On Vector-valued Inequalities of the Marcinkiewicz-Zygmund, Herz and Krivine Type

Dedicated to Professor MISCHA COTLAR on the occasion of his 80th birthday

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(Received February 8, 1993)

Abstract. Any continuous linear operator $T: L^p \rightarrow L^q$ has a natural vector-valued extension $T_E: L^p(I_n) \rightarrow L^q(I_n)$ which is automatically continuous. Relations between the norms of these operators in the cases of $p = q$ and $r = 2$ were considered by MARCINKIEWICZ-ZYGMUND [28], HERZ [14] and KRIVINE [19]–[21]. In this paper we study systematically these relations and given some applications. It turns out that some known results can be proved in a simple way as a consequence of these developments.

0. Introduction

Exact relations between the norm of an operator and its complexification are very important in various branches of mathematics, in particular in interpolation theory. The fact that these norms are not always the same, even between finite-dimensional spaces, was observed already by M. RIESZ [35] and G. O. THORIN [44]. Let T be a bounded linear operator mapping from the real space $L^p(\mu)$ into another real space $L^q(v)$ with the norm $\|T\|$. Let T_C be the complex extension of T , i.e.,

$$T_C(f + ig) := Tf + iTg,$$

and let $\|T_C\|$ be the norm of T_C from the complex extension $L_C^p(\mu)$ into the complex extension $L_C^q(v)$.

Of course $\|T\| \leq \|T_C\|$, but it is interesting to know the reverse estimate

$$\|T_C\| \leq \gamma \|T\|$$

for some $\gamma = \gamma(L^p(\mu), L^q(v))$.

What does this inequality mean? We have

$$\begin{aligned} \|T_C(f + ig)\|_q &= \|Tf + iTg\|_q = \|(|Tf|^2 + |Tg|^2)^{1/2}\|_q \\ &\leq \|T_C\| \|f + ig\|_p \leq \gamma \|T\| \|f + ig\|_p \\ &= \gamma \|T\| \|(|f|^2 + |g|^2)^{1/2}\|_p. \end{aligned}$$

Therefore γ is the best constant in the inequality

$$(0) \quad \|(|Tf|^2 + |Tg|^2)^{1/2}\|_q \leq \gamma \|T\| \|(|f|^2 + |g|^2)^{1/2}\|_p$$

for all $f, g \in L^p(\mu)$ and any T .

We will consider the following generalizations of the complexification problem. For $1 \leq p, q, r \leq \infty$, $n = 2, 3, \dots$ and for any linear bounded operator between real spaces $T: L^p(X, \mu) \rightarrow L^q(Y, \nu)$ let us consider the following expressions:

$$K_{L^p(\mu), L^q(\nu)}^{(n)}(r) = \inf \left\{ C \geq 1: \left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_q \leq C \|T\| \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_p \right.$$

$$\left. \text{for all } f_1, f_2, \dots, f_n \in L^p(X, \mu) \text{ and any } T: L^p(X, \mu) \rightarrow L^q(Y, \nu) \right\},$$

$$K_{p,q}^{(n)}(r) = \sup \{ K_{L^p(\mu), L^q(\nu)}^{(n)}(r): \mu, \nu, \sigma\text{-finite measures} \}$$

and

$$K_{p,q}(r) = \sup_{n \geq 2} K_{p,q}^{(n)}(r).$$

We can also consider the constants $K_{p,q}^{(n)}(r)$ as well as the relationship between the norms of natural vector-valued extension T_E and T . More precisely, we can consider the vector-valued operator (or, equivalently, the operator between mixed-norm spaces) $T_E: L^p(\mu, l_n^r) \rightarrow L^q(\nu, l_n^r)$ defined in the following way:

$$T_E(f) = T_E((f_1, f_2, \dots, f_n)) := (Tf_1, Tf_2, \dots, Tf_n)$$

for $f = (f_1, f_2, \dots, f_n) \in L^p(\mu, l_n^r)$.

Then the supremum over the non-zero maps $T \in B(L^p(\mu), L^q(\nu))$ of the norm of T_E in $B(L^p(\mu, l_n^r), L^q(\nu, l_n^r))$ over the norm of T in $B(L^p(\mu), L^q(\nu))$ is equal to $K_{p,q}^{(n)}(r)$, and thus

$$K_{p,q}^{(n)}(r) = \sup \{ \|T_E\|_{L^p(\mu, l_n^r) \rightarrow L^q(\nu, l_n^r)} / \|T\|_{L^p \rightarrow L^q} : T \in B(L^p(\mu), L^q(\nu)), T \neq 0 \}.$$

Of course with our notations we have $\gamma = K_{L^p(\mu), L^q(\nu)}^{(2)}(2)$.

It is easy to see that if $p = q = r$, then the three above expressions are equal to 1.

MARCINKIEWICZ and ZYGMUND [28] proved that $K_{p,p}(2) = 1$. KRIVINE proved in [20] and [21] that $K_{\infty,1}^{(2)}(2) = \sqrt{2}$, and in [19] that $K_{p,q}^{(n)}(2) \leq K_{\infty,1}^{(n)}(2) := K_G(n) \leq \lim_{n \rightarrow \infty} K_G(n) := K_G$,

where K_G is the celebrated Grothendieck constant.

The purpose of this paper is to study systematically the numbers $K_{p,q}^{(n)}(r)$ especially when $r \neq 2$. This paper is divided into seven parts.

Section 1 contains some preliminaries with definitions and some results. These preparations are necessary for later calculations.

In Section 2 our main results are the monotonicity properties of the expressions $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r)$. First we prove that the functions $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r)$ are increasing in p and decreasing in q . In particular from this it follows that if $p \leq r \leq q$, then $K_{p,q}(r) = 1$. Second we prove the duality formulas $K_{p,q}^{(n)}(r) = K_{q',p'}^{(n)}(r')$ and $K_{p,q}(r) = K_{q',p'}(r')$. It is also proved that $K_{p,q}(r)$ as a function of r is decreasing on $[1, 2]$ and increasing on $[2, \infty]$.

In Section 3 we investigate the cases when the constants $K_{p,q}(r)$ are equal to 1. As note above, MARCINKIEWICZ and ZYGMUND [28] proved that $K_{p,p}(2) = 1$, while HERZ [14] proved

that $K_{p,p}(r) = 1$ for $1 < p < \infty$ and $\min(p, 2) \leq r \leq \max(p, 2)$. We give here a little more information on the equality $K_{p,q}(r) = 1$. Such equality is mainly connected with the assumption that $p \leq q$. In this section we also give estimates of $K_{p,q}(r)$ in the case when $p > q$; sometimes we obtain even equalities like $K_{2,q}(2) = d_2(2)/d_q(2)$. For $r \neq 2$ there also possible situations such that $K_{p,q}(r) = \infty$.

In Section 4 we prove estimates for the case $n = 2$. Our main result, namely that $K_{\infty,1}^{(2)}(r) = \max(2^{1/r}, 2^{1-1/r})$, is a generalization of the KRIVINE result [21] which was proved by him only for $r = 2$.

In Section 5 we give estimates of $K_{p,q}^{(n)}(2)$. In particular, the relation between the norms of the real operator and its complexification appears. These types of questions are important in the real-version of the Riesz-Thorin interpolation theorem (see [3], [35], [44], and [46]). In the “lower triangle” the real version of this theorem still holds with the constant 1 and in the “upper triangle” with the constant not bigger than $\sqrt{2}$.

In Section 6 we consider instead of the constants $K_{p,q}^{(n)}(r)$ the more general ones, namely $K_{X,Y}^{(n)}(r)$, where L^p and L^q spaces are replaced by arbitrary Banach lattices X and Y . It is interesting to note that we always have $K_{X,Y}^{(2)}(r) \leq \max(2^{1-1/r}, 2^{1/r})$ and that there exists a symmetric two-dimensional Banach space X such that $K_X^{(2)}(2) > 1$.

Finally, Section 7 is reserved for some additional results and remarks. Specifically we discuss the case when the Banach spaces are complex spaces and present some alternative proofs and possible generalizations. Moreover, we consider and discuss some known estimates of the (famous) Grothendieck constants K_G , $K_G(n)$ in the real case and K_G^C , $K_G^C(n)$ in the complex case.

Some open problems are posed in Sections 2–6.

1. Preliminaries

A) For $0 < p \leq \infty$ and a_1, a_2 arbitrary real numbers it is easy to prove that

$$(1) \quad \left(\int_0^1 |a_1 \cos 2\pi t + a_2 \sin 2\pi t|^p dt \right)^{1/2} = d_p^{(2)}(a_1^2 + a_2^2)^{1/2},$$

$$\text{where } d_p^{(2)} = \left(\int_0^1 |\cos 2\pi t|^p dt \right)^{1/p}.$$

The equality (1) for $0 < p < \infty$ was generalized by STECHKIN [41], in the following lemma:

Lemma 1. *Let $0 < p < \infty$ and let $n = 2, 3, \dots$ be fixed. Then there exists a sequence of orthogonal functions (spherical coordinates) $\{\alpha_k\}_{k=1}^n$ such that*

$$(2) \quad \left(\int_0^1 \left| \sum_{k=1}^n a_k \alpha_k(t) \right|^p dt \right)^{1/p} = d_p^{(n)} \left(\sum_{k=1}^n a_k^2 \right)^{1/2},$$

for every sequence of real numbers $\{a_k\}_{k=1}^n$, where

$$d_p^{(n)} = [\Gamma((p+1)/2) \Gamma(n/2) / (\sqrt{\pi} \Gamma((p+n)/2))]^{1/p}.$$

Observe that $d_p^{(n)}$ is increasing in p , $\lim_{p \rightarrow \infty} d_p^{(n)} = 1$, and $1 = d_p^{(1)} \geq d_p^{(2)} \geq \dots \geq 0$. Moreover, if p is finite, then $\lim_{n \rightarrow \infty} n^{1/2} d_p^{(n)} = \sqrt{2} [\Gamma((p+1)/2) / \sqrt{\pi}]^{1/p}$.

In the case $p = \infty$ it can be shown that $L^\infty[0, 1]$ contains an isometric copy of l^2 . This means that equality (2) still holds (maybe only with different sequences of functions).

B) We also need to use the r -stable Gaussian variables. A random (Gaussian) variable ω is said to be r -stable ($0 < r \leq 2$), if its Fourier transform on $\mathbb{R}$ is equal to $\exp(-|x|^r)$. The importance of r -stable random variables becomes clear in equation (3) below. It means that the space l^r is isometric to a subspace of $L^p[0, 1]$ for $1 \leq p < r \leq 2$. This result is contained implicitly in LEVY [20] and MARCINKIEWICZ-ZYGMUND [28] (see also KAWATA [16] or PIETSCH [32], p. 289).

Lemma 2. *Let either $r = 2$ and $0 < p < \infty$ or $0 < p < r < 2$. If $\{\omega_k\}$ is a sequence of identically distributed independent r -stable random variables defined on the interval $[0, 1]$ with the Lebesgue measure, then*

$$(3) \quad \left(\int_0^1 \left| \sum_{k=1}^n a_k \omega_k(t) \right|^p dt \right)^{1/p} = d_p(r) \left(\sum_{k=1}^n |a_k|^r \right)^{1/r}$$

for every sequence of real numbers $\{a_k\}_{k=1}^n$ and any $n = 2, 3, \dots$, where

$$d_p(r) = 2[\Gamma(1 - p/r) \Gamma((p+1)/2) / (\sqrt{\pi} \Gamma((1 - p/2)))^{1/p}.$$

In particular, $d_p(2) = 2[\Gamma((p+1)/2) / \sqrt{\pi}]^{1/p}$, $d_2(2) = \sqrt{2}$, $d_1(2) = 2/\sqrt{\pi}$. Observe also that $d_p(r)$ is increasing in p and decreasing in r .

The equality (3) for $r = 2$ is the Gaussian analogue of the Khintchine inequality for the Rademacher functions.

C) Let $\{r_k\}$ be a sequence of Rademacher functions defined by

$$r_k(t) = \text{sign}(\sin 2^k \pi t), \quad k = 0, 1, \dots, 0 \leq t \leq 1.$$

Lemma 3. *The equality*

$$(4) \quad \left\| \sum_{k=1}^n a_k r_k(t) \right\|_\infty = \sum_{k=1}^n |a_k|$$

holds for every sequence of real numbers $\{a_k\}_{k=1}^n$ and any $n = 1, 2, 3, \dots$.

Proof. It is sufficient to note that $\|\sum a_k r_k(t)\|_\infty = \max_\pm |\sum \pm a_k|$ and $\max_\pm |\pm a \pm b| = |a| + |b|$.

D) Let X be a Banach lattice and $1 \leq p \leq \infty$. For $\{x_k\}_{k=1}^n \in X$ we define

$$\left(\sum_{k=1}^n |x_k|^p \right)^{1/p} = \sup \left\{ \sum_{k=1}^n \lambda_k x_k : \lambda_k \in \mathbb{R}, \sum_{k=1}^n |\lambda_k|^{p'} \leq 1 \right\},$$

where the supremum means the least upper bound for the lattice structure of X . By the Yudin theorem (cf. KRIVINE [19], Th. 1 and LINDENSTRAUSS-TZAFIRI [25], Th. 1.d.1) we have that (3) holds for $x_1, x_2, \dots, x_n \in X$, i.e.,

$$(5) \quad \left(\int_0^1 \left| \sum_{k=1}^n x_k \omega_k(t) \right|^p dt \right)^{1/p} = d_p(r) \left(\sum_{k=1}^n |x_k|^r \right)^{1/r}.$$

The statements (1) and (2) can be generalized in a similar way.

