Hardy-type Inequalities on Cones of Monotone Functions

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Key words and phrases. Integral inequalities, weights, Hardy operator, measures, Hardy-type inequalities, monotone functions, Volterra operators.

Printed by Universitetsstryckeriet, Luleå 2011

ISSN: 1402-1757
ISBN 978-91-7439-266-1

Luleå 2011

www.ltu.se

Abstract

This Licentiate thesis deals with Hardy-type inequalities restricted to cones of monotone functions.

The thesis consists of two papers (paper A and paper B) and an introduction which gives an overview to this specific field of functional analysis and also serves to put the papers into a more general frame.

We deal with positive σ -finite Borel measures on $\mathbb{R}_+ := [0, \infty)$ and the class $\mathfrak{M} \downarrow$ ($\mathfrak{M} \uparrow$) consisting of all non-increasing (non-decreasing) Borel functions $f: \mathbb{R}_+ \rightarrow [0, +\infty]$.

In paper A some two-sided inequalities for Hardy operators on the cones of monotone functions are proved. The idea to study such equivalences follows from the Hardy inequality

$$\begin{aligned} \left(\int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}} &\leq \left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} f(t) d\lambda(t) \right)^p d\lambda(x) \right)^{\frac{1}{p}} \\ &\leq \frac{p}{p-1} \left(\int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}}, \end{aligned}$$

which holds for any $f \in \mathfrak{M} \downarrow$ and $1 < p < \infty$. In the paper similar equivalences are found for some other Hardy-type operators for the full range of parameter p , $p \neq 0$. As an application of one of the results, we also obtain a new characterization of the discrete inequality for one of the most interesting cases of parameters, namely when $0 < q < p \leq 1$.

The equivalences we have proved in paper A are used in paper B to obtain necessary and sufficient conditions for some other Hardy-type inequalities on cones of monotone functions. In particular, we give a complete description for inequalities with Volterra integral operators involving Oinarov's kernel for the parameters $0 < p < \infty$, $1 \leq q < \infty$. We also study inequalities of the form

$$\left(\int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow, f \neq 0$$

and

$$\left(\int_{[0,\infty]} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^+, f \not\equiv 0$$

and find necessary and sufficient conditions not only for positive, but also for negative parameters of summation.

Preface

This Licentiate thesis consists of two papers (paper A and paper B) and an introduction which puts these papers into a more general frame.

[A] L.-E. Persson, O. V. Popova, V. D. Stepanov. Two-sided Hardy-type inequalities for monotone functions. *Complex Variables and Elliptic Equations*, **55** (2010), no. 8-10, 973-989.

[B] O. V. Popova. Hardy-type inequalities on cones of monotone functions. *Research Report, no. 6, ISSN: 1400-4003, Department of Engineering Sciences and Mathematics, Luleå University of Technology*, 22 pages, 2011.

The results of paper A have also been announced in the following paper:

[C] V. D. Stepanov, L.-E. Persson, O. V. Popova. Two-sided Hardy-type inequalities for monotone functions. *Doklady Mathematics*, **80** (2009), no. 3, 814-817.

Acknowledgement

I want to express my deep gratitude to my main supervisors Professor Lars-Erik Persson and Professor Vladimir D. Stepanov for their valuable remarks and attention to my work and their incessant support.

I also want to thank my co-supervisor Thomas Strömberg for his help and interest he took in my work and research assistant Yulia Koroleva for her help in many practical things and great support.

I thank Luleå University of Technology and Peoples' Friendship University of Russia for giving me an opportunity to participate in a partnership program in research and postgraduate education in mathematics. I also thank both universities for financial support which made my studies possible.

Finally, I am grateful to my family and friends for helping me in every possible way.

Introduction

Hardy-type inequalities and their applications compose a great part of Classical Analysis concentrated around the characterization problems for various integral and differential inequalities to hold. In particular, the inequalities restricted to the cone of monotone functions have been intensively studied during the last two decades.

The study of Hardy-type inequalities in weighted Lebesgue spaces began in 1915 with the work of G.H. Hardy and his contemporaries¹. In 1925 G.H. Hardy proved the following result [23]:

Suppose that $f(x) > 0$, $p > 1$, f is integrable over any finite interval $(0, X)$ and f^p is integrable over $(0, \infty)$. Then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx. \quad (0.1)$$

The dramatic period of almost 10 years of research, which involved such mathematicians as M. Riesz, E. Landau and G. Polya, was recently described in the paper [34].

Then the first weighted modifications appeared (see [24]), especially the ones with power weights, which more frequently can be seen in applications. Finally the inequality appeared in the form

$$\left(\int_a^b \left(\int_a^x f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p(x) v(x) dx \right)^{\frac{1}{p}}, \quad (0.2)$$

where $-\infty \leq a < b \leq \infty$, $0 < q < \infty$, $1 \leq p < \infty$ and u, v are measurable functions (weights), positive a.e. in (a, b) .

Let us mention some important results on the inequality (0.2). We notate $p' := \frac{p}{p-1}$ for $0 < p < \infty$, $p \neq 1$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ for $0 < q < p < \infty$.

Theorem 1. *(i) If $1 \leq p \leq q < \infty$, then (0.2) holds for all measurable*

¹Some facts we mention in this work can be found in the papers [33], [34].

functions $f(x) \geq 0$ on (a, b) iff

$$A_1 := \sup_{a < x < b} \left(\int_x^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty.$$

(ii) If $1 \leq q < p < \infty$, then (0.2) holds iff

$$A_2 := \left(\int_a^b \left(\int_x^b u(t) dt \right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right)^{\frac{1}{r}} < \infty.$$

(iii) If $0 < q < 1 < p < \infty$, then (0.2) holds iff

$$A_3 := \left(\int_a^b \left(\int_x^b u(t) dt \right)^{\frac{r}{p}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{r}{p'}} u(x) dx \right)^{\frac{1}{r}} < \infty.$$

(iv) If $0 < q < 1 = p$, then (0.2) holds iff

$$A_4 := \left(\int_a^b \left[\bar{v}(x) \int_x^b u(t) dt \right]^{\frac{q}{1-q}} u(x) dx \right)^{\frac{1}{q}-1} < \infty,$$

where $\bar{v}(x) = \operatorname{ess\,sup}_{a < t < x} \frac{1}{v(t)}$.

The proof of (i) can be found in [15], [38], [39], [44, Theorem 1.14], [67], [2], [74], [75] and [33, Theorem 1.1], the proof of (ii) in [38], [39], [44, Theorem 1.15], [25] and [33, Theorem 1.2]. For the the proof of (iii) see [59] and [44, Chapter 9]. The proof of the case $0 < q < p$, $1 < p < \infty$ was given in [62, Theorem 2.4], and the proof of (iv) in [62, Theorem 3.3]. The case $0 < q < p$, $p \geq 1$ was also proved by E. Sawyer in [56], where he presented an equivalent condition, and by G. Sinnamon in [59], [60], [61].

The problem of the best constant has been studied in several papers. For example, for $p < q$ it was investigated in [12] in the classical case and in [36] for (0.2).

Remark. It is well known that the conditions $A_1 < \infty$ and $A_2 < \infty$ can be replaced by different (equivalent) conditions. For example, the condition $A_2 < \infty$ can be replaced by the following one (see the paper [46] by L.-E. Persson and V.D. Stepanov):

$$A^* := \int_0^\infty \left(\int_0^x u(t) \left(\int_0^t v(s)^{1-p'} ds \right)^q dt \right)^{\frac{r}{q}} \left(\int_0^x v(t)^{1-p'} dt \right)^{-\frac{r}{q}} v(x)^{1-p'} dx < \infty.$$

More generally, the condition $A_2 < \infty$ can be replaced by four different scales of conditions (see [48]) and the condition $A_1 < \infty$ can be replaced by fourteen different scales of conditions (see [19] and the references given there).

The analogous criteria were found for the dual inequality which has the form

$$\left(\int_a^b \left(\int_x^b f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p(x) v(x) dx \right)^{\frac{1}{p}}.$$

The inequality (0.2) can also be written in a short way

$$\|Hf\|_{q,u} \leq C \|f\|_{p,v}, \quad (0.3)$$

where the Hardy operator is given by

$$(Hf)(x) := \int_a^x f(t) dt. \quad (0.4)$$

We use here the notion of a norm (quasi-norm, if $p < 1$) in a weighted Lebesgue space $L^p(a, b, u)$, where $0 < p \leq \infty$, u is a weight function on (a, b) . The space $L^p(a, b, u)$ consists of all measurable functions on (a, b) such that

$$\|f\|_{p,u} := \left(\int_a^b |f(x)|^p u(x) dx \right)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$

and

$$\|f\|_{\infty,u} := \operatorname{ess\,sup}_{a < x < b} |f(x)| < \infty.$$

It is obvious that we can also write

$$\|\tilde{H}f\|_{q,u} \leq C \|f\|_{p,v} \quad (0.5)$$

for the dual inequality, where $\tilde{H}$ is a conjugate Hardy operator:

$$(\tilde{H}f)(x) := \int_x^b f(t) dt.$$

The inequalities (0.3) and (0.5) can be extended to a more generalized form

$$\|Tf\|_{q,u} \leq C \|f\|_{p,v},$$

where T is an integral operator. The validity of this inequality is equivalent to the continuity of the mapping $T : L^p(v) \rightarrow L^q(u)$.

Also the weighted Hardy operator of the form

$$(H_{u,v}f)(x) := u(x) \int_a^x v(t) f(t) dt \quad (0.6)$$

is used. Clearly, the inequality

$$\|H_{u,v}f\|_q \leq C\|f\|_p$$

can be easily reduced to the original inequality (0.3).

The reverse Hardy-type inequality

$$\left(\int_0^\infty f(x)^p dx\right)^{\frac{1}{p}} \leq C \left(\int_0^\infty \int_0^x f(y)v(y)dy\right)^q dx\right)^{\frac{1}{q}}, \quad f > 0 \quad (0.7)$$

for $-\infty < p, q < 0$ and $0 < p, q < 1$ was studied in [3], where P.R. Beesack and H.P. Heinig obtained sufficient and necessary conditions for (0.7) to hold. However, the authors pointed out that the necessary conditions required some restrictions on the weight functions and therefore were weaker than the sufficient conditions.

D.V. Prokhorov in [53] gave the precise characterization of (0.7) for the same range of parameters p, q , as Beesack and Heinig. He also established a duality between the cases $p, q < 0$ and $0 < p, q < 1$. Specifically, he proved that the Hardy-type inequality with negative indices of integration is equivalent to the same inequality with the dual operator for conjugate indices. This principle of duality allowed to extend the conditions obtained for (0.7) in cases $-\infty < p < q < 0$ and $-\infty < q \leq p < 0$ to the case $0 < p, q < 1$. The principle of duality reads as follows.

Theorem 2 [53]. *Let $-\infty < p, q < 0$, $0 < C < +\infty$, k be a non-negative measurable function on $(a, b) \times (a, b)$ and*

$$(Tf)(x) := \int_a^b k(x, y)f(y)dy,$$

$$(T^*g)(x) := \int_a^b k(y, x)g(y)dy$$

for $x \in (a, b)$. Then the inequality

$$\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \leq C \left(\int_a^b [(Tf)(x)]^q dx\right)^{\frac{1}{q}}$$

holds for all $f \in \mathfrak{M}^+(a, b)$ iff the inequality

$$\left(\int_a^b g(x)^{q'} dx\right)^{\frac{1}{q'}} \leq C \left(\int_a^b [(T^*g)(x)]^{p'} dx\right)^{\frac{1}{p'}}$$

holds for all $g \in \mathfrak{M}^+(a, b)$. In particular, it follows that the least possible constants in these inequalities are equal.

Here $\mathfrak{M}^+(a, b)$ denotes the class of all measurable functions $f : (a, b) \rightarrow [0, +\infty]$.

The case $-\infty < q \leq p < 0$ for (0.7) was described in [31]. The curious cases $q < 0, p > 0$ and $q > 0, p < 0$ were studied in [32]. We summarize the results of [32] as follows.

Theorem 3 [32]. (i) Let $p < 0$ and $q > 0$. Then the inequality

$$\left(\int_a^b \left(\int_a^x f(t)v(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \geq C \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} \quad (0.8)$$

holds for all $f \geq 0$ iff there exists $\tau \in (a, b)$ such that

$$A(\tau) > 0,$$

where

$$A(t) := \left(\int_a^t v^{p'}(x)dx \right)^{\frac{1}{p'}} \left(\int_t^b u(x)dx \right)^{\frac{1}{q}}, \quad p' = \frac{p}{p-1}.$$

(ii) Let $0 < p < 1$ and $q < 0$. Then (0.8) holds for all $f \geq 0$ iff

$$A_* := \inf_{(a,b)} A(t) > 0.$$

(iii) Let $p \geq 1$ and $q < 0$. Then (0.8) holds for all $f \geq 0$ iff there exists $\tau \in (a, b)$ such that

$$B(\tau) < \infty,$$

where

$$B(t) := \begin{cases} \left(\operatorname{ess\,sup}_{(a,t)} v(x) \right) \left(\int_a^t u(x)dx \right)^{\frac{1}{q}}, & \text{if } p = 1, \\ \left(\int_a^t v^{p'}(x)dx \right)^{\frac{1}{p'}} \left(\int_a^t u(x)dx \right)^{\frac{1}{q}}, & \text{if } p > 1. \end{cases}$$

For the Hardy inequality with arbitrary σ -finite Borel measures there are less results than there are for the case of Lebesgue measures. In the general case the inequality has the form

$$\left(\int_{[a,b]} \left(\int_{[a,x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[a,b]} f^p w d\nu \right)^{\frac{1}{p}}, \quad (0.9)$$

where $f \in \mathfrak{M}^+$. The case $u = v = w \equiv 1$, where λ is a Lebesgue measure, was described in [39]. The more general cases for two (when $\lambda = \nu$) and three measures for $p > 1$, $q > 0$ were in full characterized in [50]. The criterion for three measures is given in [50] in terms which include (ν_a, ν_s) — the Lebesgue decomposition of measure ν relative to λ , where $d\nu_a/d\lambda$ denotes the Radon-Nikodym derivative of ν_a with respect to λ .

Similar to the weighted Hardy inequalities in Lebesgue spaces, they can also be studied in the Lorentz spaces $\Lambda^p(w)$, $0 < p < \infty$, where

$$\Lambda^p(w) := \left\{ f : \|f^*\|_{p,w} = \left(\int_0^\infty (f^*(t))^p w(t) dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Here f^* is the equimeasurable decreasing rearrangement of $|f|$ defined by

$$f^*(t) := \inf \{ y > 0 : \lambda_f(y) \leq t \},$$

where λ_f is the distribution function:

$$\lambda_f(y) := \text{mes} \{ x \in X : |f(x)| > y \}.$$

A weight characterization of classical operators in Lorentz spaces brought up the necessity to study operators defined on the cone of decreasing functions.

For instance, let the Hardy-Littlewood maximal function Mf be defined by

$$(Mf)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(z)| dz, \quad x \in \mathbb{R}^n,$$

where Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ is its Lebesgue measure. It is well known (see, for example, [4]) that

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0.$$

Thus, the characterization of weight functions u and v , for which the mapping

$$M : \Lambda^p(v) \rightarrow \Lambda^q(u), \quad 1 < p, q < \infty,$$

is bounded between Lorentz spaces, is equivalent to the characterization of weight functions u and v , for which the Hardy operator P defined by

$$(Pf)(t) := \frac{1}{t} \int_0^t f(s) ds, \quad t \geq 0,$$

is bounded from $L^p(v)$ to $L^q(u)$, $0 < p, q < \infty$ on the cone of non-negative decreasing functions. That means, we need to characterize the weight functions, for which the Hardy inequality

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s) ds \right)^q u(t) dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(t) v(t) dt \right)^{\frac{1}{p}} \quad (0.10)$$

holds for all decreasing functions $f \geq 0$.

This case clearly differs from the non-monotone one, because the inequality makes sense for all values of positive parameters of integration p and q and, naturally, the study breaks up into four cases: (I) $1 \leq p \leq q < \infty$; (II) $0 < q < p < \infty, p > 1$; (III) $0 < p \leq q, 0 < p \leq 1$ and (IV) $0 < q < p \leq 1$.

The above problem has been investigated by many authors. One of the first results was obtained by D.W. Boyd in 1967 (see [14]). He received conditions on the weight w , for which the inequality $\|Hf\|_{p,w} \leq A\|f\|_{p,w}$ holds for all $0 \leq f \downarrow$. His criterion reads as follows:

There exist positive constants $C > 0$ and $0 \leq \gamma < 1$ such that $\frac{W(t)}{W(st)} \leq Cs^{-\gamma p}$ for all $0 < s \leq 1$, where $W(x) = \int_0^x w(t) dt$.

The same case was also studied in [30].

In 1990 the necessary and sufficient condition for (0.10) in the case $1 \leq p = q < \infty$ and $u(t) = v(t)$ was proved in [1] by M. Arino and B. Muckenhoupt. The result is the following:

Let $1 \leq p < \infty$. Then the inequality

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s) ds \right)^p w(t) dt \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty f^p(t) w(t) dt \right)^{\frac{1}{p}} \quad (0.11)$$

holds for all $0 \leq f \downarrow$ if and only if there is a constant $D > 0$ such that

$$\int_t^\infty s^{-p} w(s) ds \leq Dt^{-p} \int_0^t w(s) ds \text{ for all } t > 0.$$

This condition is sometimes called the B_p -condition. It is known nowadays that the B_p -condition can be replaced by infinitely many equivalent conditions (see [20] and the references given there).

Observe that the result is valid also for $0 < p < 1$. This was proved in [64] and [17].

If we introduce

$$\|w\|_{B_p} := \sup_{r>0} \frac{\int_0^r w(t) dt + r^p \int_r^\infty \frac{w(t)}{t^p} dt}{\int_0^r w(t) dt} < \infty,$$

then, due to C.J. Neugebauer [42], for the constant C in (0.11) it yields that

$$C = \|w\|_{B_p}^{\frac{1}{p}}, \quad p \leq 1,$$

and

$$\|w\|_{B_p}^{\frac{1}{p}} \leq C \leq \|w\|_{B_p}, \quad p > 1.$$

The study of such inequalities on the cone of decreasing functions clearly differs from investigating the similar problem on the set of positive functions and has some peculiarities which require careful consideration. Note, that there is an example of weight w , for which (0.11) is true on the cone of monotone functions, but invalid for all $f \geq 0$ (see [1]).

Let $1 \leq q < \infty$. As an example we consider the function

$$W(x) := \begin{cases} 0, & 1 < x < 2, \\ x^{-\frac{1}{2}}, & 0 \leq x \leq 1 \text{ or } 2 \leq x. \end{cases}$$

This function does not satisfy the condition

$$\sup_{r>0} \left[\int_r^\infty \frac{W(x)}{x^q} dx \right] \left[\int_0^r W(x)^{-\frac{q'}{q}} dx \right]^{\frac{q}{q'}} < \infty,$$

where the second factor is taken to be $\operatorname{ess\,sup}_{[0,r]} \frac{1}{W(x)}$ in case $q = 1$ and $q' = \frac{q}{q-1}$.

It was proved by B. Muckenhoupt in [40, Theorem 1], that this condition is equivalent to the validity of inequality (0.11) with the weight W on the set all non-negative functions f on $[0, \infty)$. However, with the restriction that f is non-negative and non-increasing it can be shown that the inequality (0.11) holds.

The result of M. Arino and B. Muckenhoupt was extended by E. Sawyer [55] to the case of different weights v and w and $1 < p, q < \infty$. In his proof he used the duality principle which is also of great importance as an independent result.

