Sharp Weighted Multidimensional Integral Inequalities
for Monotone Functions

By Sorina Barza and Lars-Erik Persson of Luleå, and Javier Soria of Barcelona

(Received May 21, 1997)
(Revised Version March 9, 1998)

Abstract. We prove sharp weighted inequalities for general integral operators acting on monotone functions of several variables. We extend previous results in one dimension, and also those in higher dimension for particular choices of the weights (power weights, etc.). We introduce a new kind of conditions, which take into account the more complicated structure of monotone functions in dimension $n > 1$, and give an example that shows how intervals are not enough to characterize the boundedness of the operators (contrary to what happens for $n = 1$). We also give several applications of our techniques.

1. Introduction

Let $f(x)$ be a nonnegative, measurable and monotone function on $[0, \infty)$ and let $T_i$ be linear integral operators given by

$$T_i f(x) = \int_0^\infty k_i(x,y)f(y) \, dy, \quad i = 1, 2,$$

where $k_i(x,y)$ are nonnegative kernels on $[0, \infty) \times [0, \infty)$. We also permit the limiting cases when $k_i(x,y) = \delta_x(y)$ so that $T_i f(x) = f(x)$ for $i = 1$ and/or $i = 2$. Some inequalities of the type

$$\left( \int_0^\infty (T_1 f(x))^q \omega(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty (T_2 f(x))^p v(x) \, dx \right)^{1/p},$$

where $0 < p \leq q < \infty$ and $\omega(x)$ and $v(x)$ are weight functions on $[0, \infty)$, have recently been studied, e.g. necessary and sufficient conditions for such inequalities to hold have been proved and the best constants $C$ are found out (see e.g. [CS1], [CS2], [HM], [L], [M], [MPS] and [S1]).
The following theorem summarizes the main information of this type for the case when \(0 \leq f\) is a decreasing function.

**Theorem 1.1.** Let \(f\) be a decreasing function and let \(v(x)\) and \(\omega(x)\) be weight functions on \([0, \infty)\).

(a) If \(0 < p \leq 1 \leq q < \infty\), then
\[
\left( \int_0^\infty (T_1f(x))^q \omega(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty (T_2f(x))^p v(x) \, dx \right)^{1/p},
\]
where
\[
C = \sup_{t > 0} \left( \int_0^t k_1(x, y) \, dy \right)^q \omega(x) \, dx \right)^{1/q} \left( \int_0^t v(x) \, dx \right)^{-1/p} < \infty.
\]

(b) If \(0 < p \leq 1\) and \(p \leq q < \infty\), then
\[
\left( \int_0^\infty (T_1f(x))^q \omega(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty f^p(x) v(x) \, dx \right)^{1/p},
\]
where
\[
C = \sup_{t > 0} \left[ \left( \int_0^t k_1(x, y) \, dy \right)^q \omega(x) \, dx \right)^{1/q} \left( \int_0^t v(x) \, dx \right)^{-1/p} \] < \infty.
\]

(c) If \(0 < p \leq q < \infty\) and \(1 \leq q < \infty\), then
\[
\left( \int_0^\infty f^q(x) \omega(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty (T_2f(x))^p v(x) \, dx \right)^{1/p},
\]
where
\[
C = \sup_{t > 0} \left[ \left( \int_0^t \omega(x) \, dx \right)^{1/q} \left( \int_0^t k_2(x, y) \, dy \right)^p v(x) \, dx \right)^{-1/p} \] < \infty.
\]

(d) If \(0 < p \leq q < \infty\), then
\[
\left( \int_0^\infty f^q(x) \omega(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty f^p(x) v(x) \, dx \right)^{1/p},
\]
where
\[
C = \sup_{t > 0} \left[ \left( \int_0^t \omega(x) \, dx \right)^{1/q} \left( \int_0^t v(x) \, dx \right)^{-1/p} \] < \infty.
\]

The constant \(C\) is the least possible in all cases.
Remark 1.2. The corresponding theorem can be stated for the case when \(0 \leq f\) is measurable and increasing. In this case we only need to change the integrals over \([0, t]\) in the expression for \(C\) to integrals over \([t, \infty)\).

Remark 1.3. Part (a) is just a consequence of Minkowski's integral inequality as was pointed out in [HM, Theorem 3.2]. Part (b) is a special case of [CS1, Theorem 2.4] (c.f. also [MPS, Theorem 2.1], [HM, Theorem 3.2] and with the additional restriction \(q \leq 1\) in [L, Theorem 2.2]). Part (c) was proved in [L, Theorem 2.1], [S2] and [MPS, Theorem 2.1] (c.f. also [HM, Theorem 3.2]). Part (d) was proved in [CS2, Corollary 2.7] (c.f. also [M, Theorem 1]).

Remark 1.4. We see that in all the cases (a) – (d) we have the restriction \(0 < p \leq q < \infty\) and that we have the most restrictions on the parameters in (a). Thus Theorem 1.1 may be regarded as the study of inequalities of the type (1.2) for the most extreme cases. In fact, several papers have been devoted to the study of the inequality (1.2) when we have restricted the kernels to other intermediate cases, e.g. corresponding to some of the Hardy operators, the Volterra operator, the Liouville operator, etc. Some intermediate restrictions on the parameters (and/or loss of the control of the best constant) appear in these cases. See e.g. [CS1], [CS2], [HM], [L], [M], [MPS], [S1], [S2] and the references given in these papers.

