CALDERÓN–LOZANOVSKIĬ CONSTRUCTION FOR MIXED NORM SPACES

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Abstract. We show that the Calderón–Lozanovskiĭ construction $\varphi(\cdot)$ commutes with arbitrary mixed norm spaces, that is,

$$\varphi(E_0[F_0], E_1[F_1]) = \varphi(E_0, E_1)[\varphi(F_0, F_1)]$$

if and only if φ is equivalent to a power function. This result we obtain by giving characterizations of the corresponding embeddings of $\varphi(E_0[F_0], E_1[F_1])$ into $\varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$ and vice versa in terms of the functions $\varphi, \varphi_0, \varphi_1$. As a particular case, we get embeddings of an Orlicz space with mixed norms into an Orlicz space on a product of measure spaces. Applications to classical operators between mixed norm Orlicz spaces are also discussed.

0. Introduction

For vector-valued L_p -spaces the following Lions–Peetre formula is known (see [2]):

$$(L_{p_0}(X_0), L_{p_1}(X_1))_{\theta, p} = L_p((X_0, X_1)_{\theta, p}),$$

and also the Calderón result (see [10], [2])

$$[L_{p_0}(X_0), L_{p_1}(X_1)]_{\theta} = L_p([X_0, X_1]_{\theta}),$$

where $1 \leq p_0, p_1 < \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$, $0 < \theta < 1$. Calderón [10] proved this result even for general vector-valued spaces. He showed that

$$[E_0(X_0), E_1(X_1)]_{\theta} \hookrightarrow E_0^{\theta} E_1^{1-\theta}([X_0, X_1]_{\theta}),$$

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with equality when the space $E_0^\theta E_1^{1-\theta}$ has a continuous norm. The general equality

$$\mathcal{F}(E_0(X_0), E_1(X_1)) = \mathcal{F}(E_0, E_1)(\mathcal{F}(X_0, X_1))$$

for some functors $\mathcal{F}$ was investigated by several authors (see [9] and the references given there). We consider here the case of mixed norm spaces and the Calderón–Lozanovskiĭ construction as functor $\mathcal{F}$.

The first result, in the case of the power function $\varphi_\theta(u, v) = u^{1-\theta}v^\theta$, was proved by Bukhvalov in 1987 ([9], Theorem 3): if E_0, E_1 are Banach ideal spaces on (T, μ) and F_0, F_1 are Banach ideal spaces on (S, ν) , then

$$(1) \quad (E_0[F_0])^{1-\theta} (E_1[F_1])^\theta \stackrel{1}{\hookrightarrow} E_0^{1-\theta} E_1^\theta [F_0^{1-\theta} F_1^\theta],$$

and equality in (1) holds if either the measure μ is discrete or F_0 and F_1 have the Fatou property or F_0 has a continuous norm. Bukhvalov also proved (see [8], p. 95) that

$$(2) \quad \mathcal{F}(E_0[F], E_1[F]) = \mathcal{F}(E_0, E_1)[F]$$

for any interpolation functor $\mathcal{F}$ and since for the real method it is not true that $\mathcal{F}(E[F_0], E[F_1]) = E[\mathcal{F}(F_0, F_1)]$ he considered such an equality for the Calderón–Lozanovskiĭ construction. He also proved (see [9], Theorem 6) that the formula

$$(3) \quad \varphi(E[F_0], E[F_1]) = E[\varphi(F_0, F_1)]$$

is true if either the measure μ is discrete or if F_0 and F_1 have the Fatou property.

Of course, we would like to have the equality

$$\mathcal{F}(E_0[F_0], E_1[F_1]) = \mathcal{F}(E_0, E_1)[\mathcal{F}(F_0, F_1)],$$

but this is rarely satisfied.

The main purpose of this paper is to find necessary and sufficient conditions on $\varphi, \varphi_0, \varphi_1$ under which we have a continuous embedding

$$(4) \quad \varphi(E_0[F_0], E_1[F_1]) \stackrel{C}{\hookrightarrow} \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$$

and also the reverse embedding for any Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) . The necessary and sufficient condition for (4) is the inequality

$$(5) \quad \varphi(1, uv) \leq C \varphi_0(1, u) \varphi_1(1, v) \quad \text{for all } u, v > 0$$

or the reverse inequality to (5) in the reverse embedding case.

In particular, we will get characterizations of the embedding of Orlicz spaces $L_M(S \times T) \subset L_{M_0}(T)[L_{M_1}(S)]$ and its reverse in terms of the functions M, M_0 and M_1 . These results were considered in several papers and with different proofs (see [12], [17], [16], [26], [37]).

Moreover, we can see from the characterization of the embedding (4) and its reverse that the Calderón–Lozanovskiĭ construction $\varphi(\cdot)$ commutes with the mixed norm spaces, that is, the equality

$$(6) \quad \varphi(E_0[F_0], E_1[F_1]) = \varphi(E_0, E_1)[\varphi(F_0, F_1)]$$

holds for arbitrary Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) if and only if φ is equivalent to the power function $\varphi_\theta(u, v) = u^{1-\theta}v^\theta$. This description for power functions gives equality in (1). Equalities in (1), (2) and (4) together with the interpolation property of the Calderón–Lozanovskiĭ construction on Banach ideal spaces with the Fatou property (cf. [35], [36], [31], [4]) give interpolation theorems for mixed norm spaces.

The Calderón–Lozanovskiĭ construction on weighted Banach ideal spaces was investigated in [25].

The content of the paper is as follows. In Section 1 we define the mixed norm space and the Calderón–Lozanovskiĭ construction on the class of Banach ideal spaces. Section 2 contains the main results of the paper. The embedding (4) and its reverse are characterized in terms of the functions φ, φ_0 and φ_1 . The results obtained here are then applied to prove that the equality (6) holds if and only if φ is equivalent to the power function. In Section 3 these results are used for the mixed norm Orlicz spaces and we obtain here, as particular cases, known theorems proved before by several authors. Finally, in Section 4 we are showing how the interpolation property of the Calderón–Lozanovskiĭ construction on spaces with the Fatou property can be used in possible applications to classical operators between mixed norm spaces. More precisely, we can obtain results on boundedness between mixed norm Orlicz spaces from the boundedness on mixed norm $L_p[L_q]$ -estimates. As a concrete example, the maximal operators are presented.

1. Mixed norm spaces and the Calderón–Lozanovskiĭ construction

We recall some notions and definitions which we will need further. Let (T, Σ, μ) be a complete σ -finite measure space, and let $L^0(\mu)$ denote, as usual, the space of all equivalence classes of measurable functions on Ω with the topology of convergence in measure on μ -finite sets. The order $|x| \leq |y|$ means that $|x(t)| \leq |y(t)|$ for μ -almost all $t \in T$.

If a Banach subspace $E = (E, \|\cdot\|)$ of $L^0(\mu)$ is such that there exists $u \in E$ with $u > 0$ μ -a.e. on T and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, we say that E is a *Banach ideal space* (on T or on (T, μ)).

A Banach ideal space E has the *Fatou property* if $0 \leq x_n \uparrow x$, $x_n \in E$ and $\sup_n \|x_n\|_E < \infty$ implies $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$. A Banach ideal space is said to have *continuous norm* if $\|x_n\|_E \downarrow 0$ whenever $0 \leq x_n \in E$ and $x_n \downarrow 0$.

