THE E-FUNCTIONAL FOR SOME PAIRS OF GROUPS

by

Lech Maligranda and Lars Erik Persson

0. INTRODUCTION

The theory of the real interpolation method and the K-functional (see e.g. [3], [4], [6]) has been developed in several papers (see e.g. Maligranda [15] with more than 2000 references). There exists a close connection between the K-functional and the E-functional and, thus, between the corresponding K- and E-methods. By using these facts we can also obtain both new information about the E-functional (e.g. the E-divisibility in the Brudnyi-Krugljak sense and Holmstedt type formulas - see [5] and [19]) and about the equivalence of the E- and K-spaces (e.g. reiteration and Wolff type theorems - see [5], [17] and [19]). In this paper we work in the opposite direction. Namely, we discuss and calculate directly the E-functional for some pairs of quasi-normed groups and, in particular, we also obtain some information about the K-functional. We point out several applications of our exact descriptions. In particular, we obtain some well-known exact formulas for the K-functional. Thus our results may also be regarded as a unification and generalization of some well-known exact descriptions both of E-functionals and of K-functionals.

The paper is organized in the following way: In section 1 we present some preliminaries. Our main results are stated and proved in section 2. In section 3 we give some examples of applications of our main theorem. In section 4 we discuss the close connection between the E- and the K-functionals. We also include some examples of the fact that our results also imply exact descriptions of the K-functional. Section 5 is reserved for some concluding remarks.

1. PRELIMINARIES

By a quasi-normed Abelian group A we mean an Abelian group equipped with a quasi-norm $\|\cdot\|_A$, i.e. $\|\cdot\|_A$ satisfy

$$\|a\|_A = 0 \Leftrightarrow a = 0, \quad \|-a\|_A = \|a\|_A \text{ and } \|a+b\|_A \leq c(\|a\|_A + \|b\|_A),$$

where the constant $c \geq 1$ is independent of $a, b \in A$. If $c = 1$, then we say that A is a normed Abelian group. By a quasi-normed pair $\bar{A} = (A_0, A_1)$ we mean two quasi-normed Abelian subgroups A_0 and A_1 of some large Abelian group $\mathcal{A}$ satisfying the separability axiom, i.e. if $\|a_n - a^0\|_{A_0} \rightarrow 0$ and $\|a_n - a^1\|_{A_1} \rightarrow 0$, then $a^0 = a^1$.

We recall that the K-functional and the E-functional (the best approximation) of the element $a \in \Sigma(\bar{A}) = A_0 + A_1$ are defined, for every $t > 0$, by

$$K(t, a) = K(t, a; \bar{A}) = \inf \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1 \}$$

and

$$E(t, a) = E(t, a; \bar{A}) = \inf \{ \|a - a_1\|_{A_0} : a_1 \in A_1, a - a_1 \in A_0, \|a_1\|_{A_1} \leq t \},$$

respectively. In the last expression the usual convention $\inf \emptyset = +\infty$ is used. For measurable functions on $\mathbb{R}_+$, $\theta \in \mathbb{R}$ and $0 < p \leq \infty$ we consider the functional

$$\Phi_{\theta, p}(f) = \left(\int_0^\infty |t^{-\theta} f(t)|^p \frac{dt}{t} \right)^{1/p}.$$

If $0 < \theta < 1$ and $0 < p \leq \infty$, then we define the (Lions-Peetre) real interpolation spaces (K-spaces or K-method of interpolation) as

$$\bar{A}_{\theta, p; K} = \{ a \in \Sigma(\bar{A}) : \|a\|_{\theta, p; K} = \Phi_{\theta, p}(K(t, a)) < \infty \}$$

and if $\alpha > 0$ and $0 < p \leq \infty$, then we define the approximation spaces (E-spaces or the E-method of interpolation) as

$$\bar{A}_{\alpha, p; E} = \{ a \in \Sigma(\bar{A}) : \|a\|_{\alpha, p; E} = \Phi_{-\alpha, p}(E(t, a)) < \infty \}.$$

More general approximation spaces E_Φ where our particular function norm $\Phi_{-\alpha, p}$ is replaced by a general function norm Φ were investigated by Brudnyi-Krugljak [5], Asekritova [1], Nilsson [19] and Brudnyi-Krugljak [6].

Another type of spaces connected with the E-functional are the so called weak spaces considered by many authors and defined in the following way (cf. [7], [17] and [25]):

$$W(\bar{A}) = \left\{ a \in \Sigma(\bar{A}) : \sup_{t>0} \left(\frac{1}{t} \int_0^t E(s,a) ds - E(t,a) \right) < \infty \right\}.$$

In this paper by $S = S(I, m)$ we denote the space of all measurable real-valued functions on $I = [0, \tau]$, $0 < \tau \leq \infty$, with the Lebesgue measure m (two functions are equal if they coincide a.e.). If $a \in S$, then the distribution function d_a and the non-increasing rearrangement a^* of a are defined as

$$d_a(\lambda) = m(\{t \in I : |a(t)| > \lambda\}), \lambda \geq 0, \text{ and } a^*(t) = \inf \{\lambda > 0 : d_a(\lambda) \leq t\},$$

$\inf \emptyset = +\infty$, respectively. We note that both d_a and a^* are non-increasing and right-continuous functions on $[0, \infty)$. Moreover, if $\tau < \infty$, then $a^*(t) = 0$ for $t \geq \tau$ and if $\tau = \infty$, then both d_a and a^* may also have the value $+\infty$.