All results above are concerned with functions of one variable and our main concern in this paper is to establish the extension to the multidimensional case. For the power weighted case some results of this type have been obtained in [BPP] and [PPP]. We will below illustrate such a result (corresponding to part (d) of Theorem 1.1) from [BPP, Theorem 3.1(b)] and we therefore consider a measurable, nonnegative and decreasing function \(f: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+\) (here and in the sequel “decreasing” means that \(f\) is decreasing on each variable). We use the notation \(0 \leq f^\dowarrow\) for such a function.

Theorem 1.5. Assume that \(0 \leq f^\dowarrow\) and \(0 < p \leq q < \infty\). If \(\alpha > 0\), then
\[
\left( \int_{\mathbb{R}^n_+} (L^{\alpha-1}(x)f(x))^q \, dx \right)^{1/q} \leq \left( \frac{p^{1/p}}{q^{1/q}} \right)^n \alpha^{n(1/p-1/q)} \left( \int_{\mathbb{R}^n_+} (L^{\alpha-1}(x)f(x))^p \, dx \right)^{1/p},
\]
where \(x = (x_1, x_2, \ldots, x_n), L(x) = \prod_{i=1}^n x_i\), and the inequality is sharp.

Also a similar result for the increasing and more general cases holds (see [BPP, Theorem 3.1]). Moreover, a weighted multidimenal inequality of the desired type (corresponding to Theorem 1.1 (b)) was proved in [CS1, Theorem 2.4].

In this paper we will continue this research by proving a multidimensional inequality which generalizes and unifies all results mentioned above. Our proofs are heavily depending on the ideas introduced and developed in [CS1] and [CS2] but also the techniques and results from [BBP] and [BPP] are useful for our purposes. In order to facilitate our discussion later on we introduce already here some necessary notations and conventions:
Let \((\mathcal{M}_j, \mu_j)\), \(j = 1, 2\), denote two \(\sigma\)-finite measure spaces. Moreover, for every \(x \in \mathcal{M}_j\), let \(d\sigma_j^x(y)\) denote a positive measure on \(\mathbb{R}^n_+\), and
\[
(1.3) \quad T_j f(x) = \int_{\mathbb{R}^n_+} f(y) \, d\sigma_j^x(y),
\]
\(j = 1, 2\). In particular if \(\mathcal{M}_1 = \mathcal{M}_2 = \mathbb{R}^n_+\), \(n = 1\), and \(d\sigma_j^x(y) = k_j(x, y) \, dy\), \(j = 1, 2\), then we have the situation in Theorem 1.1(a) and if instead \(d\sigma_j^x(y) = \delta_x(y)\) (Dirac’s delta function) so that \(T_j f(x) = f(x)\) for \(j = 1\) and/or \(j = 2\), then we obtain the other cases in Theorem 1.1.

The main results of this paper (Theorems 2.2, 2.4 and 2.5) are stated and proved in Section 2. Moreover, in Section 3 we prove some further results and present some applications. It is particularly interesting to note that the expression for the best constant \(C = C_n\) (analogous to the constants \(C\) in Theorem 1.1) is of the type
\[
(1.4) \quad C_n = C_n(p, q, T_1, T_2) = \sup_{D \in \Delta_d} \left( \frac{\int_{\mathcal{M}_1} (T_1 \chi_D)^q \, d\mu_1}{\int_{\mathcal{M}_2} (T_2 \chi_D)^p \, d\mu_2} \right)^{1/p},
\]
where \(\Delta_d\) is a special type of “decreasing” set, (see Definition 2.1). In general this supremum is not easy to calculate. However, if \(w(x)\) and \(v(x)\) are product weights (i.e. of the type \(\prod_{i=1}^n v_i(x_i)\)), then \(C_n\) defined by (1.4) can be calculated just by taking supremum over sets of the “expected” type: \(\{x : 0 \leq x_i \leq a_i, i = 1, \ldots, n\}\) (see Theorem 2.5), which allows us to recover some of the results given in [BPP] and [PPP], but it is important to point out that this equivalence of constants is not true for general weights (see Remark 3.1). The main difference between one and higher dimension is the freedom we have in \(\mathbb{R}^n_+\) to choose among decreasing sets, while in \(\mathbb{R}_+\) we can only consider intervals of the form \((0, r)\).

2. Main results

**Definition 2.1.** We say that a set \(D \subset \mathbb{R}^n_+\) is decreasing (and write \(D \in \Delta_d\)) if the function \(\chi_D\) is decreasing (on each variable). Similarly, we say that a set \(I \subset \mathbb{R}^n_+\) is increasing (and write \(I \in \Delta_i\)) if the function \(\chi_I\) is increasing (on each variable).

**Theorem 2.2.** Let \(C_n\) be as in (1.4) and \(f\) be decreasing.

a) If \(0 < p \leq 1 \leq q < \infty\), then
\[
\left( \int_{\mathcal{M}_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} \leq C_n \left( \int_{\mathcal{M}_2} (T_2 f(x))^p \, d\mu_2(x) \right)^{1/p}.
\]

b) If \(T_1 = \text{Id}, 0 < \max(1, p) \leq q < \infty\), then
\[
\left( \int_{\mathbb{R}^n_+} f^q(x) \, d\mu_1(x) \right)^{1/q} \leq C_n \left( \int_{\mathcal{M}_2} (T_2 f(x))^p \, d\mu_2(x) \right)^{1/p}.
\]
c) If \( T_2 = \text{Id} \), \( 0 < p \leq \min(1, q) < \infty \), then
\[
\left( \int_{\mathcal{M}_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} \leq C_n \left( \int_{\mathbb{R}^n_+} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\]

d) If \( T_1 = T_2 = \text{Id} \), \( 0 < p \leq q < \infty \), then
\[
\left( \int_{\mathbb{R}^n_+} f^q(x) \, d\mu_1(x) \right)^{1/q} \leq C_n \left( \int_{\mathbb{R}^n_+} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\]

In all cases \( C_n \) is the best constant.