The Köthe dual or the *associate space* E' to a Banach ideal space E is the space of all $x \in L^0(\mu)$ such that $\int_T |x(t)y(t)| d\mu < \infty$ for every $y \in E$ endowed with the norm

$$\|x\|_{E'} = \sup \left\{ \int_T |x(t)y(t)| d\mu : \|y\|_E \leq 1 \right\}.$$

E' is a Banach ideal space. We have the embedding $E \subset E''$ with $\|x\|_{E''} \leq \|x\|_E$ for every $x \in E$. Moreover, $E = E''$ with equality of the norms if and only if E has the Fatou property (cf. [24], [27]).

If $0 < u \in E$, then the *principal ideal* $E(u)$ in E is

$$E(u) = \{x \in E : \exists \lambda > 0 \text{ with } |x(t)| \leq \lambda u(t) \text{ for } t \in T \mu\text{-a.e.}\}$$

with the norm

$$\|x\| = \inf \left\{ \lambda > 0 : |x(t)| \leq \lambda u(t) \text{ for } \mu\text{-a.e. } t \in T \right\}.$$

The space $E(u)$ is a Banach ideal space which is an AM-space and $\|x\| \geq \|x\|$ for $x \in E(u)$ with $\|u\| = \|u\|$. Moreover, the space $(E(u), \|\cdot\|)$ has the Fatou property since the unit ball in $E(u)$ is an interval which is closed and by [20, Lemma 5, p. 143] we obtain that $E(u)$ has the Fatou property.

Let E be a Banach ideal space on (T, μ) and let X be a Banach space. We denote by $E(X)$ the set of all strongly-measurable functions $x = x(\cdot) : T \rightarrow X$ such that $\|x(\cdot)\|_X \in E$. This is a Banach space under pointwise operations and a natural norm given by

$$\|x\|_{E(X)} := \left\| \|x(\cdot)\|_X \right\|_E.$$

The space $E(X)$ is strictly connected with the mixed norm space.

Consider two Banach ideal spaces E and F on (T, Σ, μ) and (S, Λ, ν) , respectively, and assume all the time (for measurability reasons) that either the measure ν is discrete or the norm $\|\cdot\|_F$ is semi-continuous, i.e., if $0 \leq x_n \uparrow x \in E$, then $\|x_n\|_E \uparrow \|x\|_E$.

Define a *mixed norm Banach ideal space* $E[F]$ on $S \times T$ as classes of all $\nu \times \mu$ measurable functions $K = \{K(s, t), s \in S, t \in T\}$ on $S \times T$ such that $K(\cdot, t) \in F$ for μ -almost all $t \in T$ and $\|K(\cdot, t)\|_F \in E$ with the norm

$$\|K\|_{E[F]} := \left\| \|K(s, t)\|_{F, s} \right\|_{E, t}.$$

How are the spaces $E(F)$ and $E[F]$ connected? Any vector-valued function $x(\cdot) : T \rightarrow F$ can be considered as a function of two variables $[x(t)](s) = x(s, t)$ for almost all t and s . Always we have the embedding $E(F) \subset E[F]$ and equality holds if and only if either the measure μ on T is discrete or F has a continuous norm (cf. Bukhvalov [5]).

The spaces $L_p[L_q]$ are sometimes called *spaces of Hille–Tamarkin kernels*. In particular, we have that $L_p(L_q) = L_p[L_q]$ for $1 \leq p, q < \infty$ and $L_p(L_\infty) \neq L_p[L_\infty]$.

The Köthe duality $(E[F])' = E'[F']$ with equality of the norms was proved by Blozinski ([3], Theorem 3.12) and Bukhvalov ([6], Theorem 3.1).

Now we give the definition of the *Calderón–Lozanovskii spaces* sometimes also called the *Calderón–Lozanovskii construction*. Let $\bar{E} = (E_0, E_1)$ be a pair of Banach ideal spaces on (T, μ) and let $\mathcal{U}$ denote the set of all positive concave and positively homogeneous continuous functions $\varphi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(0, 0) = 0$. Then the *Calderón–Lozanovskii ideal space* or the *Calderón–Lozanovskii construction* $\varphi(\bar{E}) = \varphi(E_0, E_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \leq \lambda \varphi(|x_0|, |x_1|)$ μ -a.e. for some $x_i \in E_i$, with $\|x_i\|_{E_i} \leq 1$, $i = 0, 1$. The space $\varphi(\bar{E})$ is a Banach ideal space equipped with the norm

$$\|x\|_\varphi = \inf \left\{ \lambda > 0 : |x| \leq \lambda \varphi(|x_0|, |x_1|), \|x_0\|_{E_0} \leq 1, \|x_1\|_{E_1} \leq 1 \right\}$$

(see [28]). In the case of the power function $\varphi_\theta(s, t) = s^{1-\theta}t^\theta$ with $0 \leq \theta \leq 1$, $\varphi_\theta(\bar{E})$ is the well known Calderón space $E_0^{1-\theta}E_1^\theta$ (see [10]). Moreover, the particular case of $E^\theta(L^\infty)^{1-\theta} = E^{(p)}$ for $\theta = \frac{1}{p}$, $1 < p < \infty$ is known as the p -convexification of E (see [27]).

The properties of $\varphi(\bar{E})$ were studied by Lozanovskii in [28] and [29] (see also [31] and [4]), where among other facts it was proved that the Köthe duality result $\varphi(E_0, E_1)' = \hat{\varphi}(E_0', E_1')$ holds with equivalent norms. Here, for $\varphi \in \mathcal{U}$, the conjugate function $\hat{\varphi}$ is defined by

$$\hat{\varphi}(u, v) := \inf \left\{ \frac{\alpha u + \beta v}{\varphi(\alpha, \beta)}; \alpha, \beta > 0 \right\}, \quad u, v \geq 0.$$

We have $\hat{\varphi} \in \mathcal{U}$ and $\hat{\varphi} = \varphi$ (see [29], [30] and [31]).

The Calderón–Lozanovskii spaces are closely related to Orlicz spaces. Let $M : [0, \infty) \rightarrow [0, \infty]$ be an *Orlicz function*, that is, a nondecreasing, convex and left-continuous function not identical to 0 or ∞ on $(0, \infty)$ with $M(0) = 0$. For the function φ defined by $\varphi(u, v) = vM^{-1}(u/v)$ if $v > 0$ and 0 if $v = 0$, where M^{-1} is the right continuous inverse of M , we have $\varphi \in \mathcal{U}$ and then for any Banach ideal space E , the Calderón–Lozanovskii space $\varphi(E, L_\infty)$ coincides isometrically with the Banach ideal space

$$E_M = \{x \in L^0(\mu); M(|x|/\lambda) \in E \text{ for some } \lambda > 0\}$$

equipped with the norm

$$\|x\|_{E_M} = \inf \left\{ \lambda > 0; \|M(|x|/\lambda)\|_E \leq 1 \right\}.$$

In particular, $\varphi(L_1, L_\infty)$ coincides isometrically with the Orlicz space L_M (see [4], [31], [36]). If the Orlicz function M satisfies the so-called Δ_2 -condition, i.e., $M(2u) \leq CM(u)$ for some $C \geq 1$ and all $u \in (0, \infty)$, then the Orlicz space L_M has a continuous norm.

