Interpolation of weakly compact operators

By

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Let $\bar{A} = (A_0, A_1)$ be a Banach pair, i.e., A_0 and A_1 are Banach spaces continuously imbedded into a Hausdorff topological vector space. Define, as usual, the spaces $\Delta(\bar{A}) = A_0 \cap A_1$ and $\Sigma(\bar{A}) = A_0 + A_1$ with the norms $\|a\|_{\Delta} = \max(\|a\|_{A_0}, \|a\|_{A_1})$ and $\|a\|_{\Sigma} = K(1, a)$, where for $t > 0$,

$$K(t, a) = \inf \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1 \}.$$

For $0 < \theta < 1$ and $1 \leq p < \infty$, the Lions-Peetre interpolation space $\bar{A}_{\theta, p} = (A_0, A_1)_{\theta, p}$ is the space of all $a \in \Sigma(\bar{A})$ for which the norm

$$\|a\|_{\theta, p} = \left(\sum_{n \in \mathbb{Z}} (2^{-\theta n} K(2^n, a))^p \right)^{1/p}$$

is finite (cf. [2]). Let us note that $\mu_n(a) \leq K(2^n, a) \leq 2\mu_n(a)$, where μ_n is the Minkowski functional of the set $V_n = B_{A_0} + 2^{-n} B_{A_1}$ (B_{A_i} is the unit ball of A_i , $i = 0, 1$), i.e., $\mu_n(a) = \inf \{ \alpha > 0 : a \in \alpha V_n \}$. Hence $\bar{A}_{\theta, p}$ consists of those $a \in \Sigma(\bar{A})$ for which the norm

$$\|a\|_{\theta, p}^1 = \left(\sum_{n \in \mathbb{Z}} (2^{-\theta n} \mu_n(a))^p \right)^{1/p}$$

is finite. If in the above definitions sums are only over natural n then we have the definition of the space $\bar{A}_{\theta, p}^+$. Always $\bar{A}_{\theta, p} \subset \bar{A}_{\theta, p}^+$ and if $A_0 \subset A_1$, then $\bar{A}_{\theta, p} = \bar{A}_{\theta, p}^+$.

Now let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach pairs. We say that a linear operator T is bounded from the pair $\bar{A}$ into the pair $\bar{B}$, and write $T: \bar{A} \rightarrow \bar{B}$, if $T: \Sigma(\bar{A}) \rightarrow \Sigma(\bar{B})$ is a bounded linear operator and the restriction of T to the space A_i is a bounded operator from A_i into B_i , $i = 0, 1$.

From the construction of the Lions-Peetre spaces we have that if $T: \bar{A} \rightarrow \bar{B}$, then $T: \bar{A}_{\theta, p} \rightarrow \bar{B}_{\theta, p}$ is bounded and

$$\|T\|_{\bar{A}_{\theta, p} \rightarrow \bar{B}_{\theta, p}} \leq \|T\|_{A_0 \rightarrow B_0}^{1-\theta} \|T\|_{A_1 \rightarrow B_1}^{\theta}.$$

The following result can be regarded as a generalization of the theorem of Beauzamy [1] about the reflexivity of $\bar{A}_{\theta, p}$. Moreover, our proof will be simpler.

Theorem 1. *Let $1 < p < \infty$ and let $T: \bar{A} \rightarrow \bar{B}$. Then $T: \bar{A}_{\theta, p} \rightarrow \bar{B}_{\theta, p}$ is weakly compact if and only if $T: \Delta(\bar{A}) \rightarrow \Sigma(\bar{B})$ is weakly compact.*

