Positive bilinear operators in Calderón-Lozanovskii spaces

By

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Abstract. A generalization of an abstract Hölder-Rogers inequality for positive bilinear operators is proved. Then it is used in the theory of interpolation of operators. An interpolation theorem for positive bilinear operators between Calderón-Lozanovskii spaces holds if and only if the parameter functions generating those spaces satisfy a generalized C-supermultiplicativity condition (2). In the case when all generating functions are the same this condition is exactly the same as the C-supermultiplicativity condition on the function.

0. Introduction. For a positive linear operator T on $L^0(\mu)$ it is well-known that an abstract Hölder-Rogers inequality

$$T(|x||y|) \leq [T(|x|^p)]^{1/p} [T(|y|^q)]^{1/q},$$

holds for any $x, y \in L^0(\mu)$, where $1/p + 1/q = 1$, $p > 1$, and it can be generalized to the “function” version

$$T(\varphi(|x|, |y|)) \leq \varphi(T(|x|), T(|y|)).$$

We will extend the last inequality to positive bilinear operators and use it in the proof of bilinear interpolation result.

There are well known bilinear interpolation results for the real method of interpolation and also for the complex method even for general Banach couples (cf. [5]). For the Calderón-Lozanovskii spaces interpolation results for arbitrary linear (not only positive) operators are also known but for positive bilinear operators there is only a recent result of Astashkin [3] with some restrictions on the function generating the space and the Fatou property on the spaces in the image.

The next purpose of this paper is to find necessary and sufficient conditions on φ under which we have interpolation results for positive bilinear operators in Calderón-Lozanovskii spaces. We will see that the additional assumptions on indices and the Fatou property on spaces, in the Astashkin result, are superfluous.

The content of the paper is as follows. In Section 1 we define the Calderón-Lozanovskiĭ spaces on the class of Banach function lattices.

Section 2 contains pointwise estimates for positive bilinear operators. Two proofs, each of independent interest, are presented there. The result obtained here is then applied to prove the interpolation property of different Calderón-Lozanovskiĭ spaces for positive bilinear operators.

In Section 3 we prove that the Calderón-Lozanovskiĭ spaces generated by different functions have the interpolation property for positive bilinear operators if and only if these parameter functions satisfy a generalized C-supermultiplicativity condition (2). In the case when all generating functions are the same $\varphi_0 = \varphi_1 = \varphi$ this condition is exactly the same as the C-supermultiplicativity condition on the function φ and our result improves the recent theorem presented by Astashkin [3].

1. Calderón-Lozanovskiĭ spaces. Let (Ω, μ) be a complete σ -finite measure space and let $L^0(\mu)$ denote, as usual, the space of all equivalence classes of measurable functions on Ω with the topology of convergence in measure on μ -finite sets. The order $|x| \leq |y|$ means that $|x(t)| \leq |y(t)|$ for μ -almost all $t \in \Omega$.

If a normed subspace E of $L^0(\mu)$ is such that there exists $u \in E$ with $u > 0$ μ -a.e. on Ω and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, we say that E is a *normed function lattice* (on Ω or on (Ω, μ)). A normed function lattice E has the *Fatou property* if $0 \leq x_n \uparrow x$, $x_n \in E$ and $\sup_n \|x_n\|_E < \infty$ implies $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$.

Recall that if $\bar{E} = (E_0, E_1)$ is a pair of Banach function lattices on Ω and $\mathcal{U}$ denotes the set of all non-negative concave and positively homogeneous continuous functions $\varphi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(0, 0) = 0$. Then the *Calderón-Lozanovskiĭ space* $\varphi(\bar{E}) = \varphi(E_0, E_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \leq \lambda \varphi(|x_0|, |x_1|)$ for some $x_i \in E_i$, with $\|x_i\|_{E_i} \leq 1$, $i = 0, 1$. The spaces $\varphi(\bar{E})$ are Banach function lattices equipped with the norms

$$\|x\|_\varphi = \inf\{\lambda > 0; |x| \leq \lambda \varphi(|x_0|, |x_1|), \|x_0\|_{E_0} \leq 1, \|x_1\|_{E_1} \leq 1\}$$

(see [13]). In the case of the power functions $\varphi_\theta(s, t) = s^{1-\theta}t^\theta$ with $0 \leq \theta \leq 1$, $\varphi_\theta(\bar{E})$ are the well known Calderón spaces $E_0^{1-\theta}E_1^\theta$ (see [7]). Moreover, the particular case of $E^{1/p}(L^\infty)^{1-1/p} = E^{(p)}$ for $1 < p < \infty$ is known as the p -convexification of E (see [12]).