The subset of functions in $\mathcal{U}$ for which $\varphi(u, 1) \rightarrow 0$ and $\varphi(1, v) \rightarrow 0$ as $u \rightarrow 0^+$ and $v \rightarrow 0^+$ is denoted by $\mathcal{U}_0$. The subsets of functions in $\mathcal{U}$ for which $\bar{\varphi}(u, 1) \rightarrow 0$ as $u \rightarrow 0^+$ are denoted by $\mathcal{U}^-$ and the other for which $\bar{\varphi}(1, v) \rightarrow 0$ as $v \rightarrow 0^+$ by $\mathcal{U}^+$, where $\bar{\varphi}(u, v) = \sup_{a, b > 0} \frac{\varphi(ua, vb)}{\varphi(a, b)}$.

Note that $\varphi \in \mathcal{U}^-$ if and only if the upper Orlicz–Matuszewska index of $\varphi(\cdot) := \varphi(1, \cdot)$ is strictly less than 1, i.e., $\beta_\varphi < 1$ and $\varphi \in \mathcal{U}^+$ if and only if the lower Orlicz–Matuszewska index of φ is strictly bigger than 0, i.e., $\alpha_\varphi > 0$ (cf. [24], [31]).

It is true that if either E_0 has a continuous norm and $\beta_\varphi < 1$ or E_1 has a continuous norm and $\alpha_\varphi > 0$, then $\varphi(E_0, E_1)$ has a continuous norm (see [31], Theorem 15.10).

2. The Calderón–Lozanovskii construction for mixed norm spaces

The first result, connected with equality (2), was mentioned in [9] but we present here the proof for the sake of completeness.

LEMMA 1. *Let $\varphi \in \mathcal{U}$. Then*

$$\varphi(E_0[F], E_1[F]) = \varphi(E_0, E_1)[F]$$

with equality of the norms for any Banach ideal spaces E_0, E_1 on (T, μ) and F on (S, ν) .

PROOF. If $K \in \varphi(E_0[F], E_1[F])$, then

$$|K(s, t)| \leq \lambda \varphi(|x_0(s, t)|, |x_1(s, t)|) \leq \lambda \frac{a|x_0(s, t)| + b|x_1(s, t)|}{\hat{\varphi}(a, b)}$$

for almost all $s \in S, t \in T$ and any $a, b > 0$ with $\|x_i\|_{E_i[F]} \leq 1, i = 0, 1$. Therefore

$$\|K(s, t)\|_F \leq \lambda \frac{a\|x_0(s, t)\|_F + b\|x_1(s, t)\|_F}{\hat{\varphi}(a, b)}$$

for almost all $t \in T$ and any $a, b > 0$. Since $\hat{\varphi} = \varphi$ it follows that

$$\|K(s, t)\|_F \leq \lambda \varphi(\|x_0(s, t)\|_F, \|x_1(s, t)\|_F) = \lambda \varphi(u_0(t), u_1(t))$$

with $u_i(t) := \|x_i(\cdot, t)\|_F \in E_i$ and $\|u_i\|_{E_i} \leq 1, i = 0, 1$. This means that $\|K(\cdot, t)\|_F \in \varphi(E_0, E_1)$ and

$$\|K(s, t)\|_{\varphi(E_0, E_1)[F]} = \|\|K(\cdot, t)\|_F\|_{\varphi(E_0, E_1)} \leq \lambda,$$

which gives the embedding

$$\varphi(E_0[F], E_1[F]) \xrightarrow{1} \varphi(E_0, E_1)[F].$$

On the other hand, if $K \in \varphi(E_0, E_1)[F]$ then $K(\cdot, t) \in F, \|K(\cdot, t)\|_F \in \varphi(E_0, E_1)$ and so

$$\|K(\cdot, t)\|_F \leq \lambda \varphi(|x_0(t)|, |x_1(t)|) \quad \text{with} \quad \|x_i\|_{E_i} \leq 1, \quad i = 0, 1.$$

Let $A = \{t \in T : \|K(\cdot, t)\|_F > 0\}$ and

$$K_i(s, t) = \frac{x_i(t)}{\|K(\cdot, t)\|_F} \chi_A(t) K(s, t), \quad i = 0, 1.$$

Then

$$|K(s, t)| \leq \frac{\lambda \varphi(|x_0(t)|, |x_1(t)|)}{\|K(\cdot, t)\|_F} \chi_A(t) K(s, t) = \lambda \varphi(|K_0(s, t)|, |K_1(s, t)|)$$

for almost all $s \in S$, $t \in T$ and $\|K_i\|_{E_i[F]} = \|\|K_i(\cdot, t)\|_F\|_{E_i} \leq \|x_i\|_{E_i} \leq 1$. Therefore, $K \in \varphi(E_0[F], E_1[F])$ and $\|K\| \leq \lambda$. From this it follows that the embedding

$$\varphi(E_0, E_1)[F] \xrightarrow{1} \varphi(E_0[F], E_1[F])$$

holds. The proof is complete.

Immediately from Lemma 1, by taking $\varphi(s, t) = \min(s, t)$ and $\varphi(s, t) = \max(s, t)$ we obtain the following result (cf. [15], Proposition 4.4 for the sum):

COROLLARY 1. *We have $E_0[F] \cap E_1[F] = (E_0 \cap E_1)[F]$ with equality of the norms and $E_0[F] + E_1[F] = (E_0 + E_1)[F]$ with equivalent norms.*

We consider now the continuous inclusions for the Calderón–Lozanovskii construction of mixed norm spaces.

THEOREM 1. *Let $\varphi, \varphi_0, \varphi_1 \in \mathcal{U}$. The embedding*

$$(7) \quad \varphi(E_0[F_0], E_1[F_1]) \xrightarrow{C} \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$$

holds for any Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) if and only if

$$(8) \quad \varphi(1, uv) \leq C\varphi_0(1, u)\varphi_1(1, v) \quad \text{for all } u, v > 0.$$

PROOF. *Sufficiency.* Assume that (8) is true. Suppose that $K \in \varphi(E_0[F_0], E_1[F_1])$ and has norm $\|K\|$. Then for any $\varepsilon > 0$ there are $K_i \in E_i[F_i]$ with $\|K_i\|_{E_i[F_i]} \leq 1$, $i = 0, 1$, such that

$$|K(s, t)| \leq (\|K\| + \varepsilon) \varphi(|K_0(s, t)|, |K_1(s, t)|).$$

Hence, by (8),

$$\begin{aligned} \frac{|K(s, t)|}{\|K\| + \varepsilon} &= \varphi\left(\|K_0(\cdot, t)\|_{F_0} \frac{|K_0(s, t)|}{\|K_0(\cdot, t)\|_{F_0}}, \|K_1(\cdot, t)\|_{F_1} \frac{|K_1(s, t)|}{\|K_1(\cdot, t)\|_{F_1}}\right) \\ &\leq C\varphi_0(\|K_0(\cdot, t)\|_{F_0}, \|K_1(\cdot, t)\|_{F_1})\varphi_1\left(\frac{|K_0(s, t)|}{\|K_0(\cdot, t)\|_{F_0}}, \frac{|K_1(s, t)|}{\|K_1(\cdot, t)\|_{F_1}}\right). \end{aligned}$$