A quasi-normed Abelian group $A = (A, \|\cdot\|_A)$ in S is called
a) solid if

$$a \in S, b \in A, |a| \leq |b| \text{ a.e.} \Rightarrow a \in A \text{ and } \|a\|_A \leq \|b\|_A.$$

b) symmetric if it is solid and if for any $b \in S$ which is equimeasurable with $a \in A$ (i.e. $d_a = d_b$) it holds that $b \in A$ and $\|b\|_A = \|a\|_A$.

Moreover, we say that a symmetric Banach space A is int-symmetric (or has the majorant property) if, for any $a \in S$ and $b \in A$, the implication

$$a \in S, b \in A, |a| \leq |b| \text{ a.e.} \Rightarrow a \in A \text{ and } \|a\|_A \leq \|b\|_A$$

holds. In particular, if a symmetric Banach space is either separable or has the Fatou property, then it is int-symmetric (see [8] and [13]). Some examples of and new information concerning symmetric quasi-normed groups can be found in [10]. The group L_0 , defined by

$$L_0 = \{a \in S : \|a\|_{L_0} = m(\text{supp } a) < \infty\}$$

is of particular interest in this paper because it may be regarded as the left end-point in the scale of L_p -spaces (cf. [20]).

2. THE MAIN RESULT

Our main result can be formulated in the following way:

THEOREM 1 Let $\bar{A} = (A_0, A_1)$ be a pair of solid quasi-normed Abelian groups in $S(I)$. Then, for any $a \in \Sigma(\bar{A})$,

(a) $E(t, a) = E(t, |a|) = \inf \{ \| |a| - a_1 \|_{A_0} : 0 \leq a_1 \in A_1, 0 \leq |a| - a_1 \in A_0, \|a_1\|_{A_1} \leq t \}$,
and

(b) $E(t, a; A_0, L_\infty) = \|(|a| - t)_+\|_{A_0}$.

If, in addition, A_0 is an int-symmetric space and A_1 is a symmetric group, then

(c) $E(t, a) = E(t, a^*)$,

and

(d) $E(t, a; A_0, L_0) = \|a^*(\cdot + t)\|_{A_0}$ for $0 < t < \tau = \infty$, $E(t, a; A_0, L_0) = \|a^*(\cdot + t)1_{[0, \tau-t]}\|_{A_0}$ for $0 < t < \tau < \infty$ and $E(t, a; A_0, L_0) = 0$ for $t \geq \tau$.

Proof: (a) We have

$$\begin{aligned} E(t, a) &= \inf \{ \|(a - a_2) \operatorname{sgn} a\|_{A_0} : a_2 \in A_1, a - a_2 \in A_0, \|a_2\|_{A_1} \leq t \} = \\ &= \inf \{ \| |a| - a_2 \operatorname{sgn} a \|_{A_0} : a_2 \operatorname{sgn} a \in A_1, (a - a_2) \operatorname{sgn} a \in A_0, \|a_2 \operatorname{sgn} a\|_{A_1} \leq t \} = \\ &= \inf \{ \| |a| - a_1 \|_{A_0} : a_1 \in A_1, |a| - a_1 \in A_0, \|a_1\|_{A_1} \leq t \} = E(t, |a|). \end{aligned}$$

Now, we consider a, a_1 such that $|a| \in \Sigma(\bar{A})$, $a_1 \in A_1$, $|a| - a_1 \in A_0$ and $\|a_1\|_{A_1} \leq t$. We define

$$a_1' = \min\{a_1^+, |a|\}$$

and find that $0 \leq a_1' \leq a_1^+ \leq |a|$ and $0 \leq |a| - a_1' \leq |a| - a_1$. Hence

$$\begin{aligned} \inf \{ \| |a| - a_1 \|_{A_0} : 0 \leq a_1 \in A_1, 0 \leq |a| - a_1 \in A_0, \|a_1\|_{A_1} \leq t \} &\leq \\ &\leq \| |a| - a_1' \|_{A_0} \leq \| |a| - a_1 \|_{A_0}. \end{aligned}$$

By taking infimum over all a_1 of the type considered above we obtain the proof of the non-trivial part of the second equality.