E) We shall also need some inequalities for real and complex numbers. First we mention the following generalized version of an inequality of Clarkson type proved by MALIGRANDA and PERSSON [27]:

Lemma 4. *If $1 \leq p, q \leq \infty$, then*

$$(|a + b|^q + |a - b|^q)^{1/q} \leq C_{p,q}(|a|^p + |b|^p)^{1/p}$$

for any complex numbers a, b , where $C_{p,q} = \max \{2^{1-1/p}, 2^{1/q}, 2^{1/q-1/p+1/2}\}$ are the best possible constants.

Next we remark that an inequality for the numbers in $[0, 1]$ in the case of $p \geq 2$ was applied by HERZ [14]. Here we state and prove that this inequality holds also in the case $1 \leq p \leq 2$.

Lemma 5. *Let $1 \leq p < \infty$. Then for any $0 \leq x \leq 1$*

$$(1 + x)^p + (1 - x)^p \leq c_p x^{\min(p, 2)} + 2,$$

where

$$c_p = 2^{p-1} \quad \text{if } p \leq 2, \quad \text{and} \quad c_p = 2^{p-2}p(p-1) \quad \text{if } p \geq 2.$$

Proof. Let

$$f(x) = (1 + x)^p + (1 - x)^p - c_p x^{\min(p, 2)}.$$

If $1 \leq p \leq 2$, then

$$f'(x) = p[(1 + x)^{p-1} - (1 - x)^{p-1} - 2^{p-1}x^{p-1}]$$

and since

$$(1 + x)^{p-1} = (1 - x + 2x)^{p-1} \leq (1 - x)^{p-1} + (2x)^{p-1}$$

it follows that $f'(x) \leq 0$. Thus $f(x) \leq f(0) = 2$.

If $p \geq 2$, then

$$f'(x) = p[(1 + x)^{p-1} - (1 - x)^{p-1} - 2^{p-1}(p-1)x]$$

and

$$f''(x) = p(p-1)[(1 + x)^{p-2} + (1 - x)^{p-2} - 2^{p-1}].$$

For the function

$$g(x) = (1 + x)^{p-2} + (1 - x)^{p-2}$$

we find that

$$g'(x) = (p-2)[(1 + x)^{p-3} - (1 - x)^{p-3}],$$

and thus

$$g'(x) \leq 0 \quad \text{if } 2 \leq p \leq 3,$$

and

$$g'(x) > 0 \quad \text{if } p > 3.$$

Therefore

$$g(x) \leq \max [g(0), g(1)] = \max (2, 2^{p-2}) \leq 2^{p-1},$$

which means $f''(x) \leq 0$, and so $f''(x) \leq f''(0) = 0$. Finally this yields

$$f(x) \leq f(0) = 2$$

and the proof is complete.

2. Some properties of $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r)$

Next we prove the monotonicity of the numbers $K_{p,q}^{(n)}(r)$ as a function of p and q . It is obvious that these numbers are increasing in n .

Theorem 1. (a) *The following duality formula hold:*

$$(b) \quad K_{p,q}^{(n)}(r) = K_{p',q'}^{(n)}(r') \quad \text{and} \quad K_{p,q}(r) = K_{q',p'}(r').$$

(b) $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r)$ are increasing in p and decreasing in q . In particular $K_{p,q}^{(n)}(r) \leq K_{\infty,1}^{(n)}(r)$.

(c) $K_{p,q}(r)$ as a function of r is decreasing on $[1, 2]$ and increasing on $[2, \infty]$.

Before the proof of this theorem let us consider the numbers $K_{p,q}^{(n)}(r)$ as the tensor-product norms of some projections.

Let

$$S_n^r = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n |x_k|^r = 1 \right\}$$

be the unit sphere in ℓ_r^n and let $C_p(S_n^r)$ be the linear space of all real-valued continuous functions defined on S_n^r with the L^p -norm, i.e., for any continuous function $f: S_n^r \rightarrow \mathbb{R}$

$$\|f\|_p = \left(\int_{S_n^r} |f(x)|^p dm_r(x) \right)^{1/p},$$

where m_r is the normalized Lebesgue measure on S_n^r , $m_r(S_n^r) = 1$.

We will calculate the numbers $K_{p,q}^{(n)}(r)$ by using the projections $P_k: S_n^r \rightarrow [-1, 1]$ and $Q_k: S_n^r \rightarrow [-1, 1]$ given by

$$P_k(x_1, \dots, x_n) = x_k \quad \text{and} \quad Q_k(y_1, \dots, y_n) = y_k \quad (k = 1, 2, \dots, n).$$

The idea of the proof is taken from KRIVINE [21, Prop. 2].

Let us recall that if K_1, K_2 are compact sets and $C(K_1), C(K_2)$ are normed spaces of continuous functions with the norms $\|\cdot\|_1, \|\cdot\|_2$, then $C(K_1) \otimes C(K_2)$ denotes the real subspace of $C(K_1 \times K_2)$ generated by linear combinations of the type $f(x)g(y)$ with $f \in C(K_1)$ and $g \in C(K_2)$.

For any $F \in C(K_1) \otimes C(K_2)$ we define its norm as

$$\|F\|_{\otimes} = \inf \sum_{k=1}^m \|f_k\|_1 \|g_k\|_2,$$

where the infimum is taken over all representations of $F(x, y)$ as

$$F(x, y) = \sum_{k=1}^m f_k(x) g_k(y),$$

$f_k \in C(K_1)$, $g_k \in C(K_2)$, $k = 1, 2, \dots, m$ and all $m = 1, 2, \dots$. The completion of $C(K_1) \otimes C(K_2)$ with respect to the above norm is the projective tensor product $C(K_1) \hat{\otimes} C(K_2)$ of $C(K_1)$ and $C(K_2)$.

The dual of this space is identified with the bounded operators T from $C(K_1)$ into $C(K_2)^*$. For such a given operator T , a linear form U_T can be defined by

$$\langle U_T, f \otimes g \rangle = \langle Tf, g \rangle \quad \text{for } f \in C(K_1), \quad g \in C(K_2).$$

For $A \subset S'_n$, $B \subset S'_n$ and for $F \in C_p(S'_n) \otimes C_q(S'_n)$ we denote by $F|_{A \times B}$ the restriction of F to $A \times B$.

Proposition 1. *Let $F \in C_p(S'_n) \hat{\otimes} C_q(S'_n)$.*

(a) *If $1 \leq p \leq \infty$ and $q = \infty$, then*

$$\|F\|_{\otimes} = \sup \{ \|F|_{S'_n \times B}\|_{\otimes} : B \subset S'_n, B \text{ finite} \}.$$

(b) *If $p = q = \infty$, then*

$$\|F\|_{\otimes} = \sup \{ \|F|_{A \times B}\|_{\otimes} : A \subset S'_n, B \subset S'_n, A \text{ finite}, B \text{ finite} \}.$$

(c) *If $1 \leq p, q < \infty$, then*

$$\|F\|_{\otimes} = \sup \{ \|F|_{A \times B}\|_{\otimes} : A \subset S'_n, B \subset S'_n, m_r(A) > 0, m_r(B) > 0 \}.$$

Proof. (a) Since S'_n is a compact subset of $\mathbb{R}^n$, it follows that for a given $k = 1, 2, \dots$, there is a finite subset D_k of S'_n such that for any $x \in S'_n$ there exists $y \in D_k$ with the distance in $\mathbb{R}^n$ not bigger than 2^{-k} . Define

$$B_1 = D_1 \quad \text{and} \quad B_k = D_1 \cup D_2 \cup \dots \cup D_k \quad \text{for } k = 2, 3, \dots$$

Then each B_k is finite and $\lim_{k \rightarrow \infty} \|F|_{S'_n \times B_k}\|_{\otimes} = \|F\|_{\otimes}$.

The proofs of (b) and (c) are similar.

Proposition 2. *The following equality holds*

$$(7) \quad K_{p,q}^{(n)}(r) = \left\| \sum_{k=1}^n P_k \otimes Q_k \right\|_{C_p(S'_n) \hat{\otimes} C_{q'}(S'_n)}.$$

Proof. First let us consider the case $p = \infty$ and $q = 1$.

By Proposition 1(a), for any $\varepsilon > 0$ there is a finite set $B \subset S'_n$ such that

$$\left\| \sum_{k=1}^n P_k \otimes Q_k \right\| - \varepsilon < \left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{S'_n \times B}.$$

By the Hahn-Banach theorem there exists a continuous linear functional

$$H: C_\infty(S'_n) \hat{\otimes} C_\infty(B) \rightarrow \mathbb{R} \quad \text{such that} \quad \|H\| = 1$$

and

$$H\left(\sum_{k=1}^n P_k \otimes Q_k\right) = \left\| \left(\sum_{k=1}^n P_k \otimes Q_k\right) \right\|_{S'_n \times B}.$$

We know that the dual of $C_\infty(S'_n) \hat{\otimes} C_\infty(B)$ can be identified with the bounded linear operators from $C_\infty(S'_n)$ into $C_\infty(B)^*$. Therefore there is $T: C_\infty(S'_n) \rightarrow C_\infty(B)^*$ such that $\|H\| = \|T\| = 1$ and $H\left(\sum_{k=1}^n P_k \otimes Q_k\right) = \sum_{k=1}^n \langle TP_k, Q_k \rangle$. Moreover, since $C_\infty(S'_n)$ is a $L^\infty(S'_n, \mu)$ space for some measure μ , we find that T is an operator on $L^\infty(S'_n, \mu)$ with the same norm, and since B is a finite set, it follows that $C_\infty(B)^* = L^1(B, m)$ with $m(\{b\}) = 1$ for each $b \in B$ and

$$\sum_{k=1}^n \langle TP_k, Q_k \rangle = \sum_{k=1}^n \int_B TP_k(b) Q_k(b) dm(b) = \int_B \sum_{k=1}^n TP_k(b) Q_k(b) dm(b).$$

Now using the facts that

$$\sum_{k=1}^n |P_k(b)|^r = 1, \quad \sum_{k=1}^n |Q_k(b)|^{r'} = 1$$

for each $b \in B$, $\|T\| = 1$ and the Hölder inequality we obtain

$$\begin{aligned} \sum_{k=1}^n \langle TP_k, Q_k \rangle &\leq \int_B \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} dm(b) = \left\| \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} \right\|_{L^1(B, m)} \\ &\leq K_{\infty, 1}^{(n)}(r) \|T\| \left\| \left(\sum_{k=1}^n |P_k|^r \right)^{1/r} \right\|_{L^\infty(S'_n, \mu)} = K_{\infty, 1}^{(n)}(r). \end{aligned}$$

Hence we conclude that

$$\left\| \sum_{k=1}^n P_k \otimes Q_k \right\| \leq K_{\infty, 1}^{(n)}(r).$$

For the proof in the opposite direction, let us consider any $T: L^\infty(X, \mu) \rightarrow L^1(Y, \nu)$, $\|T\| = 1$ and $f_k \in L^\infty(X, \mu)$, $k = 1, 2, \dots, n$.

We can suppose that X, Y are finite subsets and $\left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_\infty = 1$.

Let $X' = \bigcup_{k=1}^n \text{supp}(f_k)$ and $Y' = \bigcup_{k=1}^n \text{supp}(Tf_k)$. Define

$$J_1: L^\infty(X') \rightarrow L^\infty(X) \quad \text{by} \quad J_1(g) = g \left(\sum_{k=1}^n |f_k|^r \right)^{1/r},$$

$$J_2: L^1(Y) \rightarrow L^1(Y') \quad \text{by} \quad J_2(g) = g|_{Y'}$$

and

$$S: L^\infty(X') \rightarrow L^1(Y') \quad \text{by} \quad S = J_2 \circ T \circ J_1.$$

For $g_k = f_k / \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \in L^\infty(X')$ we have $\sum_{k=1}^n |g_k|^r = 1$, $Sg_k = J_2(T(J_1 g_k)) = J_2(T(f_k)) = 1_{Y'} \cdot T(f_k)$ and

$$\begin{aligned} \left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_1 &= \left\| \left(\sum_{k=1}^n |Sg_k|^r \right)^{1/r} \right\|_1 = \int_{Y'} \left(\sum_{k=1}^n |Sg_k|^r \right)^{1/r} d\nu \\ &= \int_{Y'} \sum_{k=1}^n |Sg_k| \sum_{k=1}^n |Sg_k|^{r-1} / \left(\sum_{k=1}^n |Sg_k|^r \right)^{1-1/r} d\nu \\ &= \int_{Y'} \sum_{k=1}^n Sg_k \cdot h_k d\nu, \end{aligned}$$

where

$$h_k = |Sg_k|^{r-1} \left(\sum_{k=1}^n |Sg_k|^r \right)^{1/r-1} \text{sign}(Sg_k) \quad \text{with} \quad \sum_{k=1}^n |h_k|^{r'} = 1.$$

Now if we consider

$$S: L^\infty(X') = C(X') \rightarrow C(Y')^* = L^1(Y'),$$

then

$$\int_{Y'} \sum_{k=1}^n Sg_k \cdot h_k d\nu = \sum_{k=1}^n \langle Sg_k, h_k \rangle \leq \|S\| \left\| \sum_{k=1}^n g_k \otimes h_k \right\|_{C(X') \otimes C(Y')}.$$

Since

$$\|S\| \leq \|T\| = 1 \quad \text{and} \quad \sum_{k=1}^n |g_k|^r = \sum_{k=1}^n |h_k|^{r'} = 1,$$

we have

$$\left\| \sum_{k=1}^n g_k \otimes h_k \right\|_{C(X') \otimes C(Y')} \leq \left\| \sum_{k=1}^n P_k \otimes Q_k \right\|_{C_\infty(\mathbb{S}_n^r) \otimes C_\infty(\mathbb{S}_n^{r'})}$$

and the reverse inequality of (6) holds for $q = 1$.