Since, for μ -almost all $t \in T$,

$$u_t(s) := \varphi_1\left(\frac{|K_0(s, t)|}{\|K_0(\cdot, t)\|_{F_0}}, \frac{|K_1(s, t)|}{\|K_1(\cdot, t)\|_{F_1}}\right) = \varphi_1(u_t^0(s), u_t^1(s))$$

and $\|u_t^i\|_{F_i} = 1$ for $i = 0, 1$ it follows that $u_t \in \varphi_1(F_0, F_1)$ and $\|u_t\|_{\varphi_1} \leq 1$. Thus

$$|K(s, t)| \leq C(\|K\| + \varepsilon) \varphi_0(\|K_0(\cdot, t)\|_{F_0}, \|K_1(\cdot, t)\|_{F_1}) \times u_t(s)$$

and

$$\begin{aligned} v(t) &:= \|K(\cdot, t)\|_{\varphi_1(F_0, F_1)} \\ &\leq C(\|K\| + \varepsilon) \varphi_0(\|K_0(\cdot, t)\|_{F_0}, \|K_1(\cdot, t)\|_{F_1}) \|u_t\|_{\varphi_1(F_0, F_1)} \\ &\leq C(\|K\| + \varepsilon) \varphi_0(\|K_0(\cdot, t)\|_{F_0}, \|K_1(\cdot, t)\|_{F_1}) \\ &= C(\|K\| + \varepsilon) \varphi_0(v_0(t), v_1(t)), \end{aligned}$$

where $v_i(t) = \|K_i(\cdot, t)\|_{F_i}$ with $\|v_i\|_{E_i} \leq 1$ for $i = 0, 1$. Therefore $v \in \varphi_0(E_0, E_1)$ with $\|v\|_{\varphi_0(E_0, E_1)} \leq C(\|K\| + \varepsilon)$ or

$$\| \|K\|_{\varphi_1(F_0, F_1)} \|_{\varphi_0(E_0, E_1)} \leq C(\|K\| + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary we obtain

$$(9) \quad \|K\|_{\varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]} \leq C\|K\|.$$

Necessity. We consider special Banach ideal spaces $E_0 = F_0 = L_\infty(I)$, $E_1 = F_1 = L_1(I)$ with $I = (0, \infty)$ and special functions

$$K(s, t) = \chi_A(s) \chi_B(t),$$

where A, B are any subsets of I of positive finite measure. Then

$$\varphi_i(L_\infty, L_1) = L_{M_i}(I), \quad \varphi(L_\infty[L_\infty], L_1[L_1]) = L_M(I \times I),$$

with $\varphi_i(s, t) = sM_i^{-1}(t/s)$, $i = 0, 1$, and $\varphi(s, t) = sM^{-1}(t/s)$ for $s > 0$. Moreover,

$$\begin{aligned} \|K(\cdot, t)\|_{\varphi_1(L_\infty, L_1)} &= \|K(\cdot, t)\|_{L_{M_1}} = \|\chi_A\|_{L_{M_1}} \chi_B(t) \\ &= \frac{1}{M_1^{-1}(1/mA)} \chi_B(t) = \frac{1}{\varphi_1(1, \frac{1}{mA})} \chi_B(t) \end{aligned}$$

and

$$\| \| K(\cdot, t) \|_{\varphi_1} \|_{\varphi_0} = \frac{1}{\varphi_1\left(1, \frac{1}{mA}\right) \varphi_0\left(1, \frac{1}{mB}\right)}.$$

Also

$$\begin{aligned} \| K(s, t) \|_{\varphi(L_\infty[L_\infty], L_1[L_1])} &= \| \chi_A(s) \chi_B(t) \|_{L_M(I \times I)} \\ &= \inf \left\{ \lambda > 0 : \int_0^\infty \int_0^\infty M(\chi_A(s) \chi_B(t) / \lambda) \leq 1 \right\} \\ &= \frac{1}{M^{-1}\left(\frac{1}{mA} \frac{1}{mB}\right)} = \frac{1}{\varphi\left(1, \frac{1}{mA mB}\right)}. \end{aligned}$$

Thus from (7) we obtain

$$\varphi\left(1, \frac{1}{mA mB}\right) \leq C \varphi_1\left(1, \frac{1}{mA}\right) \varphi_0\left(1, \frac{1}{mB}\right).$$

Since A and B were arbitrary subsets of $I = (0, \infty)$ we can take $u = \frac{1}{mB}$, $v = \frac{1}{mA}$ and then the last estimate is exactly the estimate (8), which means that also the proof of the necessity is finished.

COROLLARY 2. *If $\varphi \in \mathcal{U}$ is a C -submultiplicative function, then the embedding*

$$\varphi(E_0[F_0], E_1[F_1]) \xhookrightarrow{C} \varphi(E_0, E_1)[\varphi(F_0, F_1)]$$

holds for any Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) . In particular, we always have the embedding

$$(E_0[F_0])^{1-\theta} (E_1[F_1])^\theta \xhookrightarrow{1} E_0^{1-\theta} E_1^\theta [F_0^{1-\theta} F_1^\theta].$$

REMARK 1. In Theorem 1 the assumption (8) gives by its symmetry the embedding

$$\varphi(E_0[F_0], E_1[F_1]) \xhookrightarrow{C} \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)] \cap \varphi_1(E_0, E_1)[\varphi_0(F_0, F_1)].$$

This result is also true for quasi-Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) ; we only should mention here that the Calderón–Lozanovskiĭ spaces can be defined also in this case (cf. [33], [19]).

REMARK 2. If we know something more about the spaces F_0, F_1 , then the proof of Theorem 1 gives also the following result: Let one of the following two conditions be satisfied:

(i) $F_0 \hookrightarrow F_1$ with $\|x\|_{F_1} \leq a\|x\|_{F_0}$ for all $x \in F_0$ and

$$(*) \quad \varphi(1, uv) \leq C\varphi_0(1, u)\varphi_1(1, v)$$

for a certain $C > 0$ and all $0 < u \leq a, v > 0$,

(ii) $F_1 \hookrightarrow F_0$ with $\|x\|_{F_0} \leq b\|x\|_{F_1}$ for all $x \in F_1$ and $(*)$ holds for a certain $C > 0$ and all $u \geq \frac{1}{b}, v > 0$.

Then the embedding

$$\varphi(E_0[F_0], E_1[F_1]) \xhookrightarrow{C} \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$$

holds for any Banach ideal spaces E_0, E_1 .

Bukhvalov ([5], Theorem 2 which is without proof; [9], Theorem 6) proved that the formula

$$\varphi(E[F_0], E[F_1]) = E[\varphi(F_0, F_1)]$$

holds if either the measure μ is discrete or F_0 and F_1 have the Fatou property or F_0 has a continuous norm and $\beta_\varphi < 1$. In particular, we have the equalities $E[F_0] \cap E[F_1] = E[F_0 \cap F_1]$ and $E[F_0] + E[F_1] = E[F_0 + F_1]$ with equivalent norms (cf. [15], Proposition 4.6 for the sum). The following generalization is true:

THEOREM 2. Let $\varphi, \varphi_0, \varphi_1 \in \mathcal{U}$. Assume that one of the following four conditions is satisfied:

- (i) the measure μ is discrete,
- (ii) F_0 and F_1 have the Fatou property,
- (iii) F_0 has a continuous norm and $\beta_{\varphi_1} < 1$.
- (iv) F_1 has a continuous norm and $\alpha_{\varphi_1} > 0$.

