REITERATED HOMOGENIZATION OF NONLINEAR MONOTONE OPERATORS

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(Dedicated to Professor Jaak Peetre on the Occasion of his 65th Birthday)

Abstract

In this paper, the authors study reiterated homogenization of nonlinear equations of the form $-\operatorname{div}(a(x, x/\varepsilon, x/\varepsilon^2, Du_\varepsilon)) = f$, where a is periodic in the first two arguments and monotone in the third. It is proved that u_ε converges weakly in $W^{1,p}(\Omega)$ (and even in some multiscale sense), as $\varepsilon \rightarrow 0$ to the solution u_0 of a limit problem. Moreover, an explicit expression for the limit problem is given. The main results were also stated in [15]. This article presents the complete proofs of these results.

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§1. Introduction

Let us consider the class of partial differential equations of the form

$$-\operatorname{div}(a_\varepsilon(x, Du_\varepsilon)) = f \quad \text{on } \Omega, \quad u_\varepsilon \in W_0^{1,p}(\Omega), \quad (1.1)$$

where a_ε is increasingly oscillating as $\varepsilon \rightarrow 0$, Ω is an open bounded subset of R^n , $1 < p < \infty$, $1/p + 1/q = 1$ and $f \in W^{-1,q}(\Omega)$. The homogenization problem for (1.1) consists of the study of the asymptotic behavior of solutions u_ε , as $\varepsilon \rightarrow 0$. In many important cases u_ε converges weakly in $W_0^{1,p}(\Omega)$ to the solution u_0 of the homogenized problem

$$-\operatorname{div}(b(Du_0)) = f \quad \text{on } \Omega, \quad u_0 \in W_0^{1,p}(\Omega).$$

The first homogenization result of this kind was given by De Giorgi and Spagnolo in [13] for the linear case when a_ε is of the form $a_\varepsilon(x, \xi) = a(x/\varepsilon, \xi) = A(x/\varepsilon)\xi$, where $A(\cdot)$ is a periodic and bounded matrix. In [8] Chiado Piat and Defranceschi considered the case when a is monotone, continuous and satisfies suitable coerciveness and growth conditions in the second variable. The corresponding variational problems have been studied by many authors in the notion of Γ -convergence (see e.g. the books [4, 11]). Quite recently a new

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method has been developed, known as two-scale convergence, which is well-suited for the analysis of the convergence questions described above (see e.g. [1, 20]).

When $a_\varepsilon(x, \xi) = a(x/\varepsilon, x/\varepsilon^2, \xi)$, where a is periodic in the first two variables, one usually speaks about reiterated homogenization. This concept was introduced by Bensoussan, Lions and Papanicolaou in [3], where it was stated a result for the linear operators which later has been known as the iterated homogenization theorem. This theorem has been an indispensable tool in the construction of structures with extreme effective material properties. Concerning this topic we refer to the collection of classical papers in [7], where the introduction gives a good selection of references (see also [17]). Braides and Lukkassen considered the corresponding Γ -convergence results for convex reiterated problems in [5] and [16] and later on Müller, and subsequently Braides and Defranceschi considered the non-convex case (see [4, p.215]). These results have recently been used to construct nonlinear materials with surprisingly effective behavior (see e.g. [5, 16, 22, 24] and the references therein). In the linear case reiterated homogenization has been studied by multiscale-convergence, which is a generalization of two-scale convergence (see [2]).

In this paper we study reiterated homogenization for the case when a_ε is of the form

$$a_\varepsilon(x, \xi) = a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi\right), \quad (1.2)$$

where a is assumed to satisfy suitable monotonicity conditions in ξ and to be periodic in the second and third arguments. The results have been presented earlier in [15] without detailed proofs. In this paper we give the detailed proofs.

The paper is organized as follows. We have collected some notations and preliminary results in Section 2. In Section 3 we study the nonlinear reiterated homogenization problem corresponding to (1.2) by using G -convergence and in Section 4 by multiscale-convergence, respectively.

§2. Notation and Preliminary Results

Let Y and Z be open bounded rectangles in R^n . $|E|$ denotes the Lebesgue measure of the set $E \subset R^n$ and $(\cdot, \cdot)$ is the Euclidean scalar product on R^n . Moreover, c will be a constant that may differ from one place to another and $h \in N$. The function $\tilde{\omega} : R \rightarrow R$ is an arbitrary function which is continuous, increasing and $\tilde{\omega}(0) = 0$. By $W_{\text{per}}^{1,p}(Y)$ we denote the set of all functions $u \in W^{1,p}(Y)$ with mean value zero which have the same trace on opposite faces of Y , $W_{\text{per}}^{1,p}(Z)$ is defined in the corresponding way. Every function $u \in W_{\text{per}}^{1,p}(Y)$ can be extended by periodicity to a function in $W_{\text{loc}}^{1,p}(R^n)$ (in this paper we will not make any distinction between the original function and its extension). Let $\Omega_i \subset Y$, $i = 1, \dots, N$, be a family of disjoint open sets such that

$$\left| Y \setminus \bigcup_{i=1}^N \Omega_i \right| = 0 \quad \text{and} \quad |\partial \Omega_i| = 0.$$

