A study of Schur Multipliers and some Banach Spaces of Infinite Matrices

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Abstract

This PhD thesis consists of an introduction and five papers, which deal with some spaces of infinite matrices and Schur multipliers.

In the introduction we give an overview of the area that serves as a frame for the rest of the Thesis.

In Paper 1 we introduce the space $B_w(\ell^2)$ of linear (unbounded) operators on $\ell^2$ which map decreasing sequences from $\ell^2$ into sequences from $\ell^2$ and we find some classes of operators belonging either to $B_w(\ell^2)$ or to the space of all Schur multipliers on $B_w(\ell^2)$.

In Paper 2 we further continue the study of the space $B_w(\ell^p)$ in the range $1 < p < \infty$. In particular, we characterize the upper triangular positive matrices from $B_w(\ell^p)$.

In Paper 3 we prove a new characterization of the Bergman-Schatten spaces $L^p_a(D, \ell^2)$, the space of all upper triangular matrices such that $\|A(\cdot)\|_{L^p(D, \ell^2)} < \infty$, where

$$\|A(r)\|_{L^p(D, \ell^2)} = \left(2 \int_0^1 \|A(r)\|^{p}_{L^p} rdr\right)^{1/p}.$$  

This characterization is similar to that for the classical Bergman spaces. We also prove a duality between the little Bloch space and the Bergman-Schatten classes in the case of infinite matrices.

In Paper 4 we prove a duality result between $B_p(\ell^2)$ and $B_q(\ell^2)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, where by $B_p(\ell^2)$ we denote the Besov-Schatten space of all upper triangular matrices $A$ such that

$$\|A\|_{B_p(\ell^2)} = \left[\int_0^1 (1 - r^2)^{2p} \|A''(r)\|^{p}_{L^p} d\lambda(r)\right]^{1/p} < \infty.$$  

In Paper 5 we introduce and discuss a new class of linear operators on quasi-monotone sequences in $\ell^2$. We give some characterizations for such a class, for instance we characterize the diagonal matrices.
Preface

This PhD thesis contains the following papers:

1. Updated and slightly revised version (2010) of the paper


These papers are put to a more general frame in an introduction, which also serves as a basic overview of the field.
Acknowledgements

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Introduction

This PhD Thesis is dedicated to the study of some Banach spaces of infinite matrices. An important role is played by the upper triangular matrices called analytic matrices as well as some special operators acting on them, for instance Schur multipliers. The first research component of the thesis is strongly related to a classical mathematical object having deep implications in the development of mathematics in the last 150 years: infinite matrices.

The Bloch space has been studied for a long time in complex analysis, for the first time in 1920 by A. Bloch (general references include K. Zhu’s book [45]) regarding the boundary behavior of normal functions.

There are a few good sources for results and references about Bloch functions on the open unit disk. We mention here the paper of J. M. Anderson, J. Clunie and Ch. Pommerenke [2] and the survey papers of J. M. Anderson [1] and J. Cima [31].

The theory of Bergman spaces has evolved from several sources. A primary model is the related theory of Hardy spaces. For \(0 < p < \infty\), a function \(f\) analytic in the unit disk \(\mathbb{D}\) is said to belong to the Hardy space \(H^p\) if the integrals \(\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\) remain bounded as \(r \to 1\). It belongs to the Bergman space \(A^p\) if the area integral \(\int_{\mathbb{D}} |f(z)|^p d\sigma\) is finite. It is clear that \(H^p \subset A^p\).

The structural properties of individual functions in \(H^p\) were studied actively in the period 1915-1930, beginning with some classical papers of G. H. Hardy. With the emergence of functional analysis in the 1930’s, \(H^p\) spaces began to be viewed as examples of Banach spaces, for \(1 \leq p \leq \infty\). This point of view suggested a variety of new problems and provided effective methods for the solution of old problems. We mention here only a few mathematicians who early dealt with these spaces: A. Beurling, H. Shapiro, L. Carleson and A. L. Shields. More generally, we mention here Triebel-Lizorkin spaces, which are very important in the theory of function spaces. Details about these spaces can be found in H. Triebel’s books [42], [43] and [44]. Meanwhile, S. Bergman developed an elegant theory of Hilbert spaces of analytic functions in planar domains and in higher-dimensional complex space, relying heavily on a reproducing kernel that became known as the Bergman
kernel function \[13\]. Bergman’s work focused on spaces of analytic functions that are square-integrable over the domain with respect to Lebesgue area or volume measure. When attention was later directed to the spaces \(A^p\) over the unit disk, it was natural to call them Bergman spaces.