The properties of $\varphi(\bar{E})$ were studied by Lozanovskiĭ in [13] and [14] (see also [16]), where among other facts it is proved the Köthe duality result

$$\varphi(E_0, E_1)' = \hat{\varphi}(E_0', E_1')$$

with equivalent norms. Here, for $\varphi \in \mathcal{U}$, the conjugate function $\hat{\varphi}$ is defined by

$$\hat{\varphi}(s, t) := \inf \left\{ \frac{\alpha s + \beta t}{\varphi(\alpha, \beta)}; \alpha, \beta > 0 \right\}, s, t \geq 0.$$

We have $\hat{\varphi} \in \mathcal{U}$ and $\hat{\hat{\varphi}} = \varphi$ (see [14], Lemma 2, [15], Lemma 2 and [16], Lemma 15.8).

Observe also that if $M : [0, \infty) \rightarrow [0, \infty]$ is a nondecreasing, convex and left-continuous function not identical 0 or ∞ on $(0, \infty)$ with $M(0) = 0$, and $\varphi \in \mathcal{U}$ is defined by $\varphi(s, t) = tM^{-1}(s/t)$ if $t > 0$ and 0 if $t = 0$, where M^{-1} is the right continuous inverse of M , then, for any Banach lattice E on Ω , the Calderón-Lozanovskiĭ space $\varphi(E, L_\infty)$ coincides isometrically with the Banach lattice

$$E_M = \{x \in L^0(\mu); M(|x|/\lambda) \in E \text{ for some } \lambda > 0\}$$

equipped with the norm

$$\|x\|_{E_M} = \inf \{\lambda > 0; \|M(|x|/\lambda)\|_E \leq 1\}.$$

In particular, $\varphi(L_1, L_\infty)$ coincides isometrically with the Orlicz space L_M (see [6], [16], [19]).

The subset of functions in $\mathcal{U}$ for which $\varphi(s, 1) \rightarrow 0$ as $s \rightarrow 0^+$ and $\varphi(1, t) \rightarrow 0$ as $t \rightarrow 0^+$ is denoted by $\mathcal{U}_0$. The subset of functions in $\mathcal{U}_0$ for which $\bar{\varphi}(s, 1) \rightarrow 0$ as $s \rightarrow 0^+$ and $\bar{\varphi}(1, t) \rightarrow 0$ as $t \rightarrow 0^+$, where $\bar{\varphi}(s, t) = \sup_{u, v > 0} \frac{\varphi(su, tv)}{\varphi(u, v)}$ is denoted by $\mathcal{U}^\pm$.

Note that $\varphi \in \mathcal{U}^\pm$ if and only if the dilation indices (Orlicz-Matuszewska indices) of $\rho = \varphi(1, \cdot)$ are strictly between 0 and 1, i. e., $0 < \alpha_\rho \leq \beta_\rho < 1$ (cf. [8], [16]).

2. Pointwise estimates for positive bilinear operators. Let us start with the result for positive linear operators which was used, independently by Bereznoi (1981), Shestakov (1982) and Maligranda (1985), in the proof that the Calderón-Lozanovskiĭ spaces are exact interpolation spaces for positive linear operators (see [4], [21], [15]; see also [16], Th. 15.13). Since the result below was used but not written explicitly we include here a proof for the sake of completeness.

Lemma 1. *Let E be either $L^0(\mu_1)$ or a normed function lattice on (Ω_1, μ_1) and let $T : E \rightarrow L^0(\mu)$ be a positive linear operator. If $\varphi \in \mathcal{U}$, then*

$$(1) \quad T(\varphi(|x|, |y|)) \leq \varphi(T(|x|), T(|y|))$$

for any $x, y \in E$.

Proof. Since

$$\varphi(|x|, |y|) \leq \frac{a|x| + b|y|}{\hat{\varphi}(a, b)} \quad (a > 0, b > 0)$$

it follows that

$$T(\varphi(|x|, |y|)) \leq \frac{aT(|x|) + bT(|y|)}{\hat{\varphi}(a, b)}$$

for arbitrary $a, b > 0$ and so by the fact that operation “ $\hat{\cdot}$ ” is an involution on $\mathcal{U}$

$$T(\varphi(|x|, |y|)) \leq \hat{\hat{\varphi}}(T(|x|), T(|y|)) = \varphi(T(|x|), T(|y|)). \quad \square$$

Remark 1. As an easy consequence of Lemma 1, for $\varphi(u, v) = u^{1/p}v^{1-1/p}$ with $1 < p < \infty$, we obtain the inequality

$$T(|x|^{1/p}|y|^{1-1/p}) \leq [T(|x|)]^{1/p}[T(|y|)]^{1-1/p}.$$

We can rewrite this as

$$|T(xy)| \leq T(|x||y|) \leq [T(|x|^p)]^{1/p}[T(|y|^q)]^{1/q},$$

where $1/p + 1/q = 1$. The last estimate is an abstract Hölder-Rogers inequality for positive linear operators, which is known in interpolation theory (see [8], p. 246) and in ergodic theory (see [9], Lemma 7.4).