Let us fix a function $a : R^n \times R^n \times R^n \rightarrow R^n$ which fulfills the conditions:

- (1) $a(\cdot, z, \xi)$ is Y -periodic for every $\xi \in R^n$ and every $z \in R^n$.
- (2) $a(y, \cdot, \xi)$ is Z -periodic and Lebesgue measurable for every $\xi \in R^n$ and every $y \in R^n$.
- (3) There exists two constants $c_1, c_2 > 0$ and two constants α and β , with $0 \leq \alpha \leq \min\{1, p-1\}$ and $\max\{p, 2\} \leq \beta < \infty$ such that a satisfies the following boundedness,

continuity and monotonicity assumptions:

$$a(y, z, 0) = 0 \text{ for a.e. } y, z \in R^n, \quad (2.1)$$

$$|a(y, z, \xi_1) - a(y, z, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha, \quad (2.2)$$

$$(a(y, z, \xi_1) - a(y, z, \xi_2), \xi_1 - \xi_2) \geq c_2(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta. \quad (2.3)$$

(4) a is of the form $a(y, z, \xi) = \sum_{i=1}^N \chi_{\Omega_i}(y) a_i(y, z, \xi)$ and satisfies a continuity condition of the form

$$|a(y_1, z, \xi) - a(y_2, z, \xi)|^q \leq \omega(|y_1 - y_2|) (1 + |\xi|^p) \quad (2.4)$$

for $y_1, y_2 \in \Omega_i$, $i = 1, \dots, N$, a.e. $z \in R^n$ and every $\xi \in R^n$, where $\omega : R \rightarrow R$ is continuous, increasing and $\omega(0) = 0$.

By (2.1), (2.2), and (2.3) it follows that

$$|a(y, z, \xi)| \leq c (1 + |\xi|^{p-1}), \quad (2.5)$$

$$|\xi|^p \leq c (1 + (a(y, z, \xi), \xi)) \quad (2.6)$$

hold for $y \in R^n$, a.e. $z \in R^n$ and every $\xi \in R^n$. The proof of (2.5) follows immediately by (2.1) and (2.2). To see (2.6), let a, b, η be positive constants. By using Young's inequality,

$$ab \leq \frac{p}{\beta} \eta^{-\frac{\beta}{p}} a^{\frac{\beta}{p}} + \frac{\beta-p}{\beta} \eta^{\frac{\beta}{\beta-p}} b^{\frac{\beta}{\beta-p}},$$

(2.1) and (2.3) it follows that

$$\begin{aligned} |\xi|^p &= (1 + |\xi|)^{\frac{p(p-\beta)}{\beta}} |\xi|^p (1 + |\xi|)^{-\frac{p(p-\beta)}{\beta}} \\ &\leq \frac{p}{\beta} \eta^{-\frac{\beta}{p}} (1 + |\xi|)^{p-\beta} |\xi|^\beta + \frac{\beta-p}{\beta} \eta^{\frac{\beta}{\beta-p}} (1 + |\xi|)^p \\ &\leq \frac{p}{\beta} \eta^{-\frac{\beta}{p}} (a(y, z, \xi), \xi) + \frac{\beta-p}{\beta} \eta^{\frac{\beta}{\beta-p}} 2^{p-1} (1 + |\xi|^p). \end{aligned}$$

Now (2.6) follows by choosing η sufficiently small.

If Ω is an open bounded convex set and $1 \leq p < \infty$ then the Poincare-Wirtinger inequality states that there exists a constant $K > 0$ such that

$$\|u - M_\Omega(u)\|_{L^p(\Omega)} \leq K \|Du\|_{L^p(\Omega, R^n)}$$

for every $u \in W^{1,p}(\Omega)$, where $M_\Omega(u)$ denotes the mean value of u over Ω . A corollary of this is that $\|D\cdot\|_{L^p(Y, R^n)}$ defines an equivalent norm on $W_{\text{per}}^{1,p}(Y)$, i.e. there exists constants $k_1, k_2 > 0$ such that

$$k_1 \|D\cdot\|_{L^p(Y, R^n)} \leq \|\cdot\|_{W_{\text{per}}^{1,p}(Y)} \leq k_2 \|D\cdot\|_{L^p(Y, R^n)}. \quad (2.7)$$

For the sake of completeness we now formulate two well-known lemmas. The proof of the first lemma can be found in e.g. [21] and the second lemma is a simple case of compensated compactness of F. Murat (see [18, 19]).