Secondly let $p = \infty$ and $1 < q < \infty$. Let us write

$$a = \left\| \sum_{k=1}^n P_k \otimes Q_k \right\|_{C_\infty(\mathbb{S}_n^r) \otimes C_{q'}(\mathbb{S}_n^{r'})}.$$

According to Proposition 1(c), for any $\varepsilon > 0$ there are subsets $A \subset \mathbb{S}_n^r$, $B \subset \mathbb{S}_n^{r'}$ with $m_r(A) > 0$, $m_{r'}(B) > 0$ such that

$$\begin{aligned} a - \varepsilon &< \left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{A \times B} \Big\|_{C_\infty(\mathbb{S}_n^r) \otimes C_{q'}(\mathbb{S}_n^{r'})} \\ &= \left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{A \times B} \Big\|_{C_\infty(\mathbb{S}_n^r) \otimes L^{q'}(\mathbb{S}_n^{r'})}, \end{aligned}$$

where the last equality holds because $C_{q'}(\mathbb{S}_n^{r'})$ is dense in $L^{q'}(\mathbb{S}_n^{r'})$. There exists also a linear bounded functional

$$H: C_\infty(\mathbb{S}_n^r) \hat{\otimes} L^{q'}(\mathbb{S}_n^{r'}) \rightarrow \mathbb{R}$$

and an operator

$$T: C_\infty(S'_n) \rightarrow L^{q'}(S'_n)^* = L^q(S'_n) \quad \text{such that} \quad \|H\| = \|T\| = 1$$

and

$$\begin{aligned} & \left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{A \times B} \Big|_{C_\infty(S'_n) \otimes L^{q'}(S'_n)} = H \left(\left(\sum_{k=1}^n P_k \otimes Q_k \right) \right)_{A \times B} \\ &= \sum_{k=1}^n \langle TP_k, Q_k \rangle_{A \times B} = \int_B \sum_{k=1}^n TP_k(t) Q_k(t) dm_{r'} \\ &\leq \int_B \left(\sum_{k=1}^n |TP_k(t)|^r \right)^{1/r} \left(\sum_{k=1}^n |Q_k(t)|^{r'} \right)^{1/r'} dm_{r'} \\ &\leq \left(\int_B \left(\sum_{k=1}^n |TP_k(t)|^r \right)^{q/r} dm_{r'} \right)^{1/q} \left(\int_B \left(\sum_{k=1}^n |Q_k(t)|^{r'} \right)^{q'/r'} dm_{r'} \right)^{1/q'} \\ &\leq \left\| \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} \right\|_{L^q(B, m_{r'})} \leq K_{\infty, q}^{(n)}(r) \|T\| \left\| \left(\sum_{k=1}^n |P_k|^r \right)^{1/r} \right\|_\infty \leq K_{\infty, q}^{(n)}(r). \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily we conclude that $a \leq K_{\infty, q}^{(n)}(r)$.

Conversely, for any linear bounded operator $T: L^\infty(X, \mu) \rightarrow L^q(Y, \nu)$ we shall prove that for all $f_1, f_2, \dots, f_n \in L^\infty(X, \mu)$ the following inequality holds:

$$\left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_q \leq a \|T\| \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_\infty.$$

We can assume that

$$\|T\| = 1 = \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_\infty$$

and that X and Y are finite subsets. Using, as before, the functions f_k, g_k ($k = 1, 2, \dots, n$) and the operators J_1, J_2 and S with $L^q(Y')$ instead of $L^1(Y')$ we have

$$\begin{aligned} \left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_q &= \left\| \left(\sum_{k=1}^n |Sg_k|^r \right)^{1/r} \right\|_q = \left(\int_{Y'} \left(\sum_{k=1}^n |Sg_k|^r \right)^{q/r} dv \right)^{1/q} \\ &= \left(\int_{Y'} \left(\sum_{k=1}^n Sg_k \cdot h_k \right)^q dv \right)^{1/q} \\ &= \left(\sum_{y \in Y'} \left(\sum_{k=1}^n Sg_k(y) \cdot h_k(y) \right)^q \right)^{1/q} \leq \left(\sum_{y \in Y'} \sum_{k=1}^n Sg_k(y) \cdot h_k(y) \right) \\ &= \int_{Y'} \left(\sum_{k=1}^n Sg_k \cdot h_k \right) dv = \sum_{k=1}^n \langle Sg_k, h_k \rangle \\ &\leq \|S\| \left\| \sum_{k=1}^n g_k \otimes h_k \right\|_{L^\infty(X') \otimes L^{q'}(Y')} \\ &\leq \left\| \sum_{k=1}^n g_k \otimes h_k \right\|_{L^\infty(X') \otimes L^{q'}(Y')} \leq a, \end{aligned}$$

which implies the reverse inequality $K_{\infty, q}^{(n)}(r) \leq a$.

Thirdly let $1 \leq p < \infty$ and $1 < q < \infty$. Let us write

$$a = \left\| \sum_{k=1}^n P_k \otimes Q_k \right\|_{C_p(\mathbf{S}_n^r) \otimes C_{q'}(\mathbf{S}_n^{r'})} = \left\| \sum_{k=1}^n P_k \otimes Q_k \right\|_{L^p(\mathbf{S}_n^r) \otimes L^{q'}(\mathbf{S}_n^{r'})}.$$

The last equality holds true because $C_p(\mathbf{S}_n^r)$ and $C_{q'}(\mathbf{S}_n^{r'})$ are dense in $L^p(\mathbf{S}_n^r)$ and $L^{q'}(\mathbf{S}_n^{r'})$, respectively.

By Proposition 1(c), for any $\varepsilon > 0$ there are $A \subset \mathbf{S}_n^r$, $B \subset \mathbf{S}_n^{r'}$ with $m_r(A) > 0$, $m_{r'}(B) > 0$ such that

$$a - \varepsilon < \left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{A \times B} \Big|_{L^p(A) \otimes L^{q'}(B)}.$$

Then, similarly as in (a), there exists a continuous linear functional

$$H: L^p(A, m_r) \otimes L^{q'}(B, m_{r'}) \rightarrow \mathbb{R}$$

and a continuous linear operator

$$T: L^p(A, m_r) \rightarrow L^{q'}(B, m_{r'})^* = L^q(B, m_{r'})$$

such that $\|H\| = \|T\| = 1$ and

$$\left\| \left(\sum_{k=1}^n P_k \otimes Q_k \right) \right\|_{A \times B} = H \left(\sum_{k=1}^n P_k \otimes Q_k \right) = \sum_{k=1}^n \langle TP_k, Q_k \rangle.$$

Since

$$\begin{aligned} \sum_{k=1}^n \langle TP_k, Q_k \rangle &= \int_B \sum_{k=1}^n TP_k(b) Q_k(b) dm_{r'} \\ &\leq \int_B \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} \left(\sum_{k=1}^n |Q_k(b)|^{r'} \right)^{1/r'} dm_{r'} \\ &\leq \int_B \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} dm_{r'} \\ &\leq \left(\int_B \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{q/r} dm_{r'} \right)^{1/q} m_{r'}(B)^{1/q'} \\ &\leq \left\| \left(\sum_{k=1}^n |TP_k(b)|^r \right)^{1/r} \right\|_{L^q(B, m_{r'})} \\ &\leq K_{p,q}^{(n)}(r) \|T\| \left\| \left(\sum_{k=1}^n |P_k|^r \right)^{1/r} \right\|_{L^p(A)} \leq K_{p,q}^{(n)}(r), \end{aligned}$$

it follows that $a \leq K_{p,q}^{(n)}(r)$. The proof of the reverse inequality is similar as in the second case.

The proofs in the remaining cases $1 \leq p \leq \infty$, $q = \infty$ and $1 \leq p < \infty$, $q = 1$ are similar and thus omitted.

Proof of the Theorem 1. (a) The duality formula for $K_{p,q}^{(n)}(r)$ follows immediately from (7). Moreover, since the sequence $\{K_{p,q}^{(n)}(r)\}$ is increasing in n ,

$$K_{p,q}(r) = \lim_{n \rightarrow \infty} K_{p,q}^{(n)}(r) = \lim_{n \rightarrow \infty} K_{q',p'}^{(n)}(r') = K_{q',p'}(r').$$

(b) Equality (7) also means that

$$K_{p,q}^{(n)}(r) = \inf \left\{ \sum_{i=1}^m \|f_i\|_{C_p(\mathbf{S}_n)} \|g_i\|_{C_{q'}(\mathbf{S}_n)} : \sum_{k=1}^n P_k \otimes Q_k = \sum_{i=1}^m f_i g_i, m = 1, 2, \dots \right\}$$

and in this formula we have L^p - and $L^{q'}$ -norms on probability measures spaces. Therefore, $K_{p,q}^{(n)}(r)$ increases with p and decreases with q . Then, similarly as in (a), we have also that $K_{p,q}^{(n)}(r)$ increases with p and decreases with q .

(c) First let us prove that $C_r \leq K_{p,q}(r)$, where C_r is the smallest constant $C \geq 1$ such that

$$(8) \quad \left\| \left(\int_0^1 \left| \sum_{k=1}^n T f_k g_k(t) \right|^r dt \right)^{1/r} \right\|_q \leq C \|T\| \left\| \left(\int_0^1 \left| \sum_{k=1}^n f_k g_k(t) \right|^r dt \right)^{1/r} \right\|_p$$

for all $f_1, f_2, \dots, f_n \in L^p(X, \mu)$ and some $g_1, g_2, \dots, g_n \in L^r(0, 1)$, with $n = 1, 2, \dots$ arbitrary. We will use in the proof the KRIVINE's idea (cf. [19], Lemma 1). For any $\varepsilon > 0$ there are simple functions s_k such that $\|g_k - s_k\|_r < \varepsilon$, $k = 1, 2, \dots, n$. Then

$$\begin{aligned} \left| \left\| \sum_{k=1}^n f_k g_k(\cdot) \right\|_r - \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right| &\leq \left\| \sum_{k=1}^n f_k [g_k(\cdot) - s_k(\cdot)] \right\|_r \\ &\leq \sum_{k=1}^n \|f_k\|_p \|g_k - s_k\|_r < \varepsilon \sum_{k=1}^n \|f_k\|_p \end{aligned}$$

and so

$$\left\| \sum_{k=1}^n f_k g_k(\cdot) \right\|_r \leq \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r + \varepsilon \sum_{k=1}^n \|f_k\|_p \leq \left\| \sum_{k=1}^n f_k g_k(\cdot) \right\|_r + 2\varepsilon \sum_{k=1}^n \|f_k\|_p.$$

The monotonicity of the norm $\|\cdot\|_p$ gives

$$\begin{aligned} \left\| \left\| \sum_{k=1}^n f_k g_k(\cdot) \right\|_r \right\|_p &\leq \left\| \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right\|_p + \varepsilon \left\| \sum_{k=1}^n \|f_k\|_p \right\|_p \\ &\leq \left\| \left\| \sum_{k=1}^n f_k g_k(\cdot) \right\|_r \right\|_p + 2\varepsilon \left\| \sum_{k=1}^n \|f_k\|_p \right\|_p. \end{aligned}$$

In the same way, by putting Tf_k instead of f_k , we obtain

$$\begin{aligned} \left\| \left\| \sum_{k=1}^n T f_k s_k(\cdot) \right\|_r \right\|_q &\leq \left\| \left\| \sum_{k=1}^n T f_k g_k(\cdot) \right\|_r \right\|_q + \varepsilon \left\| \sum_{k=1}^n \|T f_k\|_q \right\|_q \\ &\leq \left\| \left\| \sum_{k=1}^n T f_k g_k(\cdot) \right\|_r \right\|_q + 2\varepsilon \left\| \sum_{k=1}^n \|T f_k\|_q \right\|_q. \end{aligned}$$

But

$$s_k = \sum_{i=1}^m c_{i,k} 1_{A_i}(t),$$

where $c_{i,k} \in \mathbb{R}$ and A_i are measureable subsets of $[0, 1]$ ($i = 1, 2, \dots, m; k = 1, 2, \dots, n$), therefore

$$\begin{aligned} \left\| \left\| \sum_{k=1}^n T f_k g_k(\cdot) \right\|_r \right\|_q &= \left\| \left(\sum_{i=1}^m \left| \sum_{k=1}^n T f_k c_{i,k} m(A_i)^{1/r} \right|^r \right)^{1/r} \right\|_q \\ &= \left\| \left(\sum_{i=1}^m \left| T \left(\sum_{k=1}^n f_k c_{i,k} m(A_i)^{1/r} \right) \right|^r \right)^{1/r} \right\|_q \\ &\leq K_{p,q}(r) \|T\| \left\| \left(\sum_{i=1}^m \left| \sum_{k=1}^n f_k c_{i,k} m(A_i)^{1/r} \right|^r \right)^{1/r} \right\|_q \\ &= K_{p,q}(r) \|T\| \left\| \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right\|_p. \end{aligned}$$

Hence

$$\begin{aligned} \left\| \left\| \sum_{k=1}^n T f_k g_k(\cdot) \right\|_r \right\|_q &\leq \left\| \left\| \sum_{k=1}^n T f_k s_k(\cdot) \right\|_r \right\|_q + \varepsilon \left\| \sum_{k=1}^n |T f_k| \right\|_q \\ &\leq K_{p,q}(r) \|T\| \left\| \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right\|_p + \varepsilon \left\| \sum_{k=1}^n |T f_k| \right\|_q \\ &\leq K_{p,q}(r) \|T\| \left(\left\| \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right\|_p + \varepsilon \left\| \sum_{k=1}^n |f_k| \right\|_p \right) \\ &\quad + \varepsilon \left\| \sum_{k=1}^n |T f_k| \right\|_q \\ &\leq K_{p,q}(r) \|T\| \left(\left\| \left\| \sum_{k=1}^n f_k s_k(\cdot) \right\|_r \right\|_p + \varepsilon \sum_{k=1}^n \|f_k\|_p \right) \\ &\quad + \varepsilon \|T\| \left(\sum_{k=1}^n \|f_k\|_p \right), \end{aligned}$$

and since $\varepsilon > 0$ was chosen arbitrarily, we get $C_r \leq K_{p,q}(r)$.

