Landau type theorem for Orlicz spaces

Lech Maligranda^{1,*} and Witold Wnuk²

¹ Departamento de Matemáticas, IVIC, Apartado 21827, Caracas 1020-A, Venezuela

² Institute of Mathematics, Polish Academy of Sciences, Mielżyńskiego 27/29, 61-725 Poznań, Poland

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Introduction

In 1907 Edmund Landau [9] proved the reverse Hölder inequality, i.e., if $(x_n y_n)_{n=1}^{\infty} \in l^1$ for all sequences $(x_n) \in l^p$ ($1 < p < \infty$), then $(y_n) \in l^q$, where q is the conjugate exponent of p . In his original proof he used the following Dini theorem: if $a_n \geq 0$ and $\sum_1^{\infty} a_n = \infty$ then $\sum_1^{\infty} (a_n/s_n) = \infty$ and $\sum_1^{\infty} (a_n/s_n^{1+\varepsilon}) < \infty$ for any $\varepsilon > 0$, where s_n is the n -th partial sum. Now, if there were $\sum_1^{\infty} |y_n|^q = \infty$, then by the Dini theorem with $x_n = |y_n|^{q-1} \left/ \sum_{k=1}^n |y_k|^q \right.$ we would have $(x_n y_n) \notin l^1$ but $(x_n) \in l^p$.

Today everybody is able to prove the above result at once using the uniform boundedness principle or the form of linear continuous functionals on l^p . Landau's theorem belongs to the group of representation theorems of the Köthe dual E^x (=associate space in the other terminology) of a Banach function space E . The case $E = L^M(\mu)$ (i.e., E is an Orlicz space) was thoroughly investigated and it is well-known that $E^x = L^{M^*}(\mu)$ (see for example [14]), where M^* denotes the conjugate function of M (=complementary function in the sense of Young). This fact was originally proved (for convex functions M) by Birnbaum and Orlicz [3]. The purpose of the paper is to present a direct short and elementary proof of Birnbaum's and Orlicz's result and extend it to Orlicz spaces over atomless or counting measures and generated by finite-valued (not necessarily convex) functions. At the end of the paper one can find an example of an Orlicz space $L^M(\mu)$ over purely atomic measure μ and generated by non-convex function M such that its Köthe dual is not isomorphic to any space of the form $L^{M^*}(\nu)$.

We refer readers interested in Orlicz spaces to [8, 11, 14].

Let (S, Σ, μ) be a σ -finite measure space. Moreover, assume that $M: [0, \infty) \rightarrow [0, \infty]$ is a non-decreasing function and $M(0) = 0$.

* Current address: Department of Mathematics, Luleå University of Technology, S-95187 Luleå, Sweden

If $L^0 = L^0(S, \Sigma, \mu)$ denotes the space of Σ -measurable functions (with the usual identification of functions equal μ -almost everywhere) then

$$L^M(\mu) = \{f \in L^0 : m_M(rf) = \int_S M(r|f(s)|) d\mu < \infty \text{ for some } r > 0\}$$

is a linear subspace of the space L^0 (called the *Orlicz space*) and the functional

$$\|f\| = \inf\{a > 0 : m_M(f/a) \leq a\}$$

is a monotone group semi-norm on the space $L^M(\mu)$, i.e., $\|0\| = 0$, $\|f+g\| \leq \|f\| + \|g\|$ and $|f| \leq |g|$ μ -almost everywhere implies $\|f\| \leq \|g\|$. It is clear that the topology determined by $\|\cdot\|$ is Hausdorff iff M is not identically zero. Moreover, $\|\cdot\|$ is an F -norm on $L^M(\mu)$ iff M is continuous at zero or $M(r) = \infty$ for $r > 0$. In papers devoted to Orlicz spaces authors usually assume that M is left continuous, continuous at zero, non-decreasing and $M(0) = 0$. If M is convex continuous at zero and $M \not\equiv 0$, then the topology determined by $\|\cdot\|$ is equivalent to norm topology generated by $\|f\| = \inf\{a > 0 : m_M(f/a) \leq 1\}$. The F -norm $\|\cdot\|$ is always complete.