Lemma 2.1. *Let $g \in L^q(Y, R^n)$ be a function such that $\int_Y (g, Dv) dx = 0$ for every $v \in H_{\text{per}}^{1,p}(Y)$. Then g can be extended by periodicity to an element of $L_{\text{loc}}^q(R^n, R^n)$, still denoted by g , such that $-\text{div} g = 0$ in $\mathcal{D}'(R^n)$.*

Lemma 2.2. *Let $1 < p < \infty$. Moreover, let (v_h) be a sequence in $L^q(\Omega, R^n)$ which converges weakly to v , $(-\text{div} v_h)$ converges strongly to $-\text{div} v$ in $W^{-1,q}(\Omega)$ and let (u_h) be a*

sequence which converges weakly to u in $W^{1,p}(\Omega)$. Then

$$\int_{\Omega} (v_h, Du_h) \phi \, dx \rightarrow \int_{\Omega} (v, Du) \phi \, dx$$

for every $\phi \in C_0^\infty(\Omega)$.

§3. G-Convergence Results

Let us now consider the weak formulation of (1.1):

$$\begin{cases} \int_{\Omega} \left(a\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_\varepsilon\right), D\phi \right) = \langle f, \phi \rangle & \text{for every } \phi \in W_0^{1,p}(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \quad (3.1)$$

where $f \in W^{-1,q}(\Omega)$. By the Hartman-Stampancchia Theorem^[14], there exist a unique solution for each ε . Below we state a G-convergence result, which is a generalization of the classical iterated homogenization theorem.

Theorem 3.1. *Let a satisfy (2.1)–(2.4). Moreover, let (u_ε) be the solutions of (3.1). Then*

$$\begin{aligned} u_\varepsilon &\rightarrow u_0 \text{ weakly in } W_0^{1,p}(\Omega), \\ a\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_\varepsilon\right) &\rightarrow b(Du_0) \text{ weakly in } L^q(\Omega, R^n), \end{aligned}$$

as $\varepsilon \rightarrow 0$, where u_0 is the unique solution of

$$\begin{cases} \int_{\Omega} (b(Du_0), D\phi) \, dx = \langle f, \phi \rangle & \text{for every } \phi \in W_0^{1,p}(\Omega), \\ u_0 \in W_0^{1,p}(\Omega). \end{cases} \quad (3.2)$$

The operator $b : R^n \rightarrow R^n$ is defined as

$$b(\xi) = \frac{1}{|Y|} \int_Y b_1(y, \xi + Dv^\xi(y)) \, dy,$$

where v^ξ is the unique solution of the cell-problem

$$\begin{cases} \int_Y (b_1(y, \xi + Dv^\xi(y)), D\phi) \, dy = 0 & \text{for every } \phi \in W_{\text{per}}^{1,p}(Y), \\ v^\xi \in W_{\text{per}}^{1,p}(Y). \end{cases}$$

The operator $b_1 : Y \times R^n \rightarrow R^n$ is defined as

$$b_1(y, \xi) = \frac{1}{|Z|} \int_Z a(y, z, \xi + Dv^{\xi,y}(z)) \, dz,$$

where $v^{\xi,y}$ is the unique solution of the cell-problem

$$\begin{cases} \int_Z (a(y, z, \xi + Dv^{\xi,y}(z)), D\phi) \, dz = 0 & \text{for every } \phi \in W_{\text{per}}^{1,p}(Z), \\ v^{\xi,y} \in W_{\text{per}}^{1,p}(Z). \end{cases} \quad (3.3)$$

Proof. Without loss of generality we assume that $Y = Z = (0, 1)^n$ and that $\varepsilon = 1/h$, $h \in N$ (the solutions $u_\varepsilon = u_{1/h}$ will be denoted by u_h). The sequence of solutions (u_h) is bounded in $W^{1,p}(\Omega)$. Indeed, let $\phi = u_h$ in (3.1) then it follows by (2.6) and (3.1) that

$$\begin{aligned} \int_{\Omega} |Du_h|^p \, dx &\leq c \int_{\Omega} 1 + (a(hx, h^2x, Du_h), Du_h) \, dx \\ &\leq c|\Omega| + \|f\|_{W^{-1,q}(\Omega)} \|u_h\|_{W^{1,p}(\Omega)} \leq c(1 + \|u_h\|_{W^{1,p}(\Omega)}). \end{aligned} \quad (3.4)$$

If $\|u_h\|_{W^{1,p}(\Omega)} \leq 1$, it is clear that the sequence of solutions (u_h) is bounded in $W^{1,p}(\Omega)$. So let us assume that $\|u_h\|_{W^{1,p}(\Omega)} \geq 1$, then (3.4) implies that

$$\int_{\Omega} |Du_h|^p \, dx \leq c \|u_h\|_{W^{1,p}(\Omega)}. \quad (3.5)$$

Due to the Poincaré inequality the norms $\|D\cdot\|_{L^p(\Omega, R^n)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$ are equivalent. The desired boundedness then follows by taking (3.5) into account. Thus there exists a subsequence, still denoted by (u_h) , such that

$$u_h \rightarrow u_* \text{ weakly in } W_0^{1,p}(\Omega).$$

Let us now define

$$\eta_h = a(hx, h^2x, Du_h).$$

By (2.1), (2.2), and (3.5) we have that η_h is bounded in $L^q(\Omega, R^n)$. Indeed

