPa $\nu = 0.35$

Glass: $E = 70$ GPa $\nu = 0.2$.

The stiffness matrix is defined by

$$\begin{bmatrix} C_{1111} & C_{1122} & 0 \\ C_{1122} & C_{2222} & 0 \\ 0 & 0 & C_{1212} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

The corresponding stiffness matrices for epoxy and glass are

$$\begin{bmatrix} 5.6 & 3.0 & 0 \\ 3.0 & 5.6 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 77.8 & 19.4 & 0 \\ 19.4 & 77.8 & 0 \\ 0 & 0 & 29.2 \end{bmatrix},$$

Fig. 2. The three different types of cell geometries that have been used in the numerical calculations for computing the homogenized coefficients.

respectively. We now present our numerical results in terms of the homogenized stiffness matrices $\bar{C}^i$, where i is the number of the cell geometry. Both methods have been applied to geometry 1 and we obtained the same result namely

$$\bar{C}^1 = \begin{bmatrix} 11.7 & 4.5 & 0 \\ 4.5 & 11.7 & 0 \\ 0 & 0 & 2.4 \end{bmatrix}.$$

The corresponding result for geometry 2 (with both methods) is

$$\bar{C}^2 = \begin{bmatrix} 12.0 & 4.3 & 0 \\ 4.3 & 12.0 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}.$$

For the (unsymmetrical) geometry 3 we have only used “the force method” and obtained

$$\bar{C}^3 = \begin{bmatrix} 12.0 & 4.9 & 0 \\ 4.9 & 11.5 & 0 \\ 0 & 0 & 2.8 \end{bmatrix}.$$

In addition, we emphasize that, by using the results obtained from the FEW computations and the expressions (6) and (11) for the lowest order approximation of the stress field, we get important information about the local stress variations. In Fig. 3 below σ_{11} is illustrated for pure tension in the x_1 direction. Observe that for the geometries 1 and 2 the shown stress is $C_{11kl}e_{kl}(U^{11})$ and for geometry 3 it is $C_{11kl}(W^{11})$. The applied displacements are 0.1% of the displacements that are prescribed by the “displacement method” and the applied force is 1% of the force prescribed by the “force method”. The shown stresses are given in Pa.

Fig. 3. Continued opposite

Fig. 3. The local variations of σ_{11} for pure axial tension for the three cell geometries considered.

7. COMPARISON OF NUMERICAL RESULTS AND THE HASHIN-SHTRIKMAN BOUNDS

We study a unidirectional fiber composite consisting of carbon and epoxy. We use the above described homogenization method to compute the effective plane strain bulk modulus $\bar{k}$ and the effective transverse shear modulus $\bar{\mu}$, where $\bar{k}$ and $\bar{\mu}$ are defined by

$$\bar{k} = \frac{1}{2}(\bar{C}_{1111} + \bar{C}_{1122}) \quad \text{and} \quad \bar{\mu} = \frac{1}{2}(\bar{C}_{1111} - \bar{C}_{1122}).$$

We use the following data on k_i and μ_i , where the indices denote the material number.

$$\begin{aligned} \text{Carbon: } k_1 &= 10.42 \text{ GPa} & \mu_1 &= 6.25 \text{ GPa} \\ \text{Epoxy: } k_2 &= 3.70 \text{ GPa} & \mu_2 &= 1.11 \text{ GPa} \end{aligned} \quad (14)$$

In our numerical calculations we use a hexagonal fiber array to model the composite, this fiber-packing is known to be transversely isotropic. The considered materials (14) are well ordered. This means that both the plane strain bulk modulus and the transverse shear modulus are bigger for material 1

$$k_1 \geq k_2 \quad \text{and} \quad \mu_1 \geq \mu_2.$$

The effective moduli were computed for six different volume fractions of fiber (m_1). The results from these calculations are presented in Table 1.