We will often write L^M instead of $L^M(\mu)$ and l^M when μ is the counting measure on subsets of natural numbers, i.e., l^M is an Orlicz sequence space.

Let us distinguish the ideal $L_A^M \subset L^M$ consisting of elements with order continuous norm, i.e., $L_A^M = \{f \in L^M : |f| \geq f_n \downarrow 0 \text{ implies } \|f_n\| \rightarrow 0\}$. If (S_n) is a sequence of atoms of finite measure μ such that μ restricted to measurable subsets of $S \setminus \bigcup_{n=1}^{\infty} S_n$ is atomless, then

$$L_A^M = \left\{ f \in L^M : \text{supp } f \subset S \setminus \bigcup_n S_n \text{ and } m_M(rf) < \infty \text{ for all } r\text{'s} \right\} \\ \oplus \overline{\text{span}\{1_{S_n} : n \in N\}},$$

where 1_{S_n} is the characteristic function of the set S_n .

For a function M we define the *conjugate function* M^* of M by

$$M^*(r) = \sup\{sr - M(s) : s \geq 0\},$$

(we put $x - \infty = -\infty$ for every real number x). The function M^* is always convex and left continuous. The notation M^{**} means the function $M^{**} = (M^*)^*$. After simple computations we obtain a useful inequality

$$(*) \quad M^{**}(M^*(u)/u) \leq M^*(u)$$

(we understand $M^{**}(\infty)$ as ∞). We always have $M^{**} \leq M$ and $M^* = M^{***}$.

Since we also consider Orlicz spaces $L^M(\mu)$ which are not Banach spaces let us quote the exact definition of the *Köthe dual* $(L^M(\mu))^x = (L^M)^x$ of $L^M(\mu)$:

$$(L^M)^x = \{g \in L^0(S, \Sigma, \mu) : f \cdot g \in L^1(\mu) \text{ for all } f \in L^M(\mu)\}.$$

The space $(L^M)^x$ is always a Banach space with respect to the norm

$$\|g\| = \sup\{\|f \cdot g\|_{L^1} : m_M(f) \leq 1\}.$$

Main results

Theorem 1 *If a function $M \not\equiv 0$ is convex and left continuous on $(0, \infty)$ then $(L^M)^x = L^{M^*}$.*

Proof. It is sufficient to prove $(L^M)^x \subset L^{M^*}$ because the reverse inclusion is a consequence of the inequality $rs \leq M(r) + M^*(s)$.

Suppose $g \in L^0$ is such that $f \cdot g \in L^1$ for all $f \in L^M$. Therefore g determines a continuous linear functional G on L^M by the equality $G(f) = \int_S f \cdot g \, d\mu$. Put $c = \|G\|$.

Let us consider two cases:

1. $M^*(r) < \infty$ for all r 's.

Define

$$g_0(s) = \begin{cases} \frac{c+1}{|g(s)|} M^*\left(\frac{|g(s)|}{c+1}\right) & \text{for } g(s) \neq 0, \\ 0 & \text{for } g(s) = 0. \end{cases}$$

There is no loss of generality to assume $M(u)$ is finite at some point $u > 0$ (if $M(r) = \infty$ for $r > 0$ then $(L^M)^x \subset LM^* = L^0$ and we are done). It is clear that the support of L^M equals S and so we can choose a sequence (S_n) of measurable sets with the following properties: $S_n \subset S_{n+1}$, $S = \bigcup_{n=1}^{\infty} S_n$, $0 < \mu(S_n) < \infty$ and $g_0 \cdot 1_{S_n} \in L^M$ (see [7, p. 136, Corollary 1]).