The proof will follow from two Propositions, which themselves are of independent interest. Before we do this let us recall the DFJP construction. Let X be a Banach space and let W be a convex symmetric, norm bounded subset of X . For each n put $U_n = W + 2^{-n} B_X$ and denote by j_n the Minkowski functional of U_n . For $0 < \theta < 1$ and $1 < p < \infty$ set $Y = Y_{\theta,p} = \{x \in X: \|x\| = (\sum_{n \in \mathbb{N}} (2^{-\theta n} j_n(x))^p)^{1/p} < \infty\}$, and let $J: Y \rightarrow X$ denote the natural inclusion. Then (cf. [4], where in fact the space $Y_{1/2,2}$ is considered:

1. Y is a Banach space and J is continuous;
2. W is a subset of $J(B_Y)$;
3. $J^{**}: Y^{**} \rightarrow X^{**}$ is one to one and $J^{**^{-1}}(X) = Y$;
4. The $\sigma(X^{**}, X^*)$ -closure of $J(B_Y)$ is $J^{**}(B_{Y^{**}})$;
5. Y is reflexive if and only if W is a relatively weakly compact subset of X .

We refer to Y as the space of Davis, Figiel, Johnson and Pelczynski or DFJP space in short. Another important property associated with the space Y , needed for our work, is described in the next lemma.

Lemma 2. *Let T be a continuous operator from X into any Banach space Z . Then $S = TJ: Y \rightarrow Z$ is weakly compact if and only if $T(W)$ is relatively weakly compact in Z .*

P r o o f. Since $W \subset J(B_Y)$ it follows that $T(W) \subset TJ(B_Y) = S(B_Y)$ and if S is weakly compact then $T(W)$ is relatively weakly compact.

Suppose now that $T(W)$ is relatively weakly compact in Z .

First of all, the $\sigma(Z^{**}, Z^*)$ -closure of $S(B_Y)$ is $S^{**}(B_{Y^{**}})$. In fact,

$$\begin{aligned} \overline{S(B_Y)} &= \overline{S^{**}(B_Y)} \subset \overline{S^{**}(B_{Y^{**}})} = S^{**}(B_{Y^{**}}) \subset T^{**}(\overline{J(B_Y)})^{\sigma(X^{**}, X^*)} \\ &= T^{**}(\overline{J^{**}(B_Y)})^{\sigma(X^{**}, X^*)} \quad [T^{**} \text{ is } \omega^* \text{ continuous}] \\ &\subset \overline{S^{**}(B_Y)} = \overline{S(B_Y)}. \end{aligned}$$

Now, we have $J(B_Y) \subset 2^{\theta n} U_n = 2^{\theta n} W + 2^{(\theta-1)n} B_X$ and so

$$\begin{aligned} TJ(B_Y) &\subset T(2^{\theta n} U_n) = 2^{\theta n} T(W) + 2^{(\theta-1)n} T(B_X) \quad [\text{from the continuity of } T] \\ &\subset 2^{\theta n} T(W) + 2^{(\theta-1)n} M B_Z \\ &\subset 2^{\theta n} \overline{T(W)}^{\sigma(Z, Z^*)} + 2^{(\theta-1)n} M B_{Z^{**}} := W_n \end{aligned}$$

for every n . Since $\overline{T(W)}^{\sigma(Z, Z^*)}$ is weakly compact then it is also compact in the $\sigma(Z^{**}, Z^*)$ topology and the sets W_n are $\sigma(Z^{**}, Z^*)$ closed and contain $S(B_Y)$, therefore they contain $S^{**}(B_{Y^{**}})$. Thus

$$\begin{aligned} S^{**}(B_{Y^{**}}) &\subset \bigcap_n (2^{\theta n} \overline{T(W)} + 2^{(\theta-1)n} M B_{Z^{**}}) \\ &\subset \bigcap_n (Z + 2^{(\theta-1)n} M B_{Z^{**}}) = \overline{Z}^{Z^{**}} = Z. \end{aligned}$$

Hence $S^{**}(Y^{**}) \subset Z$ and so $S = TJ$ is weakly compact.