The Sawyer duality principle. *Let $1 < p < \infty$, g, v be non-negative measurable functions on $(0, \infty)$ with v locally integrable. Then*

$$\begin{aligned} & \sup_{0 \leq f \downarrow} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}}} \\ & \approx \left(\int_0^\infty \left(\int_0^x g(t)dt \right)^{p'} \left(\int_0^x v(t)dt \right)^{-p'} v(x)dx \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x)dx}{\left(\int_0^\infty v(x)dx \right)^{\frac{1}{p}}}. \end{aligned} \quad (0.12)$$

We note, that if $\int_0^\infty v(x)dx = \infty$, then the second term on the right hand side of (0.12) can be omitted.

Theorem 4 [55]. (i) Let $1 < p \leq q < \infty$, $V(t) := \int_0^t v(s)ds$. Then (0.10) holds for all non-increasing functions $f \geq 0$ iff

$$A_0 := \sup_{x>0} \left(\int_0^x u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x v(t)dt \right)^{-\frac{1}{p}} < \infty,$$

$$A_1 := \sup_{x>0} \left(\int_x^\infty t^{-q}u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x t^{p'}V^{-p'}(t)v(t)dt \right)^{\frac{1}{p'}} < \infty,$$

and $C \approx A_0 + A_1$.

(ii) Let $1 < q < p < \infty$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. Then (0.10) holds for all non-increasing functions $f \geq 0$ iff

$$B_0 := \left(\int_0^\infty \left[\left(\int_0^x u(t)dt \right)^{\frac{1}{p}} \left(\int_0^x v(t)dt \right)^{-\frac{1}{p}} \right]^r u(x)dx \right)^{\frac{1}{r}} < \infty,$$

$$B_1 := \left(\left[\left(\int_x^\infty t^{-q}u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x t^{p'}V^{-p'}(t)v(t)dt \right)^{\frac{1}{q'}} \right]^r x^{p'}V^{-p'}(x)v(x)dx \right)^{\frac{1}{r}} < \infty,$$

and $C \approx B_0 + B_1$.

V.D. Stepanov [64] gave an alternative proof of the theorem above and also extended it to the cases $0 < q < 1 < p < \infty$ and $0 < p \leq q < \infty$, $0 < p < 1$.

Theorem 5 [64]. (i) Assertion (ii) of Theorem 4 holds if $0 < q < 1 < p < \infty$.

(ii) Let $0 < p \leq q < \infty$, $0 < p < 1$. Then (0.10) holds for all non-increasing functions $f \geq 0$ iff

$$A_0 < \infty$$

and

$$\mathcal{A}_1 := \sup_{x>0} x \left(\int_x^\infty t^{-q}u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x v(t)dt \right)^{-\frac{1}{p}} < \infty$$

and also $C \approx A_0 + \mathcal{A}_1$.

The duality principle for the case $0 < p \leq 1$ was proved in [17, Theorem 3.2], [27] and [64].

The duality principle (the case $0 < p \leq 1$). Let $0 < p \leq 1$, g, v be non-negative measurable functions on $(0, \infty)$ with v locally integrable. Then

$$\sup_{0 \leq f \downarrow} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}} = \sup_{t>0} \left(\int_0^t g(x)dx\right) \left(\int_0^t v(x)dx\right)^{-\frac{1}{p}}.$$

The case $0 < q < p < 1$ of (0.10) was studied in [22] and later in [8]. Here we cite the following criterion.

Theorem 6 [8]. Let $0 < q < p \leq 1$. Then the inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}$$

holds for all non-increasing functions $f \geq 0$ on $(0, \infty)$ iff

$$\mathcal{B}_0 := \int_0^\infty x^q w(x) \left(\frac{1}{V(x)} \int_0^x t^q w(t)dt\right)^{\frac{r}{p}} dx < \infty$$

and

$$\mathcal{B}_1 := \int_0^\infty w(x) \left(\int_x^\infty w(t)dt\right)^{\frac{r}{p}} \sup_{0 < t \leq x} \left(\frac{t^p}{V(t)}\right)^{\frac{r}{p}} dx < \infty.$$

Numerous publications dedicated to inequalities on the cones of monotone functions also include [16], [26], [21], [18], [57], [10], [11], [41], [9], [63] and most recently [29].

The next step in the process of generalization of inequalities restricted to the cones of monotone functions is inequalities with measures. As we proceed from Lebesgue measures to arbitrary positive Borel σ -finite measures, in general case we receive the inequality with three measures

$$\left(\int_{[0,\infty)} (H_s f)^q v d\mu\right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}$$

which holds for all $f \in \mathfrak{M} \downarrow$ ($f \in \mathfrak{M} \uparrow$), where

$$(H_s f)(x) := \left(\int_{[0,x]} f^s u d\lambda\right)^{\frac{1}{s}}$$

and $\mathfrak{M} \downarrow$ ($\mathfrak{M} \uparrow$) is a subclass of all non-increasing (non-decreasing) functions $f \in \mathfrak{M}^+$. Note that by making the substitution $f^s \rightarrow f$ this inequality

can be reduced to the equivalent one with new parameters p and q . Thus, we need only to study the inequality

$$\left(\int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p w d\nu \right)^{\frac{1}{p}}, \quad (0.13)$$

where $(Hf)(x) := \int_{[0,x]} f u d\lambda$.

It is important to mention that this inequality, unlike the inequality on the set of positive measurable functions, cannot be reduced to the one with two measures because of the condition of monotonicity, and the general case includes three measures.

The inequality (0.13) on monotone functions for $d\lambda = d\nu$ and $u \equiv v \equiv w \equiv 1$ was investigated by G. Sinnamon [58]. Specifically, in this case he established the equivalence of (0.13) for $f \in \mathfrak{M}^+$ to the similar inequality restricted to $f \in \mathfrak{M} \downarrow$.

Theorem 7 [58]. *Let $1 < p < \infty$, $0 < q < \infty$, λ, μ be measures on $[0, \infty)$, $\Lambda(x) := \int_{[0,x]} d\lambda$. Then the inequality*

$$\left(\int_0^\infty \left(\int_0^x f d\lambda \right)^q d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p d\lambda \right)^{\frac{1}{p}} \quad (0.14)$$

holds for functions $f \geq 0$ if and only if the inequality

$$\left(\int_0^\infty F^q \Lambda^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty F^p d\lambda \right)^{\frac{1}{p}} \quad (0.15)$$

holds for all non-increasing functions $F \geq 0$.

The similar theorem was proved for non-decreasing functions, but it requires introducing the following notion.

Definition 1 [58]. *Let $1 < p < \infty$. We say that $\lambda \in \mathcal{T}_p(\infty)$ provided*

$$\Lambda(x)^{1-p} - \Lambda(\infty)^{1-p} \leq C \int_{[x,\infty)} \Lambda^{-p} d\lambda$$

for some constant C .

Theorem 8 [58]. *Let $1 < p < \infty$, $0 < q < \infty$, λ, μ be measures on $[0, \infty)$, $\Lambda(x) := \int_{[0,x]} d\lambda$. Then the inequality (0.14) holds for all functions $f \geq 0$ whenever*

$$\left(\int_0^\infty F^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty F^p \Lambda^{-p} d\lambda \right)^{\frac{1}{p}} \quad (0.16)$$

holds for all non-decreasing functions $F \geq 0$. If $\lambda \in \mathcal{T}_p(\infty)$ and $\Lambda(\infty) = \infty$, then the condition (0.16) is also necessary for (0.14).

In [58] G. Sinnamon also obtained the characterization of the inequality

$$\left(\int_0^\infty F^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}, \quad F \downarrow$$

for $p, q > 0$. This result has been used in paper B of this thesis and obviously it can be extended to the case of non-decreasing functions F .

Theorem 9 [58]. (i) If $0 < p \leq q < \infty$, then

$$\sup_{F \downarrow} \frac{\left(\int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left(\int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} = \sup_{x \geq 0} \frac{\left(\int_{[0,x]} d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,x]} d\lambda \right)^{\frac{1}{p}}}.$$

(ii) If $0 < q < p < \infty$ and $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, then

$$\sup_{F \downarrow} \frac{\left(\int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left(\int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\int_{[x,\infty)} \frac{d\mu}{\Lambda} \right)^{\frac{r}{q}} d\lambda(x) \right)^{\frac{1}{r}}.$$

(iii) Suppose that $\lambda \in \mathcal{T}_2(\infty)$. If $0 < q < p < \infty$, then

$$\sup_{F \downarrow} \frac{\left(\int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left(\int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\frac{1}{\Lambda(x)} \int_{[0,x]} d\mu + \frac{1}{\Lambda(\infty)} \int_0^\infty d\mu \right)^{\frac{r}{q}} d\lambda(x) \right)^{\frac{1}{r}}.$$

Note that assertion (ii) of Theorem 9 with $q = 1$ gives an analog of the Sawyer duality principle with general Borel measures.

The general case of (0.13) was studied in [28]. Here we summarize the main results of this paper, and first we introduce the following definition.

Definition 2 [28]. Let $f \in \mathfrak{M} \downarrow$ and be continuous on the left. It is known (see [54]) that there exists a Borel measure η_w , such that $w(x) = \int_{[x,\infty)} d\eta_w + w(+\infty)$. We say that $w \in \mathcal{S}(0)$ if there exists a constant $C \geq 1$ such that

$$\frac{1}{w(x)} - \frac{1}{w(0)} \leq C \int_{[0,x]} \frac{d\eta_w}{w^2}, \quad x > 0.$$

Theorem 10 [28]. (i) Let $0 < p \leq q < \infty$, $0 < p \leq 1$ and $\Lambda(x) := \int_{[0,x]} d\lambda$. Then (0.13) holds for all $f \in \mathfrak{M} \downarrow$ iff

$$A_0 := \sup_{t \in [0, \infty)} \left(\int_{[0,t]} w d\nu \right)^{-\frac{1}{p}} \left(\int_{[0,t]} \Lambda^q v d\mu \right)^{\frac{1}{q}} < \infty,$$

$$A_1 := \sup_{t \in [0, \infty)} \Lambda(t) \left(\int_{[0,t]} w d\nu \right)^{-\frac{1}{p}} \left(\int_{[t, \infty)} v d\mu \right)^{\frac{1}{q}} < \infty,$$

and $C \approx A_0 + A_1$.

(ii) Let $0 < q < 1 = p$ and $W(x) := \int_{[0,x]} w d\nu$. Then (0.13) holds for all $f \in \mathfrak{M} \downarrow$ iff

$$B_0 := \left(\int_{[0, \infty)} w(y) \left(\int_{[y, \infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{1}{1-q}} d\nu(y) \right)^{\frac{1-q}{q}} < \infty,$$

$$B_1 := \left(\int_{[0, \infty)} \left(\int_{[0,x]} \operatorname{ess\,sup}_{s \in [0,t]} \frac{\Lambda(s)}{W(s)} v(t) d\mu(t) \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} < \infty,$$

and $C \approx B_0 + B_1$.

(iii) Let $0 < q < p < 1$ and $\mathcal{V}_p(t) := \operatorname{ess\,sup}_{s \in [0,t]} \frac{\Lambda^p(s)}{W(s)}$. Then (0.13) holds for all $f \in \mathfrak{M} \downarrow$ if

$$\mathcal{B}_0 := \left(\int_{[0, \infty)} w(y) \left(\int_{[y, \infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{p}{p-q}} d\nu(y) \right)^{\frac{p-q}{pq}} < \infty,$$

$$\mathcal{B}_1 := \left(\int_{[0, \infty)} \left(\int_{[0,x]} \mathcal{V}_p(t) v(t) d\mu(t) \right)^{\frac{q}{p-q}} v(x) d\mu(x) \right)^{\frac{p-q}{pq}} < \infty$$

and only if $\mathcal{B}_0 + \mathcal{B}_1 < \infty$, provided $\mathcal{V}_p(t)$ is continuous on $(0, \infty)$ and $\frac{1}{\mathcal{V}_p(t)} \in \mathcal{S}(0)$. Then $C \approx \mathcal{B}_0 + \mathcal{B}_1$.

(iv) Let $1 < p \leq q < \infty$, $k(x, y) := \int_{[y,x]} W^{-1} u d\lambda$ and $f \in \mathfrak{M} \downarrow$. Then (0.13) holds iff $\mathbb{A} = \max \{\mathbb{A}_0 + \mathbb{A}_1\} < \infty$, where

$$\mathbb{A}_0 := \sup_{t \in [0, \infty)} \left(\int_{[0,t]} w(y) k(t, y)^{p'} d\nu(y) \right)^{\frac{1}{p'}} \left(\int_{[t, \infty)} v d\mu \right)^{\frac{1}{q}},$$

$$\mathbb{A}_1 := \sup_{t \in [0, \infty)} \left(\int_{[0, t]} w d\nu \right)^{\frac{1}{p'}} \left(\int_{[t, \infty)} v(x) k(x, t)^q d\mu(x) \right)^{\frac{1}{q}}.$$

Moreover, if C is the best constant in (0.13), then $C = \mathbb{A}$.

(v) Let $1 < q < p < \infty$ and $f \in \mathfrak{M} \downarrow$. Then (0.13) holds iff $\mathbb{B} = \max \{\mathbb{B}_0 + \mathbb{B}_1\} < \infty$, where

$$\mathbb{B}_0 := \left(\int_{[0, \infty)} \left(\int_{[0, t]} w(y) k(t, y)^{p'} d\nu(y) \right)^{\frac{r}{p'}} \left(\int_{[t, \infty)} v d\mu \right)^{\frac{r}{p}} v(t) d\mu(t) \right)^{\frac{1}{r}},$$

$$\mathbb{B}_1 := \left(\int_{[0, \infty)} \left(\int_{[0, t]} w d\nu \right)^{\frac{r}{q'}} \left(\int_{[t, \infty)} v(x) k(x, t)^q d\mu(x) \right)^{\frac{r}{q}} w(t) d\nu(t) \right)^{\frac{1}{r}}.$$

Moreover, if C is the best constant in (0.13), then $C = \mathbb{B}$.

Paper A of this thesis (see [45] and also [73]) is dedicated to studying Hardy-type operators on the cones of monotone functions with general positive σ -finite measure. This study was strongly influenced by the paper [58] by G. Sinnamon, where he obtained the inequality

$$\left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} f(t) d\lambda(t) \right)^p d\lambda(x) \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}}$$

for all λ -measurable functions $f \geq 0$ in the case $1 < p < \infty$. It can be easily checked that this inequality also holds on the cone of non-increasing functions. Therefore, in this case we obtain the equivalence

$$\left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} f(t) d\lambda(t) \right)^p d\lambda(x) \right)^{\frac{1}{p}} \approx \left(\int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}}.$$

The aim of paper A is to establish the similar equivalences for some other Hardy-type operators and also for the full range of parameter p by means of proving two-sided inequalities with measures. For example, for $p > 0$, $f \in \mathfrak{M} \downarrow$ we prove the following result.

Theorem 11 [45]. Let $p > 0$ and $\Lambda(x) := \int_{[0, x]} d\lambda$. Then for all $f \in \mathfrak{M} \downarrow$ we have

$$\left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t)^{\frac{1}{p}} f(t) d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \right)^{\frac{1}{p}} \approx \left(\int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}}.$$

The next equivalence is concerned with the dual Hardy operator.

Theorem 12 [45]. *Let $1 \leq p < \infty$. Then for all $f \in \mathfrak{M} \downarrow$, $f \not\equiv 0$ it holds that*

$$\frac{1}{(p+1)^{\frac{1}{p}}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq p.$$

In the case $0 < p < 1$ we have

$$0 < c(p) \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq \frac{1}{p}.$$

Proving an analogous result for the operator involving suprema instead of the integral, we obtain the following theorem.

Theorem 13 [45]. *Let $0 < p < \infty$. Then for all $f \in \mathfrak{M} \downarrow$, $f \not\equiv 0$*

$$1 \leq \frac{\left(\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{y \geq x} f^p(y) \Lambda^p(y) \right] d\lambda(x) \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq c(p),$$

where

$$c^p(p) = \max(1, 2^{p-1}) \left(1 + \frac{\max(1, p)}{\min(1, p)} \right).$$

For an arbitrary function $f \in \mathfrak{M} \downarrow$, which is continuous on the right (denote by $\mathfrak{M} \downarrow (+0)$), we introduce a Borel measure μ_f such that $\frac{1}{f(x)} = \int_{[0,x]} d\mu_f$. Such measure exists and it is unique, because the function $\frac{1}{f} \in \mathfrak{M} \uparrow$ is also continuous on the right (see [54]). The following two-sided inequality for the measure μ_f was proved in [45].

Theorem 14 [45]. *Let $0 < p < \infty$. Then for all $f \in \mathfrak{M} \downarrow (+0)$*

$$1 \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{p+1} d\mu_f(x) \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}} \leq (p+1)^{\frac{1}{p}}.$$

Finally, combining the results of Theorems 13 and 14, we obtain the following result.

Theorem 15 [45]. *Let $0 < p < \infty$. Then for all $f \in \mathfrak{M} \downarrow (+0)$ it yields that*

$$c(p) \leq \frac{\left(\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{y \geq x} f(y) \Lambda(y) \right]^p d\lambda(x) \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{y \geq x} f(y) \Lambda(y) \right]^{p+1} d\mu_f(x) \right)^{\frac{1}{p}}} \leq \left(\frac{\max(1, p)}{\min(1, p)} \right)^{\frac{1}{p}},$$

where $c(p) = \left(\frac{p}{(p+1)2^{p+1}} \right)^{\frac{1}{p}}$.

Remark. Note that Theorems 11-15 have complete analogues for non-decreasing functions in case we replace $\Lambda(x)$ by $\Lambda_*(x) := \int_{[x,\infty)} d\lambda$.

For negative values of parameter p the following result was proved in [45].

Theorem 16 [45]. *Let $-1 < p < 0$. Then for all $f \in \mathfrak{M} \uparrow$, $f(x) > 0$ λ - a.e.*

$$(-p)^{\frac{1}{p}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq 1.$$

For $p \leq -1$ it yields that

$$0 < c_p \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq 1.$$

Remark. Theorem 16 has complete analogue for non-increasing functions with the changes mentioned above.

As an application, using Theorem 2, we obtain a new characterization of the discrete Hardy inequality

$$\left(\sum_n w_n^q \left(\sum_{k \geq n} \alpha_k \right)^q \right)^{\frac{1}{q}} \leq C \left(\sum_n \phi_n^p \alpha_n^p \right)^{\frac{1}{p}},$$

which was also studied in [22], in the case $0 < q < p \leq 1$ and thus establish its equivalence to a simpler inequality on monotone sequences.

The equivalences we have derived in paper A are used in paper B of this thesis (see [49]) to obtain necessary and sufficient conditions for some other Hardy-type inequalities on cones of monotone functions for the full range of parameters of summation p and q . In particular, in Section 3.3 of paper B we give a full characterization of the inequalities

$$\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow, f \not\equiv 0 \quad (0.17)$$

and

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \uparrow, f \not\equiv 0$$

both when $0 < p, q < \infty$ and $-\infty < p, q < 0$.

For instance, for (0.17) in the case $0 < p, q < \infty$ we obtain the following criteria:

$$C = \sup_{x \in [0,\infty)} \frac{\left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,x]} d\mu \right)^{\frac{1}{p}}}, \quad 0 < p \leq q < \infty,$$

and

$$C \approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{r}{q}} \left(\int_{[0,x]} d\mu \right)^{-\frac{r}{q}} d\mu(x) \right)^{\frac{1}{r}}, \quad 0 < q < p < \infty,$$

provided

$$\int_{[0,\infty)} d\mu = \infty, \quad \left(\int_{[0,x]} d\mu \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} d\mu \right)^{-2} d\mu(x).$$

As the inequality (0.17) has been studied in [22] by M.L. Goldman, we also compare our criteria with the criteria from [22] (see paper B, Section 3.2).