In the last twenty years the interest concerning these spaces has increased. The pointwise multipliers of the Bloch space and the little Bloch space are characterized by J. Arazy \[4\] in the case of open disk and by K. Zhu \[45\] in the case of the open unit ball. Coefficient multipliers of Bloch functions are described by J. M. Anderson and A. L. Shields in their paper \[3\]. J. Arazy remarked in \[5\] for the first time a similarity between functions and infinite matrices. This similarity has been the source of many results and conjectures, one of the main results obtained for Schur multipliers is due to G. Bennett in his paper \[12\].

**Theorem 1** (Bennett’s Theorem). The Toeplitz matrix \(M\) is a multiplier if and only if there exists a bounded and complex Borel measure \(\mu\) on (the circle group) \(\mathbb{T}\) with Fourier coefficients
\[
\hat{\mu}(n) = c_n \text{ for } n = 0, \pm 1, \pm 2, \cdots.
\]
Moreover, we then have
\[
\|M\|_{(2,2)} = \|\mu\| = \|M\|_{(\infty,1)}.
\]

An infinite matrix \(M\) is a Toeplitz matrix if it is on the form
\[
m_{j,k} = c_{j-k} \text{ (} j, k = 0, 1, 2, \cdots\).
\]

We call a matrix \(M\) a \((p,q)\)-multiplier, \(1 \leq p, q \leq \infty\), if \(M * A\) maps \(\ell^p\) into \(\ell^q\) whenever \(A\) does. The set \(\mathcal{M}(p,q)\) of all such multipliers becomes a Banach space when it is endowed with the norm
\[
\|M\|_{(p,q)} = \sup\{\|M * A\|_{p,q} : \|A\| \leq 1\}.
\]

Another direction of research was that to study vector valued analytic functions, but considered from a Banach space point of view.

In this way appeared a series of papers e.g. by O. Blasco, J. L. Arregui, J. G. Cuerva, A. Pelczynski, Q. H. Xu dedicated to Hardy, Bergman, Bloch and BMO vector valued spaces (see e.g. \[6\], \[7\], \[14\]-\[25\], \[26\], \[27\] and \[28\]). These papers continue the previous work of D. L. Burkholder \[29\] and J. Garcia Cuerva and J. L. Rubio de Francia \[32\] concerning singular operators and vector valued Hardy spaces.

We define the matricial analogue of these spaces according to the corresponding definitions for analytic functions and we use the powerful device of Schur multipliers and its characterization in the case of Toeplitz matrices. Different spaces of infinite matrices, analytic matrices and Schur multipliers can be found in \[8\], \[9\], \[10\], \[12\], \[36\], \[37\], \[39\], \[40\] and \[41\].
One alternative to study the proprieties of some Banach spaces of infinite matrices was to understand better the properties of sequence spaces and to find another characterizations for them. Of course Hardy type inequalities plays an important role here. There is a huge literature in this field but we mention here only the books [11], [30], [34] and [35] and the references given there.

In this PhD thesis we complement the theory and prove some new results in this fascinating field in five papers:

**Paper 1**

In the first paper of this PhD Thesis we introduce the space of infinite matrices $B_w(\ell^2)$. This Banach space of infinite matrices can be regarded as a weaker version of the space $B(\ell^2)$, the space of all bounded operators from $\ell^2$ into $\ell^2$. It is a weaker version because it consists of those operators which maps the sequences which are decreasing in modulus from $\ell^2$ into $\ell^2$. This space actually appeared in the study of matriceal analogue of classical function spaces like $C(T)$ (the continuous functions on the torus), the Wiener algebra $A(T)$ and the Lebesgue space $L^1(T)$.

