The K -functional for p -convexifications

Lech Maligranda

Received: 4 July 2012 / Accepted: 20 August 2012 / Published online: 7 September 2012
© Springer Basel AG 2012

Abstract The K -functional of Peetre is calculated for p -convexifications of normed ideal space as an answer to Lozanovskiĭ problem.

Keywords Normed ideal spaces · Banach function spaces · p -convexification · K -functional

Mathematics Subject Classification (2000) 46E30 · 46B30 · 46B42 · 46M35

1 Introduction

The *normed ideal space* $X = (X, \|\cdot\|_X)$ on a complete σ -finite measure space (Ω, Σ, μ) is the normed subspace of $L^0 = L^0(\Omega, \Sigma, \mu)$ (classes of all real-valued Σ -measurable functions on Ω with identification of functions equal μ -almost everywhere on Ω) with the ideal property

$$\text{if } x \in L^0, y \in X, |x(t)| \leq |y(t)|\mu - \text{a.e. on } \Omega, \text{ then } x \in X \text{ and } \|x\|_X \leq \|y\|_X. \quad (1)$$

For $1 < p < \infty$ the p -convexification $X^{(p)}$ of X is defined by

$$X^{(p)} = \{x \in L^0 : |x|^p \in X\} \text{ with the norm } \|x\|_{X^{(p)}} = \| |x|^p \|_X^{1/p}. \quad (2)$$

L. Maligranda (✉)
Department of Engineering Sciences and Mathematics, Luleå University of Technology,
971 87 Luleå, Sweden
e-mail: lech.maligranda@ltu.se

This is also a normed ideal space and it was investigated by Lozanovskiĭ [8–10]. For more details and proofs concerning p -convexifications and various generalizations of them see [17] (see also [7]). For two normed ideal spaces X_0, X_1 on the same measure space (Ω, Σ, μ) consider the normed ideal space

$$X_0 + X_1 = \{x \in L^0 : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

with the natural norm $\|x\|_{X_0+X_1} = K(1, x; X_0, X_1)$, where $K(t, x; X_0, X_1)$ for $x \in X_0 + X_1$ and $t > 0$ denotes the K-functional of Peetre defined by

$$K(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}. \quad (3)$$

The following property of the K-functional for normed ideal spaces is well-known:

Lemma 1 For any $t > 0$ and $x \in X_0 + X_1$ we have

$$\begin{aligned} K(t, x; X_0, X_1) &= K(t, |x|; X_0, X_1) \\ &= \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : |x| = x_0 + x_1, 0 \leq x_0 \in X_0, 0 \leq x_1 \in X_1\}. \\ &= \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : |x| \leq x_0 + x_1, 0 \leq x_0 \in X_0, 0 \leq x_1 \in X_1\}. \end{aligned}$$

Proof of all three equalities can be found in ([12], Proposition 3), ([14], pp. 167–169) and ([5], pp. 89–91). Moreover, the first two equalities are also proved in ([3], Proposition 3.1.15).

The main result concerning the K -functional for p -convexifications was proved in the unpublished paper ([12], Theorem 7) and appeared only as the Remark 5 in [13] on page 176. More precisely, in the paper [13] the result is formulated for symmetric spaces but the proof works, in fact, for normed ideal spaces.

Theorem 1 If $1 < p < \infty$, then $X_0^{(p)} + X_1^{(p)} = (X_0 + X_1)^{(p)}$. Moreover, for any $t > 0$ and arbitrary $x \in X_0^{(p)} + X_1^{(p)}$ we have

$$K(t^p, |x|^p; X_0, X_1)^{1/p} \leq K(t, |x|; X_0^{(p)}, X_1^{(p)}) \leq 2^{1-1/p} K(t^p, |x|^p; X_0, X_1)^{1/p}. \quad (4)$$