$$\begin{aligned} \int_{\Omega} |\eta_h|^q dx &= \int_{\Omega} |a(hx, h^2x, Du_h)|^q dx \\ &\leq c \int_{\Omega} (1 + |Du_h|)^{q(p-1-\alpha)} |Du_h|^{q\alpha} dx \\ &\leq c \int_{\Omega} (1 + |Du_h|)^p dx \leq c, \end{aligned}$$

where c is a constant independent of h . This means that there exists a subsequence, still denoted by (η_h) , such that

$$\eta_h \rightarrow \eta_* \text{ weakly in } L^q(\Omega, R^n).$$

From our original problem (3.1) we have that

$$\begin{cases} \int_{\Omega} (a(hx, h^2x, Du_h), D\phi) dx = \langle f, \phi \rangle & \text{for every } \phi \in W_0^{1,p}(\Omega), \\ u_h \in W_0^{1,p}(\Omega). \end{cases}$$

In the limit we get

$$\int_{\Omega} (\eta_*, D\phi) dx = \langle f, \phi \rangle \text{ for every } \phi \in W_0^{1,p}(\Omega).$$

Especially this means that

$$\int_{\Omega} (\eta_*, D\phi) dx = \langle f, \phi \rangle \text{ for every } \phi \in C_0^\infty(\Omega).$$

If we now could show that

$$\eta_* = b(Du_*) \text{ for a.e } x \in \Omega, \quad (3.6)$$

then it follows by the uniqueness of the solution of the homogenized problem (3.2) that $u_* = u_0$.

Fix ξ and let u_h^ξ be defined as the solution of the auxiliary problem:

$$\begin{cases} \int_Y (a(x, hx, \xi + Du_h^\xi), D\phi) = 0 & \text{for every } \phi \in W_{\text{per}}^{1,p}(Y), \\ u_h^\xi \in W_{\text{per}}^{1,p}(Y). \end{cases} \quad (3.7)$$

Then, by using the ideas in [6] and [23], it is possible to show that

$$u_h^\xi \rightarrow u_0^\xi \text{ weakly in } W_{\text{per}}^{1,p}(Y),$$

$$a(x, hx, \xi + Du_h^\xi) \rightarrow b_1(x, \xi + Du_0^\xi) \text{ weakly in } L^q(Y, R^n),$$

as $h \rightarrow \infty$, where u_0^ξ is the unique solution of

$$\begin{cases} \int_Y (b_1(x, \xi + Du_0^\xi), D\phi) dx = 0 & \text{for every } \phi \in W_{\text{per}}^{1,p}(Y), \\ u_0^\xi \in W_{\text{per}}^{1,p}(Y). \end{cases}$$

Let us now define the function

$$w_h^\xi(x) = (\xi, x) + \frac{1}{h} u_h^\xi(hx).$$

We note that, by periodicity and the fact that if (v_h) is a sequence of Y -periodic functions in $L^p_{\text{loc}}(R^n)$ such that $v_h \rightarrow v$ weakly in $L^p(Y)$ and if w_h is defined as $w_h(x) = v_h(hx)$, then $w_h \rightarrow \frac{1}{|Y|} \int_Y v(x) dx$, we have the following convergences:

$$\begin{aligned} w_h^\xi &\rightarrow (\xi, \cdot) \text{ weakly in } W^{1,p}(\Omega), \\ Dw_h^\xi &\rightarrow \xi \text{ weakly in } L^p(\Omega, R^n), \\ a(hx, h^2x, Dw_h^\xi) &\rightarrow \int_Y b_1(x, \xi + Du_0^\xi) dx = b(\xi) \text{ weakly in } L^q(\Omega, R^n). \end{aligned}$$

By the monotonicity of a we have for a fixed ξ that

$$\int_{\Omega} (a(hx, h^2x, Du_h) - a(hx, h^2x, Dw_h^\xi), Du_h - Dw_h^\xi) \phi dx \geq 0,$$

for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. By (3.1), we have that

$$\int_{\Omega} (a(hx, h^2x, Du_h), D\phi) dx = \langle f, \phi \rangle \text{ for every } \phi \in C_0^\infty(\Omega),$$

which implies that

$$-\operatorname{div}(a(hx, h^2x, Du_h)) = -\operatorname{div}\eta_h \rightarrow -\operatorname{div}\eta_* \text{ in } H^{-1,q}(\Omega).$$

From the definition of u_h^ξ (see (3.7) and Lemma 2.1 we have that

$$-\operatorname{div}(a(hx, h^2x, Dw_h^\xi)) = 0 \text{ on } \Omega.$$

The compensated compactness lemma, Lemma 2.2, then implies that we have in the limit

$$\int_{\Omega} (\eta_*(x) - b(\xi), Du_*(x) - \xi) \phi(x) dx \geq 0,$$

for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. Hence, for every $\xi \in R^n$ we have that

$$(\eta_*(x) - b(\xi), Du_*(x) - \xi) \geq 0 \text{ for a.e. } x \in \Omega. \quad (3.8)$$