(b) Let $E(t, a) < \infty$. Assume that $a_1 \in L_\infty$, $a - a_1 \in A_0$ and $\|a_1\|_{L_\infty} \leq t$. Then $|a_1| \leq t$ a.e. and it follows that $|a - a_1| \geq (|a| - t)_+$ a.e. and we conclude that $\|(|a| - t)_+\|_{A_0} \leq E(t, a)$. Next we assume that $(|a| - t)_+ \in A_0$ and define

$$a^{(t)}(s) = \min \{ |a(s)|, t \} \operatorname{sgn} a(s).$$

We note that $a^{(t)} \in L_\infty$, $\|a^{(t)}\|_{L_\infty} \leq t$, $a - a^{(t)} = (|a| - t)_+ \operatorname{sgn} a \in A_0$ which means that

$$E(t, a) \leq \|a - a^{(t)}\|_{A_0} = \|(|a| - t)_+ \operatorname{sgn} a\|_{A_0} = \|(|a| - t)_+\|_{A_0}.$$

By using the above arguments the proofs of the cases $(|a| - t)_+ \notin A_0$ and $E(t, a) = +\infty$ follow in a similar way so we omit the details.

(c) Let $E(t, a) < \infty$ and choose $\varepsilon > 0$. We consider $a_1 \in A_1$ such that $a - a_1 \in A_0$, $\|a_1\|_{A_1} \leq t$ and $\|a - a_1\|_{A_0} < E(t, a) + \varepsilon$. Then $\|a_1^*\|_{A_1} = \|a_1\|_{A_1} \leq t$ and, according to the Lorentz-Shimogaki inequality (see [14, Prop. 1] and [13, Th. 3.1]),

$$(2.1) \quad \int_0^t (a^* - a_1^*)^*(s) \, ds \leq \int_0^t (a - a_1)^*(s) \, ds \quad \forall t \in I$$

and the assumption that A_0 is an int-symmetric space, we see that $a^* - a_1^* \in A_0$ and

$$\|a^* - a_1^*\|_{A_0} \leq \|a - a_1\|_{A_0} < E(t, a) + \varepsilon.$$

We conclude that $E(t, a^*) \leq E(t, a)$.

Next, we assume that $E(t, a^*) < \infty$. In view of the equalities in (a) we can, without loss of generality, assume that $a \geq 0$ and, for any $\varepsilon > 0$, choose $0 \leq a_1 \in A_1$ such that $a^* - a_1 \in A_0$, $\|a_1\|_{A_1} \leq t$ and $\|a^* - a_1\|_{A_0} < E(t, a^*) + \varepsilon$. Moreover, if either $\tau < \infty$ or $\tau = \infty$ and $a^*(\infty) := \lim_{t \rightarrow \infty} a^*(t) = 0$, then, by the Ryff theorem (see [24, Prop. 3] or [3, Th. 7.5]), we know that $a = a^* \circ \sigma$, where $\sigma: I \rightarrow I$ is a measure preserving transformation. We put

$$a_2 = a_1 \circ \sigma$$

and note that $d_{a_2} = d_{a_1}$, $d_{a-a_2} = d_{a^*-a_1}$. Thus $\|a_2\|_{A_1} = \|a_1\|_{A_1} \leq t$ and $\|a - a_2\|_{A_0} = \|a^* - a_1\|_{A_0} < E(t, a^*) + \varepsilon$. For the case when $a^*(\infty) > 0$ we have not the pointwise equality $a = a^* \circ \sigma$ but we can use Lemma 2.1 from [13] saying that for any $\varepsilon > 0$ there exists a measure preserving transformation $\sigma: R_+ \rightarrow R_+$ such that

$$\|a_3 - a^* \circ \sigma\|_{L_1 \cap L_\infty} \leq \varepsilon, \text{ where } a_3(s) = \max\{a(s), a^*(\infty)\}.$$

Then for $\alpha(s) = a(s)/a_3(s)$ we have $0 \leq \alpha(s) \leq 1$ and

$$\|a - \alpha a^* \circ \sigma\|_{L_1 \cap L_\infty} = \|\alpha(a_3 - a^* \circ \sigma)\|_{L_1 \cap L_\infty} \leq \|a_3 - a^* \circ \sigma\|_{L_1 \cap L_\infty} \leq \varepsilon.$$

Therefore

$$\begin{aligned} \|a - \alpha a_1 \circ \sigma\|_{A_0} &\leq \|a - \alpha a^* \circ \sigma\|_{A_0} + \|\alpha a^* \circ \sigma - \alpha a_1 \circ \sigma\|_{A_0} \leq c \|a - \alpha a^* \circ \sigma\|_{L_1 \cap L_\infty} + \\ &+ \|a^* - a_1\|_{A_0} < E(t, a^*) + c\varepsilon + \varepsilon, \end{aligned}$$

and we conclude that $E(t, a) \leq E(t, a^*)$.

By using similar arguments as above we also find that if one of $E(t, a)$ or $E(t, a^*)$ is equal to infinity then the other is also equal to infinity and the proof is complete.