According to the Hashin–Shtrikman bounds the plane strain bulk modulus and the transverse shear modulus for a well ordered transversely isotropic must satisfy the following conditions.

$$k_l \leq \bar{k} \leq k_u,$$

$$\mu_l \leq \bar{\mu} \leq \mu_u,$$

where

$$k_l = k_2 + \frac{m_1}{\frac{1}{k_1 - k_2} + \frac{m_2}{k_2 + \mu_2}},$$

$$k_u = k_1 + \frac{m_2}{\frac{1}{k_2 - k_1} + \frac{m_1}{k_1 + \mu_1}},$$

$$\mu_l = \mu_2 + \frac{m_1}{\frac{1}{\mu_1 - \mu_2} + \frac{m_2(k_2 + 2\mu_2)}{2\mu_2(k_2 + \mu_2)}},$$

$$\mu_u = \mu_1 + \frac{m_2}{\frac{1}{\mu_2 - \mu_1} + \frac{m_1(k_1 + 2\mu_1)}{2\mu_1(k_1 + \mu_1)}},$$

and m_i is the volume fraction of material i . For a proof see the original paper by Hashin (1965).

Now we compare the numerical results with the Hashin–Shtrikman bounds, see Figs 4 and 5.

Table 1.

m_1	0.35	0.45	0.55	0.65	0.75	0.85
$\bar{k}$ (GPa)	4.94	5.41	5.97	6.64	7.46	8.50
$\bar{\mu}$ (GPa)	1.75	2.04	2.42	2.91	3.54	4.38

Fig. 4. Numerical results and Hashin-Shtrikman bounds for the bulk modulus.

Fig. 5. Numerical results and Hashin-Shtrikman bounds for the shear modulus.

We observe that the numerical results are close to the lower bounds. A physical explanation of this can be given for the plane strain bulk modulus. The Hashin-Shtrikman bounds are optimal and for $\bar{k}$ the optimal structure is the so-called Hashin-Shtrikman structure, see the front page of this journal. In the numerical calculations we used a hexagonal fiber array which means that the body is filled with regular hexagons with a circular inclusion of material 1 instead of the coated cylinders in the Hashin-Shtrikman structure. Due to the similar geometries, see Fig. 6, it is reasonable to believe that the results for these structures would be close to each other. If we interchange the materials we would come close to the upper bound.

For more information about bounds we refer to Milton and Kohn (1988), Milton (1990) and Cherkaev and Gibiansky (1993).

Fig. 6. Coated regular hexagon and coated cylinder.

8. FINAL DISCUSSION AND CONCLUDING REMARKS

We have modelled the composite by different cell geometries, see Sections 6 and 7. We observe that the differences in the stiffness matrices are small. This indicates that the method is not very sensitive for the choice of cell geometry, for reasonable choices, if we want to compute the effective stiffness. Additional information about the composite can of course be used, e.g. if we know that the composite is transversely isotropic it is preferable to use a hexagonal fiber array which is known to be transversely isotropic. On the other hand our numerical experiments show that the local stress variations depend strongly on the choice of cell geometry.

The homogenization method gives us important information about the model problem (1). An effective homogenization algorithm for the computation of effective stiffness matrices and microstress variations can be extracted and applied. The homogenization method is mathematically based and therefore we have good control of the actual error estimates, stability and convergence questions. Here we also remark that our numerical experiments agree very well with both theoretical predictions, see Section 7, and the information we have got from the research institute SICOMP (Piteå, Sweden).

The homogenization algorithm presented in this paper is easy to apply and interpret in physical terms. The crucial point in the homogenization method is to interpret and solve a corresponding cell problem over one fixed typical Y -cell. Here we have derived and explained two concrete methods, “the displacement method” and “the force method”, for solving this crucial problem. Both methods are easy to implement on computers. In general, “the displacement method” seems to be the most natural to apply. Though, if the cell geometry is unsymmetrical, e.g. geometry 3, it is easier to handle our special type of boundary condition by “the force method” in some FEM softwares.

We remark that the effective stiffness computed by the homogenization method is not dependent on the RVE size. In standard mechanics the effective stiffness predictions will change as the number of repeating units contained in the RVE changes. When the number of repeating units is increased the effective stiffness will converge to the effective stiffness obtained by using the homogenization method where the RVE consists of only one unit cell. More about these things and other comparisons of homogenization and other methods can be found in Hollister and Kikuchi (1992).

In this paper we have used the homogenization theory to solve a linear elasticity problem. It is however, important to observe that the method is not restricted to linear problems and that it can be applied to other partial differential equations as well. Other types of problem that have been studied using homogenization can be found in Bakhvalov and Panasenko (1989), Dal Maso (1993), Persson *et al.* (1993) and Sanchez-Palencia (1980). In particular, the book of Dal Maso contains approximately 1100 references and the book by Persson *et al.* is meant to be a self-contained introduction to the subject both for Engineers and Mathematicians.

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