It is easy to see that $B(\ell^2) \subset B_w(\ell^2)$ and that the inclusion is proper. It is interesting that these two spaces coincides on the subspace consisting of Toeplitz matrices. More precisely, we prove that the Toeplitz matrix $A$ belongs to $B(\ell^2)$ if and only if $A$ belongs to $B_w(\ell^2)$. Using this result we prove a theorem which is similar to G. Bennett’s Theorem 1 above.

One main theorem of this paper is the following, which is in fact a new characterization of the positive Toeplitz upper triangular infinite matrices from $B(\ell^2)$:

**Theorem 2.** Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ 0 & 0 & 0 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ be an upper triangular Toeplitz matrix. Then $A \in B(\ell^2)$ if and only if the sublinear operator $T_A$ is bounded from $\ell^2$ into $\ell^2$ where

$$T_A(b)(j) = \frac{1}{j} \sum_{m=0}^{j} |(a \ast b)(m)|, \quad T_A(b)(0) = |a_0 b_0|$$

and $(a \ast b)(m) = \sum_{k+l=m} a_k b_l$, $a = (a_k)_{k \geq 0}$, $b = (b_k)_{k \geq 0} \in \ell^2$. 

**Paper 2**

In the second paper we deal with the Banach space of infinite matrices $B_w(\ell^p)$, $1 < p < \infty$. In the study of this space an important role is played by the following old result that we obtain with a completely new proof: Let

$$d(q)^\times = ces(p)$$

where $d(p) = \{x = \{x_k\}_{k=1}^\infty : (\sum_{k=1}^\infty \sup_{n\geq k} |x_n|^p)^\frac{1}{p} < \infty\}$ and $ces(p) = \{x = \{x_k\}_{k=1}^\infty : \sum_{n=1}^\infty (\frac{1}{n} \sum_{k=1}^n |x_k|^p)^{1/p} < \infty\}$.

Here $d(q)^\times$ is the associate space of $d(q)$, that is

$$d(q)^\times = \{a = (a_n)_n ; \text{ such that } \sum_{n=1}^\infty |a_n x_n| < \infty \text{ for all } (x_n)_n \in d(q)\}.$$

This result which gives us the Köthe dual of $d(p)$ was obtained also by G. Bennett in [11] by using more technical methods, like factorization of some classical inequalities. This problem was first investigated by A. A. Jagers in 1974 in the paper [33].

In particular, in one important theorem from this paper we characterize some upper triangular operators from $\ell^p$ into another sequence spaces like $d(q)$ and $g(p) = \{x = \{x_k\}_{k=1}^\infty : \sup_{n\geq 1} (\frac{1}{n} \sum_{k=1}^n |x_k|^p)^{1/p} < \infty\}$ in terms of some special multipliers:

**Theorem 3.** Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $B$ be an upper triangular matrix. Then

1. $B \in B(\ell^p, d(p))$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in ces(q)$.
2. $B \in B(\ell^p, \ell^p)$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q$.
3. $B \in B(\ell^p, g(p))$ if and only if $B * [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q \cdot d(p)$.

**Paper 3**

In the third paper we deal with analytic matrices. We continue the study of Bergman-Schatten spaces and we give a characterization in terms of Taylor coefficients namely:

**Theorem 4.** Let $A$ be an analytic matrix. Then $A \in L^p_0(D, \ell^2)$ if and only if

$$\sum_{n=0}^\infty \frac{1}{(n+1)^2} \|\sigma_n(A)\|_p^p < \infty.$$
This characterization is similar with that of M. Mateljevic and M. Pavlovic in [38] for the classical Bergman spaces. Another important result we prove in this paper is a duality between the Bergman-Schatten space and the little Bloch space. In the proofs we use very often the powerful tools of Schur multipliers and some techniques from the theory analytic functions theory as well.