Another question we investigate in paper B is Hardy-type inequalities with kernels restricted to the cone of monotone functions. In particular, we give a complete description for the inequalities with Volterra integral operators involving so called Oinarov's kernels. Specifically, we study the inequality of the form

$$\left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}, \quad (0.18)$$

where $0 < p, q < \infty$,

$$Kf(x) := \int_{[0,x]} k(x, y) f(y) d\nu(y), \quad (0.19)$$

λ, μ and ν are positive σ -finite Borel measures on $\mathbb{R}_+ := [0, \infty)$ and a measurable kernel $k(x, y) \geq 0$ satisfies the Oinarov condition, i.e. that there exists a constant $D \geq 1$, for which

$$D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)), x \geq z \geq y. \quad (0.20)$$

The problem is to obtain a sharp two-sided estimate of the least possible positive constant C provided the inequality (0.18) is fulfilled for any function f from a prescribed set.

The study of inequalities for the class of Volterra-Hardy operators and also the class of general integral operators with a given non-negative kernel $k(x, t)$ has initially started from investigating the Riemann-Liouville operator and naturally included only Lebesgue measures. The two-weighted inequality

$$\|(Kf)u\|_q \leq C\|fv\|_p, \quad f \leq 0, \quad (0.21)$$

where

$$Kf(x) := \int_{[0,x]} k(x, y) f(y) dy, \quad (0.22)$$

has been studied by many authors. As we have shown, the problem (0.21), (0.22) with K replaced by the Hardy operator having kernel $k(x, y) \equiv 1$ has now been almost completely characterized.

V.D. Stepanov obtained necessary and sufficient conditions for (0.21) to hold and for the operator (0.22) to be compact in the case of Riemann-Liouville operator with kernel

$$k(x, y) = \frac{1}{\Gamma(r)}(x - y)^{r-1}, \quad r \geq 1$$

for $1 < p, q < \infty$ (see, for example, [68], [69], [70]).

During the same period P.J. Martin-Reyes and E. Sawyer in [37] by different methods characterized (0.21) in the case $1 < p \leq q < \infty$ for a class of integral operators given by

$$K_\phi f(x) := \int_{[0,x]} \phi\left(\frac{t}{x}\right) f(t) dt,$$

where $\phi : (0, 1) \rightarrow (0, \infty)$ is non-increasing and satisfies the condition $\phi(ab) \leq D(\phi(a) + \phi(b))$ for all $0 < a, b < 1$.

Similar criteria were obtained by V.D. Stepanov ([71], [72]) for Volterra convolution operators of the form

$$K_k f(x) := \int_{[0,x]} k(x-t)f(t)dt \quad (0.23)$$

for the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$, but with different conditions on the kernel. For instance, in [72] he studied the operator with the kernel k which satisfies the following conditions:

- (i) $k(x) \geq 0$ is non-decreasing on $(0, \infty)$,
- (ii) $k(x+y) \leq D(k(x) + k(y))$ for all $x, y \in (0, \infty)$.

Also, the case $0 < q < 1 < p < \infty$ with a kernel satisfying only condition (i) was investigated, and necessary and sufficient conditions were obtained, but they did not coincide. Moreover, the compactness criteria for the operator (0.23) for the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$ were proved.

In 1991 S. Bloom and R. Kerman [13] obtained criteria for (0.21) to hold for the operators (0.22) when $1 < p \leq q < \infty$ under assumption that not only the condition (0.20) is satisfied, but also $k(x, y) \geq 0$ for $0 < y < x$ and the kernel $k(x, y)$ is both non-decreasing in x and non-increasing in y . Specifically, they considered kernels $\phi(x, y)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with the following properties:

- (i) $\phi(x, y) > 0$ if $x > y$,
- (ii) $\phi(x, y)$ is non-decreasing in x and non-increasing in y ,
- (iii) $\phi(x, y) \approx \phi(x, z) + \phi(z, y)$ if $y < z < x$.

Moreover, the following operators were defined for $r \geq 0$:

$$(T_r f)(x) := \int_{[0,x]} \phi(x, y)^r f(y) dy,$$

$$(T_r^* g)(x) := \int_{[x,\infty)} \phi(y, x)^r g(y) dy,$$

$$(Tf)(x) := (T_1 f)(x)$$

and

$$(If)(x) := (T_0 f)(x).$$

Here we cite the principal result obtained in [13].

Theorem 17 [13]. *Let $1 < p \leq q < \infty$, u and v be non-negative, measurable functions on $(0, \infty)$ with $0 < u, v < \infty$ a.e. Then*

$$\left(\int_{[0,\infty)} (uTf)^q \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} (vf)^p \right)^{\frac{1}{p}} \text{ for all } f \geq 0$$

if and only if

$$I^* \left[(v^{-1} T^* u^q)^{p'} \right] \leq C (I^* u^q)^{\frac{p'}{q}} < \infty$$

and

$$I^* \left[(v^{-1} T_q^* u^q)^{p'} \right] \leq C (T_q^* u^q)^{\frac{p'}{q}} < \infty$$

a.e. on $(0, \infty)$.

In 1993 R. Oinarov [43] introduced the final form of condition for the kernel k and proved corresponding necessary and sufficient conditions for (0.21) with operator of the form (0.22) to hold. The condition he used reads as follows: $k(x, y) > 0$ for $x > y$ and there exists a constant $D > 0$ such that (0.20) holds. This condition is usually called the Oinarov condition.

The results obtained in papers [13], [37], [71], [72] were generalized in [66] by V.D. Stepanov for the cases $1 < p \leq q < \infty$, $1 < q < p < \infty$ with less strict conditions on the kernel. The criteria below keep the symmetric form as in [68] and differ from the criteria in [13] and [37].

Before turning to the main theorem of [66], we need to introduce the following notations. For $1 < p, q < \infty$ we assume that $u(x) \geq 0$ and $v(x) \geq 0$ are such that the dual spaces $(L_v^p)^*$ and $(L_u^q)^*$ can be identified with $L_{1/v}^{p'}$ and $L_{1/u}^{q'}$. We also define for $s \geq 0$

$$K_s f(x) := \int_{[0, x]} k(x, y)^s f(y) dy,$$

$$K_s^* g(y) := \int_{[y, \infty)} k(x, y)^s g(x) dx,$$

$$K_0 f(x) := \int_{[0, x]} f(y) dy$$

and

$$K_0^* g(y) := \int_{[y, \infty)} g(x) dx.$$

Theorem 18 [66]. (i) Let $1 < p \leq q < \infty$ and an operator K be of the form (0.22) with kernel k satisfying the following conditions:

- 1) $k(x, y) \geq 0$ for $0 < y < x$ and is non-decreasing in x or non-increasing in y ,
 - 2) $D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)),$
- $0 < x < z < y.$

(0.24)

Then the inequality (0.21) holds iff $A = \max(A_0, A_1) < \infty$, where

$$A_0 := \sup_{t>0} \left(K_q^* u^q(t) \right)^{\frac{1}{q}} \left(K_0 v^{-p'}(t) \right)^{\frac{1}{p'}},$$

$$A_1 := \sup_{t>0} \left(K_0^* u^q(t) \right)^{\frac{1}{q}} \left(K_{p'} v^{-p'}(t) \right)^{\frac{1}{p'}}.$$

(ii) Let $1 < q < p < \infty$ and $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. If operator K has the form (0.22) with kernel k satisfying (0.24), then the inequality (0.21) holds iff $B = \max(B_0, B_1) < \infty$, where

$$B_0 := \left(\int_{[0,\infty)} \left[\left(K_q^* u^q(t) \right)^{\frac{1}{q}} \left(K_0 v^{-p'}(t) \right)^{\frac{1}{p'}} \right]^r v(t)^{-p'} dt \right)^{\frac{1}{r}},$$

$$B_1 := \left(\int_{[0,\infty)} \left[\left(K_0^* u^q(t) \right)^{\frac{1}{q}} \left(K_{p'} v^{-p'}(t) \right)^{\frac{1}{p'}} \right]^r u(t)^q dt \right)^{\frac{1}{r}}.$$

Remark. The criteria for compactness of operator K for the mentioned range of p and q are also obtained in [66].

In 1999 Q. Lai [35] gave a characterization of (0.21), (0.22) which had an essentially different form. He imposed the following conditions on the kernel $k(x, y)$:

- (i) $k(x, y) > 0$ if $x > y$,
- (ii) $k(x, y)$ is non-decreasing in x and non-increasing in y ,
- (iii) there exists a constant $D > 0$ such that $k(x, y) \leq D(k(x, z) + k(z, y))$ for all $0 \leq y \leq z \leq x$.

The case of (0.18) with arbitrary Borel measures was fractionally presented in [40], [39], for discrete inequalities in [5], [6], [7] and for three measures in [50]. The full characterization of the inequality (0.18) for the operators with the Oinarov kernels in the case $1 < p, q < \infty$ was obtained by D.V. Prokhorov in [52]. Here we summarize the results of [52] in Theorem 19 for the case of two measures and in Theorem 20 for the case of three measures.

Theorem 19 [52]. (i) Let $1 < p \leq q < +\infty$, λ, μ be positive Borel σ -finite measures on $[a, b]$, $u, v \in \mathfrak{M}^+$, $w \equiv 1$, k a $\mu \times \lambda$ -measurable non-negative function satisfying condition (0.20). Then for the existence of a constant $C \geq 0$, such that inequality

$$\left(\int_{[a,b]} (Kf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[a,b]} f^p w d\lambda \right)^{\frac{1}{p}} \quad (0.25)$$

with

$$(Kf)(x) := \int_{[a,x]} k(x,y)u(y)f(y)d\lambda(y)$$

holds for all $f \in \mathfrak{M}^+$, it is necessary and sufficient that $A := \max \{A_1, A_2\} < +\infty$, where

$$A_1 := \sup_{t \in [a,b]} \left(\int_{[t,b]} v(x)k(x,t)^q d\mu(x) \right)^{\frac{1}{q}} \left(\int_{[a,t]} u^{p'} d\lambda \right)^{\frac{1}{p'}},$$

$$A_2 := \sup_{t \in [a,b]} \left(\int_{[t,b]} v d\mu \right)^{\frac{1}{q}} \left(\int_{[a,t]} k(t,y)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}.$$

(ii) Let $1 < q < p < +\infty$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, λ, μ be positive Borel σ -finite measures on $[a, b]$, $u, v \in \mathfrak{M}^+$, $w \equiv 1$, then for the existence of a constant $C \geq 0$ such that inequality (0.25) holds, it is necessary and sufficient, that $B := \max \{B_1, B_2\} < +\infty$, where

$$B_1 := \left(\int_{[a,b]} \left(\int_{[t,b]} v(x)k(x,t)^q d\mu(x) \right)^{\frac{r}{q}} \left(\int_{[a,t]} u^{p'} d\lambda \right)^{\frac{r}{q'}} u(t)^{p'} d\lambda(t) \right)^{\frac{1}{r}},$$

$$B_2 := \left(\int_{[a,b]} \left(\int_{[t,b]} v d\mu \right)^{\frac{r}{p}} \left(\int_{[a,t]} k(t,y)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{r}{p'}} v(t) d\mu(t) \right)^{\frac{1}{r}}.$$

Theorem 20 [52]. (i) Let $1 < p \leq q < +\infty$, μ, λ and ν be positive Borel σ -finite measures on $[a, b]$, $u, v, w \in \mathfrak{M}^+$, (ν_a, ν_s) the Lebesgue decomposition of ν relative to λ , and $d\nu_a/d\lambda$ the Radon-Nikodym derivative of ν_a with respect to λ , k a $\mu \times \lambda$ -measurable non-negative function satisfying condition (0.20). Put

$$\omega := u^{p'} \left(w \frac{d\nu_a}{d\lambda} \right)^{1-p'}.$$

Then for the existence of a constant $C \geq 0$ such that inequality

$$\left(\int_{[a,b]} (Kf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[a,b]} f^p w d\nu \right)^{\frac{1}{p}} \quad (0.26)$$

with

$$(Kf)(x) := \int_{[a,x]} k(x,y)u(y)f(y)d\lambda(y)$$

holds, it is necessary and sufficient, that $A := \max \{A_1, A_2\} < \infty$, where

$$A_1 := \sup_{t \in [a, b]} \left(\int_{[t, b]} v(x) k(x, t)^q d\mu(x) \right)^{\frac{1}{q}} \left(\int_{[a, t]} \omega d\lambda \right)^{\frac{1}{p'}},$$

$$A_2 := \sup_{t \in [a, b]} \left(\int_{[t, b]} v d\mu \right)^{\frac{1}{q}} \left(\int_{[a, t]} k(t, y)^{p'} \omega(y) d\lambda(y) \right)^{\frac{1}{p'}}.$$

(ii) Let $1 < q < p < +\infty$, then for the existence of a constant $C \geq 0$ such that inequality (0.26) holds, it is necessary and sufficient, that $B := \max \{B_1, B_2\} < +\infty$, where

$$B_1 := \left(\int_{[a, b]} \left(\int_{[t, b]} v(x) k(x, t)^q d\mu(x) \right)^{\frac{r}{q}} \left(\int_{[a, t]} \omega d\lambda \right)^{\frac{r}{q'}} \omega(t) d\lambda(t) \right)^{\frac{1}{r}},$$

$$B_2 := \left(\int_{[a, b]} \left(\int_{[t, b]} v d\mu \right)^{\frac{r}{p}} \left(\int_{[a, t]} k(t, y)^{p'} \omega(y) d\lambda(y) \right)^{\frac{r}{p'}} v(t) d\mu(t) \right)^{\frac{1}{r}}.$$

The results of Theorem 20 are used in [52] to characterize the inequality

$$\left(\int_{[a, b]} \int_{[a, x]} f u d\lambda \right)^q v(x) d\mu(x) \Big)^{\frac{1}{q}} \leq C \left(\int_{[a, b]} f^p u d\nu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^+$$

for $1 < p, q < \infty$.

The case $0 < p < 1$, $0 < q < \infty$ was in full characterized by D.V. Prokhorov in [51]. We omit the details here.

Inequalities of the form (0.18) for operator (0.19) on the cones of monotone functions have not been studied as actively as classical inequalities of Hardy type.

The weighted inequalities with Hardy-type operators restricted to the cones of monotone functions were investigated in [16], [17], [65]. In particular, in [65] a number of classical operators was explicitly described and criteria of boundedness for the Hardy-Littlewood maximal operator in the Lorentz space $\Gamma_p(v)$ were obtained.

In [47] some of the classical results obtained for Hardy-type inequalities, especially results from [58], were generalized for Volterra integral operators with the kernel satisfying some conditions of monotonicity. Moreover, some equivalences between inequalities on the sets of non-negative and non-increasing functions for different forms of integral operators were established.

In Section 3.1 of paper B we characterize the Hardy-type inequality (0.18) on the cone of non-increasing functions $\mathfrak{M} \downarrow$.

Theorem 21 [49]. *Let $0 < p < \infty, 1 \leq q < \infty$. Let*

$$\bar{k}(x) := \int_{[0,x]} k(x,y) d\nu(y),$$

$$\tilde{k}(x) := \int_{[x,\infty)} k(y,x) d\mu(y),$$

$$\bar{\nu}(x) := \int_{[0,x]} d\nu(y).$$

Then the inequality (0.18) holds for all functions $f \in \mathfrak{M} \downarrow$ iff

$$A_0 < \infty, \text{ if } p \leq q = 1,$$

$$B_0 < \infty, \text{ if } 1 = q < p,$$

$$A_1 + A_{2,1} + A_{2,1}^* + A_{2,2} < \infty, \text{ if } 1 < p \leq q,$$

$$B_1 + B_{2,1} + B_{2,1}^* + B_{2,2} < \infty, \text{ if } 1 < q < p,$$

$$D_1 + D_2 + D_3 < \infty, \text{ if } 0 < p \leq 1 < q,$$

where

$$A_0 := \sup_{x \geq 0} \left(\int_{[0,x]} \tilde{k}(y) d\nu(y) \right) \left(\int_{[0,x]} d\lambda \right)^{-\frac{1}{p}},$$

$$B_0 := \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{\tilde{k}(t) d\nu(t)}{\int_{[0,t]} d\lambda} \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}},$$

$$A_1 := \sup_{t \geq 0} \left(\int_{[t,\infty)} \Lambda(x)^{-p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[0,t]} \bar{k}(z)^q d\mu(z) \right)^{\frac{1}{q}},$$

$$A_{2,1} := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu(x) \right)^{\frac{1}{q}},$$

$$A_{2,1}^* := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{1}{q}},$$

$$A_{2,2} := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{k}(x)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu(z) \right)^{\frac{1}{q}};$$

$$\begin{aligned}
B_1^r &:= \int_{[0,\infty)} \left(\int_{[0,x]} \bar{k}(z)^q d\mu(z) \right)^{\frac{r}{q}} \left(\int_{[x,\infty)} \Lambda(x)^{-p'} d\lambda(x) \right)^{\frac{r}{q'}} \Lambda(x)^{-p'} d\lambda(x), \\
B_{2,1}^r &:= \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{r}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{r}{p}} d\mu(t), \\
B_{2,1}^{*r} &:= \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} d\lambda(x) \right)^{\frac{r}{q'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{r}{q}} \Lambda(t)^{-p'} \bar{\nu}(t)^{p'} d\lambda(t), \\
B_{2,2}^r &:= \int_{[0,\infty)} \left(\int_{[x,\infty)} d\mu(z) \right)^{\frac{r}{q}} \left(\int_{[0,x]} \Lambda(t)^{-p'} \bar{k}(t)^{p'} d\lambda(t) \right)^{\frac{r}{q'}} \Lambda^{-p'}(x) \bar{k}^{p'}(x) d\lambda(x), \\
\text{where } \frac{1}{r} &= \frac{1}{q} - \frac{1}{p};
\end{aligned}$$

$$\begin{aligned}
D_1 &:= \sup_{x \geq 0} \left(\int_{[0,x]} \bar{k}^q(z) d\mu(z) \right)^{\frac{1}{q}} \Lambda^{-\frac{1}{p}}(x), \\
D_2 &:= \sup_{x \geq 0} \left(\int_{[x,\infty)} k^q(y,x) d\mu(y) \right)^{\frac{1}{q}} \bar{\nu}(x) \Lambda^{-\frac{1}{p}}(x), \\
D_3 &:= \sup_{x \geq 0} \left(\int_{[x,\infty)} d\mu(z) \right)^{\frac{1}{q}} \bar{k}(x) \Lambda^{-\frac{1}{p}}(x).
\end{aligned}$$

Remark. In paper B we also obtain a characterization of the inequality

$$\left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}} \quad (0.27)$$

for all $f \in \mathfrak{M} \downarrow$ with $0 < p, q < \infty, q \geq 1$ (see [49, Theorem 8]).

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Paper A

Two-sided Hardy-type inequalities for monotone functions[†]

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Communicated by M. Lanza de Cristoforis

(Received 11 April 2009; final version received 17 June 2009)

We consider Hardy-type operators on the cones of monotone functions with general positive σ -finite Borel measure. Some two-sided Hardy-type inequalities are proved for the parameter $-\infty < p < \infty$. It is pointed out that such equivalences, in particular, imply a new characterization of the discrete Hardy inequality for the (most difficult) case $0 < q < p \leq 1$.

Keywords: integral inequalities; weights; Hardy operator; monotone functions; measures

AMS Subject Classifications: Primary 26D10; Secondary 26D15; 26D07

1. Introduction

For the last three decades, the Hardy-type inequalities have been extensively studied, see, e.g. the monographs [1–3] and references given there. In particular, much attention was paid to the inequalities restricted to the cones of monotone functions. Here we just mention the papers [4–16], which have been sources of inspiration to study the questions presented in this article.