Then the embedding

$$(10) \quad \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)] \xhookrightarrow{C} \varphi(E_0[F_0], E_1[F_1])$$

holds for any Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) if and only if

$$(11) \quad \varphi_0(1, u)\varphi_1(1, v) \leq C\varphi(1, uv) \quad \text{for all } u, v > 0.$$

PROOF. *Necessity.* We can get (11) by the same considerations as in the proof of Theorem 1.

Sufficiency. First case. Assume that (11) holds and assume additionally that E_0 , E_1 , F_0 and F_1 have the Fatou property. Then

$$\hat{\varphi}(1, uv) \leq C \hat{\varphi}_0(1, u) \hat{\varphi}_1(1, v) \quad \text{for all } u, v > 0.$$

In fact, by (11), we have

$$\begin{aligned} \hat{\varphi}_0(1, u) \hat{\varphi}_1(1, v) &= \inf_{\alpha, \beta > 0} \frac{\alpha + \beta u}{\varphi_0(\alpha, \beta)} \cdot \inf_{\gamma, \delta > 0} \frac{\gamma + \delta v}{\varphi_1(\gamma, \delta)} \\ &\geq \frac{1}{C} \inf_{\alpha, \beta, \gamma, \delta > 0} \frac{\alpha\gamma + \beta\gamma u + \alpha\delta v + \beta\delta uv}{\varphi(\alpha\gamma, \beta\delta)} \geq \frac{1}{C} \hat{\varphi}(1, uv). \end{aligned}$$

By using Theorem 1 we obtain

$$\hat{\varphi}(E'_0[F'_0], E'_1[F'_1]) \stackrel{C}{\hookrightarrow} \hat{\varphi}_0(E'_0, E'_1)[\hat{\varphi}_1(F'_0, F'_1)]$$

and by taking the Köthe dual

$$G = (\hat{\varphi}_0(E'_0, E'_1)[\hat{\varphi}_1(F'_0, F'_1)])' \stackrel{C}{\hookrightarrow} (\hat{\varphi}(E'_0[F'_0], E'_1[F'_1]))' = H.$$

Now, using the Blozinski–Bukhvalov and Lozanovskii duality theorems we obtain

$$\begin{aligned} G &= \hat{\varphi}_0(E'_0, E'_1)'[\hat{\varphi}_1(F'_0, F'_1)'] = \varphi_0(E''_0, E''_1)[\varphi_1(F''_0, F''_1)] \\ &= \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)] \end{aligned}$$

and

$$H = \varphi((E'_0[F'_0])', (E'_1[F'_1])') = \varphi(E''_0[F''_0], E''_1[F''_1]) = \varphi(E_0[F_0], E_1[F_1]),$$

which gives (10).

Second case. Assume that (11) holds but we do not have additional assumptions of the Fatou property about the spaces. Suppose that $K \in \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$ with the norm $\|K\|$. Since for μ -almost all $t \in T$ we have $K(\cdot, t) \in \varphi_1(F_0, F_1)$ it follows that

$$(12) \quad |K(s, t)| = (1 + \varepsilon) \|K(\cdot, t)\|_{\varphi_1(F_0, F_1)} \varphi_1(|x_0(s, t)|, |x_1(s, t)|), \quad \varepsilon > 0,$$

where $\|x_i(\cdot, t)\|_{F_i} \leq 1$, $i = 0, 1$.

A priori the functions $x_i(s, t)$ do not have to be measurable functions of two variables, which does not make it possible to prove (10) in the general case.

Suppose that the functions $x_i(s, t)$ in (12) can be chosen measurable. We will prove below this possibility in cases (i), (iii) and (iv).

Since $K \in \varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$ it follows that

$$(13) \quad \|K(\cdot, t)\|_{\varphi_1(F_0, F_1)} = (1 + \varepsilon)\|K\|\varphi_0(|y_0|, |y_1|),$$

where $\|y_i\|_{E_i} \leq 1$ for $i = 0, 1$.

Now under the measurability assumption on $x_i(s, t)$ we can put

$$K_i(s, t) = (1 + \varepsilon)^2\|K\|y_i(t)x_i(s, t), \quad i = 0, 1.$$

Then, by the assumption (11), we have

$$\begin{aligned} |K(s, t)| &= (1 + \varepsilon)\|K(\cdot, t)\|_{\varphi_1(F_0, F_1)}\varphi_1(|x_0(s, t)|, |x_1(s, t)|) \\ &= (1 + \varepsilon)^2\|K\|\varphi_0(|y_0(t)|, |y_1(t)|)\varphi_1(|x_0(s, t)|, |x_1(s, t)|) \\ &\leq (1 + \varepsilon)^2\|K\|C\varphi(|y_0(t)x_0(s, t)|, |y_1(t)x_1(s, t)|) \\ &= C\varphi(|K_0(s, t)|, |K_1(s, t)|), \end{aligned}$$

where

$$\|K_i(\cdot, t)\|_{F_i} \leq (1 + \varepsilon)^2\|K\|\|y_i(t)\|\|x_i(\cdot, t)\|_{F_i} \leq (1 + \varepsilon)^2\|K\|\|y_i(t)\|$$

and so

$$\|K\|_{E_i[F_i]} \leq (1 + \varepsilon)^2\|K\|, \quad \|y_i\|_{E_i} \leq (1 + \varepsilon)^2\|K\|.$$

Hence, $K \in \varphi(E_0[F_0], E_1[F_1])$ and its norm is not bigger than $C(1 + \varepsilon)^2\|K\|$.

We now return to the measurability assumption we have made. If μ is discrete, then the functions are clearly measurable. If either F_0 has a continuous norm and $\beta_{\varphi_1} < 1$ or F_1 has a continuous norm and $\alpha_{\varphi_1} > 0$, then $\varphi_1(E_0, E_1)$ has a continuous norm (see [31], Theorem 15.10). Thus the mixed norm space $\varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$ is equal to the vector-valued space $\varphi_0(E_0, E_1)(\varphi_1(F_0, F_1))$ in which countably-valued functions are dense (see [24], Lemma 4, p. 211). Therefore functions of the form

$$(14) \quad K(s, t) = \sum_{n=1}^{\infty} \chi_{A_n}(t)y_n(s),$$

where the sets A_n are μ -measurable and pairwise disjoint, $\chi_{A_n} \in E_0 \cap E_1$, $y_n \in \varphi_1(F_0, F_1)$, are dense in $\varphi_0(E_0, E_1)[\varphi_1(F_0, F_1)]$.

Now, it is enough to show the inclusion (10) for functions of the form (14). We have

$$|y_n| = (1 + \varepsilon) \|y_n\|_{\varphi_1(F_0, F_1)} \varphi_1(|x_{0n}|, |x_{1n}|)$$

with $\|x_{in}\|_{F_i} \leq 1$, $i = 0, 1$. Put

$$x_i(s, t) = \sum_{n=1}^{\infty} \chi_{A_n}(t) x_{in}(s).$$

Then $x_i(s, t)$ are measurable and $\|x_i(\cdot, t)\|_{F_i} \leq 1$, $i = 0, 1$. The rest of the proof is the same as above.