Proof If $x \in X_0^{(p)} + X_1^{(p)}$, then for any decomposition $|x| = x_0 + x_1$ with $0 \leq x_0 \in X_0^{(p)}, 0 \leq x_1 \in X_1^{(p)}$, using the fact that $X = X_0 + t^p X_1$ is a normed space and also its p -convexification functional $\|(\cdot)^{(p)}\|_X^{1/p}$ is a norm as well, we have

$$\begin{aligned} K(t^p, |x|^p; X_0, X_1)^{1/p} &= \| |x|^p \|_{X_0+t^p X_1}^{1/p} \\ &\leq \|x_0^p\|_{X_0+t^p X_1}^{1/p} + \|x_1^p\|_{X_0+t^p X_1}^{1/p} \\ &\leq \|x_0^p\|_{X_0}^{1/p} + (t^p \|x_1^p\|_{X_1})^{1/p} \\ &= \|x_0\|_{X_0^{(p)}} + t\|x_1\|_{X_1^{(p)}}, \end{aligned}$$

that is, $x \in (X_0 + X_1)^{(p)}$ and $K(t^p, |x|^p; X_0, X_1)^{1/p} \leq K(t, |x|; X_0^{(p)}, X_1^{(p)})$.

On the other hand, if $x \in (X_0 + X_1)^{(p)}$, then $|x|^p \in X_0 + X_1$ and for any decomposition $|x|^p = y_0 + y_1$ with $0 \leq y_0 \in X_0$, $0 \leq y_1 \in X_1$ we have

$$|x| = (y_0 + y_1)^{1/p} \leq y_0^{1/p} + y_1^{1/p} = z_0 + z_1$$

with $z_0 \in X_0^{(p)}$, $z_1 \in X_1^{(p)}$. Thus, $x \in X_0^{(p)} + X_1^{(p)}$ and using Lemma 1 we obtain

$$\begin{aligned} K(t, |x|; X_0^{(p)}, X_1^{(p)}) &\leq \|z_0\|_{X_0^{(p)}} + t\|z_1\|_{X_1^{(p)}} \\ &= \|y_0\|_{X_0}^{1/p} + t\|y_1\|_{X_1}^{1/p} \\ &\leq 2^{1-1/p}(\|y_0\|_{X_0} + t\|y_1\|_{X_1})^{1/p}, \end{aligned}$$

where in the last estimate we used an algebraic inequality $a^{1/p} + tb^{1/p} \leq 2^{1-1/p}(a + t^p b)^{1/p}$ which is true for any $a, b, t \geq 0$. Therefore, $K(t, |x|; X_0^{(p)}, X_1^{(p)}) \leq 2^{1-1/p} K(t^p, |x|^p; X_0, X_1)^{1/p}$, and the theorem is proved.

Remark 1 The constant $2^{1-1/p}$ is, in general, best possible. In fact, let $X_0 = L^1(0, \infty)$, $X_1 = L^\infty(0, \infty)$. For the family of functions $x_\alpha(s) = \chi_{(0,1]} + \alpha\chi_{(1,\infty)}$, $0 < \alpha < 1$ and $t \geq 1$ we have

$$\frac{K(t, x_\alpha; L^p, L^\infty)}{K(t^p, x_\alpha^p; L^1, L^\infty)^{1/p}} = \frac{1 - \alpha + \alpha t}{(1 - \alpha^p + \alpha^p t^p)^{1/p}}$$

and letting $t \rightarrow \infty$ with $\alpha t = 1$ we obtain in the limit $2^{1-1/p}$ (see also [2]).

Remark 2 The above proofs works also for complex normed ideal spaces. We can also prove Lemma 1 and Theorem 1 for *quasi-normed ideal spaces*. The quasi-norm means that the functional $\|\cdot\|_X$ satisfies instead of the triangle inequality the “triangle inequality with some constant $C \geq 1$ ”, that is, there is $C = C_X \geq 1$ such that $\|x + y\|_X \leq C(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

Remark 3 Theorem 1 gives an equality $(X_0 + X_1)^{(p)} = X_0^{(p)} + X_1^{(p)}$, which is a positive answer to a problem posed by Lozanovskii (cf. [16], Problem 1269).

Remark 4 Avni and Cwikel [1] were considering the problem if the p -convexified Banach ideal spaces form the Calderón couple. In their investigations the notion of the “disjoint” K -functional for ideal normed spaces on the same measure space appeared

$$\begin{aligned} K_d(t, x; X_0, X_1) &= \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_i \in X_i, \\ &\quad i = 0, 1, x = x_0 + x_1, x_0 \cdot x_1 = 0\}, \end{aligned}$$

where $x \in X_0 + X_1$ and $t > 0$. Of course, $K(t, x; X_0, X_1) \leq K_d(t, x; X_0, X_1)$ but also the reverse inequality $K_d(t, x; X_0, X_1) \leq 2 K(t, x; X_0, X_1)$ is true and this fact was important in their proof. The notion of $K_d(1, x; X_0, X_1)$ and the inequality $K_d(1, x; X_0, X_1) \leq 2 K(1, x; X_0, X_1)$ appeared already in the Lozanovskii paper ([11], Lemma 4.2). I have sent to them my proof of Theorem 1, which improved the constant in their result (cf. [1], Theorem 14) and they convinced me to publish the proof of Theorem 1.