Suppose that λ is a positive Borel σ -finite measure on $\mathbb{R}_+ := [0, \infty)$. Let $\mathfrak{M}^+$ be the class consisting of all Borel functions $f: [0, \infty) \rightarrow [0, +\infty]$ and $\mathfrak{M} \downarrow$ ($\mathfrak{M} \uparrow$) be a subclass of $\mathfrak{M}^+$ which consists of all non-increasing (non-decreasing) functions $f \in \mathfrak{M}^+$. Put $\Lambda(x) := \int_{[0,x]} d\lambda$ and suppose $\Lambda(x) < \infty$ for all $x \in \mathbb{R}_+$. It follows from the Hardy inequality [13] that for any $f \in \mathfrak{M} \downarrow$ and $1 < p < \infty$

$$\left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}} \leq \left(\int_{[0,\infty)} \left(\frac{1}{\Lambda(x)} \int_{[0,x]} f(t) d\lambda(t) \right)^p d\lambda(x) \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}. \quad (1)$$

The aim of this article is to extend the equivalence (1) for similar Hardy-type operators and the full range of parameter p . The case of a positive parameter is studied in Section 2 and a negative in Section 3. In particular, in Section 2 we extend the discrete result of [5, Theorem 1] for any positive σ -finite Borel measures. As an

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[†]Dedicated to Professor Gérard Bourdaud on the occasion of his 60th birthday.

application, we obtain a new characterizations of the discrete Hardy inequality in the most difficult case of parameters $0 < q < p \leq 1$ (see Section 4 and cf. [4,8]).

Throughout this article products of the form $0 \cdot \infty$ are taken to be equal to 0. The relation $A \ll B$ means that $A \leq cB$ with some constant c depending only on the parameter of summation. We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ denotes the set of all integers and χ_E stands for the characteristic function (indicator) of a subset $E \subset \mathbb{R}_+$. Moreover, we make use of the marks $:=$ and $=:$ for introducing new quantities and denote $p' := p/(p-1)$ for $0 < p < \infty$, $p \neq 1$.

2. Positive parameter

Let $0 \leq x < y \leq \infty$. We put $[x, y) := [x, y]$ if $y < \infty$ and $[x, y) := [x, y]$ if $y = \infty$. For $f \in \mathfrak{M}^+$, we denote

$$\Lambda_f(x, y) := \int_{[x, y)} f \, d\lambda. \quad (2)$$

We need the following statement which easy follows from [17, Lemmas 1 and 2]):

LEMMA 1 *If $\gamma > 0$ and Λ_f is defined by (2), then the inequalities*

$$\frac{\Lambda_f^{\gamma+1}(x, y)}{\max\{1, \gamma + 1\}} \leq \int_{[x, y)} f(z) \Lambda_f^\gamma(x, z) \, d\lambda(z) \leq \frac{\Lambda_f^{\gamma+1}(x, y)}{\min\{1, \gamma + 1\}} \quad (3)$$

and

$$\frac{\Lambda_f^{\gamma+1}(x, y)}{\max\{1, \gamma + 1\}} \leq \int_{[x, y)} f(z) \Lambda_f^\gamma(z, y) \, d\lambda(z) \leq \frac{\Lambda_f^{\gamma+1}(x, y)}{\min\{1, \gamma + 1\}} \quad (4)$$

hold. If $\gamma \in (-1, 0)$ and $\Lambda_f(x, y) < +\infty$, then (3) and (4) also hold.

THEOREM 1 *Let $p > 0$. Then for all $f \in \mathfrak{M} \downarrow$*

$$\left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t)^{\frac{1}{p}} f(t) \, d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \right)^{\frac{1}{p}} \approx \left(\int_{[0, \infty)} f^p \, d\lambda \right)^{\frac{1}{p}}. \quad (5)$$

Proof For the lower bound applying the left-hand side of (3), we find that

$$\begin{aligned} \left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t)^{\frac{1}{p}} f(t) \, d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \right)^{\frac{1}{p}} &\geq \int_{[0, \infty)} f^p(x) \left(\frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t)^{\frac{1}{p}} d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \\ &= \left(\frac{p}{p+1} \right)^p \int_{[0, \infty)} f^p \, d\lambda. \end{aligned}$$

To prove the upper bound, we observe that the characterization of the inequality

$$\left(\int_{[0, \infty)} \left(\frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t)^{\frac{1}{p}} f(t) \, d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \right)^{\frac{1}{p}} \leq C \left(\int_{[0, \infty)} f^p \, d\lambda \right)^{\frac{1}{p}},$$

with a least possible constant C governed by Theorems 3.4 and 4.2 in [9] for the cases $0 < p \leq 1$ and $1 < p < \infty$, respectively. The case $p = 1$ is trivial. Let $0 < p < 1$. According to [9, Theorem 3.4], it yields

$$C \ll \sup_{t>0} A_0(t) + \sup_{t>0} A_1(t),$$

where

$$A_0(t) := \Lambda(t)^{-\frac{1}{p}} \left(\int_{[0,t]} \left(\int_{[0,s]} \Lambda^{\frac{1}{p}} d\lambda \right)^p \Lambda(s)^{-p-1} d\lambda(s) \right)^{\frac{1}{p}}$$

and

$$A_1(t) := \left(\int_{[0,t]} \Lambda^{\frac{1}{p}} d\lambda \right) \Lambda(t)^{-\frac{1}{p}} \left(\int_{[t,\infty)} \Lambda(x)^{-p-1} d\lambda \right)^{\frac{1}{p}}.$$

Moreover, by Lemma 1, we see that $\sup_{t>0} A_0(t) \approx 1$. Observe that for $\alpha > 0$

$$\begin{aligned} \int_{[t,\infty)} \Lambda(x)^{-\alpha-1} d\lambda(x) &= \int_{[t,\infty)} \left(\int_{\Lambda(x)}^{\infty} (\alpha+1)s^{-\alpha-2} ds \right) d\lambda(x) \\ &= \int_{\Lambda(t)}^{\infty} (\alpha+1)s^{-\alpha-2} ds \int_{\{x \leq t, \Lambda(x) \leq s\}} d\lambda(x) \leq \frac{\alpha+1}{\alpha} \Lambda(t)^{-\alpha}. \end{aligned} \quad (6)$$

It follows from this and Lemma 1 that $\sup_{t>0} A_1(t) \ll 1$.

Now let $1 < p < \infty$. According to [9, Theorem 4.2], we have

$$C \approx \sup_{t>0} A_3(t) + \sup_{t>0} A_4(t),$$

where

$$A_3(t) := \left(\int_{[0,t]} \left(\int_{[y,t]} \Lambda^{-\frac{1}{p'}} d\lambda \right)^{p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} \Lambda^{-p-1} d\lambda \right)^{\frac{1}{p}}$$

and

$$A_4(t) := \Lambda^{\frac{1}{p}} \left(\int_{[t,\infty)} \Lambda^{-p-1}(x) \left(\int_{[t,x]} \Lambda^{-\frac{1}{p'}} d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}.$$

By (6) and Lemma 1, it yields

$$\left(\int_{[t,\infty)} \Lambda^{-p-1} d\lambda \right)^{\frac{1}{p}} \ll \Lambda(t)^{-1}, \quad \int_{[y,t]} \Lambda^{-\frac{1}{p'}} d\lambda \approx \Lambda^{1-\frac{1}{p'}}(t).$$

Hence, $\sup_{t>0} A_3(t) \ll 1$. Analogously, $\sup_{t>0} A_4(t) \ll 1$ and we conclude that (5) holds. The proof is complete.

We make use of the following result which follows from [12, Lemma 1.2] (see also [13, Proposition 1.5]).

LEMMA 2 ([9, Lemma 2.4]) *Let $f \in \mathfrak{M} \downarrow$ and $f(\infty) = 0$. Then there exist $f_0 \in \mathfrak{M} \downarrow$, $\{h_n\} \subset \mathfrak{M}^+$ such that*

- (a) $f_0(x) \leq f(x)$ for all $x \in \mathbb{R}_+$, $f_0(x) = f(x)$ λ -a.e.,
- (b) $f_n(x) := \int_{[x,\infty)} h_n d\lambda \leq f_0(x)$ for all $x \in \mathbb{R}_+$,

- (c) $\{f_n(x)\}$ is non-decreasing in n for every $x \in \mathbb{R}_+$ and $f_0(x) = \lim_{n \rightarrow \infty} f_n(x)$ λ -a.e.

The next result is concerned with the dual Hardy operator.

THEOREM 2 *Let $1 \leq \gamma < \infty$. Then for all $f \in \mathfrak{M} \downarrow$, $f \not\equiv 0$*

$$\frac{1}{(\gamma+1)^{\frac{1}{\gamma}}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f \, d\lambda \right)^\gamma d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq \gamma. \quad (7)$$

If $0 < \gamma < 1$, then $\forall f \in \mathfrak{M} \downarrow$, $f \not\equiv 0$

$$0 < c(\gamma) \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f \, d\lambda \right)^\gamma d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq \frac{1}{\gamma}. \quad (8)$$

Proof The right-hand side of (7) is the Hardy inequality dual to (1). For the left-hand side, without loss of generality assume that $f(\infty) = 0$. Then by Lemma 2 we can find a function $f_0 \in \mathfrak{M} \downarrow$ and a sequence $\{h_n\} \subset \mathfrak{M}^+$ such that

- (1a) $f_0^\gamma(x) \leq f^\gamma(x)$ for all $x \in \mathbb{R}_+$, $f_0^\gamma(x) = f^\gamma(x)$ λ -a.e.,
 (1b) $f_n^\gamma(x) := \int_{[x,\infty)} h_n \, d\lambda \leq f_0^\gamma(x)$ for all $x \in \mathbb{R}_+$,
 (1c) $f_0^\gamma(x) = \lim_{n \rightarrow \infty} f_n^\gamma(x)$ λ -a.e. and $\{f_n^\gamma(x)\}$ is non-decreasing in n for every $x \in \mathbb{R}_+$.

Now we write, using these properties, Fatou's lemma, Lemma 1 and Fubini's theorem

$$\begin{aligned} \int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda &\stackrel{(1a)}{=} \int_{[0,\infty)} f_0^\gamma \Lambda^\gamma d\lambda \stackrel{(1c)}{=} \lim_{n \rightarrow \infty} \int_{[0,\infty)} f_n^\gamma \Lambda^\gamma d\lambda \\ &\stackrel{(1b)}{=} \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[x,\infty)} h_n(y) \, d\lambda(y) \right) \Lambda^\gamma(x) \, d\lambda(x) \\ &= \lim_{n \rightarrow \infty} \int_{[0,\infty)} h_n(y) \left(\int_{[0,y]} \Lambda^\gamma(x) \, d\lambda(x) \right) d\lambda(y) \\ &\leq \lim_{n \rightarrow \infty} \int_{[0,\infty)} h_n(y) \Lambda^{\gamma+1}(y) \, d\lambda(y) \\ &\leq (\gamma+1) \lim_{n \rightarrow \infty} \int_{[0,\infty)} h_n(y) \left(\int_{[0,y]} \left(\int_{[s,y]} d\lambda \right)^\gamma d\lambda(s) \right) d\lambda(y) \\ &= (\gamma+1) \lim_{n \rightarrow \infty} \int_{[0,\infty)} d\lambda(s) \left(\int_{[s,\infty)} \left(\int_{[s,y]} d\lambda \right)^\gamma h_n(y) \, d\lambda(y) \right) \end{aligned}$$

[by Minkowskii's inequality]

$$\begin{aligned} &\leq (\gamma+1) \lim_{n \rightarrow \infty} \int_{[0,\infty)} d\lambda(s) \left(\int_{[s,\infty)} d\lambda(t) \left(\int_{[t,\infty)} h_n \, d\lambda \right)^{\frac{1}{\gamma}} \right)^\gamma \\ &\stackrel{(1b)}{\leq} (\gamma+1) \int_{[0,\infty)} \left(\int_{[s,\infty)} f_0(t) \, d\lambda(t) \right)^\gamma d\lambda(s) \stackrel{(1a)}{=} (\gamma+1) \int_{[0,\infty)} \left(\int_{[s,\infty)} f \, d\lambda \right)^\gamma d\lambda(s) \end{aligned}$$

and (7) is proved.

For the proof of the right-hand side of (8), we suppose without loss of generality that $\int_{[0,\infty)} (\int_{[x,\infty)} f d\lambda)^\gamma d\lambda(x) < \infty$, otherwise we use truncation. By Lemma 1

$$\begin{aligned} \int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma d\lambda(x) &\leq \int_{[0,\infty)} \int_{[x,\infty)} \left(\int_{[x,y]} f d\lambda \right)^{\gamma-1} f(y) d\lambda(y) d\lambda(x) \\ &\leq \int_{[0,\infty)} \int_{[x,\infty)} \left(\int_{[x,y]} d\lambda \right)^{\gamma-1} f^\gamma(y) d\lambda(y) d\lambda(x) \\ &= \int_{[0,\infty)} f^\gamma(y) \int_{[0,y]} \left(\int_{[x,y]} d\lambda \right)^{\gamma-1} d\lambda(x) d\lambda(y) \\ &\leq \frac{1}{\gamma} \int_{\mathbb{R}_+} f^\gamma(y) \Lambda^\gamma(y) d\lambda(y), \end{aligned}$$

and the right-hand side of (8) is proved.

For the left-hand side of (8), we note that $\Lambda(x)$ is a non-decreasing right continuous function. Without loss of generality we can suppose that $\Lambda(0)=0$, $\Lambda(x) > 0$ for $x > 0$, $\Lambda(\infty)=\infty$. For $x > 0$ we define

$$k(x) := \max\{k \in \mathbb{Z} : \Lambda(x) \geq 2^k\}.$$

Then $k : (0, \infty) \rightarrow \mathbb{Z}$ is an integer-valued non-decreasing right-continuous function and

$$2^{k(x)} \leq \Lambda(x) < 2^{k(x)+1} \quad (9)$$

for all $x \in (0, \infty)$. Let the range of $k(x)$ be $\{k_i\}$, $k_i < k_{i+1}$ and

$$\Delta_i := \{x \in \mathbb{R}_+ : k(x) = k_i\}.$$

Then $\Delta_i = [x_i, x_{i+1})$, where $x_i := \inf \Delta_i$ and (9) implies that

$$2^{k_i} \leq \Lambda(x) < 2^{k_i+1}, \quad x \in \Delta_i. \quad (10)$$

Put

$$I_1 := \{i \in \mathbb{Z} : k_i - k_{i-1} \geq 2\}, \quad I_2 := \{i \in \mathbb{Z} : k_i = k_{i-1} + 1\}.$$

It is clear that $\{x_i\}$ is an atom of λ when $i \in I_1$ and $\lambda\{x_i\} = \Lambda(x_i) - \Lambda(x_i - 0)$. It follows from (10) that $\lambda\{x_i\} \leq 2 \cdot 2^{k_i}$ and $\Lambda(x_i) \geq 2^{k_i}$, $\Lambda(x_i - 0) \leq 2^{k_{i-1}+1}$, so that $\lambda\{x_i\} \geq 2^{k_i} - 2^{k_{i-1}+1} \geq \frac{1}{2} 2^{k_i}$. Hence,

$$\frac{1}{2} \cdot 2^{k_i} \leq \lambda\{x_i\} \leq 2 \cdot 2^{k_i}, \quad i \in I_1.$$

This and (10) imply that

$$\frac{1}{2} \cdot 2^{k_i} \leq \lambda\{x_i\} \leq \int_{\Delta_i} d\lambda \leq \sup_{x \in [x_i, x_{i+1})} \Lambda(x) \leq 2 \cdot 2^{k_i}. \quad (11)$$

Now we write

$$\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda = \sum_{i \in I_1} \int_{\Delta_i} f^\gamma \Lambda^\gamma d\lambda + \sum_{i \in I_2} \int_{\Delta_i} f^\gamma \Lambda^\gamma d\lambda =: J_1 + J_2$$

and begin with the estimation of J_1 . By (11) we have

$$J_1 \ll \sum_{i \in I_1} f^\gamma(x_i) 2^{k_i \gamma} \cdot 2^{k_i}$$

as well as

$$f(x_i) 2^{k_i} \approx \int_{\{x_i\}} f(x) d\lambda \leq \int_{\Delta_i} f d\lambda, \quad i \in I_1.$$

Hence,

$$\begin{aligned} J_1 &\ll \sum_{i \in I_1} \left(\int_{\Delta_i} f d\lambda \right)^\gamma 2^{k_i} \leq \sum_{i \in I_1} \left(\sum_{m \geq i} \int_{\Delta_m} f d\lambda \right)^\gamma 2^{k_i} \\ &\approx \sum_{i \in I_1} \left(\int_{[x_i, \infty)} f d\lambda \right)^\gamma \lambda\{x_i\} \leq \int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^\gamma d\lambda(x). \end{aligned}$$

By using (10) and (11), it yields

$$J_2 \leq \sum_{i \in I_2} f^\gamma(x_i) \int_{\Delta_i} \Lambda^\gamma d\lambda \leq 2^{\gamma+1} \sum_{i \in I_2} f^\gamma(x_i) 2^{k_i(\gamma+1)}.$$

Moreover, by (10),

$$\int_{[x_{i-2}, x_i]} d\lambda = \Lambda(x_i) - \Lambda(x_{i-2}) \geq 2^{k_i} - 2^{k_{i-2}+1} \geq \frac{1}{2} \cdot 2^{k_i}.$$

By this and Lemma 1 we find that

$$2^{k_i(\gamma+1)} \ll \left(\int_{[x_{i-2}, x_i]} d\lambda \right)^{\gamma+1} \approx \int_{[x_{i-2}, x_i]} \left(\int_{[x, x_i]} d\lambda \right)^\gamma d\lambda.$$

Hence,

$$\begin{aligned} J_2 &\ll \sum_{i \in I_2} f^\gamma(x_i) \int_{[x_{i-2}, x_i]} \left(\int_{[x, x_i]} d\lambda \right)^\gamma d\lambda(x) \\ &\leq \sum_{i \in I_2} \int_{[x_{i-2}, x_i]} \left(\int_{[x, x_i]} f d\lambda \right)^\gamma d\lambda(x) \\ &\leq \sum_{i \in I_2} \int_{[x_{i-2}, x_i]} \left(\int_{[x, \infty)} f d\lambda \right)^\gamma d\lambda(x) \\ &\leq \sum_{i \in I_2} \left[\int_{[x_{i-2}, x_{i-1})} + \int_{[x_{i-1}, x_i)} + \int_{[x_i, x_{i+1})} \right] \left(\int_{[x, \infty)} f d\lambda \right)^\gamma d\lambda(x) \\ &\leq 3 \int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^\gamma d\lambda(x) \end{aligned}$$

and also (8) is proved. So the proof is complete.

Now we deal with operators involving suprema.

THEOREM 3 *Let $0 < \gamma < \infty$. Then for all $f \in \mathfrak{M} \downarrow$, $f \neq 0$*

$$1 \leq \frac{\left(\int_{[0, \infty)} \left[\operatorname{ess\,sup}_{y \geq x} f^\gamma(y) \Lambda^\gamma(y) \right] d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0, \infty)} f^\gamma \Lambda^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq c(\gamma), \quad (12)$$

where

$$c^\gamma(\gamma) = \max(1, 2^{\gamma-1}) \left(1 + \frac{\max(1, \gamma)}{\min(1, \gamma)} \right).$$

Proof Since the product of two monotone functions is a Borel function, elementary arguments show that $f(x)\Lambda(x) \leq \text{ess sup}_{\lambda, y \geq x} f(y)\Lambda(y)$ for λ -a.e. $x \in \mathbb{R}_+$ and the left-hand side of (12) follows. For the proof of the right-hand side of (12), we suppose that

$$\int_{[0, \infty)} f^\gamma \Lambda^\gamma d\lambda < \infty$$

and without loss of generality assume that $f(\infty) = 0$. Then, by Lemma 2, there exist a function $f_0 \in \mathfrak{M} \downarrow$ and a sequence $\{h_n\} \subset \mathfrak{M}^+$ such that

- (2a) $f_0(x) \leq f^\gamma(x)$ for all $x \in \mathbb{R}_+$, $f_0(x) = f^\gamma(x)$ λ -a.e.,
- (2b) $f_n(x) := \int_{[x, \infty)} h_n d\lambda \leq f_0(x)$ for all $x \in \mathbb{R}_+$,
- (2c) $f_0(x) = \lim_{n \rightarrow \infty} f_n(x)$ λ -a.e. and $\{f_n(x)\}$ is non-decreasing in n for every $x \in \mathbb{R}_+$.