In particular, if (ξ_m) is a countable dense subset in R^n , then (3.8) implies that

$$(\eta_*(x) - b(\xi_m), Du_*(x) - \xi_m) \geq 0 \text{ for a.e. } x \in \Omega.$$

By the continuity of b (see (3.10)), it follows that

$$(\eta_*(x) - b(\xi), Du_*(x) - \xi) \geq 0 \text{ for a.e. } x \in \Omega \text{ and for every } \xi \in R^n.$$

Since b is monotone and continuous (see Proposition 4.1), we have that b is maximal monotone and the crucial relation (3.6) follows.

We have now proved the theorem up to a subsequence of (u_h) . By the uniqueness of the solution of the homogenized equation it follows that it is true for the whole sequence and the proof is complete.

Remark 3.1. The extension of this result to the case with m scales, i.e. when a is on the form $a(x/\varepsilon, x/\varepsilon^2, \dots, x/\varepsilon^m, \xi)$ is straightforward.

The homogenized operator b in Theorem 3.1 has the following structure properties:

Proposition 3.1. *Let b be the homogenized operator defined in Theorem 3.1. Then*

(i) $b(\cdot)$ is strictly monotone, more precisely

$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq c(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \quad (3.9)$$

for every $\xi_1, \xi_2 \in R^n$.

(ii) $b(\cdot)$ is Lipschitz continuous, more precisely

$$|b(\xi_1) - b(\xi_2)| \leq c(1 + |\xi_1| + |\xi_2|)^{p-1-\gamma} |\xi_1 - \xi_2|^\gamma, \quad (3.10)$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$, where $\gamma = \alpha/(\beta(\beta - \alpha))$.

(iii) $b(0) = 0$.

Proof. Without loss of generality we assume that $Y = Z = (0, 1)^n$ and that $\varepsilon = 1/h$, $h \in N$ (we write the indices as h instead of $1/h$).

(i) Let $\xi_j \in \mathbb{R}^n$, $j = 1, 2$ and define for a.e. $y \in \mathbb{R}^n$,

$$w_h^{\xi_j, y}(z) = (\xi_j, z) + \frac{1}{h} v^{\xi_j, y}(hz) .$$

By the periodicity of $v^{\xi_j, y}$ we have that

$$w_h^{\xi_j, y} \rightarrow (\xi_j, z) \text{ weakly in } W^{1,p}(Y), \quad (3.11)$$

$$Dw_h^{\xi_j, y} \rightarrow \xi_j \text{ weakly in } L^p(Y, \mathbb{R}^n), \quad (3.12)$$

$$a(y, hz, Dw_h^{\xi_j, y}(z)) \rightarrow b_1(y, \xi_j) \text{ weakly in } L^p(Y, \mathbb{R}^n). \quad (3.13)$$

The monotonicity condition (2.3) on a and Hölder's reversed inequality imply that

$$\begin{aligned} & \int_Y (a(y, hz, Dw_h^{\xi_1, y}) - a(y, hz, Dw_h^{\xi_2, y}), Dw_h^{\xi_1, y} - Dw_h^{\xi_2, y}) \phi(z) dz \\ & \geq c \int_Y \left(1 + |Dw_h^{\xi_1, y}| + |Dw_h^{\xi_2, y}|\right)^{p-\beta} |Dw_h^{\xi_1, y} - Dw_h^{\xi_2, y}|^\beta \phi(z) dz \\ & \geq c \left(\int_Y \left(1 + |Dw_h^{\xi_1, y}|^p + |Dw_h^{\xi_2, y}|^p\right) \phi(z) dz \right)^{\frac{p-\beta}{p}} \\ & \quad \times \left(\int_Y |Dw_h^{\xi_1, y} - Dw_h^{\xi_2, y}|^p \phi(z) dz \right)^{\frac{\beta}{p}} \end{aligned}$$

for every $\phi \in C_0^\infty(Y)$ such that $\phi \geq 0$. We apply $\liminf_{h \rightarrow \infty}$ on both sides of this inequality and obtain

$$\begin{aligned} & \int_Y (b_1(y, \xi_1) - b_1(y, \xi_2), \xi_1 - \xi_2) \phi(z) dz \\ & \geq \liminf_{n \rightarrow \infty} c \left(\int_Y \left(1 + |Dw_h^{\xi_1, y}|^p + |Dw_h^{\xi_2, y}|^p\right) \phi(z) dz \right)^{\frac{p-\beta}{p}} \\ & \quad \times \left(\int_Y |Dw_h^{\xi_1, y} - Dw_h^{\xi_2, y}|^p \phi(z) dz \right)^{\frac{\beta}{p}} \\ & \geq \liminf_{n \rightarrow \infty} c \left(\int_Y \left(1 + |Dw_h^{\xi_1, y}|^p + |Dw_h^{\xi_2, y}|^p\right) \phi(z) dz \right)^{\frac{p-\beta}{p}} \\ & \quad \times \liminf_{n \rightarrow \infty} \left(\int_Y |Dw_h^{\xi_1, y} - Dw_h^{\xi_2, y}|^p \phi(z) dz \right)^{\frac{\beta}{p}} \\ & \geq c \left(\int_Y (1 + |\xi_1|^p + |\xi_2|^p) \phi(z) dz \right)^{\frac{p-\beta}{p}} \left(\int_Y |\xi_1 - \xi_2|^p \phi(z) dz \right)^{\frac{\beta}{p}} \\ & \geq c \int_Y (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \phi(z) dz \end{aligned}$$