(d) If $t \geq \tau$, then we obviously have that $E(t, a; A_0, L_0) = 0$. Now, let $0 < t < \tau < \infty$. According to (a) and (c) we can, without loss of generality, assume that $a = a^*$ and $0 \leq a_1 \leq a^*$. By using the Lorentz-Shimogaki inequality (2.1) and the fact that A_0 is an int-symmetric space again, we obtain that

$$\begin{aligned} E(t, a, A_0, L_0) &= E(t, a^*, A_0, L_0) = \inf \{ \|a^* - a_1\|_{A_0} : a_1^*(t) = 0, 0 \leq a_1 \leq a^* \} = \\ &= \inf \{ \|a^* - a_1^*\|_{A_0} : a_1^*(t) = 0, 0 \leq a_1 \leq a^* \} = \\ &= \inf \{ \|(a^* - a_1^*)1_{[0,t]} + a^*1_{[t,\tau]}\|_{A_0} : 0 \leq a_1 \leq a^* \} = \end{aligned}$$

$$= \|a^*1_{[t,\tau]}\|_{A_0} = \|(a^*1_{[t,\tau]})^*\|_{A_0} = \|a^*(\cdot+t)1_{[0,\tau-t]}\|_{A_0}.$$

The proof of the case $0 < t < \tau = \infty$ is the same.

Remark Let A_0 be an int-symmetric space on I and $A_0^{(p)} = \{a \in S : |a|^p \in A_0\}$, $0 < p < 1$, with the p -homogeneous norm $\|a\|_{A_0^{(p)}} = \| |a|^p \|_{A_0}$. By making some obvious modifications of the proof we find that

$$E(t,a;A_0^{(p)},L_0) = E(t,a^*;A_0^{(p)},L_0)$$

and also that our Theorem 1 (d) holds with A_0 replaced by $A_0^{(p)}$. Let us note that $L_p \cap L_\infty \subset A_0^{(p)} \subset L_p + L_\infty$ and that $L_{p,\infty}$ is not a space of the type $A_0^{(p)}$. Moreover, it was recently observed in [2] that $L_{p,\infty}$ is a p -symmetric space and that $L_{p,\infty} \subsetneq L_p + L_\infty$.

3. SOME CONCRETE EXAMPLES

In this section we present some concrete examples of applications of our main theorem in the previous section.

Example 1 Let us consider the simplest possible pair (R,R) with the norms $\|a\|_{A_0} = \|a\|_{A_1} = |a|$. Then $E(t,a;R,R) = (|a| - t)_+$. If the (quasi-)norm for $A_1 = R$ is changed to $\|a\|_{A_1} = c|a|$, $c > 0$, or $\|a\|_{A_1} = |a|^p$, $p > 0$, then $E(t,a) = (|a| - t/c)_+$ and $E(t,a) = (|a| - t^{1/p})_+$, respectively. We also note that the equality in Theorem 1 (b) implies that the formula

$$E(t,a;A_0,L_\infty) = \|E(t,a(\cdot);R,R)\|_{A_0}$$

holds for every quasi-normed Abelian group A_0 in S .

Example 2 Our Theorem 1 (b) and (d) imply in particular that

$$(3.1) \quad E(t,a;L_0,L_\infty) = \|(|a| - t)_+\|_{L_0} = d_a(t).$$

and

$$(3.2) \quad E(t,a,L_\infty,L_0) = a^*(t).$$

Remark The identity (3.1) was proved in [6] and [10] and (3.2) by Peetre-Sparr in [21].

The formula (3.2) implies that the Lorentz $L_{p,q}$ -spaces coincide with some approximation spaces, i.e.

$$(3.3) \quad E_{1/p,q}(L_\infty, L_0) = L_{p,q}.$$

It is also well-known that

$$E(t, a, \text{BMO}, L_0) \approx (M_{0,\alpha}^\#(a))^*(t),$$

where the (maximal) function $M_{0,\alpha}^\#$ is a sublinear operator bounded in L_0 and from BMO into L_∞ (see [12, Th 3.2]). By also using the imbedding $L_\infty \subset \text{BMO}$, (3.3) and interpolation we therefore find that

$$E_{1/p,q}(\text{BMO}, L_0) = L_{p,q}.$$

See also [12, Th. 3.1]. We also mention the following interesting identification due to Milman-Sagher [17, Th 4]:

$$E_{1/p,q}(W, L_0) = L_{p,q}.$$

In the proof they used reiteration and the Wolff extrapolation theorem. It seems to be of interest to give a direct proof of this statement by first computing the E-functional.

Problem 1 Compute $E(t, a; W, L_0)$.

Problem 2 Characterize A for which $E_{1/p,q}(A, L_0) = L_{p,q}$.

Example 3 Let $a \in L_p + L_0$, $p > 0$. Then, according to our Theorem 1 (d), we have

$$E(t, a, L_p, L_0) = \left(\int_t^\infty a^*(s)^p ds \right)^{1/p}.$$

In particular, by using this identity together with a generalization of Hardy's inequality (applicable to non-increasing functions and for exponent strictly less

than 1- see e.g. [21]) we find that if $0 < \alpha, p, q < \infty$, then

$$(3.4) \quad E_{\alpha, q}(L_p, L_0) = L_{r, q}, \quad r = p/(\alpha p + 1).$$

Remark For the case $q = 1$, $\alpha = 1 - 1/p$ the identity (3.4) was also observed in [17].