Moreover, we make use of the formula

$$\text{ess sup}_\lambda F = \sup_{E \subset \mathbb{R}_+, 0 < \lambda E < \infty} \frac{1}{\lambda E} \int_E F d\lambda, \quad (13)$$

which is valid for any σ -finite measure λ and a λ -measurable function F and a set E . Now we write

$$\begin{aligned} \text{ess sup}_{\lambda, y \geq x} f^\gamma(y) \Lambda^\gamma(y) &\stackrel{(2a)}{=} \text{ess sup}_{\lambda, y \geq x} f_0(y) \Lambda^\gamma(y) \\ &\stackrel{(13)}{=} \sup_{E \subset [x, \infty), 0 < \lambda E < \infty} \frac{1}{\lambda E} \int_E f_0 \Lambda^\gamma d\lambda \\ &\stackrel{(2b)}{=} \sup_{E \subset [x, \infty), 0 < \lambda E < \infty} \frac{1}{\lambda E} \int_E \left(\sup_n f_n \right) \Lambda^\gamma d\lambda \end{aligned}$$

[by Monotone Convergence Theorem]

$$\begin{aligned} &= \sup_{E \subset [x, \infty), 0 < \lambda E < \infty} \sup_n \frac{1}{\lambda E} \int_E f_n \Lambda^\gamma d\lambda \\ &= \sup_n \sup_{E \subset [x, \infty), 0 < \lambda E < \infty} \frac{1}{\lambda E} \int_E f_n \Lambda^\gamma d\lambda \\ &\stackrel{(13)}{=} \sup_n \text{ess sup}_{\lambda, y \geq x} f_n(y) \Lambda^\gamma(y) \\ &\stackrel{(2b)}{=} \sup_n \text{ess sup}_{\lambda, y \geq x} \left(\int_{[y, \infty)} h_n d\lambda \right) \Lambda^\gamma(y) \\ &\leq \sup_n \text{ess sup}_{\lambda, y \geq x} \int_{[y, \infty)} h_n \Lambda^\gamma d\lambda \\ &= \sup_n \int_{[x, \infty)} h_n \Lambda^\gamma d\lambda. \end{aligned} \quad (14)$$

Now, by Lemma 1 and an elementary inequality with $c_0(\gamma) := \max(1, 2^{\gamma-1})$, we have

$$\begin{aligned} \int_{[x,\infty)} h_n \Lambda^\gamma d\lambda &\leq c_0(\gamma) \left(\Lambda^\gamma(x) \int_{[x,\infty)} h_n d\lambda + \int_{[x,\infty)} h_n(y) \left(\int_{[x,y]} d\lambda \right)^\gamma d\lambda(y) \right) \\ &\leq c_0(\gamma) \left(f_n(x) \Lambda^\gamma(x) + \max(1, \gamma) \int_{[x,\infty)} h_n(y) \left(\int_{[x,y]} \Lambda^{\gamma-1} d\lambda \right) d\lambda(y) \right) \\ &= c_0(\gamma) \left(f_n(x) \Lambda^\gamma(x) + \max(1, \gamma) \int_{[x,\infty)} \Lambda^{\gamma-1}(z) \left(\int_{[z,\infty)} h_n d\lambda \right) d\lambda(z) \right) \\ &\stackrel{(2b),(2a)}{\leq} c_0(\gamma) \left(f^\gamma(x) \Lambda^\gamma(x) + \max(1, \gamma) \int_{[x,\infty)} \Lambda^{\gamma-1}(z) f^\gamma(z) d\lambda(z) \right). \end{aligned}$$

By (14), this implies that

$$\operatorname{ess\,sup}_{y \geq x} f^\gamma(y) \Lambda^\gamma(y) \leq c_0(\gamma) \left(\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda + \max(1, \gamma) \int_{[0,\infty)} \left(\int_{[x,\infty)} \Lambda^{\gamma-1} f^\gamma d\lambda \right) d\lambda(x) \right)$$

[changing the order of integration in the second term and applying Lemma 1]

$$\leq c_0(\gamma) \left(1 + \frac{\max(1, \gamma)}{\min(1, \gamma)} \right) \int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda$$

and also the right-hand side of (12) is proved. The proof is complete.

Denote by $\mathfrak{M} \downarrow (+0)$ the cone of all non-increasing functions $f \in \mathfrak{M} \downarrow$ continuous on the right and such that $f(x) > 0$ for all $x \in \mathbb{R}_+$. Then $\frac{1}{f} \in \mathfrak{M} \uparrow$ is continuous on the right and there exists a unique Borel measure μ_f such that

$$\frac{1}{f(x)} = \int_{[0,x]} d\mu_f \quad (15)$$

(see, e.g. [18, Chapter 12, Section 3]).

THEOREM 4 *Let $0 < \gamma < \infty$. Then, for all $f \in \mathfrak{M} \downarrow (+0)$,*

$$1 \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{\gamma+1} d\mu_f(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma d\lambda(x) \right)^{\frac{1}{\gamma}}} \leq (\gamma+1)^{\frac{1}{\gamma}}. \quad (16)$$

Proof Using (15) for the left-hand side of (16) we write

$$\begin{aligned} \int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma d\lambda(x) &= \int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma \left(\int_{[0,x]} d\mu_f \right) f(x) d\lambda(x) \\ &= \int_{[0,\infty)} \left(\int_{[y,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma f(x) d\lambda(x) \right) d\mu_f(y) \\ &\leq \int_{[0,\infty)} \left(\int_{[y,\infty)} f d\lambda \right)^\gamma d\mu_f(y). \end{aligned}$$

For the right-hand side of (16), by Lemma 1 we find

$$\begin{aligned} \int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{\gamma+1} d\mu_f(x) &\leq (\gamma+1) \int_{[0,\infty)} d\mu_f(x) \int_{[x,\infty)} \left(\int_{[y,\infty)} f d\lambda \right)^{\gamma} f(y) d\lambda(y) \\ &= (\gamma+1) \int_{[0,\infty)} \left(\int_{[y,\infty)} f d\lambda \right)^{\gamma} d\lambda(y). \end{aligned}$$

This completes the proof.

THEOREM 5 Let $0 < \gamma < \infty$. Then for all $f \in \mathfrak{M} \downarrow (+0)$

$$c(\gamma) \leq \frac{\left(\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{y \geq x} f(y) \Lambda(y) \right]^{\gamma} d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{y \geq x} f(y) \Lambda(y) \right]^{\gamma+1} d\mu_f(x) \right)^{\frac{1}{\gamma}}} \leq \left(\frac{\max(1, \gamma)}{\min(1, \gamma)} \right)^{\frac{1}{\gamma}}, \quad (17)$$

where $c(\gamma) = \left(\frac{\gamma}{(\gamma+1)2^{\gamma+1}} \right)^{\frac{1}{\gamma}}$.

Proof Denote

$$W(x) := \operatorname{ess\,sup}_{y \geq x} f(y) \Lambda(y)$$

and without loss of generality assume that $W(\infty) = 0$. Then, by Lemma 2, there exist a function $W_0 \in \mathfrak{M} \downarrow$ and a sequence $\{w_n\} \subset \mathfrak{M}^+$ such that

- (3a) $W_0(x) \leq W(x)$ for all $x \in \mathbb{R}_+$, $W_0(x) = W(x) \lambda - \text{a.e.}$,
- (3b) $W_n(x) := \int_{[x,\infty)} w_n d\lambda \leq W_0(x)$ for all $x \in \mathbb{R}_+$,
- (3c) $W_0(x) = \lim_{n \rightarrow \infty} W_n(x) \lambda - \text{a.e.}$ and $\{W_n(x)\}$ is non-decreasing in n for every $x \in \mathbb{R}_+$.

Using these properties for the right-hand side of (17), by Lemma 1 we write

$$\begin{aligned} \int_{[0,\infty)} W^{\gamma} d\lambda &\stackrel{(3a)}{=} \int_{[0,\infty)} W_0^{\gamma} d\lambda \stackrel{(3c)}{=} \lim_{n \rightarrow \infty} \int_{[0,\infty)} W_n^{\gamma} d\lambda \\ &\stackrel{(3b)}{\leq} \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[x,\infty)} W_n^{\gamma-1} w_n d\lambda \right) d\lambda(x) \\ &\leq \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[x,\infty)} W_n^{\gamma-1} \frac{W}{f\Lambda} w_n d\lambda \right) d\lambda(x) \\ &= \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} W_n^{\gamma-1}(y) \frac{W(y)}{f(y)} w_n(y) d\lambda(y) \\ &= \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[0,y]} d\mu_f \right) W_n^{\gamma-1}(y) W(y) w_n(y) d\lambda(y) \\ &= \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} d\mu_f(z) \int_{[z,\infty)} W_n^{\gamma-1} W w_n d\lambda \\ &\leq \max(1, \gamma) \lim_{n \rightarrow \infty} \int_{[0,\infty)} W(z) d\mu_f(z) \int_{[z,\infty)} W_n^{\gamma-1} w_n d\lambda \\ &\leq \frac{\max(1, \gamma)}{\min(1, \gamma)} \lim_{n \rightarrow \infty} \int_{[0,\infty)} W W_n^{\gamma} d\mu_f \\ &\stackrel{(3b), (3a)}{\leq} \frac{\max(1, \gamma)}{\min(1, \gamma)} \int_{[0,\infty)} W^{\gamma+1} d\mu_f. \end{aligned}$$

For the left-hand side, we use an idea from the proof [5, Theorem 1]. Suppose

$$\int_{[0,\infty)} W^\gamma(x) d\lambda(x) < \infty.$$

Without loss of generality we assume that $0 < f(0) < \infty$, $\Lambda(\infty) = \infty$. Then $f(0)\Lambda(0) = 0$, $W(x)$ is bounded and $W(\infty) = 0$. Let $t_0 = 0$ and suppose that we find a sequence $0 < t_1 < t_2 < \dots < t_{k-1}$. Then put

$$t_k := \sup \left\{ t \geq t_{k-1} : f(t)\Lambda(t) \geq \frac{1}{2} \operatorname{ess\,sup}_{\lambda, y \geq t_{k-1}} f(y)\Lambda(y) \right\}.$$

For the sake of brevity, we omit analysis of pathological cases and assume that $t_k > t_{k-1}$ and $\lim_{k \rightarrow \infty} t_k = \infty$. Put $f(t_0 - 0)\Lambda(t_0 - 0) := 0$, and for $k > 0$ we have

$$f(t_k - 0)\Lambda(t_k - 0) \geq \frac{1}{2} \operatorname{ess\,sup}_{\lambda, y \geq t_{k-1}} f(y)\Lambda(y). \quad (18)$$

Since f and Λ are monotone, the left limits $f(t_k - 0)$ and $\Lambda(t_k - 0)$ exist. Moreover,

$$\int_{[t_{k-1}, t_k)} d\mu_f = \frac{1}{f(t_k - 0)} - \frac{1}{f(t_{k-1} - 0)}, \quad \int_{[t_{k-1}, t_k)} d\lambda = \Lambda(t_k - 0) - \Lambda(t_{k-1} - 0).$$

Then

$$\begin{aligned} \int_{[0,\infty)} W^{\gamma+1} d\mu_f &= \sum_{k=1}^{\infty} \int_{[t_{k-1}, t_k)} W^{\gamma+1} d\mu_f \\ &\leq \sum_{k=1}^{\infty} \int_{[t_{k-1}, t_k)} \left[\operatorname{ess\,sup}_{\lambda, y \geq t_{k-1}} f(y)\Lambda(y) \right]^{\gamma+1} d\mu_f \\ &\stackrel{(18)}{\leq} 2^{\gamma+1} \sum_{k=1}^{\infty} [f(t_k - 0)\Lambda(t_k - 0)]^{\gamma+1} \int_{[t_{k-1}, t_k)} d\mu_f \end{aligned}$$

[applying [5, Corollary 1]

$$\begin{aligned} &\leq \gamma^{-1}(\gamma + 1)2^{\gamma+1} \sum_{k=1}^{\infty} [f(t_k - 0)\Lambda(t_k - 0)]^\gamma \int_{[t_{k-1}, t_k)} d\lambda \\ &\leq \gamma^{-1}(\gamma + 1)2^{\gamma+1} \sum_{k=1}^{\infty} \int_{[t_{k-1}, t_k)} W^\gamma d\lambda = \gamma^{-1}(\gamma + 1)2^{\gamma+1} \int_{[0,\infty)} W^\gamma d\lambda \end{aligned}$$

and the proof is completed.

Remark 1 For the counting measure λ , Theorems 2–4 have been established in [5, Theorem 1]. In analogy with this result, it follows from Theorems 2–4 that the following inequalities are equivalent:

- (1) $\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^\gamma d\lambda(x) < \infty$,
- (2) $\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{\gamma+1} d\mu_f < \infty$,
- (3) $\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{\lambda, y \geq x} f(y)\Lambda(y) \right]^\gamma d\lambda(x) < \infty$,
- (4) $\int_{[0,\infty)} \left[\operatorname{ess\,sup}_{\lambda, y \geq x} f(y)\Lambda(y) \right]^{\gamma+1} d\mu_f(x) < \infty$,
- (5) $\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda < \infty$

in a sense that

$$(2) \stackrel{f \in \mathfrak{M} \downarrow (+0)}{\Longleftrightarrow} (1) \stackrel{f \in \mathfrak{M} \downarrow}{\Longleftrightarrow} (5) \stackrel{f \in \mathfrak{M} \downarrow}{\Longleftrightarrow} (3) \stackrel{f \in \mathfrak{M} \downarrow (+0)}{\Longleftrightarrow} (4). \quad (19)$$

Definition [15]: Let $0 < \gamma < \infty$ and a function $f \in \mathfrak{M} \downarrow$ be continuous on the right. We say that f belongs to $\mathcal{T}_\gamma$, if

$$f^\gamma(x) - f^\gamma(\infty) \leq C \int_{[x, \infty)} f^{\gamma+1} d\mu_f \quad (20)$$

for some constant $C > 0$.

We can supplement (19) by an additional equivalence, which replaces the exact equality

$$\int_0^\infty [f\Lambda]^\gamma d\lambda = \frac{1}{\gamma+1} f^\gamma(\infty) \Lambda^{\gamma+1}(\infty) + \frac{\gamma}{\gamma+1} \int_0^\infty [f\Lambda]^{\gamma+1} d\left(\frac{1}{f(x)}\right)$$

for absolutely continuous Λ and f [7, Theorem 3.1(ii)].

PROPOSITION Let $0 < \gamma < \infty$ and $f \in \mathcal{T}_\gamma$. Then

$$f^\gamma(\infty) \Lambda^{\gamma+1}(\infty) + \int_{[0, \infty)} [f\Lambda]^{\gamma+1} d\mu_f \approx \int_{[0, \infty)} [f\Lambda]^\gamma d\lambda. \quad (21)$$

Proof Since the reverse inequality to (20) is always true [13], we have

$$f^\gamma(x) - f^\gamma(\infty) \approx \int_{[x, \infty)} f^{\gamma+1} d\mu_f, \quad (22)$$

provided $f \in \mathcal{T}_\gamma$. Using this, Lemma 1 and (22), we write

$$\begin{aligned} \int_{[0, \infty)} [f\Lambda]^{\gamma+1} d\mu_f &\approx \int_{[0, \infty)} \left(\int_{[0, x]} \Lambda^\gamma d\lambda \right) f^{\gamma+1}(x) d\mu_f(x) \\ &= \int_{[0, \infty)} \left(\int_{[y, \infty)} f^{\gamma+1} d\mu_f \right) \Lambda^\gamma(y) d\lambda(y) \\ &\approx \int_{[0, \infty)} (f^\gamma(y) - f^\gamma(\infty)) \Lambda^\gamma(y) d\lambda(y) \\ &= \int_{[0, \infty)} [f\Lambda]^\gamma d\lambda - f^\gamma(\infty) \Lambda^{\gamma+1}(\infty). \end{aligned}$$

Remark 2 Theorems 1–5 have the complete analogues for non-decreasing functions and Λ replaced by $\Lambda_*(t) := \int_{[t, \infty)} d\lambda$. For instance, the following inequalities:

- (1) $\int_{[0, \infty)} \left(\int_{[0, x]} f d\lambda \right)^\gamma d\lambda(x) < \infty$,
- (2) $\int_{[0, x]} \left(\int_{[x, \infty)} f d\lambda \right)^{\gamma+1} d\mu_f < \infty$,
- (3) $\int_{[0, \infty)} \left[\text{ess sup}_{\lambda_0 \leq y \leq x} f(y) \Lambda_*(y) \right]^\gamma d\lambda(x) < \infty$,
- (4) $\int_{[0, \infty)} \left[\text{ess sup}_{\lambda_0 \leq y \leq x} f(y) \Lambda_*(y) \right]^{\gamma+1} d\mu_f(x) < \infty$,
- (5) $\int_{[0, \infty)} f^\gamma \Lambda_*^\gamma d\lambda < \infty$

are equivalent, that is

$$(2) \stackrel{f \in \mathfrak{M} \uparrow(-0)}{\Longleftrightarrow} (1) \stackrel{f \in \mathfrak{M} \uparrow}{\Longleftrightarrow} (5) \stackrel{f \in \mathfrak{M} \uparrow}{\Longleftrightarrow} (3) \stackrel{f \in \mathfrak{M} \uparrow(-0)}{\Longleftrightarrow} (4)$$

with the corresponding changes in definitions.