**Paper 4**

In Paper 4 we continue the study of infinite matrix valued functions, namely we consider the Besov-Schatten spaces in the framework of matrices. We define the Besov-Schatten matrix space \( B_p(\ell^2) \) to be the space of all upper triangular infinite matrices \( A \) such that
\[
\| A \|_{B_p(\ell^2)} := \left[ \int_0^1 (1 - r^2)^{2p} \| A''(r) \|_{C_p}^p \, d\lambda(r) \right]^{\frac{1}{p}} < \infty,
\]
where \( d\lambda \) is a positive measure on \([0, 1)\) given by
\[
d\lambda(r) := \frac{2r \, dr}{(1 - r^2)^2}.
\]
We develop this theory by using in particular the powerfull device of Schur multipliers and its characterizations in the case of Toeplitz matrices and prove some new results. One of the main results in this paper is the following duality result:

**Theorem 5.** Under the pairing
\[
< A, B > = \int_0^1 \text{tr}(V(A)[V(B)]^*) \, d\lambda(r)
\]
we have the following dualities:

1. \( B_p(\ell^2)^* \approx B_q(\ell^2) \) if \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \);
2. \( B_{0,c}(D, \ell^2)^* \approx B_1(\ell^2) \) and \( B_1(\ell^2)^* \approx B(D, \ell^2) \).

**Paper 5**

In Paper 5 we consider a class of linear operators on quasi-monotone sequences in \( \ell^2 \) and its Schur multipliers. We define
\[
B_\alpha^w(\ell^2) = \{ A \text{ infinite matrix} : Ax \in \ell^2 \text{ for every} \}
\]
\[
x = (x_n)_n \in \ell^2 \text{ with } \frac{|x_n|}{n^\alpha} \downarrow 0, \alpha \geq 0 \}.
\]
In particular, for \( \alpha = 0 \) we have that \( B_0^w(\ell^2) = B_w(\ell^2) \) which was studied in [36] and [37]. First we recall some definitions concerning infinite matrices and Schur multipliers mainly from [12] and [10] and then we give some preliminaries regarding Sawyer’s duality Theorem. We also include some additional definitions that we need in the sequel. We give a characterization for diagonal matrices to belong in \( B_0^w(\ell^2) \), \( \alpha \geq 0 \). Moreover, we consider the Schur product of matrices and remark that for \( \alpha \geq 0 \), \( B_0^w(\ell^2) \) is not closed under this product. One of the main results is that linear and bounded operators on \( \ell^2 \) are Schur multipliers on \( B_0^w(\ell^2) \) with \( \alpha > 0 \), a result which is not obvious since \( B_0^w(\ell^2) \) is not a Schur algebra. For \( \alpha = 0 \) it is known that this result holds (see e.g. [36]). In what follows we present two of the main Theorems in this paper.

**Theorem 6.** Let \( \alpha \geq 0 \), \( A = A_0 \) be given by the sequence \( a = (a_n)_n \). Then \( A \in B_0^w(\ell^2) \) if and only if

\[
\sup_{n \geq 1} \left( \frac{\sum_{k=1}^{n} |a_k|^2 k^{2\alpha}}{\sum_{k=1}^{n} k^{2\alpha}} \right)^{\frac{1}{2}} < \infty.
\]

Moreover, the norm

\[
\|A\|_{B_0^w(\ell^2)} = \sup_{n \geq 1} \left( \frac{\sum_{k=1}^{n} |a_k|^2 k^{2\alpha}}{\sum_{k=1}^{n} k^{2\alpha}} \right)^{\frac{1}{2}}.
\]

Another main result is that linear and bounded operators on \( \ell^2 \) are Schur multipliers on \( B_0^w(\ell^2) \) when \( \alpha > 0 \).

**Theorem 7.** The space \( B(\ell^2) \) is included in the space of all Schur multipliers from \( B_0^w(\ell^2) \) to \( B_0^w(\ell^2) \) where \( \alpha \geq 0 \).

*Short description of the main contributions by L. G. Marcoci:* This introduction and Paper 5 are written by me. In all other papers I have substantially contributed to all parts, both concerning ideas and proofs. I mean that I am the main author of Papers 2 and 3. Moreover, for example, both the proof and idea of Theorem 10 in Paper 1 (see Theorem 2) and Theorem 3.2 in Paper 4 (see Theorem 5) are mainly due to me.
Bibliography

8 BIBLIOGRAPHY


A NEW CLASS OF LINEAR OPERATORS ON $\ell^2$ AND ITS SCHUR MULTIPLIERS

A.-N. MARCOCI AND L.-G. MARCOCI

Abstract. We introduce the space $B_w(\ell^2)$ of linear (unbounded) operators on $\ell^2$ which map decreasing sequences from $\ell^2$ into sequences from $\ell^2$ and we find some classes of operators belonging either to $B_w(\ell^2)$ or to the space of all Schur multipliers on $B_w(\ell^2)$. For instance we show that the space $B(\ell^2)$ of all bounded operators on $\ell^2$ is contained in the space of all Schur multipliers on $B_w(\ell^2)$.