Remark 2. Lemma 1 is true also for some nonlinear operators and functionals. An operator $T : E \rightarrow L^0(\mu)$ is called a *positive sublinear operator* if it is monotone, subadditive positively homogeneous, i. e., for any $x, y \in E$

$$|x| \leq y \Rightarrow |T(x)| \leq T(y), \quad |T(x+y)| \leq |T(x)| + |T(y)| \quad \text{and} \quad |T(\lambda x)| \leq \lambda |T(x)|.$$

We can easily see that for such operator T the estimate (1) is still true. Similarly, for the positive sublinear functional T on E (the definition is analogous) estimate of type (1) is again true. In particular, by taking $Tx = \|x\|_E$ we obtain that if $x, y \in E$, then $\varphi(|x|, |y|) \in E$ and

$$\|\varphi(|x|, |y|)\|_E \leq \varphi(\|x\|_E, \|y\|_E)$$

(see [15], Lemma 3(a), [16], Th. 15.9(a) and [11], p. 203; cf. also [20]).

Remark 3. Lemma 1 can be written also in the following form: Let E_0, E_1 and F_0, F_1 be normed function lattices on (Ω_1, μ_1) and (Ω, μ) , respectively. If $T : E_0 + E_1 \rightarrow F_0 + F_1$ is a positive linear operator, then estimate (1) holds for any $x \in E_0, y \in E_1$.

Before we extend Lemma 1 to the bilinear case we prove some properties of functions $\varphi \in \mathcal{U}$.

Lemma 2. If $\varphi \in \mathcal{U}$, then

- (i) $\varphi(s_2, t) - \varphi(s_1, t) \leq \varphi(s_2 - s_1, t) - \varphi(0, t)$ for all $s_2 > s_1 > 0$ and $t > 0$, where $\varphi(0, t) = \lim_{s \rightarrow 0^+} \varphi(s, t)$.
- (ii) $\varphi(s, t_2) - \varphi(s, t_1) \leq \varphi(s, t_2 - t_1) - \varphi(s, 0)$ for all $t_2 > t_1 > 0$ and $s > 0$, where $\varphi(s, 0) = \lim_{t \rightarrow 0^+} \varphi(s, t)$.

Proof. (i). If $\varphi \in \mathcal{U}_0$, then

$$\varphi(0, t) = \lim_{s \rightarrow 0^+} \varphi(s, t) = t \lim_{s \rightarrow 0^+} \varphi(s/t, 1) = 0$$

and the estimate follows directly from the subadditivity of φ in the first variable because the function φ is concave.

Assume now that $\varphi \in \mathcal{U} \setminus \mathcal{U}_0$. Put

$$\varphi_0(s, t) = \varphi(s, t) - a_\varphi s - b_\varphi t,$$

where

$$a_\varphi = \lim_{t \rightarrow 0^+} \varphi(1, t) \text{ and } b_\varphi = \lim_{s \rightarrow 0^+} \varphi(s, 1).$$

Then $\varphi \in \mathcal{U}_0$. The important here is to prove that $\varphi(s, t) \geq a_\varphi s + b_\varphi t$ for all $s, t > 0$. In fact, for $0 < u < v < w$ we have by the concavity of φ that

$$\frac{\varphi(1, w) - \varphi(1, v)}{w - v} \leq \frac{\varphi(1, v) - \varphi(1, u)}{v - u}.$$

Taking $u \rightarrow 0^+$ we obtain

$$\frac{\varphi(1, w) - \varphi(1, v)}{w - v} \leq \frac{\varphi(1, v) - a_\varphi}{v}$$

or

$$\varphi\left(\frac{1}{w}, 1\right) - \varphi\left(\frac{1}{w}, \frac{v}{w}\right) \leq \frac{w - v}{w} \frac{\varphi(1, v) - a_\varphi}{v}.$$