Example 4 Let A be an int-symmetric space on $[0, \infty)$ such that $\|1_{[0, t]}\|_A \leq c t^{1/p}$, $p > 1$. Then

$$(3.5) \quad L_{p, \infty} \subset W(A, L_0).$$

In fact, according to our Theorem 1 (d), we have $E(t, a; A, L_0) = \|a^* 1_{[t, \infty)}\|_A$ and, therefore,

$$\begin{aligned} E_{\#}(t, a) &= \frac{1}{t} \int_0^t \|a^* 1_{[s, \infty)}\|_A \, ds - \|a^* 1_{[t, \infty)}\|_A \leq \frac{1}{t} \int_0^t \|a^* 1_{[s, \infty)} - a^* 1_{[t, \infty)}\|_A \, ds = \\ &= \frac{1}{t} \int_0^t \|a^* 1_{[s, t)}\|_A \, ds. \end{aligned}$$

Hence, if $a \in L_{p, \infty}$ so that $a^*(u) \leq c u^{-1/p}$, we find that

$$E_{\#}(t, a) \leq c \frac{1}{t} \int_0^t \|u^{-1/p} 1_{[s, t)}(u)\|_A \, ds \leq c_1 \frac{1}{t} \int_0^t s^{-1/p} t^{1/p} \, ds \leq c_2$$

and (3.5) is proved. For the case $A = L_1$ we simply have that

$$E_{\#}(t, a) = \frac{1}{t} \int_0^t \left(\int_s^t a^*(u) \, du \right) \, ds = \frac{1}{t} \int_0^t u a^*(u) \, du$$

and, thus,

$$(3.6) \quad W(L_1, L_0) = L_{1, \infty} \text{ with } \|a\|_{L_{1, \infty}} \leq 2 \|a\|_{W(L_1, L_0)} \leq 2 \|a\|_{L_{1, \infty}}.$$

Remark For the cases when $A = L_{p, 1}$ and $A = L_p$ the imbedding (3.5) was also proved in [7, Th. 2, Th. 3]. In this connection we also remark that

$$a(t) = a^*(t) = t^{-1/p} (\ln 1/t)^{1-1/p} 1_{[0, 1]}(t) \in W(L_p, L_0) \setminus L_{p, \infty}, \quad p > 1.$$

Example 5 Let φ be an increasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and φ' is a non-increasing function. The Lorentz space Λ_φ is the int-symmetric space determined by the norm (cf. [13])

$$\|a\|_{\Lambda_\varphi} = \int_0^\infty a^*(s) \varphi'(s) ds.$$

According to our Theorem 1 (b) we have

$$E(t, a; \Lambda_\varphi, L_\infty) = \|(a^* - t)_+\|_{\Lambda_\varphi}.$$

Moreover,

$$\begin{aligned} \|(a^* - t)_+\|_{\Lambda_\varphi} &= \int_0^\infty (a^*(s) - t)_+ \varphi'(s) ds = \int_0^\infty \left(\int_{t < \lambda < a^*(s)} d\lambda \right) \varphi'(s) ds = \\ &= \int_t^\infty \left(\int_{a^*(s) > \lambda} \varphi'(s) ds \right) d\lambda = \int_t^\infty \varphi(d_{a^*}(\lambda)) d\lambda = \int_t^\infty \varphi(d_a(\lambda)) d\lambda \end{aligned}$$

and we find that

$$(3.7) \quad E(t, a; \Lambda_\varphi, L_\infty) = \int_t^\infty \varphi(d_a(\lambda)) d\lambda.$$

As a special case of (3.7) we obtain the formula

$$(3.8) \quad E(t, a; L_1, L_\infty) = \int_t^\infty d_a(\lambda) d\lambda,$$

and $E_{p-1,1}(L_1, L_\infty) = L_p$ with $\|a\|_{p-1,1,E} \approx \|a\|_{L_p}^p$ for $1 < p < \infty$.

Remark The formula (3.8) was also stated (without proof) in [11].

4. ON THE CONNECTION BETWEEN THE E- AND THE K-FUNCTIONALS

We consider the functional

$$K_\infty(t, a; A_0, A_1) = K_\infty(t, a) = \inf_{a=a_0+a_1} \max(\|a_0\|_{A_0}, t\|a_1\|_{A_1})$$

and note that

$$K_{\infty}(t,a) \leq K(t,a) \leq 2K_{\infty}(t,a).$$

The following well-known statement give us exact information about the close connections between the E- and K- functionals:

PROPOSITION 1 *If $a \in \Sigma(\bar{A})$, then*

- (a) $K(t,a) = \inf_{s>0} (st + E(s,a)),$
- (b) $\bar{E}(t,a) = \sup_{s>0} (K(s,a) - st),$ where $\bar{E}$ is the convex minorant of E,
- (c) *the right-continuous inverse of $E(t,a)/t$ is $K_{\infty}(t,a)/t$.*

For the sake of completeness we include a simple proof of proposition 1.