Proof. The main property we are going to use is the following identity for the operator \( T_j \), which is easily proved using Fubini’s theorem:
\[
T_j f(x) = \int_0^\infty \left( \int_{\{y : f(y) > t\}} d\sigma_j^x(y) \right) dt = \int_0^\infty T_j \chi_{D_t}(x) \, dt,
\]
where \( D_t = \{ y : f(y) > t \} \) (which is a decreasing set).

(a) In the proof we use Minkowski’s integral inequality twice, for \( q \geq 1 \) and for \( 1/p \geq 1 \):
\[
\left( \int_{\mathcal{M}_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} = \left( \int_{\mathcal{M}_1} \left( \int_0^\infty T_1 \chi_{D_t}(x) \, dt \right)^q \, d\mu_1(x) \right)^{1/q} \\
\leq \int_0^\infty \left( \int_{\mathcal{M}_1} (T_1 \chi_{D_t}(x))^q \, d\mu_1(x) \right)^{1/q} \, dt \\
\leq C_n \int_0^\infty \left( \int_{\mathcal{M}_2} (T_2 \chi_{D_t}(x))^p \, d\mu_2(x) \right)^{1/p} \, dt \\
= C_n \left( \int_{\mathcal{M}_2} (T_2 f(x))^p \, d\mu_2(x) \right)^{1/p}.
\]

(b) In view of (a), we only need to prove the case \( 1 \leq p \leq q < \infty \). We use now [BBP, Theorem 2 (b)] (i.e. the embedding \( L^{p/q} \hookrightarrow L^1 \) with best constant) applied to the decreasing function \( \mu_1(D_t) \):
\[
\left( \int_{\mathbb{R}^n_+} f^q(x) \, d\mu_1(x) \right)^{1/q} = q^{1/q} \left( \int_0^\infty t^{q/p} \mu_1(D_t) \, dt \right)^{1/q} \\
\leq q^{1/q} \left[ \left( \frac{p}{q} \right)^{q/p} \left( \int_0^\infty (t^{q/p} \mu_1(D_t))^{p/p} \, dt \right)^{q/p} \right]^{1/q} \\
= p^{1/p} \left( \int_0^\infty t^{p-1} (\mu_1(D_t))^{p/q} \, dt \right)^{1/p} \leq \]
Using again [BBP, Theorem 2 (b)] (i.e. the embedding \( L^1 \hookrightarrow L^p \), \( p \geq 1 \)) for the decreasing function \( T_2 \chi_{\mathcal{D}t}(x) \) we get

\[
\left( \int_{\mathbb{R}^n_+} f^q(x) \, d\mu_1(x) \right)^{1/q} \leq C_0 \left( \int_{\mathcal{M}_2} \int_0^\infty T_2 \chi_{\mathcal{D}t}(x) \frac{dt}{t} \right)^p \underbrace{\mu_2(x)}_{1/p}.
\]

\( (c) \) In this case, again according to (a), it suffices to prove the case \( 0 < p \leq q \leq 1 \). We use now twice [BBP, Theorem 2 (b)]; first for the embedding \( L^q \hookrightarrow L^1 \) and then for the embedding \( L^1 \hookrightarrow L^{q/p} \):

\[
\left( \int_{\mathcal{M}_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} = \left( \int_{\mathcal{M}_1} \left( \int_0^\infty T_1 \chi_{\mathcal{D}t}(x) \, dt \right)^q \, d\mu_1(x) \right)^{1/q} \leq \left( \int_{\mathcal{M}_1} \left( q \int_0^\infty t^{q-1} (T_1 \chi_{\mathcal{D}t}(x))^q \, dt \right) \, d\mu_1(x) \right)^{1/q} = \left( q \int_0^\infty t^{q-1} \left( \int_{\mathcal{M}_1} (T_1 \chi_{\mathcal{D}t}(x))^q \, d\mu_1(x) \right) \, dt \right)^{1/q} \leq \left( q \int_0^\infty t^{q-1} C_n^q (\mu_2(D_t))^q \, dt \right)^{1/q} \leq C_n q^{1/q} \left( \frac{q}{p} \right)^{p-1/q} \left( \int_0^\infty t^{q-1} \mu_2(D_t) \, dt \right)^{1/p} = C_n \left( p \int_0^\infty t^{p-1} \mu_2(D_t) \, dt \right)^{1/p} = C_n \left( \int_{\mathbb{R}^n_+} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\]

\( (d) \) As before, we use [BBP, Theorem 2 (b)] for the embedding \( L^1 \hookrightarrow L^{q/p} \):

\[
\left( \int_{\mathbb{R}^n_+} f^q(x) \, d\mu_1(x) \right)^{1/q} = \left( q \int_0^\infty t^{q-1} \mu_1(D_t) \, dt \right)^{1/q} \leq \left( q \mu_2^q \int_0^\infty t^{q-1} \mu_2(D_t) \, dt \right)^{1/q} \leq
\]
\[
\begin{align*}
\leq & \quad C_n q^{1/q} \left( \left( \frac{q}{p} \right)^{1-p/q} p^{1-p/q} \int_0^\infty t^{p-1} \mu_2(D_t) \, dt \right)^{1/p} \\
= & \quad C_n \left( p \int_0^\infty t^{p-1} \mu_2(D_t) \, dt \right)^{1/p} \\
= & \quad C_n \left( \int_{\mathbb{R}_+^n} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\end{align*}
\]

To finish the proof, we observe that \( C_n \) is the best constant in all cases (a) – (d) simply by evaluating the operators for the decreasing function \( f = \chi_D \).