Since M is convex and left continuous we have $M = M^{**}$ in virtue of the Fenchel-Moreau theorem ([2, p. 86, Theorem 1.4] or [6, p. 186, Theorem 1]) and therefore using inequality (*) we obtain

$$m_M(g_0 \cdot 1_{S_n}) \leq m_{M^*}\left(\frac{|g(\cdot)|}{c+1}\right) = \int_{S_n} g_0(s) |g(s)| (c+1)^{-1} \, d\mu < \infty$$

because $g_0 \cdot 1_{S_n} \in L^M$. We claim $m_M(g_0) < \infty$. If $m_M(g_0)$ were infinite then $b = m_M(g_0 \cdot 1_{S_n}) > 2$ for some n in virtue of $m_M(g_0 \cdot 1_{S_n}) \uparrow m_M(g_0)$. Moreover, the convexity of M implies $b^{-1} \|g_0 \cdot 1_{S_n}\| \leq 1$ and thus

$$\begin{aligned} b &= m_M(g_0 \cdot 1_{S_n}) \leq \int_{S_n} g_0(s) |g(s)| (c+1)^{-1} \, d\mu \\ &= b(c+1)^{-1} \int_{S_n} (b^{-1} g_0(s)) |g(s)| \, d\mu \leq bc(c+1)^{-1} < b \end{aligned}$$

and we have got a contradiction. Therefore, it has to be $m_M(g_0) < \infty$ which implies $g \in L^{M^*}$.

2. There exists $r_0 > 0$ such that $M^*(r) < \infty$ for $r < r_0$ and $M^*(r) = \infty$ for $r > r_0$. We claim $g \in L^\infty$. Indeed, let $r > r_0$ be fixed. We have

$$\infty = \sup \{sr - M(s) : s \geq 0\} = \sup \{sr - M(s) : s > n\}$$

and so there is a sequence (s_n) increasing to infinity with $rs_n - M(s_n) > 0$, i.e., $M(s_n)/s_n < r$. For a number $s > 0$ choose an index n_0 such that $s < s_{n_0}$. Since

the function $M(x)/x$ is non-decreasing we obtain $M(s)/s \leq M(s_{n_0})/s_{n_0} < r$. Finally $M(s) \leq rs$. Hence $L^1 \subset L^M$ and this inclusion implies $g \in L^\infty$. Putting

$$g_0(s) = \begin{cases} \frac{d}{|g(s)|} M^*\left(\frac{|g(s)|}{d}\right) & \text{for } g(s) \neq 0 \\ 0 & \text{for } g(s) = 0, \end{cases}$$

where $d = \max(c+1, (\|g\|_\infty + 1)r^{-1})$, and repeating the arguments used in part 1 we will also obtain $g \in L^{M^*}$.

Remark 1 Let us note that the method used in the first part of the proof is similar to that in [10, Theorem 4].

Theorem 2 *If $M \not\equiv 0$ is finite-valued, left continuous, continuous at zero, non-decreasing and μ is atomless, then $(L^M)^x = L^{M^*}$.*

Proof. We have only to prove $(L^M)^x \subset L^{M^*}$. The inequality $M^{**} \leq M$ implies $L^M \subset L^{M^{**}}$. Let us consider two possibilities:

1. There exists a positive number r with $M^{**}(r) > 0$.

Let $g \in (L^M)^x$ and let G be the functional determined by $|g|$, i.e., $G(f) = \int_S f |g| d\mu$.

This functional restricted to L_A^M remains continuous with respect to the topology τ^{**} induced from $L_A^{M^{**}}$ ([4, Theorem 2]). The support of L_A^M equals S , and so every positive function $f \in L^{M^{**}}$ is the supremum of some increasing sequence (f_n) of positive functions from L_A^M . Since (f_n) is τ^{**} -bounded and G is τ^{**} -continuous we have $\int_S f |g| d\mu = \sup_n \int_S f_n |g| d\mu < \infty$, i.e., $f \cdot g \in L^1$ for all $f \in L^{M^{**}}$. Therefore

$g \in L^{M^{***}}$ in virtue of Theorem 1 and we are done because $L^{M^{***}} = L^{M^*}$.