for every $\phi \in C_0^\infty(Y)$, where on the left hand side we have used (3.12), (3.13), Lemma 2.1 and Lemma 2.2 and on the right hand side we first used (3.12) and secondly the fact that

every norm is weakly lower semicontinuous and finally Jensen's inequality. This implies that

$$(b_1(y, \xi_1) - b_1(y, \xi_2), \xi_1 - \xi_2) \geq c(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta$$

for every $\xi_1, \xi_2 \in R^n$. Finally, (3.9) follows by using almost the same arguments with b_1 replaced with b .

(ii) First we note that by (2.9), (3.3) and (2.5) imply that

$$\begin{aligned} \int_Z |\xi + Dv^{\xi, y}|^p dz &\leq c \int_Z 1 + (a(y, z, \xi + Dv^{\xi, y}), \xi + Dv^{\xi, y}) dz \\ &= c \int_Z 1 + (a(y, z, \xi + Dv^{\xi, y}), \xi) dz \\ &\leq c \int_Z 1 + c \left(1 + |\xi + Dv^{\xi, y}|^{p-1}\right) |\xi| dz. \end{aligned}$$

By using the Young inequality we obtain that

$$\int_Z |\xi + Dv^{\xi, y}|^p dz \leq c(1 + |\xi|^p). \quad (3.14)$$

The definition of b_1 , (2.2), Hölder's inequality and (3.14) imply that

$$\begin{aligned} &|b_1(y, \xi_1) - b_1(y, \xi_2)|^q \\ &\leq \int_Z |a(y, z, \xi_1 + Dv^{\xi_1, y}) - a(y, z, \xi_2 + Dv^{\xi_2, y})|^q dz \\ &\leq c \int_Z (1 + |\xi_1 + Dv^{\xi_1, y}| + |\xi_2 + Dv^{\xi_2, y}|)^{q(p-1-\alpha)} \\ &\quad \times |\xi_1 + Dv^{\xi_1, y} - (\xi_2 + Dv^{\xi_2, y})|^{q\alpha} dz \\ &\leq c \left(\int_Z (1 + |\xi_1 + Dv^{\xi_1, y}| + |\xi_2 + Dv^{\xi_2, y}|)^p dz \right)^{\frac{p-1-\alpha}{p-1}} \\ &\quad \times \left(\int_Z |\xi_1 + Dv^{\xi_1, y} - (\xi_2 + Dv^{\xi_2, y})|^p dz \right)^{\frac{\alpha}{p-1}} \\ &\leq c(1 + |\xi_1| + |\xi_2|)^{\frac{p(p-1-\alpha)}{p-1}} \\ &\quad \times \left(\int_Z |\xi_1 + Dv^{\xi_1, y} - (\xi_2 + Dv^{\xi_2, y})|^p dz \right)^{\frac{\alpha}{p-1}}. \end{aligned} \quad (3.15)$$

The second factor in (3.15) satisfies the following estimate

$$\int_Z |\xi_1 + Dv^{\xi_1, y} - (\xi_2 + Dv^{\xi_2, y})|^p dz \leq c(1 + |\xi_1|^p + |\xi_2|^p)^{\frac{\beta-\alpha-1}{\beta-\alpha}} |\xi_1 - \xi_2|^{\frac{p}{\beta-\alpha}}. \quad (3.16)$$

For a proof of this estimate see Lemma 3.4 in [12]. By taking (3.15) and (3.16) into account we get that

$$|b(y, \xi_1) - b(y, \xi_2)| \leq c(1 + |\xi_1| + |\xi_2|)^{p-1-\gamma} |\xi_1 - \xi_2|^\gamma \quad (3.17)$$

for every $\xi_1, \xi_2 \in R^n$, where $\gamma = \alpha/(\beta - \alpha)$. Finally, (3.10) follows by using almost the same arguments with b_1 replaced with b .

(iii) Condition (2.1) implies that $b_1(y, 0) = 0$ which in turn implies that $b(0) = 0$.

§4. Multiscale Convergence Results

It is possible to extend the above result to the case when a_ε is of the form $a_\varepsilon(x, \xi) = a(x, x/\varepsilon, x/\varepsilon^2, \xi)$. In this section we assume that a_ε is of this form.