Now if $1 \leq r < s \leq 2$, then according to Lemma 2 with $(g_k) = (\omega_k)$ s -stable random variables we have

$$\begin{aligned} d_r(s) \left\| \left(\sum_{k=1}^n |T f_k|^s \right)^{1/s} \right\|_q &= \left\| \left(\int_0^1 \left| \sum_{k=1}^n T f_k \omega_k(t) \right|^r dt \right)^{1/r} \right\|_q \\ &\leq C_r \|T\| \left\| \left(\int_0^1 \left| \sum_{k=1}^n f_k \omega_k(t) \right|^r dt \right)^{1/r} \right\|_p \\ &= C_r d_r(s) \|T\| \left\| \left(\sum_{k=1}^n |f_k|^s \right)^{1/s} \right\|_p. \end{aligned}$$

Thus

$$\left\| \left(\sum_{k=1}^n |T f_k|^s \right)^{1/s} \right\|_q \leq C_r \|T\| \left\| \left(\sum_{k=1}^n |f_k|^s \right)^{1/s} \right\|_p,$$

which together with (8) implies that

$$K_{p,q}(s) \leq C_r \leq K_{p,q}(r).$$

The second case (i.e., when $2 \leq r < s \leq \infty$) follows from the duality formula.

Problem 1. Let $n \geq 2$ be fixed. Does $K_{p,q}^{(n)}(r) \geq K_{p,q}^{(n)}(2)$ hold, or more generally, is $K_{p,q}^{(n)}(r)$ a decreasing function of r on $[1, 2]$ and an increasing function of r on $[2, \infty]$?

3. On the estimates of Marcinkiewicz-Zygmund and Herz type

MARCINKIEWICZ and ZYGMUND [28] showed that if $1 \leq p < \infty$ then $K_{p,p}(2) = 1$, and if $1 \leq p < r < 2$ then $K_{p,p}(r) = 1$. HERZ [14] proved that for $1 < p < \infty$, $K_{p,p}(r) = 1$ if, and only if, $\min(p, 2) \leq r \leq \max(p, 2)$.

The purpose of this section is to utilize ideas of MARCINKIEWICZ-ZYGMUND [28] and HERZ [14] to generalize and improve these results. In particular, these results permit us to analyze $K_{p,q}(r) = 1$ also for $p \neq q$.

Theorem 2. (a) If either $p = 1$ or $q = \infty$, then $K_{p,q}(r) = 1$.

(b) If $1 < p \leq q < \infty$ and $\min(p, 2) \leq r \leq \max(q, 2)$, then $K_{p,q}(r) = 1$.

(c) If $1 < p \leq q < \infty$ and $K_{p,q}(r) = 1$, then $\min(p, q') \leq r \leq \max(p, q)$.

(d) If $1 < p < \infty$ and $K_{p,p}(r) = 1$, then $\min(p, 2) \leq r \leq \max(p, 2)$.

(e) $K_{p,q}(2) = 1$ if, and only if, $p \leq q$.

Remark 1. If $p \leq q$, then $[\min(p, 2), \max(p, 2)] \subset [\min(p, q'), \max(p', q)]$.

Before we prove Theorem 2, we investigate when the appropriate constants are 1 in the case $n = 2$.

Proposition 3. (a) The equality $K_{l_2^2, l_2^2}^{(2)}(2) = 1$ holds if, and only if, either $p \leq 2$ or $q \geq 2$.

(b) Let either $p \leq 2$ or $q \geq 2$. If $K_{p,q}^{(2)}(r) = 1$, then $\min(p, q') \leq r \leq \max(p', q)$.

(c) If $1 < p < \infty$ and $K_{p,p}^{(2)}(r) = 1$, then $\min(p, 2) \leq r \leq \max(2, p)$.

Proof. (a $\Leftarrow$) (cf. [47], [11]). Let either $p \leq 2$ or $q \geq 2$. An arbitrary operator $T: l_2^2 \rightarrow l_2^2$ is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $abcd \geq 0$, then the norm of T_c is attained on certain real vector which means that $\|T_c\|_{p,q} = \|T\|_{p,q}$.

Let $abcd < 0$ and let $q \geq 2$. Assume that the norm of the operator T_c is attained on the vector $z = (x, y)$ with $y \neq 0$. Put $y/x = \varrho(\cos \varphi + i \sin \varphi)$ and consider a function

$$\Phi_\varrho(\varphi) = (a^2 + 2ab\varrho \cos \varphi + b^2\varrho^2)^{q/2} + (c^2 + 2cd\varrho \cos \varphi + d^2\varrho^2)^{q/2}.$$

In the interval $(0, \pi)$ there is at most one extremal point of $\Phi_\varrho(\varphi)$ in which the derivative

$$\begin{aligned} \Phi'_\varrho(\varphi) = & -q\varrho \sin \varphi [ab(a^2 + 2ab\varrho \cos \varphi + b^2\varrho^2)^{q/2-1} \\ & + cd(c^2 + 2cd\varrho \cos \varphi + d^2\varrho^2)^{q/2-1}] \end{aligned}$$

changes its sign from $(-)$ to $(+)$. Therefore the maximum of $\Phi_\varrho(\varphi)$ is at $\varphi = 0$ or $\varphi = \pi$, i.e., the vector z is real.

Therefore we have proved that if $q \geq 2$, then $\|T_C\|_{p,q} = \|T\|_{p,q}$. Then case when $p \leq 2$ can be proved by using the dual operators.

(a $\Rightarrow$) To have such necessary conditions on indices p and q we should consider concrete operators.

Example 1. Let an operator $T: l_2^p \rightarrow l_2^q$ be given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. From Lemma 4 we have that the norm of the above operator between complex spaces is

$$\|T_C\|_{p,q} = \max \{2^{1-1/p}, 2^{1/p}, 2^{1/q-1/p+1/2}\}.$$

Here we also see that if either $p \leq 2$ or $q \geq 2$, then this is also the norm in the real case $\|T\|_{p,q} = \max \{2^{1-1/p}, 2^{1/q}\}$. On the other hand, if $q < 2 < p$, then some calculations show that

$$\|T\|_{p,q} = \sup_{0 \leq x \leq 1} \frac{[(x+1)^q + (1-x)^q]^{1/q}}{(x^p + 1)^{1/p}} < 2^{1/q-1/p+1/2} = \|T_C\|_{p,q}.$$

Therefore if $K_{l_2^p, l_2^q}^{(2)}(2) = 1$, then either $p \leq 2$ or $q \geq 2$.

(b) Let either $p \leq 2$ or $q \geq 2$. Then for the operator T given in Example 1 we have

$$\|T\|_{p,q} = \max \{2^{1-1/p}, 2^{1/q}\}.$$

For $f_1 = (a_1, b_1)$ and $f_2 = (a_2, b_2)$ we have

$$\begin{aligned} & \|(|Tf_1|^r + |Tf_2|^r)^{1/r}\|_q \\ &= [(|a_1 + b_1|^r + |a_2 + b_2|^r)^{q/r} + (|a_1 - b_1|^r + |a_2 - b_2|^r)^{q/r}]^{1/q} \end{aligned}$$

and

$$\|(|f_1|^r + |f_2|^r)^{1/r}\|_p = [(|a_1|^r + |a_2|^r)^{p/r} + (|b_1|^r + |b_2|^r)^{p/r}]^{1/p}.$$

Now by considering concrete f_1 and f_2 we obtain the desired results. For example if $f_1 = (0, 1)$ and $f_2 = (1, 0)$, then

$$K_{p,q}^{(2)}(r) \geq (2^{q/r} + 2^{q/r})^{1/q} / (2^{1/p} \|T\|_{p,q}) = \min \{2^{1/q+1/r-1}, 2^{1/r-1/p}\};$$

and if $f_1 = (1, 1)$ and $f_2 = (1, -1)$, then

$$K_{p,q}^{(2)}(r) \geq (2^q + 2^q)^{1/q} / [(2^{p/r} + 2^{p/r})^{1/p} \|T\|_{p,q}] = \min \{2^{1/q-1/r}, 2^{-1/p-1/r}\}.$$

Therefore, if $K_{p,q}^{(2)}(r) = 1$, then $\min(p, q') \leq r \leq \max(p', q)$.

(c) We want to prove that if $K_{p,p}^{(2)}(r) = 1$, then $r \in [\min(p, 2), \max(p, 2)]$. Again we require some specific operator. The idea in the construction below is taken from HERZ [14, Lemma 2].

Example 2. If $r < \min(p, 2)$ and $p < \infty$, then there exists an operator $T: l_2^p \rightarrow l_3^p$ such that for $T_E: l_2^p[l_2^p] \rightarrow l_3^p[l_2^p]$ we have $\|T\| < \|T_E\|$, i.e., $K_{p,p}^{(2)}(r) > 1$.

Proof. For $\alpha, \beta > 0$ let us consider the operators $T = T_{\alpha,\beta}: l_2^p \rightarrow l_3^p$ given by the matrix

$$\begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \\ 0 & \beta \end{pmatrix}.$$

Then

$$\|T_{\alpha,\beta}\|_{p,p} = \sup \{(|x + \alpha y|^p + |x - \alpha y|^p + |\beta y|^p)^{1/p} : x, y \in \mathbb{R} \text{ and } |x|^p + |y|^p = 1\}.$$

For a given α , $0 < \alpha < 1$, there exists β with $0 < \beta < 2$ such that the supremum is attained at $x = y = 2^{-1/p}$. Therefore

$$\|T_{\alpha,\beta}\|_{p,p} = 2^{-1/p}[(1 + \alpha)^p + (1 - \alpha)^p + \beta^p]^{1/p}.$$

For $f = (\alpha, 0)$ and $g = (0, \alpha)$ we have

$$\|(|f|^r + |g|^r)^{1/r}\|_p = \|(\alpha, \alpha)\|_p = \alpha 2^{1/p},$$

and

$$\begin{aligned} \|(|T_{\alpha,\beta}f|^r + |T_{\alpha,\beta}g|^r)^{1/r}\|_p &= \|((\alpha^r + \alpha^{2r})^{1/r}, (\alpha^r + \alpha^{2r})^{1/r}, \alpha\beta)\|_p \\ &= [2(\alpha^r + \alpha^{2r})^{p/r} + \alpha^p \beta^p]^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} \|(T_{\alpha,\beta})_E\|_{p,p} / \|T_{\alpha,\beta}\|_{p,p} &\geq [2(\alpha^r + \alpha^{2r})^{p/r} + \alpha^p \beta^p]^{1/p} / \{\alpha[(1 + \alpha)^p + (1 - \alpha)^p + \beta^p]^{1/p}\} \\ &= \{[2(1 + \alpha^r)^{p/r} + \beta^p] / [(1 + \alpha)^p + (1 - \alpha)^p + \beta^p]\}^{1/p}. \end{aligned}$$

Assume that $r < \min(p, 2)$ and $\|(T_{\alpha,\beta})_E\|_{p,p} = \|T_{\alpha,\beta}\|_{p,p}$. Then

$$2(1 + \alpha^r)^{p/r} + \beta^p \leq (1 + \alpha)^p + (1 - \alpha)^p + \beta^p$$

or

$$2(1 + \alpha^r)^{p/r} \leq (1 + \alpha)^p + (1 - \alpha)^p.$$

Applying Lemma 5, we obtain

$$2(1 + \alpha^r)^{p/r} \leq c_p \alpha^{\min(p,2)} + 2,$$

which cannot hold for small α since

$$\lim_{\alpha \rightarrow 0^+} [2(1 + \alpha^r)^{p/r} - 2] / \alpha^{\min(p,2)} = p \lim_{\alpha \rightarrow 0^+} [2(1 + \alpha^r)^{p/r-1}] / \alpha^{\min(p,2)-r} = \infty.$$

It was proved here that if $K_{p,p}^{(2)}(r) = 1$, then $\min(p, 2) \leq r$. The duality formula and the proved result show also that if $K_{p,p}^{(2)}(r) = 1$, then $r \leq \max(p, 2)$. Thus if $K_{p,p}^{(2)}(r) = 1$, then $\min(p, 2) \leq r \leq \max(p, 2)$.