Suppose now that F_0 and F_1 have the Fatou property. Let y_0, y_1 be from (13) and consider the principal ideals $G_i = E_i(|y_i|)$, $i = 0, 1$ which have the Fatou property. By the first case in which all the spaces have the Fatou property we have the embedding

$$\varphi_0(G_0, G_1)[\varphi_1(F_0, F_1)] \xrightarrow{C} \varphi(G_0[F_0], G_1[F_1]).$$

Therefore,

$$|K(s, t)| \leq (1 + \varepsilon) C \|K\| \varphi(K_0(s, t), K_1(s, t)) \quad \text{with} \quad \|K_i\|_{G_i[F_i]} \leq 1, \quad i = 0, 1.$$

Thus, for $i = 0, 1$, the assumption $\|K_i\|_{G_i[F_i]} \leq 1$ implies $\|K_i(\cdot, t)\|_{F_i} \leq |y_i(t)|$ and so $\|K_i(\cdot, t)\|_{F_i} \leq \|y_i\|_{E_i} \leq 1$. Hence, $K \in \varphi(E_0[F_0], E_1[F_1])$ with the required estimate of the norm. This completes the proof of Theorem 2.

COROLLARY 3. *If $\varphi \in \mathcal{U}$ is a C -supermultiplicative function and either the measure μ is discrete or both F_0 and F_1 have the Fatou property, then the embedding*

$$\varphi(E_0, E_1)[\varphi(F_0, F_1)] \xrightarrow{C} \varphi(E_0[F_0], E_1[F_1])$$

holds for any Banach ideal spaces E_0, E_1 on (T, μ) and F_0, F_1 on (S, ν) . In particular, we always have the embedding

$$E_0^{1-\theta} E_1^\theta [F_0^{1-\theta} F_1^\theta] \xrightarrow{1} (E_0[F_0])^{1-\theta} (E_1[F_1])^\theta.$$

COROLLARY 4. *Let $\varphi \in \mathcal{U}$ and either the measure μ is discrete or F_0 and F_1 have the Fatou property. Then the equality*

$$(15) \quad \varphi(E_0[F_0], E_1[F_1]) = \varphi(E_0, E_1)[\varphi(F_0, F_1)]$$

holds for any Banach ideal spaces E_0, E_1, F_0, F_1 if and only if φ is equivalent to a power function.

PROOF. By Theorems 1 and 2 (or Corollaries 2 and 3) the equality (15) holds if and only if there exist positive constants c_0 and c_1 such that for all $s, t > 0$

$$c_0 \varphi(1, s) \varphi(1, t) \leq \varphi(1, st) \leq c_1 \varphi(1, s) \varphi(1, t).$$

Hence, φ is equivalent to a power function (cf., for example, [16], p. 289 and [25], Theorem 3).

REMARK 3. Note that

$$\varphi(L_{p_0}[L_{q_0}], L_{p_1}[L_{q_1}]) = L_M[L_N],$$

where $M^{-1}(u) = \varphi(u^{1/p_0}, u^{1/p_1})$ and $N^{-1}(u) = \varphi(u^{1/q_0}, u^{1/q_1})$ for any $L_{p_0}, L_{p_1}, L_{q_0}, L_{q_1}$ spaces if and only if φ is equivalent to a power function $\varphi_\theta(u, v) = u^{1-\theta} v^\theta$ for some $0 \leq \theta \leq 1$.

The next result was proved by Bukhvalov ([9], Theorem 6) but now it will follow easily from Theorems 1 and 2.

COROLLARY 5. *Let $\varphi \in \mathcal{U}$ and let either the measure μ be discrete or let F have the Fatou property. Then the equality*

$$(16) \quad \varphi(E[F_0], E[F_1]) = E[\varphi(F_0, F_1)]$$

holds for any Banach ideal spaces E, F_0 and F_1 .

PROOF. Taking $E_0 = E_1 = E$ and $\varphi_1 = \varphi$, $\varphi_0(u, v) = \max(u, v)$ we obtain that the inequality (8) holds with $C = 1$ and by Theorem 1 we have the embedding $\varphi(E[F_0], E[F_1]) \subset E[\varphi(F_0, F_1)]$. Then changing in the above only the function $\varphi_0(u, v) = \min(u, v)$ we obtain that the inequality (11) holds with $C = 1$ and by Theorem 2 we have the reverse embedding $E[\varphi(F_0, F_1)] \subset \varphi(E[F_0], E[F_1])$, which together give the equality (16).

The following corollary was announced by Milman (cf. [32], Lemma 2.1(ii)) when both of the spaces E and F have the Fatou property.

COROLLARY 6. *Let either E or F have the Fatou property and $E[F] \xrightarrow{B} G(S \times T)$. If for the Orlicz functions M_0, M_1, M we have the estimate $M_0^{-1}(u)M_1^{-1}(v) \leq CM^{-1}(uv)$ for all $u, v > 0$, then*

$$E_{M_0}[F_{M_1}] \xrightarrow{BC} G_M(S \times T).$$

PROOF. We can rewrite this result into the following form: Let either E or F have the Fatou property and $E[F] \xrightarrow{B} G$. If $\varphi, \varphi_0, \varphi_1 \in \mathcal{U}$ satisfy $\varphi_0(u, 1)\varphi_1(v, 1) \leq C\varphi(uv, 1)$ for all $u, v > 0$, then

$$\varphi_0(E, L_\infty)[\varphi_1(F, L_\infty)] \xrightarrow{BC} \varphi(G, L_\infty)$$

and the validity of this embedding follows from Theorem 2(ii).

Cerdà and Mastysłó ([11], Theorem 2.3) proved some vector-valued continuous inclusions for the Calderón–Lozanovskii spaces in connection with the study of type and cotype. In their case the space E is a Banach sequence ideal space with $E \subset l_\infty$. We will show how, from Theorem 1, their result follows for an arbitrary Banach ideal space E .

COROLLARY 7. *Let $\varphi, \varphi_0 \in \mathcal{U}$ and let M_0, M_1, M be Orlicz functions. If $\varphi(1, uv) \leq C\varphi_0(1, u)\varphi(1, v)$, $\varphi_0(M_0^{-1}(u), M_1^{-1}(u)) = M^{-1}(u)$ for all $u, v > 0$, then*

$$\varphi(E_{M_0}[F_0], E_{M_1}[F_1]) \hookrightarrow E_M[\varphi(F_0, F_1)]$$

for any Banach ideal spaces E on (T, μ) and F_0, F_1 on (S, ν) . In particular, if $\varphi \in \mathcal{U}$ is a C -submultiplicative function and $\varphi(M_0^{-1}(u), M_1^{-1}(u)) = M^{-1}(u)$ for all $u > 0$, then $\varphi(L_{M_0}[F_0], L_{M_1}[F_1]) \hookrightarrow L_M[\varphi(F_0, F_1)]$.

PROOF. By Theorem 1 we have

$$\begin{aligned} \varphi(E_{M_0}[F_0], E_{M_1}[F_1]) &= \varphi(M_0^{-1}(E, L_\infty)[F_0], M_1^{-1}(E, L_\infty)[F_1]) \\ &\hookrightarrow \varphi_0(M_0^{-1}(E, L_\infty), M_1^{-1}(E, L_\infty))[\varphi(F_0, F_1)] \end{aligned}$$

and by the reiteration property (cf. [31], pp. 180–181) the last space is equal to

$$M^{-1}(E, L_\infty)[\varphi(F_0, F_1)] = E_M[\varphi(F_0, F_1)].$$

Note that if M satisfies the Δ_2 -condition then we get their assumption $\varphi \in \mathcal{U}(M, M_0, M_1)$ (cf. [11] for definition). We can also obtain, from our Theorem 2, their reverse result in Theorem 2.5.