Taking $w \rightarrow \infty$ we get

$$b_\varphi \leq \frac{\varphi(1, v) - a_\varphi}{v}$$

and with $v = t/s$ we have

$$\frac{t}{s} b_\varphi \leq \frac{\varphi(1, v) - a_\varphi}{v}$$

or

$$sa_\varphi + tb_\varphi \leq \varphi(s, t).$$

For $\varphi \in \mathcal{U}_0$, by the subadditivity of φ_0 , we obtain

$$\begin{aligned} \varphi(s_2, t) - \varphi(s_1, t) &= \varphi_0(s_2, t) + a_\varphi s_2 + b_\varphi t - [\varphi_0(s_1, t) + a_\varphi s_1 + b_\varphi t] \\ &= \varphi_0(s_2, t) - \varphi_0(s_1, t) + a_\varphi(s_2 - s_1) \\ &\leq \varphi_0(s_2 - s_1, t) + a_\varphi(s_2 - s_1) \\ &= \varphi(s_2 - s_1, t) - a_\varphi(s_2 - s_1) - b_\varphi t + a_\varphi(s_2 - s_1) \\ &= \varphi(s_2 - s_1, t) - b_\varphi t \\ &= \varphi(s_2 - s_1, t) - \varphi(0, t) \end{aligned}$$

and the proof of (i) is finished. The proof of (ii) is similar. $\square$

Now we extend Lemma 1 to bilinear operators. We will give even two proofs of this interesting estimate.

Let E be either $L^0(\mu_1)$ or a normed function lattice on (Ω_1, μ_1) and let F be either $L^0(\mu_2)$ or a normed function lattice on (Ω_2, μ_2) . A bilinear operator $T : E \times F \rightarrow L^0(\mu)$ is called *positive* if $T(x, y) \geq 0$ whenever $0 \leq x \in E$ and $0 \leq y \in F$.

Theorem 1. *Let $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$. Assume that either $T : L^0(\mu_1) \times L^0(\mu_2) \rightarrow L^0(\mu)$ or $T : (E + F) \times (E + F) \rightarrow L^0(\mu)$ is a positive bilinear operator. If*

$$(2) \quad \varphi_0(1, s)\varphi_1(1, t) \leq C\varphi(1, st)$$

for some positive constant $C > 0$ and all $s, t > 0$, then

$$(3) \quad T(\varphi_0(|x_0|, |x_1|), \varphi_1(|y_0|, |y_1|)) \leq C\varphi(T(|x_0|, |x_1|), T(|y_0|, |y_1|))$$

for any $x_0, y_0 \in E$ and $x_1, y_1 \in F$.

First proof. We will use here Lemma 1 and Lemma 2. The idea of the first proof is coming from Shestakov [21].

It is enough to show (3) for any $0 \leq x_0, y_0 \in E$ and $0 \leq x_1, y_1 \in F$.

Let $y = y_0 + y_1$. Suppose first that y_0 and y_1 are *y-step functions*, i. e.,

$$y_0 = \sum_{k=1}^m \alpha_k z_k, \quad y_1 = \sum_{k=1}^m \beta_k z_k,$$

where $\alpha_k, \beta_k \geq 0, k = 1, 2, \dots, m, z_1, z_2, \dots, z_m$ are pairwise disjoint with $\sum_{k=1}^m z_k = y$ (cf. [1], p. 81). For example, as z_k we can take $z_k = y\chi_{A_k}$, where A_k are measurable and pairwise disjoint subsets of Ω such that $\bigcup_{k=1}^m A_k = \Omega$.

Since T is a positive homogeneous and additive in the second variable it follows that

$$\begin{aligned} T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) &= T\left(\varphi_0(x_0, x_1), \sum_{k=1}^m \varphi_1(\alpha_k, \beta_k)z_k\right) \\ &= \sum_{k=1}^m \varphi_1(\alpha_k, \beta_k)T(\varphi_0(x_0, x_1), z_k), \end{aligned}$$

and then by using Lemma 1 to the first variable with the second variable fixed, the assumption (2) and the concavity of φ we find that

$$\begin{aligned} T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) &\leq \sum_{k=1}^m \varphi_1(\alpha_k, \beta_k)\varphi_0(T(x_0, z_k), T(x_1, z_k)) \\ &\leq C \sum_{k=1}^m \varphi(\alpha_k T(x_0, z_k), \beta_k T(x_1, z_k)) \end{aligned}$$

$$\begin{aligned} &\leq C\varphi\left(\sum_{k=1}^m \alpha_k T(x_0, z_k), \sum_{k=1}^m \beta_k T(x_1, z_k)\right) \\ &= C\varphi(T(x_0, x_1), T(y_0, y_1)). \end{aligned}$$