Proof of Theorem 2. (a) Let $q = \infty$. Then according to Lemma 3 we obtain $K_{\infty,\infty}(1) = 1$. In fact,

$$\begin{aligned} \left\| \sum_{k=1}^n |Tf_k| \right\|_{\infty} &= \left\| \left\| \sum_{k=1}^n Tf_k r_k(t) \right\|_{\infty} \right\|_{\infty} \\ &= \left\| \left\| \tilde{T} \left(\sum_{k=1}^n f_k r_k(t) \right) \right\|_{\infty} \right\|_{\infty} \\ &\leq \|T\| \left\| \left\| \sum_{k=1}^n f_k r_k(t) \right\|_{\infty} \right\|_{\infty} \\ &= \|T\| \left\| \sum_{k=1}^n |f_k| \right\|_{\infty}. \end{aligned}$$

Since $K_{\infty, \infty}(\infty) = 1$ and $K_{\infty, \infty}(1) = 1$, it follows from Theorem 1(c) that $K_{\infty, \infty}(r) = 1$ for all r , $1 \leq r \leq \infty$. Hence using Theorem 1(b) we obtain that $K_{p, \infty}(r) = 1$ for $1 \leq p, r \leq \infty$. The equality $K_{1, q}(r) = 1$ can be obtained from an immediate application of Theorem 1(a) to 2^0 .

(b) First observe that if $p \leq r \leq q$, then $K_{p, q}(r) = 1$. In fact, using Theorem 1(b) we find that

$$K_{p, q}(r) \leq K_{r, q}(r) \leq K_{r, r}(r) = 1.$$

Next we prove that if either $r = 2$ or $q < r < 2$, then $K_{p, q}(r) = 1$ (the idea of the proof here is taken from MARCINKIEWICZ-ZYGMUND [28]).

Let (ω_k) be an r -stable Gaussian sequence. If either $r = 2$ or $q < r < \infty$, then

$$\begin{aligned} d_q(r)^q & \left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_q^q && \text{[by Lemma 2]} \\ &= \int_Y \int_0^1 \left| \sum_{k=1}^n Tf_k(x) \omega_k(t) \right|^q dt dv && \text{[by Fubini theorem and since } T \text{ is linear]} \\ &= \int_0^1 \left\| T \left(\sum_{k=1}^n \omega_k(t) f_k \right) \right\|_q^q dt && \text{[by boundedness of } T\text{]} \\ &\leq \|T\|^q \int_0^1 \left\| \sum_{k=1}^n \omega_k(t) f_k \right\|_q^q dt \\ &= \|T\|^q \left(\left\| \int_X \left| \sum_{k=1}^n \omega_k(t) f_k(x) \right|^p d\mu \right\|_{L^{q/p}[0, 1]} \right)^{q/p} \\ &&& \text{[by the Minkowski inequality; } q/p \geq 1\text{]} \\ &\leq \|T\|^q \left(\int_X \left\| \sum_{k=1}^n \omega_k(t) f_k(x) \right\|_{L^{q/p}[0, 1]}^p d\mu \right)^{q/p} \\ &= \|T\|^q \left\{ \int_X \left(\int_0^1 \left| \sum_{k=1}^n f_k(x) \omega_k(t) \right|^q dt \right)^{q/p} d\mu \right\}^{q/p} && \text{[by Lemma 2]} \\ &= d_q(r)^q \|T\|^q \left\{ \int_X \left(\sum_{k=1}^n |f_k(x)|^r dt \right)^{p/r} d\mu \right\}^{q/p} \\ &= d_q(r)^q \|T\|^q \left\| \sum_{k=1}^n |f_k|^r \right\|_q^{1/r}, \end{aligned}$$

and thus $K_{p, q}(r) \leq 1$.

Finally, if $2 < r < p$, then by using the duality formula of Theorem 1(a) and the above result we obtain

$$K_{p, q}(r) = K_{q', p'}(r') = 1.$$

Therefore if $1 < p \leq q < \infty$ and $\min(p, 2) \leq r \leq \max(q, 2)$, then $K_{p, q}(r) = 1$.

(c) By using Theorem 1(c) and the obvious inequalities we have

$$K_{p, q}(r) \geq K_{p, q}(2) \geq K_{p, q}^{(2)}(2) \geq K_{l_2^2, l_2^2}^{(2)}(2) \geq 1.$$

We conclude that if $K_{p,q}(r) = 1$, then $K_{l_2^p, l_2^q}(2) = 1$, and hence according to Proposition 3(a) either $p \leq 2$ or $q \geq 2$. But in view of Theorem 1(c) $K_{p,q}(r) \geq K_{p,q}(2) \geq K_{p,q}(2)$, and then $K_{p,q}(r) = 1$ implies that $K_{p,q}(2) = 1$. Thus from Proposition 3(b) we have

$$\min(p, q') \leq r \leq \max(p', q).$$

(d) Repeating the above arguments and using Proposition 3(c) we obtain the result.

(e) From Proposition 3(a) it only follows that if $q < 2 < p$, then $K_{p,q}(2) \geq K_{p,q}^{(2)}(2) > 1$. However, if $p > q$, then we can construct an operator such that $K_{p,q}^{(2)}(2) > 1$.

Example 3. (RIESZ [35], p. 493–494). For $\varepsilon > 0$ let $T^\varepsilon: l_3^p \rightarrow l_3^q$ be given by the formula

$$T^\varepsilon(x_1, x_2, x_3) = (y_1, y_2, y_3) \quad \text{with} \quad y_k = x_k - \varepsilon(x_1 + x_2 + x_3), \\ k = 1, 2, 3.$$

For any couple of numbers (p, q) such that $p > q \geq 1$ there exists $\varepsilon > 0$ such that $\|T^\varepsilon\| < \|T_C^\varepsilon\|$.

Proof. For $x_1 = 1$, $x_2 = \exp(2\pi i/3)$ and $x_3 = \exp(-2\pi i/3)$ we have $|x_1| = |x_2| = |x_3| = 1$, $x_1 + x_2 + x_3 = 0$ and so

$$\|T_C^\varepsilon\| \geq (|y_1|^q + |y_2|^q + |y_3|^q)^{1/q} / (|x_1|^p + |x_2|^p + |x_3|^p)^{1/p} = 3^{1/q-1/p}.$$

On the other hand, let x_1, x_2 and x_3 be real numbers. Without loss of generality we can assume that $|x_1|^p + |x_2|^p + |x_3|^p = 3$. Then we consider the following five cases.

1. If $x_1 + x_2 + x_3 = 0$, then

$$|y_1|^q + |y_2|^q + |y_3|^q = |x_1|^q + |x_2|^q + |x_3|^q < 3^{1-q/p}(|x_1|^p + |x_2|^p + |x_3|^p)^{q/p} = 3.$$

In the strict inequality we can have equality only if $|x_1| = |x_2| = |x_3|$, but then (since $|x_1|^p + |x_2|^p + |x_3|^p = 3$) it follows that $|x_1| = |x_2| = |x_3| = 1$ and this is impossible according to the equality $x_1 + x_2 + x_3 = 0$.

2. If $x_1 + x_2 + x_3 \neq 0$ and $q = 1$, then for sufficiently small $\varepsilon > 0$ it yields that

$$|y_1| + |y_2| + |y_3| = |x_1 - \varepsilon(x_1 + x_2 + x_3)| + |x_2 - \varepsilon(x_1 + x_2 + x_3)| \\ + |x_3 - \varepsilon(x_1 + x_2 + x_3)| \\ \leq |x_1| + |x_2| + |x_3| + \varepsilon|x_1 + x_2 + x_3| < \varepsilon^{1-1/p}(|x_1|^p + |x_2|^p + |x_3|^p)^{1/p} = 3.$$

3. Let $x_1 + x_2 + x_3 \neq 0$ and $q > 1$.

(a) If two of the numbers are equal to zero, for example if $x_1 \neq 0$ and $x_2 = x_3 = 0$, then

$$|y_1|^q + |y_2|^q + |y_3|^q = |x_1 - \varepsilon x_1|^q + |\varepsilon x_1|^q + |\varepsilon x_1|^q \\ = |x_1|^q [(1 - \varepsilon)^q + 2\varepsilon^q] \\ = 3^{q/p}[(1 - \varepsilon)^q + 2\varepsilon^q] < 3$$

for sufficiently small $\varepsilon > 0$.

(b–c) If either one of the numbers is zero, for example $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 = 0$, or all the numbers are different from zero, then for sufficiently small $\varepsilon > 0$ we have

$$\varepsilon |x_1 + x_2 + x_3| < |x_k|, \quad k = 1, 2 \text{ or } 3,$$

and using the binomial series we obtain

$$\begin{aligned} & |y_1|^q + |y_2|^q + |y_3|^q \\ &= |x_1|^q + |x_2|^q + |x_3|^q - q\varepsilon(|x_1|^{q-1} \operatorname{sgn} x_1 + |x_2|^{q-1} \operatorname{sgn} x_2 \\ &\quad + |x_3|^{q-1} \operatorname{sgn} x_3)(x_1 + x_2 + x_3) \\ &\quad + q(q-1)\varepsilon^2/2(|x_1|^{q-2} + |x_2|^{q-2} + |x_3|^{q-2})(x_1 + x_2 + x_3)^3 + \dots \\ &= |x_1|^q + |x_2|^q + |x_3|^q - q\varepsilon(|x_1|^{q-1} \operatorname{sgn} x_1 + |x_2|^{q-1} \operatorname{sgn} x_2 + |x_3|^{q-1} \operatorname{sgn} x_3) \\ &\quad \times (x_1 + x_2 + x_3) + O(\varepsilon^2) \\ &\leq 3^{1-q/p}(|x_1|^p + |x_2|^p + |x_3|^p)^{q/p} \\ &\quad - q\varepsilon(|x_1|^{q-1} \operatorname{sgn} x_1 + |x_2|^{q-1} \operatorname{sgn} x_2 + |x_3|^{q-1} \operatorname{sgn} x_3) \\ &\quad \times (x_1 + x_2 + x_3) + O(\varepsilon^2) \\ &= 3 - q\varepsilon(|x_1|^{q-1} \operatorname{sgn} x_1 + |x_2|^{q-1} \operatorname{sgn} x_2 + |x_3|^{q-1} \operatorname{sgn} x_3) \\ &\quad \times (x_1 + x_2 + x_3) + O(\varepsilon^2) < 3. \end{aligned}$$

Thus $\|T^\varepsilon\| < 3^{1/q-1/p}$ and so $K_{p,q}^{(2)}(2) \geq \|T_C^\varepsilon\|/\|T^3\| > 1$.

Theorem 1 and Theorem 2 imply immediately the following corollaries.

Corollary 1. (ANDERSEN [1], FIGIEL-IWANIEC-PELCZYNSKI [9]). *If $1 \leq p \leq q \leq \infty$, then $K_{p,q}(2) = 1$. In particular, if $p \leq q$, then the norm of the operator $T: L^p(\mu) \rightarrow L^q(\nu)$ and the norm of its complexification T_C are the same (see also [47], [9], [31]).*

Corollary 2. (HERZ [14]). *Let $1 < p < \infty$. Then $K_{p,p}(r) = 1$ if, and only if, $\min(p, 2) \leq r \leq \max(p, 2)$.*

Corollary 3. *If $p > q$, then $K_{p,q}(r) > 1$.*

Problem 2. Prove or disprove the following conjecture:

Let $1 < p \leq q < \infty$. Then $K_{p,q}(r) = 1$ if, and only if, $\min(p, 2) \leq r \leq \max(q, 2)$.

Now we will consider the case $p > q$.

Theorem 3. *Let $1 \leq q < p < \infty$.*

- (a) *If either $r = 2$ or $p < r < 2$, then $K_{p,q}(r) \leq d_p(r)/d_q(r)$.*
- (b) *If $1 \leq q \leq 2$, then $K_{2,q}(2) = d_2(2)/d_q(2)$.*
- (c) *If $r \neq 2$ and either $q < r \leq \min(p, 2)$ or $\max(q, 2) \leq r < p$, then $K_{p,q}(r) = \infty$.*

Proof. (a) By repeating the proof in Theorem 2(a) we obtain

$$\begin{aligned}
 d_q(r)^q & \left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_q^q \leq \|T\|^q \int_0^1 \left\| \sum_{k=1}^n \omega_k(t) f_k \right\|_p^q dt \\
 & \quad \text{[by the Hölder inequality; } p/q > 1] \\
 & \leq \|T\|^q \left\{ \int_0^1 \left\| \sum_{k=1}^n \omega_k(t) f_k \right\|_p^p dt \right\}^{q/p} \\
 & \quad \text{[by Fubini theorem]} \\
 & = \|T\|^q \left\{ \int_X \int_0^1 \left| \sum_{k=1}^n f_k(x) \omega_k(t) \right|^p dt dv \right\}^{q/p} \\
 & \quad \text{[by Lemma 2]} \\
 & = d_p(r)^q \|T\|^q \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_p^p,
 \end{aligned}$$

and thus $K_{p,q}(r) \leq d_p(r)/d_q(r)$.

(a) It is enough to prove that if $1 \leq q \leq 2$, then

$$(9) \quad K_{2,q}(2) \geq d_2(2)/d_q(2).$$

To prove estimate (9) we will construct a concrete operator and then calculate their appropriate norms.