We give now a corresponding theorem for increasing functions.

**Definition 2.3.** Given \( 0 < p \leq q < \infty \) and \( T_1, T_2 \) as before, we set

\[
(2.2) \quad \tilde{C}_n = \tilde{C}_n(p, q, T_1, T_2) = \sup_{I \in \Delta_i} \left( \int_{M_1} (T_1 \chi_D)^q \, d\mu_1 \right)^{1/q} \left( \int_{M_2} (T_2 \chi_D)^p \, d\mu_2 \right)^{1/p}.
\]

**Theorem 2.4.** Let \( \tilde{C}_n \) be as in (2.2) and \( f \) increasing.

(a) If \( 0 < p \leq 1 \leq q < \infty \), then

\[
\left( \int_{M_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} \leq \tilde{C}_n \left( \int_{M_2} (T_2 f(x))^p \, d\mu_2(x) \right)^{1/p}.
\]

(b) If \( T_1 = \text{Id}, \ 0 < \max(1, p) \leq q < \infty \), then

\[
\left( \int_{\mathbb{R}_+^n} f^q(x) \, d\mu_1(x) \right)^{1/q} \leq \tilde{C}_n \left( \int_{M_2} (T_2 f(x))^p \, d\mu_2(x) \right)^{1/p}.
\]

(c) If \( T_2 = \text{Id}, \ 0 < p \leq \min(1, q) < \infty \), then

\[
\left( \int_{M_1} (T_1 f(x))^q \, d\mu_1(x) \right)^{1/q} \leq \tilde{C}_n \left( \int_{\mathbb{R}_+^n} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\]

(d) If \( T_1 = T_2 = \text{Id}, \ 0 < p \leq q < \infty \), then

\[
\left( \int_{\mathbb{R}_+^n} f^q(x) \, d\mu_1(x) \right)^{1/q} \leq \tilde{C}_n \left( \int_{\mathbb{R}_+^n} f^p(x) \, d\mu_2(x) \right)^{1/p}.
\]

In all cases \( \tilde{C}_n \) is the best constant.

Proof. To prove the sharpness of the constant, we just have to apply the hypothesis to the increasing function \( f = \chi_I, \ I \in \Delta_i \). The rest of the proof is essentially the same.
as in Theorem 2.2. One could also prove (d) by using Theorem 2.2 (d) and the change of variables \( y_i = x_i^{-1}, i = 1, \ldots, n \).

As we have already mentioned in the introduction, it is not always easy to calculate the constant defined in (1.4). However, we can show that under some restrictions on the weights, the supremum in (1.4) can be reduced to a much simpler class of decreasing sets (intervals), and this is now something much simpler to compute.

**Theorem 2.5.** If \( 0 < p \leq q < \infty \), \( u(x_1, x_2, \ldots, x_n) = u_1(x_1)u_2(x_2) \ldots u_n(x_n) \) and \( v(x_1, x_2, \ldots, x_n) = v_1(x_1)v_2(x_2) \ldots v_n(x_n) \), then

\[
(2.3) \quad \sup_{D \in \Delta_d} \left( \frac{\int_D u(x) \, dx}{\int_D v(x) \, dx} \right)^{1/q} = \sup_{a_i > 0} \left( \frac{\int_0^{a_1} \cdots \int_0^{a_n} u(x_1, \ldots, x_n) \, dx_1 \cdots dx_n}{\int_0^{a_1} \cdots \int_0^{a_n} v(x_1, \ldots, x_n) \, dx_1 \cdots dx_n} \right)^{1/p}.
\]

**Proof.** By induction, it suffices to prove (2.3) for \( n = 2 \). Also without loss of generality we may only consider the case \( q = 1 \). Assume that

\[
A = \sup_{a, b > 0} \frac{\int_0^a \int_0^b u(x_1, x_2) \, dx_1 \, dx_2}{\int_0^a \int_0^b v(x_1, x_2) \, dx_1 \, dx_2} = \sup_{a > 0} \frac{\int_0^a u_1(x_1) \, dx_1}{\int_0^a v_1(x_1) \, dx_1} \sup_{b > 0} \frac{\int_0^b u_2(x_2) \, dx_2}{\int_0^b v_2(x_2) \, dx_2} < \infty.
\]