If $0 \leq y_i \in Y_i, i = 0, 1$ are arbitrary, then we can find sequences $\{y_n^0\}, \{y_n^1\}$ of y -step functions such that

$$0 \leq y_n^i \leq y_i, \quad y_i - y_n^i \leq \frac{y}{n} \quad \text{for } n = 1, 2, \dots, \quad i = 0, 1.$$

Then, by Lemma 2,

$$\begin{aligned} 0 &\leq \varphi_1(y_0, y_1) - \varphi_1(y_n^0, y_n^1) \\ &= \varphi_1(y_0, y_1) - \varphi_1(y_n^0, y_1) + \varphi_1(y_n^0, y_1) - \varphi_1(y_n^0, y_n^1) \\ &\leq \varphi_1(y_0 - y_n^0, y_1) - \varphi_1(0, y_1) + \varphi_1(y_n^0, y_1 - y_n^1) - \varphi_1(y_0, 0) \\ &\leq \varphi_1\left(\frac{y}{n}, y_1\right) - \varphi_1(0, y_1) + \varphi_1\left(y_n^0, \frac{y}{n}\right) - \varphi_1(y_0, 0) \\ &\leq \lambda_n(y_0 + y_1), \end{aligned}$$

with

$$\lambda_n = \varphi_1\left(\frac{1}{n}, 1\right) - \varphi_1(0, 1) + \varphi_1\left(1, \frac{1}{n}\right) - \varphi_1(1, 0).$$

Thus

$$\varphi_1(y_0, y_1) \leq \varphi_1(y_n^0, y_n^1) + \lambda_n(y_0 + y_1),$$

and by the first part of the proof and the chosen sequences

$$\begin{aligned} T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) &\leq T(\varphi_0(x_0, x_1), \varphi_1(y_n^0, y_n^1)) \\ &\quad + \lambda_n T(\varphi_0(x_0, x_1), y_0 + y_1) \\ &\leq C\varphi(T(x_0, x_1), T(y_n^0, y_n^1)) + \lambda_n T(\varphi_0(x_0, x_1), y_0 + y_1) \\ &\leq C\varphi(T(x_0, x_1), T(y_0, y_1)) + \lambda_n T(\varphi_0(x_0, x_1), y_0 + y_1). \end{aligned}$$

Since by the continuity of the function φ_1 the sequence $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ it follows that

$$T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) \leq C\varphi(T(x_0, x_1), T(y_0, y_1))$$

and the proof is complete.

Second proof. The idea of the second proof is similar to the one of Lemma 1. Let $0 \leq x_0, y_0 \in E$ and $0 \leq x_1, y_1 \in F$. Then, using the definition of $\hat{\varphi}_1$, the positivity and bilinearity of T , and Lemma 1, we have for any $a, b > 0$,

$$\begin{aligned} T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) &\leq T\left(\varphi_0(x_0, x_1), \frac{ay_0 + by_1}{\hat{\varphi}_1(a, b)}\right) \\ &= \frac{a}{\hat{\varphi}_1(a, b)} T(\varphi_0(x_0, x_1), y_0) + \frac{b}{\hat{\varphi}_1(a, b)} T(\varphi_0(x_0, x_1), y_1) \end{aligned}$$

$$\begin{aligned} &\leq \frac{a}{\hat{\varphi}_1(a, b)} \varphi_0(T(x_0, y_0), T(x_1, y_0)) \\ &\quad + \frac{b}{\hat{\varphi}_1(a, b)} \varphi_0(T(x_0, y_1), T(x_1, y_1)) = A. \end{aligned}$$

Since

$$\begin{aligned} \frac{\varphi_0(u, v)}{\hat{\varphi}_1(a, b)} &= \sup_{s, t > 0} \frac{\varphi_0(u, v) \varphi_1(s, t)}{as + bt} \leq C \sup_{s, t > 0} \frac{\varphi(su, tv)}{sa + tb} \\ &= \frac{C}{\inf_{s, t > 0} \frac{sa + tb}{\varphi(su, tv)}} = \frac{C}{\hat{\varphi}(a/u, b/v)} \end{aligned}$$

it follows that

$$\begin{aligned} A &\leq C \left[\frac{a}{\hat{\varphi}(a/T(x_0, y_0), b/T(x_1, y_0))} + \frac{b}{\hat{\varphi}(a/T(x_0, y_1), b/T(x_1, y_1))} \right] \\ &\leq C \left[\frac{a_0 T(x_0, y_0) + b_0 \epsilon_0}{\hat{\varphi}(a_0, b_0)} + \frac{b_1 T(x_1, y_1) + a_1 \epsilon_1}{\hat{\varphi}(a_1, b_1)} \right], \end{aligned}$$

where $a_i = a/T(x_0, y_i)$, $b_i = b/T(x_1, y_i)$, $i = 0, 1$ and $\epsilon_0, \epsilon_1 > 0$ are arbitrary.