2. $M^{**} \equiv 0$.

In this situation $M^*(r) = \infty$ for $r > 0$, i.e., $L^{M^*} = \{0\}$. Moreover, $\liminf_{r \rightarrow \infty} M(r)/r = 0$.

The last equality gives $(L^M)^x = \{0\}$ ([4, Corollary 1]). Finally $(L^M)^x = L^{M^*}$.

Remark 2 The assumption that μ is atomless is essential. If $M(r) = r^p$ ($0 < p < 1$) then $M^*(r) = \infty$ for $r > 0$ and the Orlicz sequence space l^{M^*} is trivial, but $(l^p)^x = l^\infty$.

Now we will pay attention to the case of Orlicz sequence spaces l^M . Let us recall that functions M and N are *equivalent at zero* if there exist positive constants a, b, c, d, x such that

$$aM(bu) \leq N(u) \leq cM(du) \quad \text{for } u \in [0, x].$$

Since properties of a space l^M are determined by the behaviour of M in a neighborhood of zero we can assume that M is *finite-valued*.

Theorem 3 *If M has all properties listed in Theorem 2, then $(l^M)^x = l^{M^*}$, where $M_\infty(u) = M(u)$ for $0 \leq u \leq 1$ and $M_\infty(u) = \infty$ for $u > 1$.*

(The notation M_∞^* means the conjugate function of M_∞ – not the function $(M^*)_\infty$).

Proof. Let us consider two cases:

1. $M(u) = 0$ iff $u = 0$.

Denote by $\hat{M}$ the convex minorant of the function M in the interval $[0, 1]$, i.e., $\hat{M}(t) \leq M(t)$ for all $t \in [0, 1]$ and if N is a convex function on $[0, 1]$ and it satisfies $N(t) \leq M(t)$ for $t \in [0, 1]$ then $N(t) \leq \hat{M}(t)$ for all $t \in [0, 1]$. Putting $\hat{M}(u) = \infty$ for $u > 1$ we obtain that $\hat{M}$ is convex on the half line $[0, \infty)$ and $\hat{M} \leq M_\infty$. Thus $\hat{M} \leq M_\infty^{**}$ and $l^M = l_{M_\infty} \subset l^{M_\infty^{**}} \subset l^{\hat{M}}$.

Let (y_n) be a sequence of positive numbers belonging to $(l^M)^x$. The functional G defined on l_A^M by the formula $G((x_n)) = \sum_1^\infty x_n y_n$ is continuous with respect to the topology induced from $l_A^{\hat{M}}$ (see [5, Theorem 5.1]), and so it will remain continuous with respect to the topology induced from $l_A^{M_\infty^{**}}$ because this topology is stronger.

If $0 < x = (x(k)) \in l^{M_\infty^{**}}$, then there exists a sequence (x_n) of elements from l_A^M such that $0 \leq x_n \uparrow x$. Since the sequence (x_n) is bounded in the topology induced from $l_A^{M_\infty^{**}}$, then $c = \sup G(x_n) < \infty$. Using the famous Fatou lemma we will obtain $c = \sum_1^\infty x(k) y(k)$. Therefore $(y_n) \in (l^{M_\infty^{**}})^x = l^{M_\infty^{**x}} = l^{M_\infty^*}$ (the first equality follows from

Theorem 1). In other words $(l^M)^x \subset l^{M_\infty^*}$.

On the other hand the inclusion $l^{M_\infty} \subset l^{M_\infty^{**}}$ gives $l^{M_\infty^*} = (l^{M_\infty^{**}})^x \subset (l^{M_\infty})^x \subset (l^M)^x$. Finally $(l^M)^x = l^{M_\infty^*}$.

2. $M([0, t]) = \{0\}$ for some $t > 0$.