Then

\[
\left( \int_0^b v_2(x_2) \, dx_2 \right)^{1/p} \geq A^{-1} \left( \int_0^b u_2(x_2) \, dx_2 \right) \sup_{a > 0} \frac{\int_0^a u_1(x_1) \, dx_1}{\int_0^a v_1(x_1) \, dx_1},
\]

and, hence, since any decreasing set \( D \) can be written as

\[
D = \{(x, y) : 0 < x < r, \ 0 < y < f(x)\},
\]

for some \( r > 0 \) and some decreasing function \( f \), we have

\[
\left( \int_0^r v_1(x_1) \left( \int_0^{f(x_1)} v_2(x_2) \, dx_2 \right) \, dx_1 \right)^{1/p} \geq A^{-1} \sup_{a > 0} \frac{\int_0^a u_1(x_1) \, dx_1}{\int_0^a v_1(x_1) \, dx_1} \left( \int_0^r v_1(x_1) \left( \int_0^{f(x_1)} u_2(x_2) \, dx_2 \right)^p \, dx_1 \right)^{1/p}.
\]
Thus, using [CS2, Theorem 2.12] (with the trivial observation that the equivalence in that theorem is in fact an equality), we obtain the following set of inequalities:

\[
C_2 = \sup_{D \in \Delta_d} \frac{\int_D u(x_1, x_2) \, dx_1 \, dx_2}{\left( \int_D v(x_1, x_2) \, dx_1 \, dx_2 \right)^{1/p}}
\]

\[
= \sup_{r > 0} \left( \frac{\int_0^r u_1(x_1) \left( \int_0^{f(x_1)} u_2(x_2) \, dx_2 \right) \, dx_1}{\left( \int_0^{f(x_1)} v_1(x_1) \left( \int_0^{f(x_1)} v_2(x_2) \, dx_2 \right) \, dx_1 \right)^{1/p}} \right) \sup_{r > 0} \left( \frac{\int_0^r v_1(x_1) \left( \int_0^{f(x_1)} v_2(x_2) \, dx_2 \right) \, dx_1}{\left( \int_0^{f(x_1)} v_1(x_1) \left( \int_0^{f(x_1)} v_2(x_2) \, dx_2 \right)^p \, dx_1 \right)^{1/p}} \right) \sup_{r > 0} \left( \frac{\int_0^r u_2(x_2) \, dx_2}{\left( \int_0^{f(x_1)} v_1(x_1) \left( \int_0^{f(x_1)} v_2(x_2) \, dx_2 \right)^p \, dx_1 \right)^{1/p}} \right)
\]

\[
\leq A \left( \sup_{a > 0} \left( \frac{\int_0^a u_1(x_1) \, dx_1}{\left( \int_0^{a} v_1(x_1) \, dx_1 \right)^{1/p}} \right) \right)^{-1} \sup_{r > 0} \left( \frac{\int_0^r u_1(x_1) g(x_1) \, dx_1}{\left( \int_0^{r} v_1(x_1) g^p(x_1) \, dx_1 \right)^{1/p}} \right) \sup_{r > 0} \left( \frac{\int_0^r v_1(x_1) \, dx_1}{\left( \int_0^{r} v_1(x_1) \, dx_1 \right)^{1/p}} \right)
\]

\[
= A \left( \sup_{a > 0} \left( \frac{\int_0^a u_1(x_1) \, dx_1}{\left( \int_0^{a} v_1(x_1) \, dx_1 \right)^{1/p}} \right) \right)^{-1} \sup_{r > 0} \left( \frac{\int_0^{\min(r,s)} u_1(x_1) \, dx_1}{\left( \int_0^{\min(r,s)} v_1(x_1) \, dx_1 \right)^{1/p}} \right)
\]

\[
= A.
\]

The equality of the two constants now follows, since clearly \( A \leq C_2 \).

\[ \square \]

**Remark 2.6.** The same kind of proof can be used to show that in Theorem 2.5 we can also consider increasing sets but now the intervals are of the form \( I = [a_1, \infty) \times \cdots \times [a_n, \infty) \).

## 3. Further results and applications

We now give several applications of the main results of the previous section. First we prove that the condition we find for the constant \( C_n \) in (1.4) cannot be replaced, for general weights, by the simpler condition of Theorem 2.5.

**Remark 3.1.** For arbitrary weights we have that if we define \( I_d \) to be the class of intervals of the form \( I = [0, a_1] \times \cdots \times [0, a_n] \), then the equality

\[
\sup_{D \in \Delta_d} \frac{\int_D u(x) \, dx}{\left( \int_D v(x) \, dx \right)^{1/p}} = \sup_{I \in I_d} \frac{\int_I u(x) \, dx}{\left( \int_I v(x) \, dx \right)^{1/p}}
\]

does not hold in general. In fact, we take \( p = 1, n = 2, T = \{(x, y) \in \mathbb{R}^2_+: 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}, u(x, y) = x + y \text{ on } T, u(x, y) = 0 \text{ elsewhere, and } v(x, y) = 1 \text{ on } \mathbb{R}^2_+.
Then a simple calculation shows that

\[
\sup_{a,b \geq 0} \int_0^a \int_0^b u(x,y) \, dx \, dy \approx 0.5407895304,
\]

(which is the value at the point \( a = b = (1 + \sqrt{2})^{1/3} - 1/(1 + \sqrt{2})^{1/3} \)), and

\[
\sup_{D \in \Delta_d} \int_D u(x) \, dx \geq \int_T u(x) \, dx = 2/3.
\]

We will give now two important corollaries of Theorems 2.2, 2.4 and 2.5. In particular we obtain an improvement of Theorem 1.5.