3. Negative parameter

Let $\gamma < 0$. Then it obviously yields

$$\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda \leq \int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^\gamma d\lambda \quad (23)$$

for all $f \in \mathfrak{M} \uparrow, f(x) > 0 \lambda - \text{a.e.}$

THEOREM 6 *Let $-1 < \gamma < 0$. Then for all $f \in \mathfrak{M} \uparrow, f(x) > 0 \lambda - \text{a.e.}$*

$$(-\gamma)^{\frac{1}{\gamma}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^\gamma d\lambda \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq 1. \quad (24)$$

Proof The right-hand side of (24) follows from (23). For the left-hand side, by Lemma 2, we can find a function $f_0 \in \mathfrak{M} \uparrow$ and a sequence $\{h_n\} \in \mathfrak{M}_+$ such that

- (5a) $f_0^\gamma(x) \leq f^\gamma(x)$ for all $x \in \mathbb{R}_+, f_0^\gamma(x) = f^\gamma(x) \lambda - \text{a.e.},$
- (5b) $f_n^\gamma(x) := \int_{[x,\infty)} h_n d\lambda \leq f_0^\gamma(x)$ for all $x \in \mathbb{R}_+,$
- (5c) $f_0^\gamma(x) = \lim_{n \rightarrow \infty} f_n^\gamma(x) \lambda - \text{a.e.}$ and $\{f_n^\gamma(x)\}$ is non-decreasing in n for every $x \in \mathbb{R}_+.$

Now using the properties (5a)–(5c), Lemma 1, Fatou's lemma and Fubini's theorem, we find that

$$\begin{aligned} \int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda &\stackrel{(5a)}{=} \int_{[0,\infty)} f_0^\gamma \Lambda^\gamma d\lambda \stackrel{(5c)}{=} \lim_{n \rightarrow \infty} \int_{[0,\infty)} f_n^\gamma \Lambda^\gamma d\lambda \\ &\stackrel{(5b)}{=} \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[x,\infty)} h_n(y) d\lambda(y) \right) \Lambda^\gamma(x) d\lambda(x) \\ &= \lim_{n \rightarrow \infty} \int_{[0,\infty)} h_n(y) \left(\int_{[0,y]} \Lambda^\gamma(x) d\lambda(x) \right) d\lambda(y) \\ &\geq \lim_{n \rightarrow \infty} \int_{[0,\infty)} h_n(y) \Lambda^{\gamma+1}(y) d\lambda(y) \\ &= \lim_{n \rightarrow \infty} \int_{[0,\infty)} f^{\gamma+1}(y) \Lambda^{\gamma+1}(y) f^{-\gamma-1}(y) h_n(y) d\lambda(y) \\ &\stackrel{(5a)}{\geq} \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[0,y]} f d\lambda \right)^{\gamma+1} f_0^{-\gamma-1}(y) h_n(y) d\lambda(y) \\ &\geq \lim_{n \rightarrow \infty} \int_{[0,\infty)} \left(\int_{[0,y]} \left(\int_{[0,z]} f d\lambda \right)^\gamma f(z) d\lambda(z) \right) f_0^{-\gamma-1}(y) h_n(y) d\lambda(y) \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_{[0, \infty)} \left(\int_{[0, z]} f \, d\lambda \right)^\gamma f(z) \, d\lambda(z) \int_{[z, \infty)} f_0^{-\gamma-1}(y) h_n(y) \, d\lambda(y) \\
&\stackrel{(5b)}{\geq} \lim_{n \rightarrow \infty} \int_{[0, \infty)} \left(\int_{[0, z]} f \, d\lambda \right)^\gamma \\
&\quad \times \left(\int_{[z, \infty)} \left(\int_{[y, \infty)} h_n(x) \, d\lambda(x) \right)^{\frac{\gamma+1}{\gamma}} h_n(y) \, d\lambda(y) \right) f(z) \, d\lambda(z) \\
&\geq -\gamma \lim_{n \rightarrow \infty} \int_{[0, \infty)} \left(\int_{[0, z]} f \, d\lambda \right)^\gamma \left(\int_{[z, \infty)} h_n \, d\lambda \right)^{-\frac{1}{\gamma}} f(z) \, d\lambda(z) \\
&\stackrel{(5c)}{=} -\gamma \int_{[0, \infty)} \left(\int_{[0, z]} f \, d\lambda \right)^\gamma f_0^{-1}(z) f(z) \, d\lambda(z) \\
&\stackrel{(5a)}{=} -\gamma \int_{[0, \infty)} \left(\int_{[0, z]} f \, d\lambda \right)^\gamma \, d\lambda(z).
\end{aligned}$$

Hence, also the left-hand side of (24) is proved, so the proof is complete.

THEOREM 7 Let $\gamma \leq -1$. Then, for all $f \in \mathfrak{M}^+ \uparrow$, $f(x) > 0$, λ -a.e. it yields

$$0 < c_\gamma \leq \frac{\left(\int_{[0, \infty)} \left(\int_{[0, x]} f \, d\lambda \right)^\gamma \, d\lambda \right)^{\frac{1}{\gamma}}}{\left(\int_{[0, \infty)} f^\gamma \Lambda^\gamma \, d\lambda \right)^{\frac{1}{\gamma}}} \leq 1. \quad (25)$$

Proof The right-hand side of (25) follows from (23). For the left-hand side of (25), we utilize the sequences of points $\{x_k\}$ and intervals Δ_k constructed in the proof of Theorem 2. We have

$$\begin{aligned}
J &:= \int_{[0, \infty)} \left(\int_{[0, x]} f \, d\lambda \right)^\gamma \, d\lambda = \sum_i \int_{\Delta_i} \left(\int_{[0, x]} f \, d\lambda \right)^\gamma \, d\lambda(x) \\
&\leq \sum_i \int_{\Delta_i} \left(\int_{[x_{i-2}, x]} f \, d\lambda \right)^\gamma \, d\lambda(x) \leq \sum_i f^\gamma(x_{i-2}) \int_{\Delta_i} \left(\int_{[x_{i-2}, x]} d\lambda \right)^\gamma \, d\lambda(x).
\end{aligned}$$

If $x \in \Delta_i$, then by (10)

$$\int_{[x_{i-2}, x]} d\lambda = \Lambda(x) - \Lambda(x_{i-2}) \geq 2^{k_i} - 2^{k_{i-2}+1} \geq 2^{k_i-1}.$$

Applying this and (11), we find that

$$\int_{\Delta_i} \left(\int_{[x_{i-2}, x]} d\lambda \right)^\gamma \, d\lambda(x) \leq 2^{1-\gamma} 2^{k_i(\gamma+1)} \leq 2^{1-\gamma} 2^{k_{i-3}(\gamma+1)}.$$

Hence,

$$J \leq 2^{1-\gamma} \sum_i f^\gamma(x_{i-2}) 2^{k_{i-3}(1+\gamma)}. \quad (26)$$

Observe that if $x \in [x_{i-5}, x_{i-2})$, then by (10) we have $\Lambda(x) \leq 2 \cdot 2^{k_{i-3}}$. This implies that

$$\int_{[x_{i-5}, x_{i-2})} \Lambda^\gamma \, d\lambda \geq 2^\gamma 2^{\gamma k_{i-3}} \int_{[x_{i-5}, x_{i-2})} d\lambda \geq 2^{\gamma-1} 2^{(1+\gamma)k_{i-3}}.$$

Applying this and (26), we obtain

$$\begin{aligned} J &\leq 2^{2(1-\gamma)} \sum_i f^\gamma(x_{i-2}) \int_{[x_{i-5}, x_{i-2})} \Lambda^\gamma d\lambda \leq 2^{2(1-\gamma)} \sum_i \int_{[x_{i-5}, x_{i-2})} f^\gamma \Lambda^\gamma d\lambda \\ &\leq 2^{2(1-\gamma)} \sum_i \left(\int_{[x_{i-5}, x_{i-4})} f^\gamma \Lambda^\gamma d\lambda + \int_{[x_{i-4}, x_{i-3})} f^\gamma \Lambda^\gamma d\lambda + \int_{[x_{i-3}, x_{i-2})} f^\gamma \Lambda^\gamma d\lambda \right) \\ &\leq 3 \cdot 2^{2(1-\gamma)} \int_{[0, \infty)} f^\gamma \Lambda^\gamma d\lambda \end{aligned}$$

and also the left-hand side of (16) is proved, so the proof is complete.

Remark 3 Theorems 6 and 7 have the complete analogues for non-increasing functions with the changes mentioned in Remark 2.

4. Applications

Let $0 < q < p \leq 1$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ and

$$H(p, q) := \sup_{\alpha_n \geq 0} \left(\sum_n w_n^q \left(\sum_{k \geq n} \alpha_k \right)^q \right)^{\frac{1}{q}} \left(\sum_n \varphi_n^p \alpha_n^p \right)^{-\frac{1}{p}}. \quad (27)$$

Put

$$\Phi_n := \inf_{k \geq n} \varphi_k, \quad W_n := \left(\sum_{k \leq n} w_k^q \right)^{\frac{1}{q}} \quad (28)$$

and

$$G_0(p, q) := \left(\sum_n \Phi_n^{-r} (W_n^r - W_{n-1}^r) \right)^{\frac{1}{r}}.$$

The following result is due to Goldman [8, Proof of Theorem 2.1, page 128].

THEOREM 8 *If $0 < q < p \leq 1$, then*

$$H(p, q) \approx G_0(p, q).$$

Observe that if $\gamma > 0$, $0 \leq a \leq b$, then

$$b^\gamma - a^\gamma \approx b^{\gamma-1}(b - a). \quad (29)$$

According to (29), we have

$$W_n^r - W_{n-1}^r \approx \left(\sum_{k \leq n} w_k^q \right)^{\frac{r}{p}} w_n^q.$$

Hence,

$$G_0(p, q) \approx \left(\sum_n \Phi_n^{-r} W_n^{\frac{rq}{p}} w_n^q \right)^{\frac{1}{r}} =: G(p, q)$$

and, thus, by Theorem 8,

$$H(p, q) \approx G(p, q).$$

Since $\Phi_n \uparrow$ there exists $z_k \geq 0$ such that

$$\Phi_n^p = \sum_{k \leq n} z_k, \quad (30)$$

we have with $p \in (0, 1)$

$$\sum_n \varphi_n^p \alpha_n^p \geq \sum_n \Phi_n^p \alpha_n^p \geq \sum_n \left(\sum_{k \leq n} z_k \right) \alpha_n^p \geq \sum_k z_k \sum_{n \geq k} \alpha_n^p \geq \sum_k \left(\sum_{n \geq k} \alpha_n \right)^p z_k.$$

Hence,

$$H(p, q) \leq \varkappa(p, q) := \sup_{0 \leq \beta_n \downarrow} \frac{(\sum_n w_n^q \beta_n^q)^{\frac{1}{q}}}{(\sum_n \beta_n^p z_n)^{\frac{1}{p}}}. \quad (31)$$

Moreover, it is known [13] that

$$\varkappa(p, q) \approx \left(\sum_n \left(\sum_{k \geq n} \Phi_k^{-p} w_k^q \right)^{\frac{p}{q}} z_n \right)^{\frac{1}{r}} =: F_0(p, q).$$

By Lemma 1, it yields

$$\begin{aligned} F_0^r(p, q) &\approx \sum_n \left[\sum_{k \geq n} \left(\sum_{l \geq k} \Phi_l^{-p} w_l^q \right)^{\frac{p}{q}} \Phi_k^{-p} w_k^q \right] z_n \\ &= \sum_k \left(\sum_{l \geq k} \Phi_l^{-p} w_l^q \right)^{\frac{p}{q}} w_k^q =: F^r(p, q). \end{aligned} \quad (32)$$

If we put

$$\lambda\{k\} = w_k^q; \quad f(k) = \Phi_k^{-p}; \quad \gamma = \frac{r}{p},$$

then, by Theorem 2,

$$F(p, q) \approx G(p, q).$$

Summing up, we see that

$$G(p, q) \approx H(p, q) \leq \varkappa(p, q) \approx F(p, q) \approx G(p, q).$$

Thus, we have proved the following new characterization of the discrete Hardy inequality.

THEOREM 9 *Let $0 < q < p \leq 1$ and $H(p, q)$ is defined by (27). Then*

$$H(p, q) \approx F(p, q),$$

where $F(p, q)$ is determined by (32).

In particular, the above results imply the equivalence of the discrete Hardy inequality to a simpler inequality on monotone sequences.

COROLLARY *The discrete Hardy inequality*

$$\left(\sum_n \left(\sum_{k \geq n} \alpha_k \right)^q w_n^q \right)^{\frac{1}{q}} \leq C \left(\sum_n \varphi_n^p \alpha_n^p \right)^{\frac{1}{p}}, \quad \alpha_n \geq 0$$

holds if and only if

$$\left(\sum_n (\beta_n w_n)^q \right)^{\frac{1}{q}} \leq C \left(\sum_n \beta_n^p z_n \right)^{\frac{1}{p}}, \quad 0 \leq \beta_n \downarrow,$$

where z_n is given by (28) and (30).

Acknowledgements

The authors express their deep gratitude to Dr Dmitry Prokhorov and Prof. Gordon Sinnamon for helpful remarks. The works of O.V. Popova and V.D. Stepanov were partially supported by the Russian Fund for Basic Research (Projects 09-01-00093 and 09-01-00586) and V.D. Stepanov was also supported by grants of Far Eastern Branch of the Russian Academy of Sciences.

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Paper B

HARDY-TYPE INEQUALITIES ON CONES OF MONOTONE FUNCTIONS

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O.V.Popova. *Hardy-type inequalities on cones of monotone functions.*
Research Report No. 6, Department of Engineering Sciences and Mathematics, Luleå University of Technology, 2011.

ABSTRACT. We study Hardy-type inequalities on cones of monotone functions. First we find the necessary and sufficient conditions for the inequalities to hold and compare the new ones with already known. In particular, we give a complete description for the corresponding Hardy-type inequalities with Volterra integral operators involving Oinarov's kernel.

2000 *Mathematics Subject Classification.* 26D10, 26D15, 26D07, 44A05, 45D99.

Key words and phrases. Hardy-type inequalities, monotone functions, measurable kernel, Oinarov's condition.

Note. This report will be submitted for publication elsewhere.

ISSN: 1400-4003

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1. INTRODUCTION

Hardy inequalities with their numerous applications constitute a part of Classical Analysis concentrated around the characterization problems for various integral and differential inequalities to hold (see, for instance, the recent monographs [16] and [17] to be introduced into the area). In particular, the integral inequalities with Oinarov's kernel and the inequalities restricted to the cone of monotone functions were intensively studied during the last two decades (see [5, 6, 10, 12, 15, 23, 28 - 34]). Specifically, we study the inequality of the form

$$(1) \quad \left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}},$$

where $0 < p, q < \infty$,

$$(2) \quad Kf(x) := \int_{[0,x]} k(x,y)f(y)d\nu(y),$$

λ, μ and ν are positive σ -finite Borel measures on $\mathbb{R}_+ := [0, \infty)$ and a measurable kernel $k(x, y) \geq 0$ satisfies the Oinarov condition: there exists a constant $D \geq 1$, for which

$$(3) \quad D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)), x \geq z \geq y.$$

The problem is to obtain a sharp two-sided estimate of the least possible positive constant C provided the inequality (1) is fulfilled for any function f from a prescribed set. There is a long story of the case $k(x, y) \equiv 1$ and $f \in \mathfrak{M}^+$ – the class of all measurable functions $f: \mathbb{R}_+ \rightarrow [0, +\infty]$ with absolutely continuous measures λ and μ (ν can be without a loss of generality taken as the Lebesgue measure). The first result for $1 < p = q < \infty, d\mu(x) = x^{-p}dx, d\lambda(x) = d\nu(x) = dx$ with the best constant $C = \frac{p}{p-1}$ (classical Hardy's inequality) was proved in 1925 (see [13]) and one of the latest in a chain (the case of parameters $0 < q < 1 = p$) was characterised in [29]. The initial two-weighted cases of (1) for $1 \leq p, q < \infty$ have been started in [7, 21, 35] with a supplement for $0 < q < p, 1 < p < \infty$ in [18]. The case of arbitrary Borel measures in the Hardy inequality was fractionally already presented in the works of B. Muckenhoupt [20] and V.G.Mazya [19], for discrete inequalities in the papers of G. Bennett [2, 3, 4] and, finally, for three measures it was done by D.V.Prokhorov [24, 25]. The inequality (1) in full was characterized in [26].

Motivated by some challenging problems of Harmonic Analysis and partially parallel to the previous line the Hardy inequality was studied on the cone of monotone functions. The initial one-weighted [1] and two-weighted cases [28] gave rise to several publications (see, for instance, [33, 8, 9, 14, 11, 10, 12, 30, 31, 5, 15, 23, 4] and literature given there). This case differs from the non-monotone by two main features.

The first is that the inequality makes sense for all values of positive parameters of integration p and q and, naturally, the study used to break up into four cases: (I) $1 \leq p \leq q < \infty$; (II) $0 < q < p < \infty, p > 1$; (III) $0 < p \leq q, 0 < p \leq 1$ and (IV) $0 < q < p \leq 1$. We exclude the border cases, when p or q is equal to infinity. The second feature is that the general case includes three measures.

In § 3.1 we study the Hardy-type inequalities (1) on the cones of monotone functions $\mathfrak{M} \downarrow$, where $\mathfrak{M} \downarrow$ ($\mathfrak{M} \uparrow$) consists of all non-negative non-increasing (non-decreasing) functions, and also give the criteria for the inequality

$$(4) \quad \left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}}$$

for all $f \in \mathfrak{M} \downarrow$ with $0 < p, q < \infty, q \geq 1$.

In § 3.2, using the results of [22], we receive criteria for the inequality

$$(5) \quad \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow$$

and compare them with the criteria from [12].

In § 3.3 the inequalities for negative parameters p and q are studied. The weighted Hardy inequalities with negative parameters of summation have been described in [25]. We give criteria for the inequalities with measures

$$\left(\int_{[0,\infty]} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \uparrow$$

as for positive, so as for negative values p, q . This result supplements and generalizes the criteria from [25] and [31].

Throughout the paper expressions $\frac{\infty}{\infty}, \frac{0}{0}$ are taken to be equal to 0. The relation $A \ll B$ means that $A \leq cB$ with a constant c depending only on the parameter of summation. We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. Moreover, p' denotes $p/(p-1)$ for $0 < p < \infty, p \neq 1$; $L^p(\lambda)$ denotes the set of all λ -measurable functions f on $\mathbb{R}_+$ such that $\|f\|_{p,\lambda} := \left(\int_{[0,\infty)} |f|^p d\lambda \right)^{\frac{1}{p}} < \infty$; $\Lambda(x) := \int_{[0,x]} d\lambda$. We suppose that $\Lambda(x) < \infty$ for all $x \in \mathbb{R}_+$. The constant C may be different at different occurrences. We also use the notation $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ for $0 < q < p < \infty$.

2. PRELIMINARIES

For proving the main results we need the following statements due to G. Sinnamon.

Theorem 1 [31, Theorem 2.1]. *For $0 < p \leq q < \infty$ the equality*

$$\sup_{F \downarrow} \frac{\left(\int_{[0,\infty)} F^q d\omega \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p d\nu \right)^{\frac{1}{p}}} = \sup_{x \geq 0} \frac{\left(\int_{[0,x]} d\omega \right)^{\frac{1}{q}}}{\left(\int_{[0,x]} d\nu \right)^{\frac{1}{p}}}$$

holds.

Theorem 2 [31, Theorem 2.2]. *Let $0 < q < p < \infty$. Then*

$$\sup_{F \downarrow} \frac{\left(\int_{[0,\infty)} F^q d\omega \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p d\nu \right)^{\frac{1}{p}}} \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{d\omega(t)}{\int_{[0,t]} d\nu} \right)^{\frac{r}{q}} d\nu(x) \right)^{\frac{1}{r}}.$$

Theorem 3 [31, Theorem 2.3]. *Let $0 < q < p < \infty$. Then*

$$\sup_{F \downarrow} \frac{\left(\int_{[0,\infty)} F^q d\omega \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} F^p d\nu \right)^{\frac{1}{p}}} \approx \left(\int_{[0,\infty)} \left(\frac{1}{\int_{[0,x]} d\nu} \int_{[0,x]} d\omega + \frac{1}{\int_{[0,\infty)} d\nu} \int_{[0,\infty)} d\omega \right)^{\frac{r}{q}} d\nu(x) \right)^{\frac{1}{r}},$$

if

$$\left(\int_{[0,x]} d\nu \right)^{-1} - \left(\int_{[0,\infty]} d\nu \right)^{-1} \leq C \int_{[x,\infty)} \left(\int_{[0,t]} d\nu \right)^{-2} d\nu(t)$$

for some constant $C \in (0, \infty)$.

The next portion of preliminaries is taken from the paper [22].

Theorem 4 [22, Theorem 2]. *Let $\gamma > 0$. Then for all $f \in \mathfrak{M} \downarrow$, $f \neq 0$*

$$(6) \quad \frac{1}{(\gamma + 1)^{\frac{1}{\gamma}}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{\gamma} d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^{\gamma} \Lambda^{\gamma} d\lambda \right)^{\frac{1}{\gamma}}} \leq \gamma,$$

if $\gamma \geq 1$ and

$$(7) \quad 0 < c(\gamma) \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^{\gamma} d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^{\gamma} \Lambda^{\gamma} d\lambda \right)^{\frac{1}{\gamma}}} \leq \frac{1}{\gamma},$$

if $0 < \gamma < 1$.