References

1. Avni, E., Cwikel, M.: Calderón couples of p -convexified Banach lattices (2012). <http://arxiv.org/abs/1107.3238>
2. Bergh, J.: A generalization of Steffensen's inequality. *J. Math. Anal. Appl.* **41**, 187–191 (1973)
3. Brudnyi, Yu.A., Krugljak, N.Ya.: *Interpolation Functors and Interpolation Spaces I*. North-Holland, Amsterdam (1991)
4. Cwikel, M., Nilsson, P.G.: *Interpolation of Weighted Banach Lattices. A Characterization of Relatively Decomposable Banach Lattices*, vol. 165. *Memoirs of the American Mathematical Society*, Providence (2003)
5. Karlovich, A.Yu., Maligranda, L.: On the interpolation constant for subadditive operators in Orlicz spaces. In: Kato, M., Maligranda, L. (eds.) *Banach and Function Spaces II*. Proceedings of the International Symposium on Banach and Function Spaces II (ISBFS2006) (14–17 September 2006, Kitakyushu, Japan), pp. 85–101. Yokohama Publishers, Yokohama (2008)
6. Krein, S.G., Petunin, YuI, Semenov, E.M.: *Interpolation of Linear Operators*. American Mathematical Society, Providence (1982)
7. Lindenstrauss, J., Tzafriri, L.: *Classical Banach Spaces. II. Function Spaces*. Springer, Berlin (1979)
8. Lozanovskii, G.Ja.: Topologically reflexive KB-spaces. *Dokl. Akad. Nauk SSSR* **158**, 516–519 (1964); English transl. *Soviet Math. Dokl.* **5**, 1253–1256 (1964)
9. Lozanovskii, G.Ja.: On reflexive spaces generalizing the reflexive space of Orlicz. *Dokl. Akad. Nauk SSSR* **163**, 573–576 (1965); English transl. *Soviet Math. Dokl.* **6**, 968–971 (1965)
10. Lozanovskii, G.Ja.: On some Banach lattices. *Sibirsk. Mat. Zh.* **10**(3), 584–599 (1969); English transl. *Sib. Math. J.* **10**(3), 419–431 (1969)
11. Lozanovskii, G.Ja.: The complex method of interpolation in Banach lattices of measurable functions. *Probl. Mat. Anal.* **7**. Leningrad. Univ., 83–99, Leningrad (1979); English transl. *Selecta Math. Soviet.* **5**, 17–28 (1986)
12. Maligranda, L.: The K -functional for function spaces. Research report, 33 pages. Poznań (1983)
13. Maligranda, L.: The K -functional for symmetric spaces. *Lect. Notes Math.* **1070**, 169–182 (1984)
14. Maligranda, L.: *Orlicz Spaces and Interpolation*. *Seminars in Math.* **5**. Univ. of Campinas, Campinas SP, Brazil (1989)
15. Maligranda, L.: Type, cotype and convexity properties of quasi-Banach spaces. In: Kato, M., Maligranda, L. (eds.) *Banach and Function Spaces*. Proc. of the Internat. Symp. on Banach and Function Spaces (ISBFS2006) (2–4 Oct. 2003, Kitakyushu-Japan), pp. 83–120. Yokohama Publishers (2004)
16. Maligranda, L.: Comments to problems 1269 and 1270. In: Wójtowicz, M. (ed.) *Lozanovsky's Notebooks, Part II: Problems 610–1469*, pp. 209–210. Wydawnictwo Uniwersytetu Kazimierza Wielkiego, Bydgoszcz (2012)
17. Maligranda, L., Persson, L.E.: Generalized duality of some Banach function spaces. *Indag. Math.* **51**(3), 323–338 (1989)
18. Nilsson, P.: Interpolation of Banach lattices. *Studia Math.* **82**(2), 135–154 (1985)