**Corollary 3.2.** (a) If \( f \downarrow, 0 < p \leq q \) and \(-1 < a_i, b_i < \infty, i = 1, \ldots, n\), then

\[
\left( \int_{\mathbb{R}_+^n} f^q(x)x_{a_1}^{a_1} \cdots x_{a_n}^{a_n} \, dx \right)^{1/q} \leq \prod_{i=1}^n \frac{(b_i + 1)^{1/p}}{(a_i + 1)^{1/q}} \left( \int_{\mathbb{R}_+^n} f^p(x)x_{b_1}^{b_1} \cdots x_{b_n}^{b_n} \, dx \right)^{1/p},
\]

if and only if,

\[
\frac{n + \sum_{i=1}^n a_i}{q} = \frac{n + \sum_{i=1}^n b_i}{p}.
\]

(b) If \( f \uparrow, 0 < p \leq q \) and \(-\infty < a_i, b_i < -1, i = 1, \ldots, n\), then

\[
\left( \int_{\mathbb{R}_+^n} f^q(x)x_{a_1}^{a_1} \cdots x_{a_n}^{a_n} \, dx \right)^{1/q} \leq \prod_{i=1}^n \frac{(-b_i - 1)^{1/p}}{(-a_i - 1)^{1/q}} \left( \int_{\mathbb{R}_+^n} f^p(x)x_{b_1}^{b_1} \cdots x_{b_n}^{b_n} \, dx \right)^{1/p},
\]

if and only if,

\[
\frac{n + \sum_{i=1}^n a_i}{q} = \frac{n + \sum_{i=1}^n b_i}{p}.
\]

**Proof.** (a) A simple calculation shows that

\[
\sup_{r_i > 0} \left( \frac{\int_0^{r_1} \cdots \int_0^{r_n} x_{a_1}^{a_1} \cdots x_{a_n}^{a_n} \, dx}{\int_0^{r_1} \cdots \int_0^{r_n} x_{b_1}^{b_1} \cdots x_{b_n}^{b_n} \, dx} \right)^{1/q} \leq \prod_{i=1}^n \frac{(b_i + 1)^{1/p}}{(a_i + 1)^{1/q}},
\]

exactly when \( (n + \sum_{i=1}^n a_i)/q = (n + \sum_{i=1}^n b_i)/p \) and, thus, the proof follows by applying Theorems 2.2 and 2.5.

The proof of (b) is similar, we only need to use Theorem 2.4 instead of Theorem 2.2.

\[\square\]

**Remark 3.3.** If \( a_i = -\alpha q - 1 \) and \( b_i = -\alpha p - 1 \), \( i = 1, \ldots, n \), we obtain Theorem 3.1 (a–b) in [BPP].

**Corollary 3.4.** ([PPP, Theorem 1.1].) Let \( a, b, x \in \mathbb{R}_+^n \), \( a < b \), and suppose \( g(x) = \prod_{i=1}^n g_i(x_i) \) is continuous, \( f = f(x) \) and \( f, g \geq 0 \).
(a) Suppose that $f$ is decreasing and $g$ is increasing, and $\lim_{x_i \to a_i^+} g_i(x_i) = 0$, $i = 1, \ldots, n$. Then, for any $0 < p \leq 1$,
\[ \int_a^b f(x) \, dg(x) \leq \left( \int_a^b f^p(x) \, d(g^p(x)) \right)^{1/p}. \]
If $1 \leq p < \infty$, then the inequality holds in the reversed direction.

(b) Suppose that $f$ is increasing and $g$ is decreasing, and $\lim_{x_i \to b_i^-} g_i(x_i) = 0$, $i = 1, \ldots, n$. Then, for any $0 < p \leq 1$,
\[ \int_a^b f(x) \, d(-g(x)) \leq \left( \int_a^b f^p(x) \, d(-g^p(x)) \right)^{1/p}. \]
If $1 \leq p < \infty$, then the inequality holds in the reversed direction.
Both inequalities are sharp.

Proof. (a) Let $0 < p \leq 1$. It is enough to prove the inequality for $a = (0, \ldots, 0)$. Since
\[ \chi_{[0,b]} \, dg(x) = \chi_{[0,b_1]} \cdots \chi_{[0,b_n]} \, dg_1(x_1) \cdots dg_n(x_n), \]
the result follows from Theorems 2.2 and 2.5, because the constant is given by the expression
\[ C = \sup_{t > 0} \frac{\int_0^t \, dg(x)}{\prod_{i=1}^n \left( \int_0^{t_i} \, dg_i^p(x_i) \right)^{1/p}} = 1. \]
The proof of the case $p \geq 1$ is similar.
(b) Use Theorem 2.4 instead of Theorem 2.2 and argue as in the proof of (a). □

To finish, we want to consider some new applications to a certain class of multidimensional operators. We begin with the following restriction operator. We fix a curve $g : \mathbb{R}_+ \to \mathbb{R}^2_+$, and for a function $f : \mathbb{R}^2_+ \to \mathbb{R}_+$, we define
\[ Tf(t) = f(g(t)) = \int_{\mathbb{R}^2_+} f(x,y) \, d\mu_t(x,y), \]
where $d\mu_t(x,y) = \delta_{g(t)}(x,y)$. We want to study the boundedness of
\[ T : \mathcal{L}_p^\text{dec} (\mathbb{R}^2_+, v) \to \mathcal{L}^q(\mathbb{R}_+, u), \]
for the range $0 < p \leq \min(1,q)$, where $\mathcal{L}_p^\text{dec} (\mathbb{R}^2_+, v)$ is the cone of decreasing and positive functions in $\mathcal{L}^p(\mathbb{R}^2_+, v)$. By Theorem 2.2, we know that the norm of $T$ is given by
\[ \|T\| = \sup_{D \in \Delta_d} \frac{\|T\chi_D\|_{\mathcal{L}^q(\mathbb{R}_+, u)}}{(\int_D v)^{1/p}}, \]
and hence we need to study $T_{\chi_D}$: We assume that $g(t) = (g_1(t), g_2(t))$, where $g_j : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and bijective function. Thus, if we fix $0 < a < \infty$ and $h : (0, a) \to \mathbb{R}_+$ decreasing, and consider