Example 4. Let $1 \leq q \leq 2$ and let (g_i) be a sequence of independent standard Gaussian variables. Consider a linear operator $S: l^2 \rightarrow L^q[0, 1]$ defined by $Se_i = g_i$, for all i . Then

$$\|S\|_{l^2 \rightarrow L^q} = d_q(2),$$

and for any $n \in \mathbb{N}$ and all $f_1, f_2, \dots, f_n \in l^2$

$$\left\| \left(\sum_{k=1}^n |Sf_k|^2 \right)^{1/2} \right\|_q \leq d_2(2) \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_2.$$

In fact, if $f \in l^2$, then $f = \sum_i c_i e_i$ and by Lemma 2 we have

$$\|Sf\|_q = \left\| \sum_i c_i Se_i \right\|_q = \left\| \sum_i c_i g_i \right\|_q = d_q(2) \left(\sum_i c_i^2 \right)^{1/2} = d_q(2) \|f\|_2,$$

i.e., $\|S\|_{l^2 \rightarrow L^q} = d_q(2)$.

Furthermore, if $f_1, f_2, \dots, f_n \in l^2$, then $f_k = \sum_i c_{k,i} e_i$ ($k = 1, 2, \dots, n$) and since (g_i) is an orthogonal sequence in $L^2[0, 1]$, it follows that

$$\begin{aligned}
 & \left\| \left(\sum_{k=1}^n |Sf_k|^2 \right)^{1/2} \right\|_q^q = \int_0^1 \left| \left(\sum_{k=1}^n |Sf_k(t)|^2 \right)^{1/2} \right|^q dt \\
 & = \int_0^1 \left(\sum_{k=1}^n \left| \sum_i c_{k,i} g_i(t) \right|^2 \right)^{q/2} dt \leq \left(\int_0^1 \sum_{k=1}^n \left| \sum_i c_{k,i} g_i(t) \right|^2 dt \right)^{q/2}
 \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{k=1}^n \int_0^1 \left| \sum_i c_{k,i} g_i(t) \right|^2 dt \right)^{q/2} = \left(d_2(2)^2 \sum_{k=1}^n \sum_i c_{k,i}^2 \right)^{q/2} \\
&= d_2(2)^q \left(\sum_i \sum_{k=1}^n c_{k,i}^2 \right)^{q/2} = d_2(2)^q \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_2^q.
\end{aligned}$$

Therefore

$$\left\| \left(\sum_{k=1}^n |Sf_k|^2 \right)^{1/2} \right\|_q \leq d_2(2) \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_2.$$

Since $g_1^2, g_2^2, \dots$ are i.i.d. (independent identically distributed) random variables with the common mean equal to $d_2(2)$, it follows by the Kolmogorov strong law of large numbers that

$$\frac{1}{n} \sum_{k=1}^n g_k^2(t) \rightarrow d_2(2)^2 \quad \text{a.e.,}$$

and then by the Fatou Lemma we have

$$\begin{aligned}
&\liminf_{n \rightarrow \infty} \left\| \left(\sum_{k=1}^n |Se_k|^2 \right)^{1/2} \right\|_q / \left\| \left(\sum_{k=1}^n |e_k|^2 \right)^{1/2} \right\|_2 \\
&= \liminf_{n \rightarrow \infty} \left\| \left(\sum_{k=1}^n g_k^2 \right)^{1/2} \right\|_q / n^{1/2} = \liminf_{n \rightarrow \infty} \left[\int_0^1 \left(\frac{1}{n} \sum_{k=1}^n g_k^2(t) \right)^{q/2} dt \right]^{1/2} \\
&\geq \left[\int_0^1 \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{k=1}^n g_k^2(t)^{q/2} dt \right)^{1/q} \right]^{1/q} = \left[\int_0^1 (d_2(2)^2)^{q/2} dt \right]^{1/q} = d_2(2).
\end{aligned}$$

Thus formula (9) is proved.

(c) We just consider a concrete operator for which it yields $K_{r,q}(r) = \infty$ (concerning the existence of such an operator see Example 5 below). The rest of the cases follows from Theorem 1.

Example 5. Let $1 \leq q < r < 2$ and let (g_i) be a sequence of independent r -stable Gaussian variables. Consider a linear operator $S: l^r \rightarrow L^q[0, 1]$ defined by $Se_i = g_i$ for all i . Then

$$\|S\|_{l^r \rightarrow L^q} = d_q(r)$$

and

$$\begin{aligned}
K_{r,q}(r) &= \lim_{n \rightarrow \infty} K_{r,q}^{(n)}(r) \\
&\geq \liminf_{n \rightarrow \infty} \left\| \left(\sum_{k=1}^n |Se_k|^r \right)^{1/r} \right\|_q / \left[\left\| \left(\sum_{k=1}^n |e_k|^r \right)^{1/r} \right\|_r d_q(r) \right] \\
&= \liminf_{n \rightarrow \infty} \left\| \left(\sum_{k=1}^n |g_k|^r \right)^{1/r} \right\|_q / [n^{1/r} d_q(r)] \\
&= \liminf_{n \rightarrow \infty} \left[\int_0^1 \left(\frac{1}{n} \sum_{k=1}^n |g_k(t)|^r \right)^{q/r} dt \right]^{1/q} / d_q(r) = \infty,
\end{aligned}$$

and so by the Kolmogorov strong law of large numbers we have

$$\frac{1}{n} \sum_{k=1}^n |g_k(t)|^r \rightarrow \|g_1\|_r = \infty \quad \text{a.e.}$$

We complete this section by presenting two corollaries and a problem related to Theorem 3.

Corollary 4 (PISIER [34], Th. 5.4). $K_{2,1}(2) = \sqrt{\frac{\pi}{2}}$.

Corollary 5. *If $1 \leq q < p < \infty$, then*

$$(10) \quad K_{p,q}(2) \leq \min \{d_p(2)/d_q(2), d_q(2)/d_p(2)\}.$$

If either $p = 2$ or $q = 2$, then equality holds.

Problem 3. Give a complete characterization of the equality $K_{p,q}(r) = \infty$ in terms of p , q and r .

4. On the estimates of Krivine type

KRIVINE [21] proved that $K_{\infty,1}^{(2)}(2) = \sqrt{2}$. We generalize this result for the case $r \neq 2$. Let us first prove some useful facts about the expression $K_{p,q}^{(2)}(r)$.

Proposition 4. *The numbers $K_{p,q}^{(2)}(r)$ are the norms of the function*

$$t_r(x, y) = \cos(x - y) / [(|\sin x|^r + |\cos x|^r)^{1/r} (|\sin y|^{r'} + |\cos y|^{r'})^{1/r'}]$$

in $C_p([-\pi, \pi]) \hat{\otimes} C_q([-\pi, \pi])$.

Proof. By using Proposition 2 we find, as in KRIVINE paper ([21], Prop. 3), that for any representation

$$\sum_{k=1}^n f_k \otimes g_k \quad (f_k \text{ continuous on } S_m^r, g_k \text{ continuous on } S_n^{r'})$$

of $P_1 \otimes Q_1 + P_2 \otimes Q_2$, i.e.,

$$\sum_{k=1}^2 P_k \otimes Q_k(x, y) = x_1 y_1 + x_2 y_2 = \sum_{k=1}^n f_k \otimes g_k,$$

with $x = (x_1, x_2) \in S_m^r$, $y = (y_1, y_2) \in S_n^{r'}$, we can parametrize x and y as follows:

$$\begin{aligned} x_1 &= (|\sin x|^r + |\cos x|^r)^{-1/r} \cos x, & x_2 &= (|\sin x|^r + |\cos x|^r)^{-1/r} \sin x, \\ y_1 &= (|\sin y|^{r'} + |\cos y|^{r'})^{-1/r'} \cos y, & y_2 &= (|\sin y|^{r'} + |\cos y|^{r'})^{-1/r'} \sin y, \end{aligned}$$

and then we obtain the result.

We need also

Lemma 4. *If $1 \leq r \leq \infty$, then*

$$\max(2^{1/r}, 2^{1-1/r}) \leq K_{\infty,1}^{(2)}(r) \leq 2.$$

Proof. Consider the operator $T: l_2^\infty \rightarrow l_2^1$ given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then (cf. our Example 1) $\|T\| = 2$, and by considering $f_1 = (0, 1)$ and $f_2 = (1, 0)$ we find

$$K_{\infty,1}^{(2)}(r) \geq (2^{1/r} + 2^{1/r})/2 = 2^{1/r},$$

and by considering $f_1 = (1, 1)$ and $f_2 = (1, -1)$, we obtain

$$K_{\infty,1}^{(2)}(r) \geq (2 + 2)/(2^{1+1/r}) = 2^{1-1/r}.$$

It is easy to prove the second inequality using the inequalities

$$\max(a, b) \leq (a^r + b^r)^{1/r} \leq a + b \quad \text{for } a, b \geq 0.$$

The main result in this section is the following theorem:

Theorem 4. *If $1 \leq r \leq \infty$, then*

$$(11) \quad K_{\infty,1}^{(2)}(r) = \max(2^{1/r}, 2^{1-1/r}).$$

Proof. We already proved in Lemma 4 that $K_{\infty,1}^{(2)}(r) = 2$ in the cases $r = 1$ or $r = \infty$. Therefore we assume that $1 < r < \infty$. As it was proved by KRIVINE ([21], Th. 1), there exists $f \in C^\infty([-\pi, \pi])$ with the following properties.

- 1) f is an even function and $f(x + \pi) = -f(x)$ for $x \in [-\pi, \pi]$,
- 2) $f(x) = \sum_{k=0}^{\infty} \alpha_{2k+1} \cos(2k+1)x$, where $\alpha_1 > \sum_{k=1}^{\infty} |\alpha_{2k+1}|/(2k+1)$,
- 3) if $g(x) = \text{sign}(\cos x)$ and $F(x) = \int_{-\pi}^{\pi} f(t-x)g(t)dt/(2\pi) = \sum_{k=0}^{\infty} a_{2k+1} \cos(2k+1)x$,

then

$$\cos x = \sum_{k=0}^{\infty} b_{2k+1} F((2k+1)x),$$

where the numbers b_{2k+1} are obtained by taking the inverse of the Dirichlet series

$$D(s) = \sum_{k=0}^{\infty} a_{2k+1} (2k+1)^{-s} \quad (s \geq 0),$$

$$4) \quad \|f\|_\infty \sum_{k=0}^{\infty} |b_{2k+1}| \leq \sqrt{2}.$$

Now since

$$\min(2^{1/r-1/2}, 1) \leq (|\sin x|^r + |\cos x|^r)^{1/r} \quad \text{for each } x \in [-\pi, \pi],$$

it follows that

$$\begin{aligned}
 & \|t_r(x, y)\|_{\otimes} \\
 & \leq \sum_{k=0}^{\infty} |b_{2k+1}| \|F((2k+1)(x-y))(|\sin x|^r + |\cos x|^r)^{-1/r} \\
 & \quad \times (|\sin y|^{r'} + |\cos y|^{r'})^{-1/r'}\|_{\otimes} \\
 & \leq \sum_{k=0}^{\infty} |b_{2k+1}| \|f\|_{\infty} \|g\|_{\infty} \max(2^{1/2-1/r}, 1) \max(2^{1/2-1/r'}, 1) \\
 & \leq \max(2^{1/2-1/r}, 2^{1/r-1/2}) \|f\|_{\infty} \sum_{k=0}^{\infty} |b_{2k+1}| \leq \max(2^{1-1/r}, 2^{1/r}).
 \end{aligned}$$

Hence by using Proposition 3 and Lemma 4 we obtain the desired equality (11).

Problem 4. Calculate $K_{\infty,1}^{(n)}(r)$ for $n = 3, 4, \dots$

KRIVINE [21] proved that $K_{\infty,1}^{(3)}(2) < 1.517$ and $K_{\infty,1}^{(4)}(2) \leq \pi/2$.

5. On the estimates of $K_{p,q}^{(n)}(2)$

We wish to give the best estimates or, if possible, the exact formulae for the numbers $K_{p,q}^{(n)}(2)$ strictly connected with the complexification constants. The next estimates are our main result in this section.

Theorem 5. If $1 \leq q < p \leq \infty$, then

$$(12) \quad \max\{1, (d_2^{(n)})^2/[d_p^{(n)}d_q^{(n)}]\} \leq K_{p,q}^{(n)}(2) \leq \min\{d_p^{(n)}/d_q^{(n)}, d_q^{(n)}/d_p^{(n)}\}.$$

In particular, $K_{\infty,2}^{(n)}(2) = K_{2,1}^{(n)}(2) = d_2^{(n)}/d_1^{(n)}$.

Proof. In the same way as in Theorem 2 we can prove, by using Lemma 2 instead of Lemma 3, that

$$K_{p,q}^{(n)}(2) \leq d_p^{(n)}/d_q^{(n)}.$$

From which follows the duality formula $K_{p,q}^{(n)}(2) = K_{p',q'}^{(n)}(2) \leq d_q^{(n)}/d_p^{(n)}$ and the second inequality in (12). For the estimate from below we will construct again an special operator.