3. Mixed norm Orlicz spaces

The interest in mixed norm Orlicz spaces started already in the sixties and seventies (cf. [37] and [12], [17], [26]). Clearly, for the mixed norm L_p -spaces we have the imbedding $L_p(T)[L_q(S)] \subset L_r(S \times T)$ for any σ -finite measure spaces (T, μ) and (S, ν) if and only if $p = q = r$. Theorems 1 and 2 allow us to generalize this observation by characterization of the embeddings between Orlicz spaces with mixed norms and Orlicz spaces defined on a product of measure spaces.

THEOREM 3. (i) *The embedding*

$$L_{M_0}(T)[L_{M_1}(S)] \xhookrightarrow{C} L_M(S \times T)$$

holds for any σ -finite measure spaces (T, μ) and (S, ν) if and only if

$$M(Cuv) \leq M_0(u)M_1(v) \quad \text{for all } u, v > 0.$$

Moreover, if additionally $\mu(T) < \infty$ and $\nu(S) < \infty$, then the above embedding holds if and only if the estimate on the functions holds for all $u, v \geq u_0 > 0$.

(ii) *The embedding*

$$L_M(S \times T) \xhookrightarrow{C} L_{M_0}(T)[L_{M_1}(S)]$$

holds for any σ -finite measure spaces (T, μ) and (S, ν) if and only if

$$M(Cuv) \geq M_0(u)M_1(v) \quad \text{for all } u, v > 0.$$

Moreover, if additionally $\mu(T) < \infty$ and $\nu(S) < \infty$, then the above embedding holds if and only if the estimate on the functions holds for all $u, v \geq u_0 > 0$.

PROOF. The results follow directly from Theorems 1 and 2 by considering $E_0 = L_1(\mu)$, $E_1 = L_\infty(\mu)$, $F_0 = L_1(\nu)$ and $F_1 = L_\infty(\nu)$ in these theorems.

Theorem 3(i), in the case of non-atomic finite measures, was proved independently by Donaldson [12], Firley–Matuszewska [17] and Lacroix [26]. The case of non-atomic infinite measures was done by Lacroix [26], and then also by Milman [32].

Theorem 3 (ii), in the case of non-atomic finite measures, was proved by Rutickiĭ [37] and Firley–Matuszewska [17].

COROLLARY 8. *The equality*

$$L_{M_0}(T)[L_{M_1}(S)] = L_M(S \times T)$$

holds for any σ -finite measure spaces (T, μ) and (S, ν) if and only if there exists $p \in [1, \infty]$ such that all the Orlicz spaces are L_p -spaces for a certain $p \geq 1$.

PROOF. We can see that the above mentioned equality gives, by Theorem 3, the equivalences $M_0(u) \approx M_1(u) \approx M(u)$ for all $u > 0$ and both c -supermultiplicativity and C -submultiplicativity of M . Therefore, $M_0(u) \approx M_1(u) \approx M(u) \approx u^p$ for a certain $p \geq 1$, and all the spaces are L_p -spaces.

Observe that if $\mu(T) < \infty$ and $\nu(S) < \infty$ in Corollary 8, then the equality of the spaces holds if and only if the equivalences of the functions holds for all $u, v \geq u_0 > 0$.

Note also that the first part of Corollary 8, i.e., the equivalences $M_0(u) \approx M_1(u) \approx M(u)$ were already showed in [17], Theorem 3.3, and it was fully proved in [16], Theorem 2. In [12], Proposition 1.3 it is proved that $L_M(T)[L_M(S)] = L_M(S \times T)$ if and only if function M is equivalent to a power function u^p for a certain $p \geq 1$.

Taking concrete functions in Theorem 3 we obtain particular imbeddings.

COROLLARY 9. *The embeddings*

$$(L_1 \cap L_\infty)[L_M] \xrightarrow{1} L_M(S \times T) \xrightarrow{2} (L_1 + L_\infty)[L_M]$$

and

$$L_M[L_1 \cap L_\infty] \xrightarrow{1} L_M(S \times T) \xrightarrow{2} L_M[L_1 + L_\infty]$$

hold. Moreover, if $\nu(S) < \infty$ then

$$L_M[L_\infty] \xrightarrow{C} L_M(S \times T) \xrightarrow{2C} L_M[L_1] \quad \text{with} \quad C = \max\{1, \nu(S)\}$$

and if $\mu(T) < \infty$ then

$$L_\infty[L_M] \xrightarrow{C} L_M(S \times T) \xrightarrow{2C} L_1[L_M] \quad \text{with} \quad C = \max\{1, \mu(T)\}.$$

PROOF. The first four embeddings follow immediately from Theorem 3. The constant 2 appeared here since we understand that the space $L_1 + L_\infty$ is with its natural norm $\|\cdot\|_{L_1+L_\infty}^0$ and the well-known equivalence in any Orlicz space L_M (cf. [23], p. 80 or [31], pp. 98–99)

$$(17) \quad \|x\|_{L_M} \leq \|x\|_{L_M}^0 \leq 2\|x\|_{L_M} \quad \text{for all } x \in L_M$$

gives this constant 2.

The second part of the proof follows also from Theorem 3 but with a good control on the embedding norm we can prove as follows.

If $K = K(s, t) \in L_M[L_\infty]$, then $k(t) = \|K(\cdot, t)\|_{L_\infty} \in L_M$ and so

$$\int_T M\left(\frac{k(t)}{\|k\|_{L_M}}\right) d\mu \leq 1.$$

Integrating with respect to $s \in S$ and using the convexity of M we obtain

$$\begin{aligned} \int_S \int_T M\left(\frac{|K(s, t)|}{C\|k\|_{L_M}}\right) d\mu d\nu &\leq \frac{1}{C} \int_S \int_T M\left(\frac{|K(s, t)|}{\|k\|_{L_M}}\right) d\mu d\nu \\ &\leq \frac{1}{C} \int_S \int_T M\left(\frac{k(t)}{\|k\|_{L_M}}\right) d\mu d\nu \leq \frac{1}{C} \int_S d\nu = \frac{\nu(S)}{\max\{1, \nu(S)\}} \leq 1. \end{aligned}$$

Therefore, $K \in L_M(S \times T)$ and $\|K\| \leq \max\{1, \nu(S)\} \|k\|_{L_M}$.

The second embedding we can get by duality. Since $L_\infty[L_{M^*}] \xrightarrow{C} L_{M^*}(S \times T)$ it follows that

$$L_M(S \times T) = (L_M(S \times T))' \xrightarrow{C} (L_\infty[L_{M^*}])' = L_1[(L_{M^*})'] = L_1[L_M].$$

The constant 2 appeared here again because the dual norm to $\|\cdot\|_{L_M^*}$ is an Orlicz norm $\|\cdot\|_{L_M}^0$ which is equivalent to $\|\cdot\|_{L_M}$ by (17). We can prove the last two embeddings similarly. These two last embeddings were also proved by Bukhvalov (cf. [7], Proposition 1).

4. Operators in mixed norm spaces

The results from Section 2 can be applied to study boundedness of some classical operators of analysis in Orlicz spaces with mixed norms. Two types of questions appear here.