1. Introduction

Let $A = (a_{ij})_{i,j \geq 1}$ be an infinite matrix. We define $B_w(\ell^2) =$

$$= \{ A \text{ infinite matrix : } Ax \in \ell^2 \text{ for every } x \in \ell^2 \text{ with } |x_k| \searrow 0 \},$$

where

$$\ell^2 = \{ x = (x_k)_{k \geq 1} : \sum_{k=1}^{\infty} |x_k|^2 < \infty \}$$

is the classical space of sequences.

The above space of matrices has appeared in the study of the matricial analogue of some well-known Banach spaces as $C(\mathbb{T})$, $M(\mathbb{T})$, $L^1(\mathbb{T})$.

A similarity between the functions defined on the unit circle $\mathbb{T}$ and the infinite matrices was remarked for the first time in 1978 by J. Arazy.

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Key words and phrases. Schur multiplier, Toeplitz matrices, matricial Wiener algebra.
in [1]. Later on, in 1983, A. Shields has exploited further this similarity starting with a few constructs used in harmonic analysis together with their matriceal analogues [12].

Recently, in [4], the Fejer’s theory developed for Fourier series was extended in the framework of matrices.

The analogy is as follows: we identify a function \( f \) with the Toeplitz matrix \( A = (a_{ij})_{i,j \geq 1} \):

\[
a_{ij} = a_{i-j} \quad \text{for all } i, j \in \mathbb{N}^*
\]

where \((a_k)_{k \in \mathbb{Z}}\) is the sequence of Fourier coefficients of the function \( f \).

The Schur product of two matrices is defined by

\[
A \ast B = (a_{ij} \cdot b_{ij})_{i,j \geq 1},
\]

where \(A = (a_{ij})_{i,j \geq 1}, B = (b_{ij})_{i,j \geq 1}\). We denote by

\[
M(\ell^2) = \{ M : M \ast A \in B(\ell^2) \text{ for every } A \in B(\ell^2) \}
\]

the space of all Schur multipliers equipped with the following norm

\[
\|M\| = \sup_{\|A\|_{B(\ell^2)} \leq 1} \|M \ast A\|_{B(\ell^2)}.
\]

Let us define \(A_k = (a'_{ij})_{i,j \geq 1}, k \in \mathbb{Z}\) to be the matrix with the elements

\[
a'_{ij} = \begin{cases} 
a_{ij} & \text{if } j - i = k, \\
0 & \text{otherwise.}
\end{cases}
\]

\(A_k\) is called the Fourier coefficient of \( k \)-order of the matrix \( A \) (see e.g. [4]). We have now a similarity between the expansion in Fourier series of a periodical function \( f \) on \( \mathbb{T} \)

\[
f = \sum_{k} a_k e^{ikt}
\]

and the decomposition of matrix in diagonal matrices

\[
A = \sum_{k \in \mathbb{Z}} A_k.
\]
In this way we can say that \( B(\ell^2) \) represents the matriceal analogue of \( L^\infty(\mathbb{T}) \) and \( M(\ell^2) \) is the analogue of \( M(\mathbb{T}) \). Moreover in [4] was introduced the space of continuous matrices denoted by \( C(\ell^2) \). A is a continuous matrix if \( \lim_{n \to \infty} \sigma_n(A) = A \), where the limit is taken in the norm of \( B(\ell^2) \) and \( \sigma_n(A) \) is the Cesaro sum associated to \( S_n(A) := \sum_{k=-n}^{n} A_k \).

This space is a Banach space with the following norm

\[
\|A\|_{C(\ell^2)} = \max(\sup_n \|\sigma_n(A)\|_{B(\ell^2)}, \|A\|_{B(\ell^2)}).
\]

In the way described earlier this space represents the matriceal analogue of the space \( C(\mathbb{T}) \).