Proof: We have

$$\begin{aligned} K(t,a) &= \inf_{s>0} (\inf \{ \|a - a_0\|_{A_0} + st : a_1 \in A_1, a - a_1 \in A_0, \|a_1\|_{A_0} \leq s \}) = \\ &= \inf_{s>0} (E(s,a) + st). \end{aligned}$$

The proof of the equality in (b) follows from (a) and the corresponding duality theorem in convex analysis. According to the definitions of the E- and the K_{∞} -functionals we find that if $s > K_{\infty}(t,a)/t$, then $E(s,a)/s \leq t$ and, thus,

$$(E(t,a)/t)^{-1} := \inf \{ s > 0 : E(s,a)/s \leq t \} \leq K_{\infty}(t,a)/t.$$

In a similar way we find that if $E(s,a)/s < t$, then $K_{\infty}(t,a)/t \leq s$ so that also $(E(t,a)/t)^{-1} \geq K_{\infty}(t,a)/t$. The proof is complete.

By using Proposition 1 and our formulas for calculation of the E-functionals we get at once also information about the corresponding K-functionals.

Example 6 We consider $E(s,a) = E(s,a,L_1,L_{\infty})$ and note that the formula (3.8) implies that $E'(s,a) = -d_a(s)$. Therefore the infimum in the expression

$$K(t,a) = \inf_{s>0} (E(s,a) + st)$$

is attained for $s = a^*(t)$ and we obtain that

$$K(t, a; L_1, L_\infty) = K(t, a) = E(a^*(t), a) + ta^*(t) = \int_{a^*(t)}^{\infty} d_a(\lambda) d\lambda + ta^*(t) = \int_0^t a^*(s) ds.$$

By using (3.7) and making a similar calculation we can generalize this (Peetre) formula to the following (Lorentz-Shimogaki-Sharpley) formula:

$$K(t, a; \Lambda_{\varphi}, L_\infty) = \int_0^{\infty} a^*(t) d \min(\varphi(s), t).$$

Compare also with [13, Th 5.9].

Example 7 Let A_0 be a symmetric Banach space on $[0, \infty)$. If $\varphi_0(t) = \|1_{[0,t]}\|_{A_0}$, $t > 0$, then (cf. [16])

$$(4.1) \quad \|a^*1_{[0,t]}\|_{A_0} \leq K(\varphi_0(t), a; A_0, L_\infty) \leq 2 \|a^*1_{[0,t]}\|_{A_0}.$$

In fact, according to Proposition 1 (a) we have

$$K(\varphi_0(t), a; A_0, L_\infty) = K(\varphi_0(t), a) = \inf_{s>0} (s\varphi_0(t) + E(s, a)).$$

Moreover, $a^*1_{[0,t]} \leq (a^* - s)_+ + s1_{[0,t]}$ and, hence, by our Theorem 1 (d) and the triangle inequality, we obtain

$$\|a^*1_{[0,t]}\|_{A_0} \leq E(s, a) + s\varphi_0(t).$$

On the other hand, the special choice of $s = a^*(t)$ implies that

$$\begin{aligned} E(a^*(t), a) + a^*(t)\varphi(t) &= \|(a^* - a^*(t))_+\|_{A_0} + a^*(t)\|1_{[0,t]}\|_{A_0} = \|(a^* - a^*(t))_+\|_{A_0} + \\ &+ \|a^*(t)1_{[0,t]}\|_{A_0} \leq 2 (\|(a^* - a^*(t))_+ 1_{[0,t]}\|_{A_0} + \|a^*(t)1_{[0,t]}\|_{A_0}) = 2 \|a^*1_{[0,t]}\|_{A_0} \end{aligned}$$

and (4.1) is proved.

Example 8. Let X be a compact metric space with the metric ρ . By $C(X)$ we denote the space of continuous functions on X with the norm $\|a\|_{C(X)} = \sup_{x \in X} |a(x)|$ and by $\text{Lip}(X)$ we denote the subspace determined by the seminorm $|a|_{\text{Lip}(X)} = \sup \{ |a(x) - a(y)| / \rho(x, y) : x, y \in X, x \neq y \}$. Then, by using an idea of Pokrovskii (see [18]), we can prove that

$$(4.2) \quad E(t, a; C(X), \text{Lip}(X)) = \frac{1}{2} \sup_{s>0} (\omega(s, a) - st),$$

in a similar way as the corresponding formula for the K -functional was proved in [18] (by Brudnyi). Here, as usual, $\omega(t, a)$ denotes the modulus of continuity, i.e. $\omega(t, a) = \sup \{ |a(x) - a(y)| : x, y \in X, \rho(x, y) \leq t \}$. By using Proposition 4.1 and (4.2) we obtain the following formula for the K -functional:

$$K(t, a; C(X), \text{Lip}(X)) = \frac{1}{2} \hat{\omega}(2t, a),$$

where $\hat{\omega}$ denotes the concave majorant of ω .

Remark If $X = [0, 2\pi]$ and the functions are 2π -periodic, then, according to the Favard inequality and (4.2) we obtain the well-known Jackson-Korneichuk inequality

$$E_n(a) \leq \frac{1}{2} \hat{\omega}\left(\frac{\pi}{n}, a\right), \quad n = 1, 2, \dots,$$

where $E_n(a) = E(n-1, a, C_{2\pi}T)$.