$$D = \{(x, y) : 0 < x < a, 0 < y < h(x)\},$$

then

$$T_{\chi_D}(t) = \chi_D(g(t)) = \chi_{(0, a_h)}(t),$$

where

$$a_h = \sup \{0 < t < g^{-1}_1(a) : 0 < g_2(t) < h(g_1(t))\}.$$

Thus we have to calculate:

$$\sup_{a > 0, h} \left( \frac{\left( \int_0^{a_h} u \right)^{1/q}}{\left( \int_0^{a} \int_0^{h(x)} v(x, y) \, dy \, dx \right)^{1/p}} \right).$$

The idea is now to fix the point $a_h$ and study which $h$ gives the greatest value:

- If $0 < a_h < g^{-1}_1(a)$, then the denominator is smallest if we choose

$$h(t) = \begin{cases} 
  g_2(a_h) & \text{if } 0 < t < g_1(a_h), \\
  0 & \text{if } g_1(a_h) \leq t < a.
\end{cases}$$

- If $a_h = g^{-1}_1(a)$, then we choose $h(t) = g_2(a_h)$, $0 < t < a$. Thus

$$\|T\| = \sup_{0 < r < \infty} \left( \frac{\left( \int_0^{r} u \right)^{1/q}}{\left( \int_0^{g_1(r)} \int_0^{g_2(r)} v \right)^{1/p}} \right).$$

In particular if we choose $g_1(t) = t^r$, $g_2(t) = t^s$, $0 < r, s < \infty$, $u(t) = t^\gamma$, $v(x, y) = x^\alpha y^\beta$, $-1 < \alpha, \beta, \gamma < \infty$, $0 < p \leq \min(1, q)$, then we obtain

$$\sup_{0 < t < \infty} \left( \frac{\left( \int_0^{t} z^\gamma \, dz \right)^{1/q}}{\left( \int_0^{t} x^\alpha \, dx \int_0^{t} y^\beta \, dy \right)^{1/p}} \right) = \sup_{0 < t < \infty} \frac{(\alpha + 1)^{1/p}(\beta + 1)^{1/p}}{(\gamma + 1)^{1/q}} \cdot \frac{t^{(\gamma + 1)/q}}{t^{(r(\alpha + 1) + s(\beta + 1))/p}}.$$ 

Thus we have proved the following result:

**Theorem 3.5.** Let $T$ be defined as in (3.1), and suppose $g_1(t) = t^r$, $g_2(t) = t^s$, $0 < r, s < \infty$, $u(t) = t^\gamma$, $v(x, y) = x^\alpha y^\beta$, $-1 < \alpha, \beta, \gamma < \infty$, $0 < p \leq \min(1, q)$. Then,

$$T : L^p_{dec}(\mathbb{R}_+^2, v) \to L^q(\mathbb{R}_+, u)$$

is bounded, if and only if,

$$\frac{\gamma + 1}{q} = \frac{r(\alpha + 1) + s(\beta + 1)}{p}. $$
In this case,
\[ \|T\| = \frac{(\alpha + 1)^{1/p}(\beta + 1)^{1/p}}{(\gamma + 1)^{1/q}}. \]

**Remark 3.6.** (a) Observe that without the restriction to decreasing functions then 
\( T \) is never a bounded operator from 
\( L^p(\mathbb{R}^2_+, v) \) to 
\( L^q(\mathbb{R}^2_+, u) \): In fact, take a decreasing 
sequence of sets \( U_n \) such that \( \bigcap_{n \in \mathbb{N}} U_n = \text{graph}(g) \). Consider any 
\( f \in L^p(\mathbb{R}^2_+, v) \), \( f \neq 0 \) and continuous, and set 
\( f_n = f\chi_{U_n} \). Then \( f_n \to 0 \) in norm (Monotone Convergence 
Theorem) but \( Tf_n \to Tf \), for every \( n \). This implies that \( Tf_n \) does not converge to \( 0 \).

(b) We could also consider the example 
\[ g(t) = (\alpha \ln(1 + t), \beta \ln(1 + t)), \quad \alpha, \beta > 0, \]
\[ u(t) = t^\alpha \text{ and } v(x, y) = e^{x+y}. \]
Now we only need to calculate 
\[ \sup_{0 < t < \infty} \frac{1}{(\gamma + 1)^{1/q}} \cdot \frac{t^{\frac{\alpha + \beta}{p}}}{((t + 1)\alpha - 1)((t + 1)\beta - 1))^{1/p}}. \]
Easy arguments show that this supremum is finite if and only if, 
\[ \frac{2}{p} \leq \frac{\gamma + 1}{q} \leq \frac{\alpha + \beta}{p}. \]