Taking the infimum over all $a_0, b_0, a_1, b_1 > 0$ and using the facts that the operation " $\hat{\cdot}$ " is an involution on $\mathcal{U}$ and φ is a concave function we obtain

$$\begin{aligned} A &\leq C[\hat{\varphi}(T(x_0, y_0), \epsilon_0) + \hat{\varphi}(\epsilon_1, T(x_1, y_1))] \\ &= C[\varphi(T(x_0, y_0), \epsilon_0) + \varphi(\epsilon_1, T(x_1, y_1))] \\ &\leq C\varphi(T(x_0, y_0) + \epsilon_0, T(x_1, y_1) + \epsilon_1). \end{aligned}$$

Since φ is continuous and $\epsilon_0, \epsilon_1 > 0$ were arbitrary we get the estimate (3) and the proof is again finished.

Remark 4. Theorem 1 can be also proved for positive multilinear (n -linear) operators with the following assumption on the functions in $\mathcal{U}$:

$$(4) \quad \varphi_0(1, s_0) \varphi_1(1, s_1) \cdots \varphi_{n-1}(1, s_{n-1}) \leq C \varphi(1, s_0 s_1 \cdots s_{n-1})$$

for some $C > 0$ and all $s_0, s_1, \dots, s_{n-1} > 0$.

3. Interpolation of positive bilinear operators on Calderón-Lozanovskiĭ spaces.

As we have already mentioned before the Calderón-Lozanovskiĭ spaces are exact interpolation spaces for positive linear operators (see Bereznoi [4], Shestakov [21], Maligranda [15]; cf. also [16], Th. 15.13). Moreover,

$$\|T\|_{\varphi(\bar{E}) \rightarrow \varphi(\bar{F})} \leq \bar{\varphi}(\|T\|_{E_0 \rightarrow F_0}, \|T\|_{E_1 \rightarrow F_1}),$$

for any $T \in L(\{E_0, E_1\}, \{F_0, F_1\})$.

For arbitrary linear operators (not necessary positive) the history of interpolation results is the following (cf. [8], [10], [16]): Lozanovskii proved in 1972 that $\varphi(\cdot)$ is not an interpolation method even for $\varphi(s, t) = s^{1-\theta}t^\theta$. Moreover, Calderón (1964), Zabreiko (1967), Shestakov (1974) and Lozanovskii (1976, 1979) have been given some conditions under which $E_0^{1-\theta}E_1^\theta$ is an interpolation method. For the case $\varphi(E_0, E_1)$ these results were proved by Ovchinnikov (1976, 1984), Berezhnoi (1980), Shestakov (1981), Maligranda (1985, 1989), Brudnyi-Krugljak (1991), Kruglyak-Maligranda-Persson (1993), and moreover for the class of quasi-Banach function lattices by Ovchinnikov (1982) and Nilsson (1985).

Astashkin [3] started in 1997 with a result for positive bilinear operators between the Calderón-Lozanovskii spaces generated by some function $\varphi \in \mathcal{U}^\pm$ and the Fatou property of the spaces in the image. In fact, he announced the bilinear case at the conference in 1995 (cf. [2], Th. 2). His method of the proof was based on a calculation of the positive orbit. We generalize his result to any $\varphi \in \mathcal{U}$ (even to three functions $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$) and to general Banach function lattices. Our proof here is different.