Theorem 5 [22, Theorem 6]. *Let $-1 < \gamma < 0$. Then for all $f \in \mathfrak{M} \uparrow$, $f(x) > 0$ λ -a.e. the two-sided inequality*

$$(8) \quad (-\gamma)^{\frac{1}{\gamma}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^{\gamma} d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^{\gamma} \Lambda^{\gamma} d\lambda \right)^{\frac{1}{\gamma}}} \leq 1$$

holds.

Theorem 6 [22, Theorem 7]. *Let $\gamma \leq -1$. Then for all $f \in \mathfrak{M} \uparrow$, $f(x) > 0$ λ -a.e. we have*

$$(9) \quad 0 < c_\gamma \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^\gamma d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^\gamma \Lambda^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq 1.$$

3. THE MAIN RESULTS

3.1. On the cone of non-increasing functions we consider the inequality (1) when $0 < p < \infty$ and $1 \leq q < \infty$ for the operator (2) with a kernel which satisfies the Oinarov condition (3), where the constant $C > 0$ is chosen the least possible.

Theorem 7. *Let $0 < p < \infty, 1 \leq q < \infty$. Let*

$$\bar{k}(x) := \int_{[0,x]} k(x,y) d\nu(y),$$

$$\tilde{k}(x) := \int_{[x,\infty)} k(y,x) d\mu(y),$$

$$\bar{\nu}(x) := \int_{[0,x]} d\nu(y).$$

Then the inequality (1) holds for all functions $f \in \mathfrak{M} \downarrow$ iff

$$A_0 < \infty, \text{ if } p \leq q = 1,$$

$$B_0 < \infty, \text{ if } 1 = q < p,$$

$$A_1 + A_{2,1} + A_{2,1}^* + A_{2,2} < \infty, \text{ if } 1 < p \leq q,$$

$$B_1 + B_{2,1} + B_{2,1}^* + B_{2,2} < \infty, \text{ if } 1 < q < p,$$

$$D_1 + D_2 + D_3 < \infty, \text{ if } 0 < p \leq 1 < q,$$

where

$$A_0 := \sup_{x \geq 0} \left(\int_{[0,x]} \tilde{k} d\nu \right) \left(\int_{[0,x]} d\lambda \right)^{-\frac{1}{p}},$$

$$B_0 := \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{\tilde{k}(t) d\nu(t)}{\int_{[0,t]} d\lambda} \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}},$$

$$A_1 := \sup_{t \geq 0} \left(\int_{[t,\infty)} \Lambda^{-p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[0,t]} \bar{k}^q d\mu \right)^{\frac{1}{q}},$$

$$A_{2,1} := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{1}{q}},$$

$$\begin{aligned}
A_{2,1}^* &:= \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda^{-p'} \bar{\nu}^{p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{1}{q}}, \\
A_{2,2} &:= \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda^{-p'} \bar{k}^{p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{1}{q}}; \\
B_1^r &:= \int_{[0,\infty)} \left(\int_{[0,x]} \bar{k}^q d\mu \right)^{\frac{r}{q}} \left(\int_{[x,\infty)} \Lambda^{-p'} d\lambda \right)^{\frac{r}{q'}} \Lambda(x)^{-p'} d\lambda(x), \\
B_{2,1}^r &:= \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda(x)^{-p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{r}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{r}{p}} d\mu(t), \\
B_{2,1}^{*r} &:= \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda^{-p'} \bar{\nu}^{p'} d\lambda \right)^{\frac{r}{q'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{r}{q}} \Lambda(t)^{-p'} \bar{\nu}(t)^{p'} d\lambda(t), \\
B_{2,2}^r &:= \int_{[0,\infty)} \left(\int_{[x,\infty)} d\mu \right)^{\frac{r}{q}} \left(\int_{[0,x]} \Lambda^{-p'} \bar{k}^{p'} d\lambda \right)^{\frac{r}{q'}} \Lambda(x)^{-p'} \bar{k}(x)^{p'} d\lambda(x), \\
\text{where } \frac{1}{r} &= \frac{1}{q} - \frac{1}{p};
\end{aligned}$$

$$\begin{aligned}
D_1 &:= \sup_{x \geq 0} \left(\int_{[0,x]} \bar{k}^q d\mu \right)^{\frac{1}{q}} \Lambda(x)^{-\frac{1}{p}}, \\
D_2 &:= \sup_{x \geq 0} \left(\int_{[x,\infty)} k(y,x)^q d\mu(y) \right)^{\frac{1}{q}} \bar{\nu}(x) \Lambda(x)^{-\frac{1}{p}}, \\
D_3 &:= \sup_{x \geq 0} \left(\int_{[x,\infty)} d\mu \right)^{\frac{1}{q}} \bar{k}(x) \Lambda(x)^{-\frac{1}{p}}.
\end{aligned}$$

Proof. First we consider the case $q = 1$:

$$\int_{[0,\infty)} (Kf) d\mu \leq C \left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}.$$

Then for the least constant C we get

$$\begin{aligned}
C &= \sup_{f \downarrow} \frac{\int_{[0,\infty)} \left(\int_{[0,x]} k(x,y) f(y) d\nu(y) \right) d\mu(x)}{\left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}} \\
&= \sup_{f \downarrow} \frac{\int_{[0,\infty)} \tilde{k}(y) f(y) d\nu(y)}{\left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}}.
\end{aligned}$$

Using Theorems 1 and 2, we obtain

$$C = \sup_{x \geq 0} \frac{\int_{[0,x]} \tilde{k} d\nu}{\left(\int_{[0,x]} d\lambda \right)^{\frac{1}{p}}}$$

for $p \leq 1$ and

$$C \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{\tilde{k}(t)d\nu(t)}{\int_{[0,t]} d\lambda} \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}}$$

for $p > 1$.

For estimating the constant C in the case $q > 1$ we use the duality principle in $L^q(\mu)$:

$$\begin{aligned} C &= \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}} = \sup_{f \downarrow} \sup_{g \geq 0} \frac{\int_{[0,\infty)} (Kf)g d\mu}{\|g\|_{q',\mu} \left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}} \\ &= \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \sup_{f \downarrow} \frac{\int_{[0,\infty)} f K^*(g) d\nu}{\left(\int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}}. \end{aligned}$$

Using Theorems 1 and 3 in the last expression, we obtain

$$(10) \quad C \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \left(\int_{[0,\infty)} \left(\frac{1}{\Lambda(x)} \int_{[0,x]} K^*(g) d\nu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}},$$

under the conditions

$$\int_{[0,\infty)} d\lambda = \infty, \quad \left(\int_{[0,x]} d\lambda \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} d\lambda \right)^{-2} d\lambda(t)$$

for $p > 1$ and

$$(11) \quad C = \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \sup_{x \geq 0} \frac{\int_{[0,x]} K^*(g) d\nu}{\left(\int_{[0,x]} d\lambda \right)^{\frac{1}{p}}}$$

for $0 < p \leq 1$.

As the dual operator $K^*(g)$ has the form

$$K^*g(y) = \int_{[y,\infty)} k(x,y)g(x)d\mu(x),$$

we obtain

$$\begin{aligned} \int_{[0,x]} K^*(g) d\nu &= \int_{[0,x]} \left(\int_{[y,\infty)} k(z,y)g(z)d\mu(z) \right) d\nu(y) \\ &= \int_{[0,x]} \left(\int_{[y,x]} k(z,y)g(z)d\mu(z) + \int_{[x,\infty)} k(z,y)g(z)d\mu(z) \right) d\nu(y) =: I_1 + I_2. \end{aligned}$$

For the first term we write

$$I_1 = \int_{[0,x]} \left(\int_{[y,x]} k(z,y)g(z)d\mu(z) \right) d\nu(y) = \int_{[0,x]} g(z)d\mu(z) \left(\int_{[0,z]} k(z,y)d\nu(y) \right)$$

and similarly for the second term

$$\begin{aligned}
 I_2 &= \int_{[0,x]} \left(\int_{[x,\infty)} k(z,y)g(z)d\mu(z) \right) d\nu(y) \\
 &= \int_{[0,x]} \left(\int_{[x,\infty)} k(z,x)g(z)d\mu(z) + \int_{[x,\infty)} k(x,y)g(z)d\mu(z) \right) d\nu(y) \\
 &= \left(\int_{[0,x]} d\nu(y) \right) \int_{[x,\infty)} k(z,x)g(z)d\mu(z) + \int_{[0,x]} k(x,y)d\nu(y) \int_{[x,\infty)} g(z)d\mu(z).
 \end{aligned}$$

Thus, we get

$$(12) \quad \int_{[0,x]} K^*(g)d\nu = I_1 + I_{2,1} + I_{2,2},$$

where

$$\begin{aligned}
 I_1 &= \int_{[0,x]} \bar{k}g d\mu, \\
 I_{2,1} &:= \bar{\nu}(x)K^*g(x), \\
 I_{2,2} &:= \bar{k}(x) \int_{[x,\infty)} g d\mu.
 \end{aligned}$$

We assume first that $p > 1$. Substituting (12) into (10), we turn our problem into consideration of three inequalities:

$$(13) \quad \left(\int_{[0,\infty)} \Lambda(x)^{-p'} \left(\int_{[0,x]} \bar{k}g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_1 \|g\|_{q',\mu},$$

$$(14) \quad \left(\int_{[0,\infty)} \Lambda(y)^{-p'} \bar{\nu}(y)^{p'} \left(\int_{[y,\infty)} k(x,y)g(x)d\mu(x) \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}} \leq C_{2,1} \|g\|_{q',\mu}$$

and

$$(15) \quad \left(\int_{[0,\infty)} \Lambda(x)^{-p'} \bar{k}(x)^{p'} \left(\int_{[x,\infty)} g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_{2,2} \|g\|_{q',\mu}.$$

According to [24, 26] we have the following criteria

$$\begin{aligned}
 C_1 &\approx A_1, \text{ if } 1 < p \leq q, \\
 C_1 &\approx B_1, \text{ if } 1 < q < p, \\
 C_{2,1} &\approx A_{2,1} + A_{2,1}^*, \text{ if } 1 < p \leq q, \\
 C_{2,1} &\approx B_{2,1} + B_{2,1}^*, \text{ if } 1 < q < p, \\
 C_{2,2} &\approx A_{2,2}, \text{ if } 1 < p \leq q, \\
 C_{2,2} &\approx B_{2,2}, \text{ if } 1 < q < p.
 \end{aligned}$$

In the case $0 < p \leq 1$ we obtain

$$C = \sup_{g \geq 0} \frac{1}{\|g\|_{q', \mu}} \sup_{x \geq 0} \frac{\int_{[0, x]} K^*(g) d\nu}{\left(\int_{[0, x]} d\lambda \right)^{\frac{1}{p}}} =: J_1 + J_{2,1} + J_{2,2}.$$

We have

$$J_1 = \sup_{x \geq 0} \sup_{g \geq 0} \frac{\int_{[0, x]} \bar{k} g d\mu}{\left(\int_{[0, x]} d\lambda \right)^{\frac{1}{p}} \|g\|_{q', \mu}},$$

$$J_{2,1} = \sup_{x \geq 0} \sup_{g \geq 0} \frac{\bar{\nu}(x) K^* g(x)}{\left(\int_{[0, x]} d\lambda \right)^{\frac{1}{p}} \|g\|_{q', \mu}}$$

and

$$J_{2,2} = \sup_{x \geq 0} \sup_{g \geq 0} \frac{\bar{k}(x) \int_{[x, \infty)} g d\mu}{\left(\int_{[0, x]} d\lambda \right)^{\frac{1}{p}} \|g\|_{q', \mu}}.$$

Observe that the constant J_1 is $L^{q'}(\mu) - L^\infty$ norm of the following operator:

$$g(z) \rightarrow \Lambda(x)^{-\frac{1}{p}} \int_{[0, x]} \bar{k}(z) g(z) d\mu(z) = \int_{[0, \infty)} \tilde{K}(x, z) g(z) d\mu(z) =: \tilde{K}g(x),$$

that is

$$\|\tilde{K}g\|_\infty \leq J_1 \|g\|_{q', \mu}.$$

Using Hölder's inequality, we find that

$$\int_{[0, \infty)} \tilde{K}(x, z) g(z) d\mu(z) \leq \left(\int_{[0, \infty)} \tilde{K}(x, z)^q d\mu(z) \right)^{\frac{1}{q}} \|g\|_{q', \mu}.$$

Thus,

$$J_1 = \|\tilde{K}\| = \sup_{x \geq 0} \left(\int_{[0, \infty)} \tilde{K}(x, z)^q d\mu(z) \right)^{\frac{1}{q}} = \sup_{x \geq 0} \left(\int_{[0, x]} \bar{k}(z)^q d\mu(z) \right)^{\frac{1}{q}} \Lambda(x)^{-\frac{1}{p}}.$$

Similarly,

$$J_{2,1} = \sup_{x \geq 0} \bar{\nu}(x) \Lambda(x)^{-\frac{1}{p}} \left(\int_{[x, \infty)} k(y, x)^q d\mu(y) \right)^{\frac{1}{q}}$$

and

$$J_{2,2} = \sup_{x \geq 0} \bar{k}(x) \Lambda(x)^{-\frac{1}{p}} \left(\int_{[x, \infty)} d\mu \right)^{\frac{1}{q}}.$$

Thus, for $0 < p \leq 1$ we get

$$C \approx D_1 + D_2 + D_3,$$

and the proof is complete. $\square$

By the same method we examine the inequality (4) on the cone of non-increasing functions for $0 < p < \infty, 1 \leq q < \infty$ for the operator (2) with the kernel satisfying the condition (3).

Theorem 8. *Let $0 < p < \infty, 1 \leq q < \infty$. We define the functions*

$$\bar{k}(x) := \int_{[0,x]} k(x,y) d\nu(y),$$

$$\tilde{k}(x) := \int_{[x,\infty)} k(y,x) d\mu(y),$$

$$\bar{\nu}(x) := \int_{[0,x]} d\nu(y).$$

Then the inequality (4) holds for all functions $f \in \mathfrak{M} \downarrow$ iff

$$\mathbf{A}_0 < \infty, \text{ if } p \leq q = 1,$$

$$\mathbf{B}_0 < \infty, \text{ if } 1 = q < p,$$

$$\mathbf{A}_1 + \mathbf{A}_{2,1} + \mathbf{A}_{2,1}^* + \mathbf{A}_{2,2} < \infty, \text{ if } 1 < p \leq q,$$

$$\mathbf{B}_1 + \mathbf{B}_{2,1} + \mathbf{B}_{2,1}^* + \mathbf{B}_{2,2} < \infty, \text{ if } 1 < q < p,$$

$$\mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3 < \infty, \text{ if } 0 < p \leq 1 < q,$$

where

$$\mathbf{A}_0 := \sup_{x \geq 0} \left(\int_{[0,x]} \tilde{k} d\nu \right) \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-\frac{1}{p}},$$

$$\mathbf{B}_0 := \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{\tilde{k}(t) d\nu(t)}{\int_{[0,t]} \Lambda^p d\lambda} \right)^{p'} \Lambda(x)^p d\lambda(x) \right)^{\frac{1}{p'}},$$

$$\mathbf{A}_1 := \sup_{t \geq 0} \left(\int_{[t,\infty)} \Lambda^{-2p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[0,t]} \bar{k}^q d\mu \right)^{\frac{1}{q}},$$

$$\mathbf{A}_{2,1} := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda(x)^{-2p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{1}{q}},$$

$$\mathbf{A}_{2,1}^* := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda^{-2p'} \bar{\nu}^{p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{1}{q}},$$

$$\mathbf{A}_{2,2} := \sup_{t \geq 0} \left(\int_{[0,t]} \Lambda^{-2p'} \bar{k}^{p'} d\lambda \right)^{\frac{1}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{1}{q}};$$

$$\mathbf{B}_1^r := \int_{[0,\infty)} \left(\int_{[0,x]} \bar{k}^q d\mu \right)^{\frac{r}{q}} \left(\int_{[x,\infty)} \Lambda^{-2p'} d\lambda \right)^{\frac{r}{q'}} \Lambda(x)^{-2p'} d\lambda(x),$$

$$\mathbf{B}_{2,1}^r := \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda(x)^{-2p'} \bar{\nu}(x)^{p'} k(x,t)^{p'} d\lambda(x) \right)^{\frac{r}{p'}} \left(\int_{[t,\infty)} d\mu \right)^{\frac{r}{p}} d\mu(t),$$

$$\mathbf{B}_{2,1}^{*r} := \int_{[0,\infty)} \left(\int_{[0,t]} \Lambda^{-2p'} \bar{\nu}^{p'} d\lambda \right)^{\frac{r}{q'}} \left(\int_{[t,\infty)} k(t,y)^q d\mu(y) \right)^{\frac{r}{q}} \Lambda(t)^{-2p'} \bar{\nu}(t)^{p'} d\lambda(t),$$

$$\mathbf{B}_{2,2}^r := \int_{[0,\infty)} \left(\int_{[x,\infty)} d\mu \right)^{\frac{r}{q}} \left(\int_{[0,x]} \Lambda^{-2p'} \bar{k}^{p'} d\lambda \right)^{\frac{r}{q'}} \Lambda(x)^{-2p'} \bar{k}(x)^{p'} d\lambda(x),$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$;

$$\mathbf{D}_1 := \sup_{x \geq 0} \left(\int_{[0,x]} \bar{k}^q d\mu \right)^{\frac{1}{q}} \Lambda(x)^{-1-\frac{1}{p}},$$

$$\mathbf{D}_2 := \sup_{x \geq 0} \left(\int_{[x,\infty)} k(y,x)^q d\mu(y) \right)^{\frac{1}{q}} \bar{\nu}(x) \Lambda(x)^{-1-\frac{1}{p}},$$

$$\mathbf{D}_3 := \sup_{x \geq 0} \left(\int_{[x,\infty)} d\mu \right)^{\frac{1}{q}} \bar{k}(x) \Lambda(x)^{-1-\frac{1}{p}}.$$

Proof. In the case $q = 1$ we have

$$C = \sup_{f \downarrow} \frac{\int_{[0,\infty)} \tilde{k}(y) f(y) d\nu(y)}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}}.$$

Let $p, q > 0$. Suppose that the auxiliary inequality

$$\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}} \leq C \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}}, \quad f \in \mathfrak{M} \downarrow, f \not\equiv 0,$$

holds on the cone of non-increasing functions. We consider the constant C to be the least possible, that is equal to J_{pq} . Using Theorem 4, we obtain

$$J_{pq} = \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}}} \approx \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}{\left(\int_{[0,\infty)} \Lambda^q f^q d\lambda \right)^{\frac{1}{q}}}.$$

Thus,

$$J_{pq} = \sup_{x \in [0,\infty)} \frac{\left(\int_{[0,x]} d\mu \right)^{\frac{1}{p}}}{\left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{1}{q}}}, \quad 0 < q \leq p < \infty,$$

$$J_{pq} \approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} d\mu \right)^{\frac{q}{q-p}} \left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{q}{p-q}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q-p}{pq}}, \quad 0 < p < q < \infty,$$

provided

$$\int_{[0,\infty)} \Lambda^q d\lambda = \infty, \quad \left(\int_{[0,x]} \Lambda^q d\lambda \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} \Lambda^q d\lambda \right)^{-2} \Lambda(t)^q d\lambda(t),$$

or

$$J_{pq} \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{d\mu(t)}{\int_{[0,t]} \Lambda^q d\lambda} \right)^{\frac{q}{q-p}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q-p}{pq}}, \quad 0 < p < q < \infty.$$

In particular, for $p = 1$ we find that

$$(16) \quad J_{1q} \approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} d\mu \right)^{\frac{q}{q-1}} \left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{q}{1-q}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q-1}{q}}$$

and

$$(17) \quad J_{1q} \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{d\mu(t)}{\int_{[0,t]} \Lambda^q d\lambda} \right)^{\frac{q}{q-1}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q-1}{q}}$$

for $1 < q < \infty$ and

$$(18) \quad J_{1q} = \sup_{x \in [0,\infty)} \frac{\left(\int_{[0,x]} d\mu \right)}{\left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{1}{q}}}$$

for $0 < q \leq 1$.