As a consequence of the Lemma 2 we have

Proposition 3. *Let $T: \Sigma(\bar{A}) \rightarrow Z$ be a continuous operator. The following are equivalent:*

- (a) $T: \Delta(\bar{A}) \rightarrow Z$ is weakly compact.
- (b) For all $0 < \theta < 1$ and all $1 < p < \infty$, $T: (\Delta(\bar{A}), \Sigma(\bar{A}))_{\theta, p} \rightarrow Z$ is weakly compact.
- (c) For all $0 < \theta < 1$ and all $1 < p < \infty$, $T: \bar{A}_{\theta, p} \rightarrow Z$ is weakly compact.

Proof. (a) $\Rightarrow$ (b). The interpolation space $(\Delta(\bar{A}), \Sigma(\bar{A}))_{\theta, p}$ is a DFJP space constructed from $X = \Sigma(\bar{A})$ and $W = I(B_{\Delta(\bar{A})})$, where I is the inclusion $I: \Delta(\bar{A}) \rightarrow \Sigma(\bar{A})$. Using Lemma 2 we have the above implication.

(b) $\Rightarrow$ (c). The interpolation space $\bar{A}_{\theta, p}$ is not, in general, a DFJP space but such as is proved in [9] they are imbedded into $(\Delta(\bar{A}), \Sigma(\bar{A}))_{\theta, p}$, i.e., if $1/2 \leq \theta < 1$ then

$$(\Delta(\bar{A}), \Sigma(\bar{A}))_{\theta, p} = (\Sigma(\bar{A}), \Delta(\bar{A}))_{1-\theta, p} = \bar{A}_{\theta, p} + \bar{A}_{1-\theta, p}$$

and so $\bar{A}_{\theta, p}$ and $\bar{A}_{1-\theta, p}$ are in $(\Delta(\bar{A}), \Sigma(\bar{A}))_{\theta, p}$.

That (c) implies (a) is obvious, because $\bar{A}_{\theta, p} \supset \Delta(\bar{A})$.

Remark 4. Lemma 2 and Proposition 3 are also true for compact, weakly precompact (= Rosenthal) and Banach-Saks operators. Moreover the equivalence between (a) and (b) is true even for surjective closed operator ideals (see [6], Prop. 1.7).

Proposition 5. *Let $1 < p < \infty$ and let $T: Z \rightarrow \bar{B}_{\theta, p}$ be a continuous operator. Then T is weakly compact if and only if $T: Z \rightarrow \Sigma(\bar{B})$ is weakly compact.*

Proof. Since the necessity is clear, we will only prove sufficiency. First, we prove that if $1 < p < \infty$ then the imbedding $J: \bar{B}_{\theta, p} \rightarrow \Sigma(\bar{B})$ is a Tauberian operator, i.e., $J^{**}b^{**} \in \Sigma(\bar{B})$ implies $b^{**} \in \bar{B}_{\theta, p}$ (see also [1], p. 28).

Let $B = \left(\sum_{n \in \mathbb{Z}} \oplus \Sigma_n \right)_p$, where $\Sigma_n = \Sigma(\bar{B})$ with the norm $\|b\|_n = 2^{-\theta n} K(2^n, b)$. Then $J = PR$, where $R: \bar{B}_{\theta, p} \rightarrow B$ is the into isometry given by $R(b) = (\dots, b, b, b, \dots)$, $P: B \rightarrow \Sigma_0 = \Sigma(\bar{B})$ is the zero coordinate projection. Then $B^{**} = \left(\sum_{n \in \mathbb{Z}} \oplus \Sigma_n^{**} \right)_p$ and $P^{**}: B^{**} \rightarrow \Sigma_0^{**}$ is again the zero coordinate projection. Suppose that $J^{**}b^{**} \in \Sigma(\bar{B})$. Then $P^{**}R^{**}b^{**} \in \Sigma_0$ and $R^{**}b^{**} = (\dots, d_{-1}^{**}, d, d_1^{**}, d_2^{**}, \dots)$ where $d \in \Sigma_0$. There exists a net (b_α) such that (Rb_α) converges in the weak* topology to $R^{**}b^{**}$. Since every coordinate of (Rb_α) is equal and each Σ_n^{**} is isomorphic to $\Sigma(\bar{B})^{**}$, $R^{**}b^{**} = (\dots, d, d, d, \dots) = Rd \in B$. Since R^{**} is one to one, $b^{**} = d \in \bar{B}_{\theta, p}$.