5. CONCLUDING REMARKS

1. If $a \in A_0$, then, by choosing $a_1 = 0$, we find that $E(t, a) \leq \|a\|_{A_0}$. Furthermore, our definition of the E -functional makes it natural to define $E(0, a) = \|a\|_{A_0}$ if $a \in A_0$ and $E(0, a) = +\infty$ if $a \notin A_0$.
2. According to our proof of Theorem 1 it is easy to see that this theorem holds also for all non-atomic measures and for the counting measure.

3. Let A_0 be a normed Banach space and let A_1 be continuously imbedded into A_0 with constant c , i.e. let $\|a\|_{A_0} \leq c\|a\|_{A_1}$ for all $a \in A_1$. Then, by the definition of the E-functional, we see and $E(t/c, a) \geq \|a\|_{A_0} - t$ if $t \geq c$. In particular, these estimates and Proposition 1 (a) imply the well-known equality $K(t, a) = \|a\|_{A_0}$ for $t \geq c$. In fact, if $t \geq c$, then

$$K(t, a) = \inf_{s>0} \left(\frac{s}{c}t + E\left(\frac{s}{c}, a\right) \right) \geq \inf_{s>0} \left(\frac{s}{c}t + \|a\|_{A_0} - s \right) \geq \|a\|_{A_0}$$

and, according to remark 1 above, the estimate in the opposite direction is obviously true for every $t > 0$.

4. By using Proposition 3.1 c) and making the change $t \rightarrow E(s, a)/s$ of variables we obtain a simple proof of the following (Peetre-Sparr [21]) equivalence formula for the E- and K-methods of interpolation:

$$(\bar{A}_{\theta, p; K})^{\alpha+1} = \bar{A}_{\alpha, p; E} \text{ with } \|a\|_{\alpha, q; E} \approx (\|a\|_{\theta, p; K})^{\alpha+1},$$

where $\theta = \alpha/(\alpha+1)$, $p = q(\alpha+1)$, $\alpha > 0$ and $0 < q \leq \infty$ (cf. also [9]).

5. If $A_1 \subset A_0$, then $E(t, a) = \inf \{ \|a - a_1\|_{A_0} : a_1 \in D_t \}$, $D_t = \{a_1 \in A_1 : \|a_1\|_{A_1} \leq t\}$ is the usual expression (the best approximation of a by the set D_t) frequently used in the theory of approximation. Therefore our results can be useful also in approximation theory and here we only note that if also $A_2 \subset A_0$, then by using the equivalence formula in remark 4 above, Proposition 3 (c) and the power theorem (see e.g. [4]), we obtain that

a) the generalized Jackson inequality

$$E(t, a; A_0, A_1) \leq C t^{-\alpha/\theta} \|a\|_{A_2} \text{ for all } a \in A_2$$

implies the direct imbedding theorem

$$(A_0, A_2)_{\theta, p, K} \subset (A_0, A_1)_{\alpha, p, E},$$

b) the generalized Bernstein inequality

$$\|a\|_{A_2} \leq C(\|a\|_{A_1})^{\alpha/\theta} \|a\|_{A_0} \text{ for all } a \in A_1$$

implies the inverse imbedding theorem

$$(A_0, A_1)_{\alpha, p, E} \subset (A_0, A_2)_{\theta, p, K}.$$

REFERENCES

- [1] I.U. Asekritova, *On some properties of general approximation spaces*, Dokl. Akad. Nauk SSSR 267 (1982), 265-268; Soviet Math. Dokl. 26 (1982), 563-566.
- [2] J. Bastero, H. Hudzik and A.M. Steinberg, *On the smallest and the largest spaces among rearrangement-invariant p -Banach function spaces* ($0 < p < 1$), Poznan 1990, 7 pp. (manuscript).
- [3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics 129, Academic Press, Boston 1988.
- [4] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer Verlag, Berlin-Heidelberg-New York 1976.
- [5] Yu.A. Brudnyi and N.Ya. Krugljak, *Real Interpolation Functors*, Book manuscript, Yaroslavl 1981 (Russian).
- [6] Yu.A. Brudnyi and N.Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam 1990 (to appear).
- [7] M. Cwikel and Y. Sagher, *Weak type classes*, J. Functional Anal. 52 (1983), 11-18.
- [8] H. Heinig and L. Maligranda, *Chebyshev inequality in function spaces*, Hamilton 1990, 49 pp. (manuscript).
- [9] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), 177-199.
- [10] H. Hudzik and L. Maligranda, *An interpolation theorem in symmetric function F -spaces*, Proc. Amer. Math. Soc. 110 (1990), 89-96.
- [11] B. Jawerth and M. Milman, *Interpolation of weak type spaces*, Math. Z. 201 (1989), 509-519.
- [12] B. Jawerth and A. Torchinsky, *Local sharp maximal functions*, J. Approx. Theory 43 (1985), 231-270.
- [13] S.G. Krein, Yu.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators*, Providence 1982.