Theorem 2. *Let $\bar{E} = (E_0, E_1)$, $\bar{F} = (F_0, F_1)$, $\bar{G} = (G_0, G_1)$ be couples of Banach function lattices on the measure spaces (Ω_1, μ_1) , (Ω_2, μ_2) and (Ω, μ) , respectively. Assume that $T : (E_0 + E_1) \times (F_0 + F_1) \rightarrow G_0 + G_1$ is a positive bilinear operator such that the restriction $T : E_i \times F_i \rightarrow G_i$ is bounded for $i = 0, 1$. If $\varphi_0, \varphi_1, \varphi \in \mathcal{U}$ are such that*

$$(5) \quad \varphi_0(1, s)\varphi_1(1, t) \leq C\varphi(1, st)$$

for some positive constant $C > 0$ and all $s, t > 0$, then

$$(6) \quad T : \varphi_0(\bar{E}) \times \varphi_1(\bar{F}) \rightarrow \varphi(\bar{G})$$

is bounded and

$$(7) \quad \|T\|_{\varphi(\bar{E}) \times \varphi(\bar{F}) \rightarrow \varphi(\bar{G})} \leq C\bar{\varphi}(\|T\|_{E_0 \times F_0 \rightarrow G_0}, \|T\|_{E_1 \times F_1 \rightarrow G_1}).$$

The result is sharp in the sense that if (7) holds for any Banach couples of function lattices $\bar{E}, \bar{F}, \bar{G}$, then condition (5) on functions must be satisfied.

Proof. First, let us observe that Theorem 1 can be also written in the following form: for the above operator T with the assumption (5) on the functions $\varphi_0, \varphi_1, \varphi$ we have that the estimate (3) holds for any $x_i \in E_i, y_i \in F_i, i = 0, 1$.

Now, let $x \in \varphi_0(\bar{E}), y \in \varphi_1(\bar{F})$ and $\|x\|_{\varphi_0(\bar{E})} < 1, \|y\|_{\varphi_1(\bar{F})} < 1$. Then there exist $x_i \in E_i, y_i \in F_i$ with $\|x_i\|_{E_i} \leq 1, \|y_i\|_{F_i} \leq 1$ for $i = 0, 1$ such that

$$|x| \leq \varphi_0(|x_0|, |x_1|) \text{ and } |y| \leq \varphi_1(|y_0|, |y_1|).$$

In view of the monotonicity of T and by using Theorem 1 (in fact, our first observation at the beginning of the proof here) we obtain

$$\begin{aligned} |T(x, y)| &\leq T(|x|, |y|) \\ &\leq T(\varphi_0(|x_0|, |x_1|), \varphi_1(|y_0|, |y_1|)) \\ &\leq C\varphi(T(|x_0|, |x_1|), T(|y_0|, |y_1|)). \end{aligned}$$

Denote $M_i = \|T\|_{E_i \times F_i \rightarrow G_i}$, $i = 0, 1$, and $z_0 = (T(|x_0|, |x_1|))/M_0$, $z_1 = T(|y_0|, |y_1|)/M_1$. Then

$$\begin{aligned} |T(x, y)| &\leq C\varphi(T(|x_0|, |x_1|), T(|y_0|, |y_1|)) \\ &= C\varphi(M_0 z_0, M_1 z_1) \\ &\leq C\bar{\varphi}(M_0, M_1)\varphi(z_0, z_1). \end{aligned}$$

Since $\|z_i\|_{G_i} \leq 1$, $i = 0, 1$ it follows that

$$T(x, y) \in \varphi(\bar{G}) \text{ and } \|T(x, y)\|_{\varphi(\bar{G})} \leq C\bar{\varphi}(M_0, M_1),$$

from which we have the estimate (7).

The necessity of inequality (5) we show by considering concrete positive bilinear operator and concrete spaces.

Let $\bar{E} = \bar{F} = (L_1, L_\infty)$ on the measure space (R_+, m_1) and $\bar{G} = (L_1, L_\infty)$ on the measure space (R_+^2, m_2) , where m_1 and m_2 are Lebesgue measures on R_+ and R_+^2 , respectively.

Consider the tensor product map $T : (L_1 + L_\infty) \times (L_1 + L_\infty) \rightarrow L_1 + L_\infty$ defined by

$$T(x, y)(u, v) = x(u)y(v), u, v \in R_+.$$

Then T is a positive bilinear operator with the norms

$$\|T\|_{L_i \times L_i \rightarrow L_i} = 1, \quad i = 1, \infty.$$

Assume that (6) holds. Then

$$\|T(x, y)\|_{\varphi(L_1, L_\infty)} \leq C\|x\|_{\varphi_0(L_1, L_\infty)}\|y\|_{\varphi_1(L_1, L_\infty)}$$

for any $x \in \varphi_0(L_1, L_\infty)$, $y \in \varphi_1(L_1, L_\infty)$.