Example 6. Let $n \geq 2$ be fixed and let $\{\alpha_k\}_{k=1}^n$ be an orthogonal sequence of functions from Lemma 2. Define the operators $T_n: L^p(0, 1) \rightarrow L^q(0, 1)$ by

$$T_n f(x) = \sum_{k=1}^n a_k \alpha_k(x),$$

with

$$a_k = n \int_0^1 f(t) \alpha_k(t) dt, \quad k = 1, 2, \dots, n, \quad \text{and} \quad f \in L^p(0, 1).$$

Then

$$(13) \quad \|T_n\|_{p,q} = d_p^{(n)} d_q^{(n)} / (d_2^{(n)})^2.$$

Proof. Since

$$\|T_n f\|_q = d_q^{(n)} \left(\sum_{k=1}^n a_k^2 \right)^{1/2} = d_p^{(n)} / d_2^{(n)} \|T_n f\|_2,$$

it follows that

$$\|T_n\|_{p,q} = d_q^{(n)} / d_2^{(n)} \|T_n\|_{p,2}.$$

But $\|T_n\|_{p,2} = \|T_n^*\|_{2,p'}$, where the conjugate operator is given by

$$T_n^* g(x) = \sum_{k=1}^n b_k \alpha_k(x),$$

where

$$b_k = n \int_0^1 g(t) \alpha_k(t) dt, \quad k = 1, 2, \dots, n.$$

But Parseval's equality implies that $\|T_n^*\|_{2,p'}$ is attained if the function is of the form

$$g(x) = \sum_{k=1}^n c_k \alpha_k(x).$$

Since the $L^2(0,1)$ norm of this function is equal to $d_2^{(n)} \left(\sum_{k=1}^n c_k \right)^{1/2}$, it follows that

$$\|T_n\|_{p,2} = \|T_n^*\|_{2,p'} = d_p^{(n)} / d_2^{(n)}$$

and hence (13) is proved.

Since

$$T_n(\alpha_k) = \alpha_k \quad \text{and} \quad \sum_{k=1}^n \alpha_k^2 = 1,$$

it follows that

$$\left\| \left(\sum_{k=1}^n |T_n(\alpha_k)|^2 \right)^{1/2} \right\|_q = \left\| \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} \right\|_p = 1$$

and so

$$\begin{aligned} K_{p,q}^{(n)}(2) &\geq \left\| \left(\sum_{k=1}^n |T_n(\alpha_k)|^2 \right)^{1/2} \right\|_q / \left[\left\| \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} \right\|_p \|T_n\|_{p,q} \right] \\ &= 1 / \|T_n\|_{p,q} = (d_2^{(n)})^2 / [d_p^{(n)} d_q^{(n)}]. \end{aligned}$$

These estimates also prove the first inequality in (12).

Proposition 5. Let $p > q \geq 1$. If $1/p + 1/q \geq 1$, then $\min \{d_p^{(n)} / d_q^{(n)}, d_q^{(n)} / d_{p'}^{(n)}\} = d_p^{(n)} / d_q^{(n)}$, and if $1/p + 1/q < 1$, then $\min \{d_p^{(n)} / d_q^{(n)}, d_q^{(n)} / d_{p'}^{(n)}\} = d_q^{(n)} / d_{p'}^{(n)}$.

Proof. According to (13) we have

$$\|T_n\|_p := \|T_n\|_{p,p} = d_p^{(n)} d_p^{(n)} / (d_2^{(n)})^2.$$

If $1/p + 1/q \geq 1$, then $q < p \leq q'$ and, by using the Riesz-Thorin interpolation theorem (see [2] or [3]), we find

$$\|T_n\|_p \leq \|T_n\|_q^\theta \|T_n\|_{q'}^{1-\theta} = \|T_n\|_q.$$

This means that

$$d_{q'}^{(n)} / d_p^{(n)} \geq d_p^{(n)} / d_q^{(n)}.$$

In the case $1/p + 1/q < 1$ the proof is similar.

In the special case $n = 2$ the estimates (12) are particularly interesting in the complexification problem. Recall that in this case the constant γ in the complexification procedure is equal to the constant $K_{L^p(\mu), L^q(\nu)}^{(2)}(2)$. Therefore, according to Theorem 2, Theorem 4 and Theorem 5, we get immediately the following results about the constant γ and $K_{p,q}^{(2)}(2) = \sup_{\mu, \nu} \gamma(L^p(\mu), L^q(\nu))$.

Corollary 6. *The following relations hold:*

$$(a) \quad 1 \leq \gamma(L^p(\mu), L^q(\nu)) \leq \sup_{\mu, \nu} \gamma(L^\infty(\mu), L^1(\nu)) = K_{\infty,1}^{(2)}(2) = \sqrt{2}.$$

(b) *If $1 \leq p \leq q \leq \infty$, then $\gamma(L^p(\mu), L^q(\nu)) = 1$ for arbitrary σ -finite measures μ and ν .*

(c) *If $1 \leq q < p < \infty$, then*

$$(14) \quad \max(1, [d_2^{(2)}]^2 / [d_p^{(2)} d_q^{(2)}]) \leq K_{p,q}^{(2)}(2) \leq \min(d_p^{(2)} / d_q^{(2)}, d_{q'}^{(2)} d_{p'}^{(2)}).$$

$$\text{In particular } K_{\infty,2}^{(2)}(2) = \frac{\pi}{2\sqrt{2}}.$$

In [43] it was shown that if $1 \leq p \leq q \leq \infty$, then $\gamma(l_n^p, l_n^q) = 1$. By using the equality (1), ZYGMUND [50, p. 181] proved that $\gamma(L^p(\mu), L^p(\nu)) = 1$. Moreover, by using an idea of ZYGMUND, SEREDA-VERBICKII [47, Th. 1] proved that $\gamma(L^p(\mu), L^q(\nu)) = 1$ for arbitrary σ -finite measures μ and ν , provided that $1 \leq p \leq q \leq \infty$. Formula (14) was also proved by VERBICKII [46].

Remark 2. The constant γ depends essentially on the measures μ and ν . For example, $\gamma(l_2^p, l_2^q) = 1$ if, and only if, either $p \leq 2$ or $2 \leq q$ (cf. Proposition 3(a)). GASCH [11] proved that $\gamma(l_2^p, l_2^q) = \gamma(l_2^\infty, l_2^q) = \max(1, 2^{1-p/2})$.

Problem 5. Let $1 \leq q < p < \infty$. Compute $\gamma(l_m^p, l_n^q)$ for any $m, n \geq 2$ and also $\gamma(l^p, l^q)$.

6. On some estimates in Banach lattices

We now make some general observation regarding the constants K where general Banach lattices X instead of the Lebesgue spaces L^p are considered.

Let $n \in \mathbb{N}$ and $1 \leq r \leq \infty$. For any real Banach lattices X, Y and any continuous linear operator $T: X \rightarrow Y$ let us denote

$$K_{X,Y}^{(n)}(r) = \inf \left\{ C \geq 1 : \left\| \left(\sum_{k=1}^n |Tx_k|^r \right)^{1/r} \right\|_Y \leq C \|T\| \left\| \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_X \right. \\ \left. \text{for all } x_1, \dots, x_n \in X \text{ and any above } T \right\},$$

and we write $K_{X,X}^{(n)}(r) = K_X^{(n)}(r)$.

Theorem 6. (a) *The following equality holds:*

$$\sup \{K_{X,Y}^{(n)}(r) : X, Y \text{ Banach lattices}\} = K_{\infty,1}^{(n)}(r).$$

(b) *If X, Y are Banach function spaces and $Y'' \equiv Y$, then $K_{X,L^\infty}^{(n)}(2) = K_{L^1,Y}^{(n)}(2) = 1$.*

(c) *There exists a two-dimensional symmetric space X for which $K_X^{(2)}(2) > 1$.*

Proof. (a) The proof in a general case of Banach lattices X, Y can be found in [19] or [25]. Here we include a proof in the case of Banach function spaces X, Y with $Y'' \equiv Y$ (see also [34]).

For $x_1, \dots, x_n \in X(\mu)$ let us put

$$\psi = \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \quad \text{and} \quad y_k = x_k/\psi$$

(with the convention $0/0 = 0$), $k = 1, 2, \dots, n$. Clearly, $y_k \in L^\infty(\mu)$.

Define $M: L^\infty(\mu) \rightarrow X(\mu)$ as a multiplication by the function ψ , i.e., $Mx = x \cdot \psi$. Then

$$\|M\|_{L^\infty \rightarrow X} \leq \|\psi\|_X, \quad My_k = x_k \quad \text{and} \quad \left\| \left(\sum_{k=1}^n |y_k|^r \right)^{1/r} \right\|_X \leq 1.$$

Let φ be an arbitrary positive function $\varphi \in Y'$ with $\|\varphi\|_{Y'} \leq 1$. Define $N: Y(v) \rightarrow L^1(v)$ as a multiplication operator by φ , i.e., $Ny = y \cdot \varphi$.

Consider now the operator $NTM: L^\infty(\mu) \rightarrow L^1(v)$. For such an operator we have

$$\left\| \left(\sum_{k=1}^n |NTMy_k|^r \right)^{1/r} \right\|_{L^1} \leq K_{\infty,1}^{(n)}(r) \|NTM\|_{L^\infty \rightarrow L^1} \left\| \left(\sum_{k=1}^n |y_k|^r \right)^{1/r} \right\|_{L^\infty} \\ \leq K_{\infty,1}^{(n)}(r) \|N\|_{Y \rightarrow L^1} \|TM\|_{L^\infty \rightarrow Y} \leq K_{\infty,1}^{(n)}(r) \|TM\|_{L^\infty} \\ \leq K_{\infty,1}^{(n)}(r) \|T\|_{X \rightarrow Y} \|M\|_{L^\infty \rightarrow X} \leq K_{\infty,1}^{(n)}(r) \|T\|_{X \rightarrow Y} \|\psi\|_X.$$

Since $NTM(y_k) = NT(x_k) \cdot \varphi$, it follows

$$\int \left(\sum_{k=1}^n |Tx_k|^r \right)^{1/r} \cdot \varphi \, dv \leq K_{\infty,1}^{(n)}(r) \|T\|_{X \rightarrow Y} \|\psi\|_X.$$

Taking the supremum over all above φ we obtain

$$\left\| \left(\sum_{k=1}^n |Tx_k|^r \right)^{1/r} \right\|_Y = \left\| \left(\sum_{k=1}^n |Tx_k|^r \right)^{1/r} \right\|_{Y''} \leq K_{\infty,1}^{(n)}(r) \|T\|_{X \rightarrow Y} \|\psi\|_X \\ = K_{\infty,1}^{(n)}(r) \|T\|_{X \rightarrow Y} \left\| \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_X,$$

and so $K_{X,Y}^{(n)}(r) \leq K_{\infty,1}^{(n)}(r)$.

(b) If $T: X \rightarrow L^\infty(v)$ is a bounded operator, then, by using Lemma 1 twice with $p = \infty$, we obtain for all $f_1, \dots, f_n \in X$

$$\begin{aligned} \left\| \left(\sum_{k=1}^n |Tf_k|^2 \right)^{1/2} \right\|_{L^\infty(v)} &= \left\| \left\| \sum_{k=1}^n Tf_k \alpha_k(t) \right\|_{L^\infty(0,1)} \right\|_{L^\infty(v)} \\ &= \left\| \left\| T \left(\sum_{k=1}^n f_k \alpha_k(t) \right) \right\|_{L^\infty(v)} \right\|_{L^\infty(0,1)} \\ &\leq \|T\|_{X \rightarrow L^\infty} \left\| \left\| \sum_{k=1}^n f_k \alpha_k(t) \right\|_X \right\|_{L^\infty(0,1)} \\ &\leq \|T\|_{X \rightarrow L^\infty} \left\| \left\| \sum_{k=1}^n f_k \alpha_k(t) \right\|_{L^\infty(0,1)} \right\|_X \\ &= \|T\|_{X \rightarrow L^\infty} \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_X, \end{aligned}$$

and so $K_{X, L^\infty}^{(n)}(2) = 1$.

Now if $T: L^1(\mu) \rightarrow Y$ is a bounded operator, then again using Lemma 1 twice with $p = 1$, the generalized Minkowski inequality (cf. [18], p. 45) and Fubini theorem, we get

$$\begin{aligned} d_1^{(n)} \left\| \left(\sum_{k=1}^n |Tf_k|^2 \right)^{1/2} \right\|_Y &= \left\| \left\| \sum_{k=1}^n Tf_k \alpha_k(t) \right\|_{L^1(0,1)} \right\|_Y \\ &\leq \left\| \left\| \sum_{k=1}^n Tf_k \alpha_k(t) \right\|_Y \right\|_{L^1(0,1)} \leq \|T\|_{L^1 \rightarrow Y} \\ &\quad \times \left\| \left\| \sum_{k=1}^n f_k \alpha_k(t) \right\|_{L^1(\mu)} \right\|_{L^1(0,1)} \\ &= \|T\|_{L^1 \rightarrow Y} \left\| \left\| \sum_{k=1}^n f_k \alpha_k(t) \right\|_{L^1(0,1)} \right\|_{L^1(\mu)} \\ &= d_1^{(n)} \|T\|_{L^1 \rightarrow Y} \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_X. \end{aligned}$$

for all $f_1, \dots, f_n \in L^1(\mu)$.

Therefore $K_{L^1, Y}^{(n)}(2) = 1$.

(c) It is proved by presenting the following two-dimensional example:

Example 7. (see also [40]). The two-dimensional space $X = \mathbb{R}^2$ with the norm of $x = (x_1, x_2)$ given by

$$\|x\| = |x_1| + |x_2| + \sqrt{2} \max(|x_1|, |x_2|)$$

is a symmetric space. For the operator

$$T: X \rightarrow X \quad \text{defined by} \quad T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we have the inequalities

$$\begin{aligned}\|Tx\| &\leq |x_1 + x_2| + |x_1 - x_2| + \sqrt{2} \max(|x_1 + x_2|, |x_1 - x_2|) \\ &\leq 2 \max(|x_1|, |x_2|) + \sqrt{2} (|x_1| + |x_2|) = \sqrt{2} \|x\|\end{aligned}$$

with the equality for $x = (1, 0)$ or $x = (x_1, x_1)$, i.e., $\|T\| = \sqrt{2}$.