A sequence u_ε is said to 3-scale converge to $u_0(x, y, z) \in L^p(\Omega, Y, Z)$ if

$$\lim_{\varepsilon \rightarrow \infty} \int_{\Omega} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx = \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} u_0(x, y, z) \varphi(x, y, z) dx dy dz \quad (4.1)$$

for all $\varphi \in \mathcal{D}(\Omega; C_{\text{per}}^\infty(Y, Z))$ (i.e. functions $\varphi(x, y, z)$ which are C^∞ with compact support in x with values of functions which are C^∞ and periodic in y and z). This definition is a variant of the one introduced in [2] for the case $p = 2$ (see also [1, 9, 20]).

Theorem 4.1. *Let u_ε be a solution of (1.1). Then as $\varepsilon \rightarrow 0$, u_ε 3-scale converges to $u_0 \in W_0^{1,p}(\Omega)$ and Du_ε 3-scale converges to $Du_0 + D_y u_1 + D_z u_2$, where $\{u_0, u_1, u_2\}$ is a solution of*

$$\begin{aligned} & \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (a(x, y, z, Du_0 + D_y u_1 + D_z u_2), Dv_0 + D_y v_1 + D_z v_2) dx dy dz \\ &= \int_{\Omega} f v_0 dx \text{ for all } v_0 \in W_0^{1,p}(\Omega), \quad v_1 \in L^p(\Omega; W_{\text{per}}^{1,p}(Y)), \quad v_2 \in L^p(\Omega; W_{\text{per}}^{1,p}(Y \times Z)). \end{aligned}$$

Remark 4.1. The theorem above can easily be extended to the case when a_ε is on the form

$$a_\varepsilon(x, \xi) = a(x, x/\varepsilon, \dots, x/\varepsilon^m, \xi).$$

Proof. The proof is divided into several steps:

Step 1. $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$, so one can extract a subsequence such that $u_\varepsilon \rightharpoonup u_0$ weakly in $W_0^{1,p}(\Omega)$.

Step 2. For each bounded sequence $\{u_\varepsilon\}$ in $L^p(\Omega)$ there exists a subsequence and $u_0 \in L^p(\Omega, Y, Z)$ such that u_ε 3-scale converges to u_0 (this is a theorem of Nguetseng for $p = 2$ and two scales). The proof is as follows: Observe that

$$\left| \int_{\Omega} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx \right| \leq \|u_\varepsilon\|_{L^p(\Omega)} \left\| \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \right\|_{L^q(\Omega)}. \quad (4.2)$$

In the two-scale case it is possible to prove that for any $\psi \in L^2(\Omega; C_{\text{per}}^\infty(Y))$ it holds that

$$\int_{\Omega} \left| \psi\left(x, \frac{x}{\varepsilon}\right) \right|^2 dx \rightarrow \frac{1}{|Y|} \int_{\Omega \times Y} |\psi(x, y)|^2 dx dy$$

as $\varepsilon \rightarrow 0$ (see e.g. Lemma 5.3 in [10]). This result is easily extendable to 3 scales. Thus we obtain that

$$\begin{aligned} \int_{\Omega} \left(\left| \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \right|^{\frac{q}{2}} \right)^2 dx &\rightarrow \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} |\varphi(x, y, z)|^q dx dy dz \\ &= \frac{1}{|Y||Z|} \|\varphi\|_{L^q(\Omega \times Y \times Z)}^q. \end{aligned}$$

Accordingly, we obtain that there exists a constant c which is independent of φ and a subsequence of $\{\varepsilon\}$ (which may depend on φ) such that

$$\left| \int_{\Omega} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx \right| \leq c \|\varphi\|_{L^q(\Omega \times Y \times Z)}. \quad (4.3)$$

By choosing a countable subset of $\mathcal{D}(\Omega; C_{\text{per}}^\infty(Y, Z))$ which is dense in $L^q(\Omega, Y, Z)$ and using a standard diagonalization process, we can extract a subsequence of $\{\varepsilon\}$ such that (4.3) holds for all φ in this subset with a constant c which is independent of φ and ε . It is possible to show that $\mathcal{D}(\Omega; C_{\text{per}}^\infty(Y, Z))$ is dense in $L^q(\Omega, Y, Z)$ so

$$F_\varepsilon(\varphi) = \int_{\Omega} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx$$

defines a bounded sequence $\{F_\varepsilon\}$ of linear functionals on $L^q(\Omega, Y, Z)$. Since $L^p(\Omega, Y, Z)$ is separable, there exists a subsequence of $\{\varepsilon\}$ and a bounded linear functional F_0 such that

$$F_\varepsilon(\varphi) \rightarrow F_0(\varphi) \quad \text{for all } \varphi \in L^q(\Omega, Y, Z).$$

Finally, Riesz Representation Theorem gives the existence of a function u_0 such that (4.1) is satisfied.