Now if $JT(B_Z)$ is relatively weakly compact in $\Sigma(\bar{B})$ then the weak*-closure of $JT(B_Z)$ in $\Sigma(\bar{B})^{**}$ is contained in $\Sigma(\bar{B})$, i.e., $\widetilde{JT(B_Z)} \subset \Sigma(\bar{B})$. Since J^{**} is weak*-to-weak* continuous,

$$J^{**}(\widetilde{TB_Z}) \subset \widetilde{JT(B_Z)} \subset \Sigma(\bar{B}).$$

Since J is Tauberian, $\widetilde{TB_Z} \subset \bar{B}_{\theta, p}$. But TB_Z is bounded and so TB_Z is relatively weakly compact in $\bar{B}_{\theta, p}$.

The last property characterizes Tauberian operators (cf. [7], Th. 3.2 and [10], Th. 1.4).

Proof of Theorem 1. If $T: \bar{A} \rightarrow \bar{B}$ is such that $T: \Delta(\bar{A}) \rightarrow \Sigma(\bar{B})$ is weakly compact, then $T: \bar{A}_{\theta,p} \rightarrow \bar{B}_{\theta,p}$ is bounded and by Proposition 3 $T: \bar{A}_{\theta,p} \rightarrow \Sigma(\bar{B})$ is weakly compact. Then by Proposition 5 $T: \bar{A}_{\theta,p} \rightarrow \bar{B}_{\theta,p}$ is weakly compact.

The converse implication is obvious because $\bar{A}_{\theta,p} \supset \Delta(\bar{A})$ and $\bar{B}_{\theta,p} \subset \Sigma(\bar{B})$.

Remarks 6. Proposition 5 and Theorem 1 are also true for weakly precompact operators. Theorem 1 is true even for properties which are preserved by Proposition 3 and Tauberian operators.

7. The techniques used above also work for the (more general) method of interpolation with a function parameter. We refer to [5] for details of this method.

If in our Theorem the operator is taken to be the identity, then we have the result of Beauzamy. Namely, $\bar{A}_{\theta,p}$ ($0 < \theta < 1$, $1 < p < \infty$) is reflexive if and only if the imbedding $I: \Delta(\bar{A}) \rightarrow \Sigma(\bar{A})$ is weakly compact.

Another proof of this result is the following:

If the imbedding I is weakly compact, then the DFJP space $(\Delta(\bar{A}), \Sigma(\bar{A}))_{1/2,p}$ is reflexive (see [4]). But from [9] we have that $(\Delta(\bar{A}), \Sigma(\bar{A}))_{1/2,p} = \bar{A}_{1/2,p}$ and from the reiteration theorem for the complex interpolation space (see [2], Th. 4.7.2)

$$\bar{A}_{\theta,p} = [\bar{A}_{\theta_0,p}, \bar{A}_{1/2,p}]_{\theta_1}$$

for some θ_0 and θ_1 from $(0, 1)$. Now, using the above and Calderón's result (if either A_0 or A_1 is reflexive space then the complex interpolation space $[A_0, A_1]_{\theta}$ is also reflexive—see [3], Th. 12.2) we have that $\bar{A}_{\theta,p}$ is reflexive.

Let us note that in the nontrivial case, i.e. when the imbedding I is nonclosed, the space $\bar{A}_{\theta,1}$ is nonreflexive. Indeed, by Levy's [8] result it contains isomorphic copy of ℓ^1 .

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