Another interesting operator is the following 
\( T : L^p_{\text{dec}}(\mathbb{R}^2_+, v) \to L^q(\mathbb{R}^2_+, u), \quad 0 < p \leq 1 \leq q < \infty, \)
(3.2) 
\[ Tf(t) = \frac{1}{t} \int_0^t f(g(s)) \, ds, \]
where \( g : \mathbb{R}_+ \to \mathbb{R}^2_+ \) is as before, an increasing and bijective function. We observe that 
\( T \) is a composition of the one defined in (3.1) and Hardy’s operator. Also notice that 
\( T \) cannot be written as in (1.3), but we can still obtain the boundedness as follows. 
By choosing \( f = \chi_D \), \( D \) decreasing, we get the necessary condition 
\[ N = \sup_{D \in \Delta_d} \frac{\|T\chi_D\|_{L^q(\mathbb{R}_+^2, u)}}{\|\chi_D\|_{L^p(e^{x+y})}} < \infty. \]
But as before, there exists an \( a_h \) such that, 
\[ T\chi_D(t) = \frac{1}{t} \int_0^t \chi_D(g(s)) \, ds = \begin{cases} 
1 & \text{if } 0 < t < a_h, \\
\frac{a_h}{t} & \text{if } t \geq a_h. 
\end{cases} \]
Take \( D = \{(x, y) : 0 < x < a, 0 < y < h(x)\} \). Therefore 
\[ \frac{\|T\chi_D\|_{L^q(\mathbb{R}_+^2, u)}}{\|\chi_D\|_{L^p(e^{x+y})}} = \left(\int_0^{a_h} u(t) \, dt + a_h \int_{a_h}^\infty u(t) \, dt \right)^{1/q} \left(\int_0^a \int_0^{h(x)} v(x, y) \, dy \, dx \right)^{1/p}. \]
If we fix $a_h$, the function $h$ which makes the denominator smallest is given by
\[ h(x) = \begin{cases} 
g_2(a_h) & \text{if } 0 < x < g_1(a_h), \\
0 & \text{if } g_1(a_h) \leq x < a. 
\end{cases} \]

Thus
\[ N = \sup_{r > 0} \left( \frac{\left( \int_0^r u(t) \, dt + r^q \int_r^\infty \frac{u(t)}{t^q} \, dt \right)^{1/q}}{\left( \int_0^{g_1(r)} \int_0^{g_2(r)} v(x, y) \, dy \, dx \right)^{1/p}} \right) \leq \|T\|. \]

As we have already pointed out, a sufficient condition for the boundedness of $T$ is that both the restriction operator and Hardy’s operator $S$ are bounded:
\[
\begin{array}{ccc}
L_p^p(\mathbb{R}_+^2, v) & \xrightarrow{f \mapsto f \circ g} & L_q^q(\mathbb{R}_+^2, u) \\
T & \quad & S \\
& \quad & L^1(\mathbb{R}_+^2, u)
\end{array}
\]

Thus we obtain
\[
\|T\| \leq \sup_{r > 0} \left( \frac{\left( \int_0^r u(t) \, dt \right)^{1/q}}{\left( \int_0^{g_1(r)} \int_0^{g_2(r)} v(x, y) \, dy \, dx \right)^{1/p}} \times \sup_{s > 0} \left( \frac{\left( \int_0^s u(t) \, dt + \int_0^\infty \frac{u(t)}{t^q} \, dt \right)^{1/q}}{(\int_0^s u(t) \, dt)^{1/q}} \right) \right) \leq \|T\|.
\]

In particular if we consider $u, v, g_1$ and $g_2$ as in Theorem 3.5, we see that
\[ N < \infty \iff \frac{\gamma + 1}{q} = \frac{r(\alpha + 1) + s(\beta + 1)}{p} < 1, \]
and in this case,
\[
\frac{q^{1/q}(\alpha + 1)^{1/p}(\beta + 1)^{1/p}}{(\gamma + 1)^{1/q}(q - \gamma - 1)^{1/q}} \leq \|T\|.
\]

Also, the right-hand side of the sufficient condition (3.3) is finite, if and only if
\[
\left( \sup_{t > 0} \left( \frac{(\alpha + 1)^{1/p}(\beta + 1)^{1/p}}{(\gamma + 1)^{1/q}} \cdot \frac{t^{(\gamma + 1)/q}}{t^{r(\alpha + 1) + s(\beta + 1)/p}} \right) \right)^{1/q} < \infty,
\]
which is equivalent to the conditions
\[ \frac{\gamma + 1}{q} = \frac{r(\alpha + 1) + s(\beta + 1)}{p} < 1, \]
and now
\[ \|T\| \leq (\alpha + 1)^{1/p}(\beta + 1)^{1/p} \frac{q^{1/q}}{(\gamma + 1)^{1/q}(q - \gamma - 1)^{1/q}}. \]
Therefore, we have proved

**Theorem 3.7.** If \(0 < p \leq 1 \leq q < \infty\), \(r, s > 0\), and \(-1 < \alpha, \beta, \gamma\), then

\[
T : L^p_{\text{dec}}(\mathbb{R}^2_+, x^\alpha y^\beta \, dx \, dy) \longrightarrow L^q(\mathbb{R}^+_+, r^\gamma \, dt)
\]

\[
f \longrightarrow \frac{1}{t} \int_0^t f(a^r, a^s) \, da,
\]

is bounded, if and only if

\[
\frac{\gamma + 1}{q} = \frac{r(\alpha + 1) + s(\beta + 1)}{p} < 1,
\]

and in this case,

\[
\|T\| = q^{1/q} (\alpha + 1)^{1/p} (\beta + 1)^{1/p} \frac{1}{(\gamma + 1)^{1/q} (q - \gamma - 1)^{1/q}}.
\]

**Acknowledgements**

The third author has been partially supported by DGICYT PB97 – 0986.

**References**