It is obvious that $l^M = l^\infty$, and so $(l^M)^x = l^1$. The proof will be finished if we show $l^{M_\infty^*} = l^1$.

Suppose first $t \geq 1$. Under this assumption we have $M_\infty^*(u) = \sup\{uv - M(v) : 0 \leq v \leq 1\} = u$, i.e., $l^{M_\infty^*} = l^1$.

If $t < 1$ then $M_\infty^*(u) \geq \sup\{uv - M(v) : 0 \leq v \leq t\} = tu$. The convexity of the function M_∞^* implies $M_\infty^*(u) \leq M_\infty^*(1)u \leq u$ for $u \in [0, 1]$. Therefore M_∞^* is equivalent at zero to the function $N(u) = u$ and thus $l^{M_\infty^*} = l^1$ and we are done.

Remark 3 If $M(1) > 0$ and $\liminf_{u \rightarrow \infty} M(u)/u > 0$, then $M_\infty^*(u) = M^*(u)$ for u belonging to some neighborhood of zero and therefore $l^{M_\infty^*} = l^{M^*}$.

Indeed, supposing $\liminf_{u \rightarrow \infty} M(u)/u > 0$ we find numbers $a > 0$, $v_0 > 1$ such that $M(v)/v > a$ for all $v \geq v_0$. Taking $v \in (1, v_0)$ we have $M(v)/v \geq M(1)/v_0 > 0$, and so $\inf\{M(v)/v : v > 1\} \geq \min(a, M(1)/v_0)$. The following inequalities are valid for $u \in [0, \min(a, M(1)/v_0)]$:

$$\begin{aligned} M^*(u) &= \sup\{uv - M(v) : v \geq 0\} = \sup\{uv - M(v) : 0 \leq v \leq 1\} \\ &= \sup\{uv - M_\infty(v) : 0 \leq v \leq 1\} = \sup\{uv - M_\infty(v) : v \geq 0\} = M_\infty^*(u). \end{aligned}$$

The assumption $M(1) > 0$ is essential: if $N(u) = 0$ for $0 \leq u \leq 2$ and $N(u) = u^2 - 4$ for $u > 2$, then $N^*(u) \geq \sup\{uv - N(v) : 0 \leq v \leq 2\} = 2u$ and $N_\infty^*(u) = u$, and so the functions N^* and N_∞^* are different, although they are equivalent at zero.

Remark 4 If $\liminf_{u \rightarrow \infty} M(u)/u = 0$ then $M^*(u) = \infty$ for $u > 0$.

We can find a sequence (u_n) tending to infinity such that $(M(u_n)/u_n)$ converges to zero. Therefore, fixing $r > 0$ and taking sufficiently large n we will obtain $M^*(r) \geq u_n(r - (M(u_n)/u_n)) \geq 2^{-1} r u_n$, i.e., $M^*(r) = \infty$.

Remark 5 If $\liminf M(u)/u = 0$ (in particular, if M is concave) then $l^{M^*} = l^\infty$. Indeed, the inequality $\liminf M(u)/u > 0$ guarantees the existence of positive numbers c and u_0 such that $M(u)/u > c$ for $0 < u \leq u_0$. Denoting $c_0 = \inf\{M(v): u_0 \leq v \leq 1\}$ we obtain $M_\infty^*(u) = 0$ for $u < \min(c, c_0)$. Moreover M_∞^* is not identically equal to zero. If there were $M_\infty^*(u) = 0$ for $u > 0$, then we would have $uv \leq M(v)$ for all $u > 0$ and all $v \in [0, 1]$. In particular, $u \leq M(1)$ for all u and we have got a contradiction because we consider finite-valued functions only. Finally, the facts that M_∞^* vanish in some neighborhood of zero but M_∞^* is not identically zero imply $l^{M_\infty^*} = l^\infty$.

Remark 6 Theorem 3 (under slightly stronger assumptions about M) was also proved (as a corollary of more general results) by M. Nowak (see [12, Theorem 3.3]) but his proof is more complicated.