Using (17) and (18), we receive for the constant C the following: if $q = 1$, then

$$C = \sup_{x \geq 0} \left(\int_{[0,x]} \tilde{k} d\nu \right) \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-\frac{1}{p}}, \quad 0 < p \leq 1$$

and

$$C \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \frac{\tilde{k}(t) d\nu(t)}{\int_{[0,t]} \Lambda^p d\lambda} \right)^{p'} \Lambda(x)^p d\lambda(x) \right)^{\frac{1}{p'}}, \quad p > 1.$$

In the case $q > 1$ we use the duality principle in $L^q(\mu)$ and obtain

$$(19) \quad \begin{aligned} C &= \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}} = \sup_{f \downarrow} \sup_{g \geq 0} \frac{\int_{[0,\infty)} (Kf) g d\mu}{\|g\|_{q',\mu} \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}} \\ &= \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \sup_{f \downarrow} \frac{\int_{[0,\infty)} f K^*(g) d\nu}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}}. \end{aligned}$$

Using (16) and (18) in (19), we find that

$$(20) \quad C \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \left(\int_{[0,\infty)} \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-p'} \left(\int_{[0,x]} K^*(g) d\nu \right)^{p'} \Lambda(x)^p d\lambda(x) \right)^{\frac{1}{p'}},$$

provided $p > 1$ and

$$\int_{[0,\infty)} \Lambda^p d\lambda = \infty, \quad \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} \Lambda^p d\lambda \right)^{-2} \Lambda(t)^p d\lambda(t).$$

If $0 < p \leq 1$, then

$$(21) \quad C = \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \sup_{x \geq 0} \frac{\int_{[0,x]} K^*(g) d\nu}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}}.$$

The dual operator $K^*(g)$ has the form

$$\int_{[0,x]} K^*(g) d\nu = I_1 + I_{2,1} + I_{2,2},$$

where

$$\begin{aligned} I_1 &:= \int_{[0,x]} \bar{k} g d\mu, \\ I_{2,1} &:= \bar{\nu}(x) K^* g(x), \\ I_{2,2} &:= \bar{k}(x) \int_{[x,\infty)} g d\mu. \end{aligned}$$

Now, let $p > 1$. Similar to the proof of the previous inequality, we consider three inequalities

$$(22) \quad \left(\int_{[0,\infty)} \Lambda(x)^{-2p'} \left(\int_{[0,x]} \bar{k} g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_1 \|g\|_{q',\mu},$$

$$(23) \quad \left(\int_{[0,\infty)} \Lambda(y)^{-2p'} \bar{\nu}(y)^{p'} \left(\int_{[y,\infty)} k(x,y) g(x) g\mu(x) \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}} \leq C_{2,1} \|g\|_{q',\mu}$$

and

$$(24) \quad \left(\int_{[0,\infty)} \Lambda(x)^{-2p'} \bar{k}^{p'}(x) \left(\int_{[x,\infty)} g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_{2,2} \|g\|_{q',\mu}.$$

By the known criteria [24, 26] we obtain the following equivalences

$$C_1 + C_{2,1} + C_{2,2} \approx \mathbf{A}_1 + \mathbf{A}_{2,1} + \mathbf{A}_{2,1}^* + \mathbf{A}_{2,2}, \text{ if } 1 < p \leq q,$$

$$C_1 + C_{2,1} + C_{2,2} \approx \mathbf{B}_1 + \mathbf{B}_{2,1} + \mathbf{B}_{2,1}^* + \mathbf{B}_{2,2}, \text{ if } 1 < q < p.$$

In the case $0 < p \leq 1 < q$ we get

$$C = \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}} \sup_{x \geq 0} \frac{\int_{[0,x]} K^*(g) d\nu}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}} = J_1 + J_{2,1} + J_{2,2},$$

where

$$J_1 := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\int_{[0,x]} \bar{k} g d\mu}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}} \|g\|_{q',\mu}},$$

$$J_{2,1} := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\bar{\nu}(x) K^* g(x)}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}} \|g\|_{q', \mu}}$$

and

$$J_{2,2} := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\bar{k}(x) \int_{[x,\infty)} g d\mu}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}} \|g\|_{q', \mu}}.$$

Similar to the previous result,

$$J_1 = \|\tilde{K}\| = \sup_{x \geq 0} \left(\int_{[0,\infty)} \tilde{K}(x, z)^q d\mu(z) \right)^{\frac{1}{q}} = \sup_{x \geq 0} \left(\int_{[0,x]} \bar{k}^q(z) d\mu(z) \right)^{\frac{1}{q}} \Lambda(x)^{-1-\frac{1}{p}},$$

$$J_{2,1} = \sup_{x \geq 0} \bar{\nu}(x) \Lambda(x)^{-1-\frac{1}{p}} \left(\int_{[x,\infty)} k(y, x)^q d\mu(y) \right)^{\frac{1}{q}}$$

and

$$J_{2,2} = \sup_{x \geq 0} \bar{k}(x) \Lambda(x)^{-1-\frac{1}{p}} \left(\int_{[x,\infty)} d\mu \right)^{\frac{1}{q}}.$$

Hence, for $0 < p \leq 1$ we obtain the criterion

$$C \approx \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3,$$

and the proof is complete. $\square$

3.2. Let $0 < p, q < \infty$. We study the problem when the inequality (25)

$$\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow, f \not\equiv 0,$$

holds. For the least constant C , which we denote J_{pq} , we write

$$J_{pq} := \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}.$$

Using Theorem 4, we get

$$(26) \quad J_{pq} = \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^q \Lambda^q d\lambda \right)^{\frac{1}{q}}} \cdot \frac{\left(\int_{[0,\infty)} f^q \Lambda^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}$$

$$\approx \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} f^q \Lambda^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}.$$

For the last term we use Theorems 1 and 3 and conclude that $J_{pq} \approx S_{pq}$, where

$$S_{pq} = \sup_{x \in [0, \infty)} \frac{\left(\int_{[0, x]} \Lambda^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0, x]} d\mu \right)^{\frac{1}{p}}}, \quad 0 < p \leq q < \infty,$$

and

$$S_{pq} \approx \left(\int_{[0, \infty)} \left(\int_{[0, x]} \Lambda^q d\lambda \right)^{\frac{r}{q}} \left(\int_{[0, x]} d\mu \right)^{-\frac{r}{q}} d\mu(x) \right)^{\frac{1}{r}}, \quad 0 < q < p < \infty,$$

provided

$$\int_{[0, \infty)} d\mu = \infty, \quad \left(\int_{[0, x]} d\mu \right)^{-1} \leq c \int_{[x, \infty)} \left(\int_{[0, t]} d\mu \right)^{-2} d\mu(t).$$

An alternate method of characterization of (25) was given in [12]. In this paper the expression

$$H_{\mathfrak{M}\downarrow}(B; p, q, \mu, \lambda) = \sup_{f \in \mathfrak{M}\downarrow} \left[\left(\int_0^\infty (Bf)^q d\gamma \right)^{\frac{1}{q}} \left(\int_0^\infty f^p d\beta \right)^{-\frac{1}{p}} \right]$$

was examined, where

$$(Bf)(t) = \left(\int_t^\infty f d\mu \right), \quad t \in \mathbb{R}_+.$$

Obviously, the finiteness of $H_{\mathfrak{M}\downarrow}(B; p, q, \mu, \lambda)$ is equivalent to a corresponding inequality with three measures, more general than (25). However, in [12] the integrals $\int_{[0, t]} d\beta$ and $\int_{[0, t]} d\gamma$ have been assumed to be continuous.

Using [12, Theorem 1.1] for the characterization of (25), we get

$$J_{pq} \approx E_{pq} + F_{pq},$$

where

$$E_{pq} = \sup_{t \in [0, \infty)} \left[\left(\int_{[0, t]} \left(\int_{[x, t]} d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \left(\int_{[0, t]} d\mu \right)^{-\frac{1}{p}} \right]$$

for $0 < p \leq q < \infty$,

$$F_{pq} = \sup_{x \in [0, \infty)} \left(\sup_{t \in [x, \infty)} \left[\left(\int_{[x, t]} d\lambda \right) \left(\int_{[0, t]} d\mu \right)^{-\frac{1}{p}} \right] \left(\int_{[0, x]} d\lambda \right)^{\frac{1}{q}} \right)$$

for $0 < p \leq q < \infty$, $p \leq 1$,

$$F_{pq} = \sup_{x \in [0, \infty)} \left[\left(\int_{[x, \infty)} \left(\int_{[x, t]} d\lambda \right)^{\frac{p}{p-1}} \left(-d \left[\left(\int_{[0, t]} d\mu \right)^{-\frac{1}{p-1}} \right] \right) \right)^{\frac{p-1}{p}} \left(\int_{[0, x]} d\lambda \right)^{\frac{1}{q}} \right]$$

for $1 < p \leq q < \infty$,

$$E_{p,q}^r = \int_{[0,\infty)} \left(\int_{[0,t]} \left(\int_{[x,t]} d\lambda \right)^q d\lambda(x) \right)^{\frac{r}{q}} \left(-d \left[\left(\int_{[0,t]} d\mu \right)^{-\frac{r}{p}} \right] \right)$$

for $0 < q < p < \infty$,

$$F_{pq}^r = \int_{[0,\infty)} \sup_{t \in [x,\infty)} \left[\left(\int_{[x,t]} d\lambda \right) \left(\int_{[0,t]} d\mu \right)^{-\frac{1}{p}} \right]^r d \left[\left(\int_{[0,x]} d\lambda \right)^{\frac{r}{q}} \right]$$

for $0 < q < p \leq 1$,

$$F_{pq}^r = \int_{[0,\infty)} \left(\int_{[x,\infty)} \left(\int_{[x,t]} d\lambda \right)^{\frac{p}{p-1}} \left(-d \left(\int_{[0,t]} d\mu \right)^{-\frac{1}{p-1}} \right) \right)^{\frac{r}{p'}} d \left[\left(\int_{[0,x]} d\lambda \right)^{\frac{r}{q}} \right]$$

for $0 < q < p < \infty$, $p > 1$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Let us compare the criteria which we obtain for (25) using Theorems 1, 3 and 4.

Using [24, Lemma 1], we find that

$$(27) \quad \left(\int_{[0,t]} \left(\int_{[x,t]} d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \approx \left(\int_{[0,t]} \Lambda^q(x) d\lambda(x) \right)^{\frac{1}{q}}.$$

Thus, for $0 < p \leq q < \infty$

$$\left(\int_{[0,t]} \left(\int_{[x,t]} d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \left(\int_{[0,t]} d\mu \right)^{-\frac{1}{p}} \approx \left(\int_{[0,t]} \Lambda^q(x) d\lambda(x) \right)^{\frac{1}{q}} \left(\int_{[0,t]} d\mu \right)^{-\frac{1}{p}},$$

which means that $S_{pq} \approx E_{pq}$. Therefore, $E_{pq} \gg F_{pq}$.

To prove the similar fact for $0 < q < p < \infty$ we suppose that the measure μ satisfies the following condition:

$$\int_{[t,\infty)} \left(\int_{[0,x]} d\mu \right)^{-\lambda} d\mu(x) \approx \left(\int_{[0,t]} d\mu \right)^{-\lambda+1}, \lambda > 1.$$

In this case, using again (27), we get

$$\begin{aligned} & \left(\int_{[0,\infty)} \left(\int_{[0,t]} \left(\int_{[x,t]} d\lambda \right)^q d\lambda(x) \right)^{\frac{r}{q}} \left(-d \left[\left(\int_{[0,t]} d\mu \right)^{-\frac{r}{p}} \right] \right) \right)^{\frac{1}{r}} \approx \\ & \left(\int_{[0,\infty)} \left(\int_{[0,x]} \Lambda^q d\lambda \right)^{\frac{r}{q}} \left(\int_{[0,x]} d\mu \right)^{-\frac{r}{q}} d\mu(x) \right)^{\frac{1}{r}}, \end{aligned}$$

which means $S_{pq} \approx E_{pq}$ and $E_{pq} \gg F_{pq}$.

3.3. We consider the inequality analogous to the one studied in Section 3.2 on the cone of non-decreasing functions for $p, q > 0$:

$$\left(\int_{[0,\infty]} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \uparrow, f \neq 0.$$

We use a statement which follows from Theorem 4:

$$\frac{1}{(\gamma + 1)^{\frac{1}{\gamma}}} \leq \frac{\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^\gamma d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left(\int_{[0,\infty)} f^\gamma \Lambda_*^\gamma d\lambda \right)^{\frac{1}{\gamma}}} \leq \gamma, \quad f \uparrow, \quad p, q > 0,$$

where $\Lambda_*(t) = \int_{[t,\infty)} d\lambda$. Using again Theorems 1 and 3, we obtain $C = J_{pq}$, where

$$J_{pq} = \sup_{f \uparrow} \frac{\left(\int_{[0,\infty]} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \cdot \left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}} \cdot \left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}} \approx \sup_{f \uparrow} \frac{\left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}.$$

Thus,

$$J_{pq} \approx \sup_{x \in [0,\infty)} \frac{\left(\int_{[x,\infty)} \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[x,\infty)} d\mu \right)^{\frac{1}{p}}}, \quad p \leq q,$$

and

$$J_{pq} \approx \left(\int_{[0,\infty)} \left(\int_{[x,\infty)} \Lambda_*^q d\lambda \right)^{\frac{r}{q}} \left(\int_{[x,\infty)} d\mu \right)^{-\frac{r}{q}} d\mu(x) \right)^{\frac{1}{r}}, \quad q < p,$$

provided

$$\int_{[0,\infty)} d\mu = \infty, \quad \left(\int_{[0,x]} d\mu \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} d\mu \right)^{-2} d\mu(t).$$

Similarly, using the Corollaries from Theorems 5 and 6, we can characterize the inequality for $p, q < 0$. We have

$$\left(\int_{[0,\infty)} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \uparrow, \quad f \not\equiv 0.$$

$$J_{pq} = \sup_{f \uparrow} \frac{\left(\int_{[0,\infty]} \left(\int_{[0,x]} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \cdot \left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}} \cdot \left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}} \approx \sup_{f \uparrow} \frac{\left(\int_{[0,\infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}}.$$

Let $\tilde{f}(x) := f(x)^{-1}$. Then

$$J_{pq} = \sup_{\tilde{f} \downarrow} \frac{\left(\int_{[0,\infty)} \tilde{f}^{-q} \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \tilde{f}^{-p} d\mu \right)^{\frac{1}{p}}} = \sup_{\tilde{f} \downarrow} \frac{\left(\int_{[0,\infty)} \tilde{f}^{-p} d\mu \right)^{-\frac{1}{p}}}{\left(\int_{[0,\infty)} \tilde{f}^{-q} \Lambda_*^q d\lambda \right)^{-\frac{1}{q}}}.$$

$$J_{pq} \approx \sup_{x \in [0, \infty)} \frac{\left(\int_{[0, x]} \Lambda^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0, x]} d\mu \right)^{\frac{1}{p}}}, \quad -\infty < p \leq q < 0,$$

$$J_{pq} \approx \left(\int_{[0, \infty)} \left(\int_{[0, x]} d\mu \right)^{\frac{r}{p}} \left(\int_{[0, x]} \Lambda^q d\lambda \right)^{-\frac{r}{p}} \Lambda(x)^q d\lambda(x) \right)^{-\frac{1}{r}}, \quad -\infty < q < p < 0,$$

provided

$$\int_{[0, \infty)} \Lambda^q d\lambda = \infty, \quad \left(\int_{[0, x]} \Lambda^q d\lambda \right)^{-1} \leq c \int_{[x, \infty)} \left(\int_{[0, t]} \Lambda^q d\lambda \right)^{-2} \Lambda(t)^q d\lambda(t).$$

We consider the similar inequality on the cone of non-increasing functions. For the inequality

$$\left(\int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \downarrow, \quad p, q < 0,$$

we have $C = J_{pq}$, where

$$\begin{aligned} J_{pq} &= \sup_{f \downarrow} \frac{\left(\int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^q d\lambda(x) \right)^{\frac{1}{q}}}{\left(\int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}} \cdot \frac{\left(\int_{[0, \infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0, \infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}} \approx \sup_{f \downarrow} \frac{\left(\int_{[0, \infty)} f^q \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}} \\ &= \sup_{\tilde{f} \uparrow} \frac{\left(\int_{[0, \infty)} \tilde{f}^{-q} \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[0, \infty)} \tilde{f}^{-p} d\mu \right)^{\frac{1}{p}}} = \sup_{\tilde{f} \uparrow} \frac{\left(\int_{[0, \infty)} \tilde{f}^{-p} d\mu \right)^{-\frac{1}{p}}}{\left(\int_{[0, \infty)} \tilde{f}^{-q} \Lambda_*^q d\lambda \right)^{-\frac{1}{q}}}. \\ J_{pq} &\approx \sup_{x \in [0, \infty)} \frac{\left(\int_{[x, \infty)} \Lambda_*^q d\lambda \right)^{\frac{1}{q}}}{\left(\int_{[x, \infty)} d\mu \right)^{\frac{1}{p}}}, \quad -\infty < p \leq q < 0, \end{aligned}$$

$$J_{pq} \approx \left(\int_{[0, \infty)} \left(\int_{[x, \infty)} d\mu \right)^{\frac{r}{p}} \left(\int_{[x, \infty)} \Lambda_*^q d\lambda \right)^{-\frac{r}{p}} \Lambda_*(x)^q d\lambda(x) \right)^{-\frac{1}{r}}, \quad -\infty < q < p < 0,$$

provided

$$\int_{[0, \infty)} \Lambda_*^q d\lambda = \infty, \quad \left(\int_{[0, x]} \Lambda_*^q d\lambda \right)^{-1} \leq c \int_{[x, \infty)} \left(\int_{[0, t]} \Lambda_*^q d\lambda \right)^{-2} \Lambda_*(t)^q d\lambda(t).$$

Finally, using the previous results, we consider the inequality

$$\left(\int_{[0, \infty)} \left(\int_{[x, \infty)} f d\mu \right)^q d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_{[0, \infty)} \left(\int_{[x, \infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow, f \not\equiv 0$$

for $p, q > 0$. By Theorem 4,

$$C = \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\mu \right)^q d\mu(x) \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} \left(\int_{[x,\infty)} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}} \approx \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} f^q \Lambda^q d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}}.$$

Hence,

$$J_{pq} \approx \sup_{f \downarrow} \frac{\left(\int_{[0,\infty)} f^q \Lambda^q d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}},$$

and we obtain

$$J_{pq} = \sup_{x \in [0,\infty)} \frac{\left(\int_{[0,x]} \Lambda^q d\mu \right)^{\frac{1}{q}}}{\left(\int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}}, \quad 0 < p \leq q < \infty$$

and

$$J_{pq} \approx \left(\int_{[0,\infty)} \left(\int_{[0,x]} \Lambda^q d\mu \right)^{\frac{r}{q}} \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-\frac{r}{q}} \Lambda(x)^p d\lambda(x) \right)^{\frac{1}{r}}, \quad 0 < q < p < \infty,$$

provided

$$\int_{[0,\infty)} \Lambda^p d\lambda = \infty, \quad \left(\int_{[0,x]} \Lambda^p d\lambda \right)^{-1} \leq c \int_{[x,\infty)} \left(\int_{[0,t]} \Lambda^p d\lambda \right)^{-2} \Lambda(t)^p d\lambda(t).$$

Remark. Inequalities similar to the one considered above can be proved for non-decreasing functions and also for negative p and q .

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