- [14] G.G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, J. Functional Anal. 2 (1968), 31-51.
- [15] L. Maligranda, *A bibliography on "Interpolation of operators and applications" (1926-1990)*, Luleå University 1990, 154 pp.
- [16] L. Maligranda, *The K-functional for symmetric spaces*, Lecture Notes in Math. 1070 (1984), 169-182.
- [17] M. Milman and Y. Sagher, *An interpolation theorem*, Ark. Mat. 22 (1984), 33-38.
- [18] B.S. Mitjagin and E.M. Semenov, *Absence of interpolation of linear operators in spaces of smooth functions*, Izv. Akad. Nauk SSSR 41 (1977), 1289-1328; Math. USSR-Izv. 11 (1977), 1229-1266.
- [19] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. 132 (1982), 291-330.
- [20] R. O'Neil, *Integral transforms and tensor products on Orlicz spaces and $L(p,q)$ -spaces*, J. Analyse Math. 21 (1968), 1-276.
- [21] J. Peetre and G. Sparr, *Interpolation of normed Abelian groups*, Ann. Mat. Pura Appl. 92 (1972), 217-262.
- [22] L.E. Persson, *On a weak-type theorem with applications*, Proc. London Math. Soc. (3) 38 (1979), 295-308.
- [23] A.V. Pokrovskii, *On a theorem of A.F. Timan*, Funkcional Anal. Prilozhen. 1 (1967), 93-94, Functional Anal. Appl. 1 (1967).
- [24] J.V. Ryff, *Measure preserving transformations and rearrangements*, J. Math. Anal. Appl. 31 (1970), 449-458.
- [25] Y. Sagher, *An application of the approximation functional in interpolation theory*, In: Proc. Conf. on Harmonic Analysis in Honour to A. Zygmund, Wadsworth, Belmont, Calif. 1982, 802-809.

Eingegangen am 29. Oktober 1990

Lech Maligranda
Departamento de
Matemáticas, IVIC
Apartado 21827
CARACAS 1020-A
VENEZUELA

Lars Erik Persson
Dept. of Appl. Math.
Luleå University
S-951 87 LULEÅ
SWEDEN

92h:46111 46M35 46E99

Maligranda, Lech (YV-IVIC); Persson, Lars-Erik (S-LUL-A)

The E -functional for some pairs of groups.

Resultate Math. 20 (1991), no. 1-2, 538-553.

Let $\bar{A} = (A_0, A_1)$ be a couple of quasi-normed abelian subgroups A_0 and A_1 of some abelian group $\mathcal{A}$ satisfying the separability axiom (i.e., if $\|a_n - a\|_{A_0} \rightarrow 0$ and $\|a_n - b\|_{A_1} \rightarrow 0$, then $a = b$). The E -functional of the element $a \in A_0 + A_1$ is defined, for every $t > 0$, by $E(t, a; \bar{A}) = \inf\{\|a - a_1\|_{A_0} : a_1 \in A_1, a - a_1 \in A_0, \|a\|_{A_1} \leq t\}$, with the usual convention $\inf \emptyset = +\infty$.

The authors calculate directly the E -functional for some couples of quasi-normed groups. In the main theorem they show that if A_0 is a solid quasi-normed abelian group in the space $S = S(I, m)$ of all equivalence classes of Lebesgue measurable real-valued functions on $I = [0, \tau]$, $0 < \tau \leq \infty$, then $E(t, a; A_0, L_\infty) = \|(|a| - t)_+\|_{A_0}$. If, in addition, A_0 is an exact interpolation space between L_1 and L_∞ , then $E(t, a; A_0, L_0) = \|a^*(\cdot + t)\|_{A_0}$ for $0 < t < \tau = \infty$ and $E(t, a; A_0, L_0) = \|a^*(\cdot + t)\chi_{[0, \tau-t]}\|_{A_0}$ for $0 < t < \tau < \infty$. Here $L_0 = \{a \in S : \|a\|_{L_0} = m(\text{supp } a) < \infty\}$ and a^* is the nonincreasing rearrangement of a . The authors give several applications of the above descriptions. In particular, certain well-known exact formulas for the K -functional of special pairs of compatible Banach spaces are given.

Mieczysław Mastyło (PL-POZN)

Zbl 839

Maligranda, Lech; Persson, Lars Erik

The E -functional for some pairs of groups.

Result. Math. 20, No. 1/2, 538-553 (1991).

The theory of the real interpolation method and the K -functional has been developed in several papers. There exists a close connection between the K -functional and the E -functional and, thus, between the corresponding K - and E -methods. By using these facts we can also obtain both new information about the E -functional (e.g. the E -divisibility in the Brudnyi-Krugljak sense and Holmstedt type formulas) and about the equivalence of the E - and K -spaces (e.g. reiteration and Wolff type theorems).

In this paper we work in the opposite direction. Namely, we discuss and calculate directly the E -functional for some pairs of quasi-normed groups and, in particular, we also obtain some information about the K -functional. We point out several applications of our exact descriptions. In particular, we obtain some well-known exact formulas for the K -functional. Thus our results may also be regarded as a unification and generalization of some well-known exact descriptions both of E -functionals and of K -functionals.

From the introduction.

46067