Let us note assumptions about measures in Theorems 2 and 3 are important. There exists an Orlicz space $L^M(\mu)$ over a purely atomic measure μ with M non-convex such that the Köthe dual $(L^M(\mu))^x$ is not isomorphic to the space $L^{M^*}(v)$ for any measure v .

Theorem 4 Let M have all properties listed in Theorem 2 and suppose M satisfies the following conditions: $M(u) = 0$ iff $u = 0$, $\limsup_{u \rightarrow \infty} M(2u)/u = \infty$ and $M^{**}(u) = u$.

There exists a purely atomic measure μ such that $(L^M(\mu))^x$ is not isomorphic to the space $L^{M^*}(v)$ for any measure v .

Proof. Using [13, Theorem 1] we can choose a sequence (a_n) of positive numbers such that defining the measure μ on subsets of natural numbers by the formula $\mu(A) = \sum_{n \in A} a_n$ we will have that $L^M(\mu)$ is order isomorphic to l^∞ . Therefore $(L^M(\mu))^x$

and l^1 are order isomorphic. If the spaces $(L^M(\mu))^x$ and $L^{M^*}(v)$ were isomorphic for some measure v , then $L^{M^*}(v)$ would be order isomorphic to l^1 on account of [1, Theorem 5]. Thus by Theorem 1 the space $L^{M^{**}}(v) = (L^{M^*}(v))^x$ would be order isomorphic to l^∞ , but this is impossible because $M^{**}(u) = u$ implies $L^{M^{**}}(v) = L^1(v)$.

Example of the function M satisfying the assumptions of Theorem 4. Let

$$a_n = \begin{cases} 2^{n(n-1)/2} & \text{for odd } n\text{'s} \\ 2^{(n-2)(n-1)/2} + 1 & \text{for even } n\text{'s} \end{cases}$$

$$b_n = \begin{cases} 2^{n(n-1)/2} + 1 & \text{for odd } n\text{'s} \\ 2^{(n+1)/2} & \text{for even } n\text{'s}. \end{cases}$$

Define a function M by the formula

$$M(u) = \begin{cases} u & \text{if } 0 \leq u \leq 1 \\ 2^{n(n+1)/2} & \text{if } a_n < u \leq b_n. \end{cases}$$

It is easy to see that M is left continuous, continuous at zero, non-decreasing and M vanish only at zero. Putting $u_n = b_{2n-1}$ we get $2u_n \in (a_{2n}, b_{2n}]$ and

$$M(2u_n)/M(u_n) = 2^{2^n} \rightarrow \infty, \text{ i.e., } \limsup_{u \rightarrow \infty} M(2u)/u = \infty.$$

Suppose that N is a convex function defined on $[0, \infty)$ satisfying the inequality $N(u) \leq M(u)$ for all u 's. If we show that $N(u) \leq u$, then the function $\tilde{M}(u) = u$ will be the convex minorant of M on account of the inequality $u \leq M(u)$. Therefore $M^{**}(u) = u$ because M^{**} is the greatest convex function on $[0, \infty)$ not exceeding M .

Assume first $u \geq 2$. Since $u \in [b_{2k-1}, b_{2k+1}]$ for some k then $u = cb_{2k-1} + (1-c)b_{2k+1}$ for some $c \in [0, 1]$. Thus

$$\begin{aligned} N(u) &\leq cN(b_{2k-1}) + (1-c)N(b_{2k+1}) \leq cM(b_{2k-1}) + (1-c)M(b_{2k+1}) \\ &= cb_{2k-1} + (1-c)b_{2k+1} = u. \end{aligned}$$

If $1 < u \leq 2$ then repeating the above arguments with $1 = b_{2k-1}$ and $2 = b_{2k+1}$ we obtain $N(u) \leq u$. Finally, we have $N(u) \leq M(u) = u$ for $0 \leq u \leq 1$. Thus $N(u) \leq u$ on the half line and we are done.

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