Taking now $x = \chi_{(0,s)}$, $y = \chi_{(0,t)}$, $s, t > 0$ and using the facts that

$$\|\chi_{(0,r)}\|_{\varphi_i(L_1, L_\infty)} = \|\chi_{(0,r)}\|_{L_{M_i}} = \frac{1}{M_i^{-1}(1/r)} = \varphi_i(1, r), \quad i = 0, 1,$$

and

$$\|\chi_{(0,s)}\chi_{(0,t)}\|_{L_M(m_2)} = \frac{1}{M^{-1}(1/(st))} = \varphi(1, st)$$

we obtain

$$\varphi(1, st) \leq C\varphi_0(1, s)\varphi_1(1, t)$$

for all $s, t > 0$. The proof is complete. $\square$

Remark 5. Theorem 2 is still true for some nonlinear operators, namely for positive sublinear operators in each variable. Moreover, if $T : (E_0 + E_1) \times (F_0 + F_1) \rightarrow G_0 + G_1$ is a bilinear operator dominated by a positive bilinear S such that $S : E_i \times F_i \rightarrow G_i$ is bounded for $i = 0, 1$, then

$$\|T\|_{\varphi(\bar{E}) \times \varphi(\bar{F}) \rightarrow \varphi(\bar{G})} \leq C\bar{\varphi}(\|S\|_{E_0 \times F_0 \rightarrow G_0}, \|S\|_{E_1 \times F_1 \rightarrow G_1}).$$

Remark 6. Theorem 2 can be easily formulated for positive multilinear (n -linear) operators and the corresponding assumption to (5) will be

$$(8) \quad \varphi_0(1, s_0) \varphi_1(1, s_1) \cdots \varphi_{n-1}(1, s_{n-1}) \leq C \varphi(1, s_0 s_1 \cdots s_{n-1})$$

for some $C > 0$ and all $s_0, s_1, \dots, s_{n-1} > 0$.

Corollary 3. Let $\bar{E} = (E_0, E_1)$, $\bar{F} = (F_0, F_1)$, $\bar{G} = (G_0, G_1)$ be couples of Banach function lattices on the measure spaces (Ω_1, μ_1) , (Ω_2, μ_2) and (Ω, μ) , respectively. Assume that $T : (E_0 + E_1) \times (F_0 + F_1) \rightarrow G_0 + G_1$ is a positive bilinear operator such that the restriction $T : E_i \times F_i \rightarrow G_i$ is bounded for $i = 0, 1$. If $\varphi \in \mathcal{U}$ is a C -supermultiplicative function, i. e.,

$$(9) \quad \varphi(1, s) \varphi(1, t) \leq C \varphi(1, st)$$

for some positive constant $C > 0$ and all $s, t > 0$, then

$$(10) \quad T : \varphi(\bar{E}) \times \varphi(\bar{F}) \rightarrow \varphi(\bar{G})$$

is bounded and

$$(11) \quad \|T\|_{\varphi(\bar{E}) \times \varphi(\bar{F}) \rightarrow \varphi(\bar{G})} \leq C \bar{\varphi}(\|T\|_{E_0 \times F_0 \rightarrow G_0}, \|T\|_{E_1 \times F_1 \rightarrow G_1}).$$

The result is sharp in the sense that if (10) holds for any Banach couples of function lattices $\bar{E}$, $\bar{F}$, $\bar{G}$, then the function φ is C -supermultiplicative for some $C > 0$.

Remark 7. Corollary 3 for the C -supermultiplicative function φ and with additional assumptions that $\varphi \in \mathcal{U}^\pm$ and G_0, G_1 have the Fatou property was proved by Astashkin [3], Th. 1. He posed the question if the additional assumption $\varphi \in \mathcal{U}^\pm$ is necessary. We see, from Corollary 3, that this additional assumption is superfluous which gives then an answer to his problem. Moreover, the Fatou property on G_0, G_1 is also not needed.

What is known about interpolation of arbitrary bilinear operators (not necessary positive) between Calderón-Lozanovskii spaces? Of course, we need the C -supermultiplicativity of φ but what else?

In the case of the power functions $\varphi_\theta(s, t) = s^{1-\theta} t^\theta$ with $0 \leq \theta \leq 1$ and on the class of Banach function lattices with continuous norms the spaces $\varphi_\theta(\bar{E})$ are equal to the complex Calderón interpolation method $[E_0, E_1]_\theta$ for which the bilinear interpolation theorem holds (see [7],[5]).

We finally formulate two problems on bilinear interpolation.

Problem 1 (cf. [16], Problem 10). When is $\varphi(\cdot, \cdot)$ an interpolation method for bilinear operators?

Problem 2. Is Theorem 2, with the assumption (5) on the functions, true for arbitrary bilinear operators?

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