The first type is to investigate boundedness of classical operators, like Hardy operator, maximal operators, integral operator or singular integral operators in mixed norm $L_p[L_q]$ -spaces or their weighted versions. Such results started just after the classical paper of Benedek–Panzzone [1], where the mixed norm $L_p[L_q]$ -spaces appeared.

Consider the maximal operator M_1 defined for $K(s, t)$ on $S \times R^n$ with a σ -finite measure μ on S by

$$M_1 K(s, t) = \sup_{Q \ni t} \frac{1}{|Q|} \int_Q |K(s, y)| dy$$

and the strong maximal operator M_S defined by

$$M_S K(s, t) = \sup_{R \ni (s, t)} \frac{1}{|R|} \int_R |K(x, y)| dx dy,$$

where $R = Q \times Q'$ and $Q \in R^m$, $Q' \in R^n$ are cubes. K. F. Andersson and R. T. John (1980) proved that if $1 < p, q < \infty$, then M_1 is bounded on $L_p(wdt)[l_q]$ spaces if and only if w satisfies the so-called A_p -condition of Muckenhoupt: there exists a positive constant c such that for all cubes Q

$$\frac{1}{|Q|} \int_Q w(y) dy \left(\frac{1}{|Q|} \int_Q w(y)^{-1/(p-1)} dy \right)^{p-1} \leq c.$$

The boundedness in $L_p(wdt)[L_q(\mu)]$ spaces was also proved in 1980 by V. Kokilashvili ([22], Theorem 1) (he announced it in 1978 in the Russian Doklady): Let $1 < p, q < \infty$. Then M_1 is bounded on $L_p(wdt)[L_q(\mu)]$ if and only if w satisfies the A_p -condition.

Kokilashvili ([22], Theorem 2) proved also the boundedness of the strong maximal operator M_S in weighted mixed norm spaces $L_p(udt)[L_q(vds)]$ with separated weights: Let $1 < p, q < \infty$ and let the weight $w(s, t)$ have separated variables, that is, $w(s, t) = v(s)u(t)$. Then the strong maximal function M_S is bounded on $L_p(udt)[L_q(vds)]$ if and only if $u \in A_p, v \in A_q$.

The unweighted case of this result was proved independently by Fernandez [14] in 1987. I should also mention here that the boundedness of the operators M_1 and M_S was proved in 1984 by Bukhvalov ([8], p. 100) in weighted Orlicz spaces $L_M(udt)[L_N]$ using the Kokilashvili results, the Kerman–Torchinsky [21] result and an interpolation technique, which we will discuss below.

Note that in the case of unseparated weight $w = w(s, t)$ the boundedness of M_S in $L_p[L_q(w)]$ is still an open problem.

The other classical operators, like integral operators, double Hilbert transform, singular integral operators, multipliers of Fourier integrals, Paley–Littlewood theory, Carleson–Hunt operators (theorem) can be considered in vector-valued spaces or in the mixed norm spaces $L_p(wds)[L_q]$ (see Fernandez [14], Bukhvalov [8], Garcia-Cuerva and Rubio de Francia [18]).

The second question is how to use the results for mixed norm $L_p[L_q]$ -spaces to extend them to mixed norm Orlicz $L_M[L_N]$ -spaces. The interpolation property of the Calderón–Lozanovskii construction on the class of mixed norm ideal spaces $E[F]$ with changed spaces E and F having the Fatou property is important but not possible except the class of $L_p[L_q]$ -spaces. This follows from the Nagumo–Kolmogorov theorem for any interpolation method (cf. [5]) or in the case of Calderón–Lozanovskii construction from our Corollary 4. We can use instead of this result the interpolation in each variable independently. Note that interpolation in E with any fixed space F is always possible but interpolation in F with fixed E is not always possible, for example, it does not work for the real method of interpolation, as was observed by M. Cwikel in 1977. Bukvalov ([8], p. 98) observed that this will be possible for the Calderón–Lozanovskii construction.

We do not need, but let us mention that the connection between various Gustavsson–Peetre’s interpolation methods, Ovchinnikov’s interpolation methods and the Calderón–Lozanovskii construction on vector-valued Banach ideal spaces, in particular also on mixed norm spaces with some additional properties like the Fatou property or continuous norms, can be found in [13], Theorem 3.2 (cf. [36] for definitions and general properties).

Let us apply the Bukhvalov method in the proof of the mixed norm version of the Ryan interpolation theorem. Ryan ([38], Theorem 3) proved that if T is a bounded linear operator in every reflexive $L_p(I)$ space, then T is bounded in any reflexive Orlicz space $L_M(I)$, $I = [0, 1]$ (cf. also [31], pp. 149–150 where I is either $[0, 1]$ or $[0, \infty)$). We can easily see that this theorem is true for any σ -finite nonatomic measure. We will keep this assumption on the measure in the next theorem.

THEOREM 4. *Let $L_M[L_N]$ be a reflexive mixed norm Orlicz space. If T is a linear or sublinear bounded operator on $L_p[L_q]$ for all $1 < p, q < \infty$, then T is defined and bounded on $L_M[L_N]$.*

PROOF. We shall only consider the case where $\mu(T) = \nu(S) = \infty$. The other cases can be treated analogously. Mixed norm Orlicz spaces $L_M[L_N]$ are reflexive if and only if the Orlicz spaces L_M and L_N are reflexive, which is also equivalent to the condition that Orlicz functions M, N and their complementary functions M^*, N^* all satisfy the Δ_2 -condition (cf. [31], Theorem 9.3).

Since both M and M^* satisfy the Δ_2 -condition it follows that $M(u) = \varphi(u^{1/p_0}, u^{1/p_1})$ for some $1 < p_0 < p_1 < \infty$ and $\varphi \in \mathcal{U}$ (cf. [31], p. 149), and so $L_M = \varphi(L_{p_0}, L_{p_1})$. Similarly since both N and N^* satisfy the Δ_2 -condition it yields that $N(u) = \psi(u^{1/q_0}, u^{1/q_1})$ for some $1 < q_0 < q_1 < \infty$ and $\psi \in \mathcal{U}$, which gives that $L_N = \psi(L_{q_0}, L_{q_1})$.

Let now T be a bounded operator in $L_p[L_q]$ for any $1 < p, q < \infty$. In particular, for any fixed $1 < p < \infty$ the operator T is bounded in $L_p[L_{q_0}]$ and $L_p[L_{q_1}]$ and by the interpolation property of the Calderón–Lozanovskii construction on spaces with the Fatou property the operator T is bounded in

$$\psi(L_p[L_{q_0}], L_p[L_{q_1}]) = L_p[\psi(L_{q_0}, L_{q_1})] = L_p[L_N]$$

for all $1 < p < \infty$. We were using here Lemma 1. In particular, T is bounded in $L_{p_0}[L_N]$ and $L_{p_1}[L_N]$, and again, by the interpolation property of the Calderón–Lozanovskii construction, T is bounded in

$$\varphi(L_{p_0}[L_N], L_{p_1}[L_N]) = \varphi(L_{p_0}, L_{p_1})[L_N] = L_M[L_N]$$

and we used here Corollary 5. The proof is complete.

Weighted version of this theorem gives the above mentioned Bukhvalov's result on the boundedness of the operators M_1 and M_S in weighted mixed norm Orlicz spaces $L_M(udt)[L_N]$.

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