Step 3. Let $u_\varepsilon \in L^p(\Omega)$ and suppose u_ε 3-scale converges to u_0 . Then

$$u_\varepsilon \rightarrow \frac{1}{|Y||Z|} \int_{Y \times Z} u_0(x, y, z) dy dz$$

weakly in $L^p(\Omega)$. This is seen by taking in (4.1) test functions φ , which depends only on x .

Step 4. Let $u_\varepsilon \rightarrow u_0$ weakly in $W_0^{1,p}(\Omega)$. Then

(i) u_ε 3-scale converges to u_0 ,

(ii) There exist $u_1 \in L^p(\Omega; W_{\text{per}}^{1,p}(Y))$ and $u_2 \in L^p(\Omega; W_{\text{per}}^{1,p}(Y \times Z))$ such that Du_ε 3-scale converges to $Du_0 + D_y u_1 + D_z u_2$.

Proof. By Step 2 (extension of Step 2 to vectors is immediate) there exist $\tilde{u}_0(x, y, z)$ and $\chi_0(x, y, z)$ such that, up to a subsequence, u_ε and Du_ε 3-scale converge to $\tilde{u}_0$ and χ_0 , respectively. Using Step 2 in the identity

$$\varepsilon^2 \int_{\Omega} (Du_\varepsilon) \Phi \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx = - \int_{\Omega} u_\varepsilon (\varepsilon^2 \operatorname{div}_x \Phi + \varepsilon \operatorname{div}_y \Phi + \operatorname{div}_z \Phi) dx,$$

we obtain in the limit that

$$0 = - \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (u_0 \operatorname{div}_z \Phi) dx dy dz.$$

Thus u_0 is independent of z . Letting Φ be independent of the third variable we obtain the identity

$$\varepsilon \int_{\Omega} (Du_\varepsilon) \Phi \left(x, \frac{x}{\varepsilon} \right) dx = - \int_{\Omega} u_\varepsilon (\varepsilon \operatorname{div}_x \Phi + \operatorname{div}_y \Phi) dx,$$

which in the limit yields the identity

$$0 = - \frac{1}{|Y|} \int_{\Omega \times Y} (u_0 \operatorname{div}_y \Phi) dx dy.$$

Thus u_0 is independent of y . Next we choose $\Phi(x, y, z)$ such that

$$\operatorname{div}_y \Phi = 0, \quad \operatorname{div}_z \Phi = 0.$$

Then, the identity

$$\int_{\Omega} u_\varepsilon \operatorname{div}_x \Phi dx = - \int_{\Omega} (Du_\varepsilon) \Phi \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx$$

gives

$$\frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} u_0 \operatorname{div}_x \Phi dx dy dz = - \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} \chi_0 \Phi(x, y, z) dx dy dz.$$

Hence we have the existence of u_1 and u_2 such that

$$\chi_0 = Du_0 + D_y u_1 + D_z u_2.$$

This is seen by using the fact that the orthogonal of divergence-free functions are exactly the gradients.

Step 5. Let w_ε be defined as

$$w_\varepsilon = w_0(x) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right).$$

Then the monotonicity assumption on a implies that

$$\int_{\Omega} \left(a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_\varepsilon\right) - a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Dw_\varepsilon\right), D(u_\varepsilon - w_\varepsilon) \right) dx \geq 0,$$

i.e.

$$\int_{\Omega} f(u_\varepsilon - w_\varepsilon) dx - \int_{\Omega} \left(a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Dw_\varepsilon\right), D(u_\varepsilon - Dw_\varepsilon) \right) dx \geq 0. \quad (4.4)$$

Since $w_0(x)$, $w_1(x, y)$, $w_2(x, y, z)$ are given, one can pass to the limit in (4.4) and we obtain that

$$\begin{aligned} \int_{\Omega} f(u_0 - w_0) dx - \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (a(x, y, z, Dw_0 + D_y w_1 \\ + D_z w_2), D(u_0 - w_0) + D_y(u_1 - w_1) + D_z(u_2 - w_2)) dx dy dz \geq 0. \end{aligned} \quad (4.5)$$

We use then the standard trick of monotonicity, we take in (4.5)

$$w_i = u_i - \lambda v_i, \quad \lambda > 0,$$

where $v_0 = v_0(x)$, $v_1 = v_1(x, y)$ and $v_2 = v_2(x, y, z)$. After dividing by λ we obtain that

$$\begin{aligned} \int_{\Omega} f v_0 dx - \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (a(x, y, z, Du_0 + D_y u_1 + D_z u_2 \\ - \lambda(Dv_0 + D_y v_1 + D_z v_2)), Dv_0 + D_y v_1 + D_z v_2) dx dy dz \geq 0. \end{aligned}$$

By letting $\lambda \rightarrow 0$ we obtain that

$$\int_{\Omega} f v_0 dx = \frac{1}{|Y||Z|} \int_{\Omega \times Y \times Z} (a(x, y, z, Du_0 + D_y u_1 + D_z u_2), Dv_0 + D_y v_1 + D_z v_2) dx dy dz$$

and the proof is completed.

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