On the other hand, for $x_0 = (1, 1 + i)$ we obtain

$$\|T_C x_0\|/\|x_0\| = [(1/\sqrt{2} + 1)\sqrt{5} + 1]/(3 + \sqrt{2}) > \sqrt{2},$$

which means that $\|T\| < \|T_C\|$ and so $K_X^{(2)}(2) > 1$.

We remark that an example of a four-dimensional non-symmetric space X for which $K_X^{(2)}(2) > 1$ was given by VERBICKII-SEREDA [47]. Namely, if $X = l_2^1 \oplus l_2^\infty$ is the direct sum of the two-dimensional spaces l_2^1 and l_2^∞ with the norm of $x = (x_1, x_2, x_3, x_4)$ equal to

$$\|x\| = |x_1| + |x_2| + \max(|x_3|, |x_4|),$$

and if operator $S: X \rightarrow X$ is given by

$$S = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then

$$\|S\|_{X \rightarrow X} = \|T\|_{\infty, 1} = 2, \quad \|S_C\|_{X_C \rightarrow X_C} = \|T_C\|_{\infty, 1} = 2\sqrt{2}$$

and thus

$$K_X^{(2)}(2) \geq \|S_C\|_{X_C \rightarrow X_C} / \|S\|_{X \rightarrow X} = \sqrt{2}.$$

Finally we present a corollary of Theorem 6(a) and a problem connected to Theorem 6(c).

Corollary 7. For arbitrary Banach lattices X and Y

$$K_{X,Y}^{(2)}(r) \leq K_{\infty,1}^{(2)}(r) = \max(2^{1-1/r}, 2^{1/r}).$$

Problem 6. Characterize the Banach lattices X for which $K_X^{(2)}(2) = 1$.

Remark 3. If we take finite dimensional subspaces $X_m \subset L^p$ and $Y_n \subset L^q$, then it is not true in general that

$$\lim_{m,n \rightarrow \infty} K_{X_m, Y_n}^{(2)}(2) = K_{p,q}^{(2)}(2).$$

VERBICKII [46] proved that if $X_{m,p}$ are subspaces of $L^p(0, 2\pi)$ of all trigonometric polynomials of degree m with the L^p -norm and similarly $X_{n,q}$ are subspaces of $L^q(0, 2\pi)$ of all trigonometric polynomials of degree n with the L^q -norm, then for $p > q \geq 1$ and $1/p + 1/q < 1$

$$K_{X_m, Y_n}^{(2)}(2) = d_p^{(2)}/d_q^{(2)} > d_q^{(2)}/d_p^{(2)} = K_{p,q}^{(2)}(2)$$

holds.

7. Some additional results and remarks

7.1. The complex case

7.1.1. The constants in the complex case

If $L^p(\mu)$ and $L^q(\nu)$ are spaces over the complex field $\mathbb{C}$, then equalities (2) and (3) hold but with different constants (cf. [16] or [32], p. 289 and p. 299, respectively):

$$d_{p,c}^{(n)} = [\Gamma((p+2)/2) \Gamma(n) / (\Gamma((p+2n)/2))]^{1/p},$$

$$d_{p,c}(r) = 2[\Gamma(1-p/r) \Gamma((p+2)/2) / \Gamma((1-p/2))]^{1/p}.$$

Therefore all our results where we have used the equalities (2) or (3) can be extended to the complex spaces. For example, in the complex case we have then

$$K_{2,1}^{\mathbb{C}}(2) = \frac{2}{\sqrt{\pi}}$$

(in the real case it was $K_{2,1}(2) = \sqrt{\frac{\pi}{2}}$, cf. Corollary 4).

7.1.2. The Grothendieck constants in the real and the complex case

The Grothendieck inequality has important applications in the theory of Banach spaces (cf. [12], [19], [24], [25], [31], [34]). Let us note that in our notations it means

$$K_{\infty,1}(2) = K_G := \lim_{n \rightarrow \infty} K_G(n) = \lim_{n \rightarrow \infty} K_{\infty,1}^{(n)}(2)$$

and

$$K_{\infty,1}^{\mathbb{C}}(2) = K_G^{\mathbb{C}} := \lim_{n \rightarrow \infty} K_G^{\mathbb{C}} = \lim_{n \rightarrow \infty} K_{\infty,1}^{(n),\mathbb{C}}(2).$$

The above complex constants are always smaller than the real ones. The exact values of K_G and $K_G^{\mathbb{C}}$ are not known although the study about these constants is intense (cf. [13] and [31]): GROTHENDIECK (1956), KAUSSER (1973), RIETZ (1974), MAUREY (1974), PISIER (1978), KRIVINE (1979), DAVIE (1984), HAAGERUP (1987), KÖNIG (1990). The best known upper estimates for the Grothendieck constant K_G and $K_G^{\mathbb{C}}$ are due to KRIVINE [21]:

$$K_G \leq \frac{\pi}{2 \ln(1 + \sqrt{2})} \approx 1.782$$

and HAAGERUP [13]:

$$K_G^{\mathbb{C}} \leq \frac{8}{\pi(k_0 + 1)} \approx 1.405,$$

where k_0 is the solution of a certain equation involving elliptic integrals. The upper estimate for K_G is conjectured to be the exact value by KRIVINE, whereas the bound for K_G^C by HAAGERUP is not considered to be precise.

Very little is known on the estimates from below. The best known of these estimates are $K_G > \pi/2$ (announced by KRIVINE [21] and unpublished of DAVIE — proof cf. [17]) and $K_G^C > 1.338$ (in particular $K_G^C > 4/\pi$). From our estimate (12) we can get that $K_G \geq \pi/2$ which was already known by Grothendieck.

The real constants $K_G(n)$ have been estimated by KRIVINE [21], using the spherical function technique (cf. our Sections 2 and 4). TONGE [45], started to estimate the numbers $K_G^C(n)$ for $n = 2$ and 3. He obtained $K_G^C(2) \leq 2/\sqrt{2} \approx 1.213$ and $K_G^C(2) \leq 3/2$. After that KÖNIG [17], by modifying and combining techniques of KRIVINE [21] and HAAGERUP [13], gave better estimates from above and also estimates from below for the constants $K_G^C(n)$.

7.1.3. An interpolation proof of a result of HERZ in the complex case

If

$$T: L^p(\mu) \rightarrow L^q(\nu), \quad 1 \leq p < \infty,$$

is a bounded operator, then the natural extension

$$T_E: L^q(I_n^r) \rightarrow L^p(I_n^r)$$

is also bounded for $r = 2$ [using the equality (3) in the complex case] and $r = p$. Using the complex interpolation method (cf. [2], Th. 5.1.2) we have

$$[L^p(I_n^2), L^p(I_n^p)]_\theta = L^p(I_n^r)$$

with equality of the norms, when $1/r = (1 - \theta)/2 + \theta/p$, $0 \leq \theta \leq 1$, and so

$$\begin{aligned} \|T_E\|_{L^p(I_n^2) \rightarrow L^p(I_n^r)} &\leq (\|T_E\|_{L^p(I_n^2) \rightarrow L^p(I_n^2)})^{1-\theta} (\|T_E\|_{L^p(I_n^p) \rightarrow L^p(I_n^p)})^\theta \\ &\leq (\|T\|_{p,p})^{1-\theta} (\|T\|_{p,p})^\theta = \|T\|_{p,p}. \end{aligned}$$

Therefore we get our Corollary 1:

$$\text{If } \min(p, 2) \leq r \leq \max(p, 2), \text{ then } K_{p,p}(r) = 1.$$

This possibility of the proof was suggested also by HERZ-RIVIERE [15], p. 512 and MCCARTHY [29], p. 210.

7.2. The B -valued extensions of operators

The estimates discussed above suggests the following general scheme. For a given Banach space B and for any bounded linear operator

$$T: L^p(\mu) \rightarrow L^q(\nu)$$

we denote by

$$T \otimes I: L^p(\mu) \otimes B \rightarrow L^q(\nu) \otimes B$$

the extension which on B -valued function of the form

$$g = \sum_{i=1}^n f_i(x) b_i, \quad f_i \in L^p(\mu), \quad b_i \in B \quad (i = 1, 2, \dots, n, n = 1, 2, \dots)$$

is given by:

$$T \otimes I \left(\sum_{i=1}^n f_i(x) b_i \right) = \sum_{i=1}^n T f_i(x) b_i.$$

If this operator $T \otimes I$ is bounded from $L^p(\mu, B)$ into $L^q(\nu, B)$, then it has an extension on all $L^p(\mu, B)$ by continuity; denote this extension by T^B . We are looking for Banach spaces B such that, this operator $T \otimes I$ has a B -valued bounded extension $T^B: L^p(\mu, B) \rightarrow L^q(\nu, B)$ and $\|T^B\| \leq C_{p,q,B} \|T\|$.

This question was considered and solved by HERZ [14], KWAPIEN [22] and RUBIO DE FRANCIA-TORREA [39] (see also [10]), but nothing is said about the constant $C_{p,q,B}$. If we know, additionally, that T is a positive operator, then for any Banach space B , $T \otimes I$ has a B -valued extension T^B and the norms are the same $\|T^B\| = \|T\|$ (cf. [36], Th. 1 or [10], Th. 1.12).

7.3. Estimates in other spaces

7.3.1. A more general version of the Marcinkiewicz-Zygmund theorem

The same proof as in the MARCINKIEWICZ-ZYGMUND paper [28] (cf. also [8], p. 203–204) giving a more general version of the Marcinkiewicz-Zygmund theorem:

If $1 \leq p < \infty$ and X is a closed subspace of $L^p(\mu)$, then $K_{X, L^p(\nu)}^{(n)}(2) = 1$ for all $n \in \mathbb{N}$.

Our proof of Theorem 2(b) establishes even more (cf. also [9], Th. 1.1):

If $1 \leq p \leq q \leq \infty$ and X is a closed subspace of $L^p(\mu)$, then $K_{X, L^q(\nu)}^{(n)}(2) = 1$ for all $n \in \mathbb{N}$.

7.3.2. Weak type inequalities

We can consider also operators from $L^p(\mu)$ -space into a bigger space than the $L^q(\nu)$ space, namely the so called weak- L^q space or Marcinkiewicz space $L^{q,\infty}$; such operators are called of weak type (p, q) . Since the weak- L^q norm of a function has an equivalent form

$$\|f\|_{q,\infty} \approx \sup_{\nu(A) < \infty} \nu(A)^{1/q-1/s} \|f\chi_A\|_s, \quad 0 < s < q \leq \infty,$$

as was observed first by COTLAR [5], Th. 2a (see also [36], p. 652), can be used to prove results about strong type operators. However, in this case we have no control over the constants. Such results for the weak-operators were given by OBERLIN [30] and RUBIO DE FRANCIA [36] (see also [10], Th. 2.9).

7.3.3. Results for other spaces

If X is a Banach function space with the Fatou norm and $1 \leq q \leq \infty$, then $X^{(q)}$ is the space of all measurable f for which $|f|^q \in X$, with the norm

$$\|f\|_{X^{(q)}} = \| |f|^q \|_X^{1/q}.$$

We can consider the case of real operators $T: L^p(\mu) \rightarrow X^{(q)}$ and we can prove, in the same way as in Theorem 2, that if either $r = 2$ or $1 \leq p < r < 2$, then

$$\left\| \left(\sum_{k=1}^n |Tf_k|^r \right)^{1/r} \right\|_{X^{(q)}} \leq d_p(r)/d_{\min(p,q)}(r) \left\| \left(\sum_{k=1}^n |f_k|^r \right)^{1/r} \right\|_p$$

(see also [1]). This generalization contains both strong (p, q) -type operators and weak (p, q) -type operators.

Some results for operators between spaces of functions with finite variation, spaces of continuous functions and C^* -algebras were obtained by ZYGMUND [49], VERBICKII-SEREDA [47] and PISIER [33].

Note 1. On the occasion of Professor MISCHA COTLAR seventy-fifth birthday a special session was organized in his honor at the Universidad de Buenos Aires, Argentina, on April 18, 1988. MISCHA COTLAR's bibliography, talks by A. P. CALDERÓN, J. L. MASSERA, M. SADOSKY about his work and a collection of 34 papers dedicated to him were presented. A list of publications and a survey of his contributions in mathematics, written by J. HORVÁTH and C. SADOSKY are published in "Analysis and Partial Differential Equations", Edited by C. SADOSKY, MARCEL DEKKER, New York and Basel 1990, xxvii + 742 pp.

Note 2. The results of Sections 2 and 4 were presented by the first named author on the XXXIX Anual Convención Anual AsoVAC held at Caracas — Venezuela, November 22, 1989 and XL Anual Convención AsoVAC held at Cumaná — Venezuela, November 20, 1990. The results of Sections 2–5 were presented by the second named author on The 21st Nordic Congress of Mathematicians held at Luleå — Sweden, June 8–12, 1992 and on The 3rd International Conference on Function Spaces held at Poznan — Poland, Aug. 30–Sept. 4, 1992.

Note 3 (added in May 1993). Referee of this paper pointed out that A. DEFANT, in the forthcoming paper "Best constants for the norm of the complexification of operators between L_p -spaces", proved the following result:

If either $1 \leq q \leq p \leq 2$ or $2 \leq q \leq p \leq \infty$, then $K_{p,q}^{(2)}(2) = d_p^{(2)}/d_q^{(2)}$.

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