Also N. Popa has defined

\[
A(\ell^2) = \left\{ A \text{ infinite matrix} : \sup_{l \in \mathbb{Z}^+} \sum_{k \in \mathbb{Z}} |a^l_k| < \infty \right\}
\]

where

\[
a^l_k = \begin{cases} 
a_{l+k} & \text{if } k \geq 0 \text{ and } l \geq 1 \\
a_{l-k} & \text{if } k < 0 \text{ and } l \geq 1 \end{cases}
\]

and

\[
A = \begin{pmatrix} a^1_0 & a^1_1 & \cdots \\ a^{-1}_1 & a^1_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{see e.g. [3]}).
\]

This was the first attempt to define the matriceal analogue of the Wiener algebra \( A(\mathbb{T}) \) and we may call it matriceal Wiener algebra.

It is easy to see that the subspace of all Toeplitz matrices from \( A(\ell^2) \) can be identified with \( A(\mathbb{T}) \) but since \( A(\ell^2) \not\subseteq C(\ell^2) \) it is necessary to find a larger space than \( B(\ell^2) \). In order to see that \( A(\ell^2) \not\subseteq C(\ell^2) \) let
us consider
\[ A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}. \]

Then \( \sup \sum_{l \in \mathbb{Z}, k \in \mathbb{Z}} |a^l_k| = 1 \) and \( A \in A(\ell^2) \) implies \( \|Ae_{n(n+1)}\|_2 = \sqrt{n} \), where \( e_k = (0, \ldots, 0, 1, 0, \ldots) \).

One solution would be to choose a larger space than \( B(\ell^2) \), and, for this reason we introduce \( B_w(\ell^2) \). Clearly \( B_w(\ell^2) \) is a Banach space with the norm
\[ \|A\|_{B_w(\ell^2)} = \sup_{\|x\|_2 \leq 1, |x_k| \downarrow 0} \|Ax\|_2. \]

The following result due to E. Sawyer [11] will be used in the sequel as it appeared in [8]. For alternative simple proof see also [13].

If \( v = (v(n))_n \) is a weight on \( \mathbb{N}^* = \mathbb{Z}^*_+ \), we put \( \tilde{v} = \sum_{n=0}^{\infty} v(n)\chi_{[n,n+1)} \)
and \( \tilde{V}(t) = \int_0^t \tilde{v}(s)ds \).

Let us mention moreover that the relation \( f \approx g \) means that there are two positive constants \( a \) and \( b \) such that \( af \leq g \leq bf \).

Then we have:

**Theorem 1.** Let \( w = (w(n))_n, v = (v(n))_n \) be weights on \( \mathbb{N}^* \) and let
\[ S = \sup_{f \in \mathcal{F}} \left( \sum_{n=0}^{\infty} f(n)^p w(n) \right)^{\frac{1}{p}}. \]

Then
(i) If $0 < p \leq 1$

$$S = \sup_{n \geq 0} \frac{V(n)}{W^{1/p}(n)},$$

with $W$ defined by $W(n) = \sum_{k=0}^{n} w(k)$ and similarly for $V$.

(ii) If $1 < p < \infty$,

$$S \approx \left( \int_{0}^{\infty} \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{p'} \tilde{v}(t) dt \right)^{\frac{1}{p'}}$$

$$\approx \left( \int_{0}^{\infty} \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{p'} \tilde{w}(t) dt \right)^{\frac{1}{p'}} + \frac{\tilde{V}(\infty)}{\tilde{W}^{1/p}(\infty)}$$

where $\frac{1}{p'} + \frac{1}{p'} = 1$.

The paper is organized as follows: in Section 2 we show that the matricial Wiener algebra $A(\ell^2)$ is a subset of $B_w(\ell^2)$ (Proposition 2) and we give some criteria for diagonal matrices to belong to $B_w(\ell^2)$. Moreover we consider the Schur product of matrices and remark that $B_w(\ell^2)$ is not closed under this product. We prove in Section 3 the main result of the paper, namely that linear and bounded operators on $\ell^2$ are Schur multipliers on $B_w(\ell^2)$, a result which is not obvious, since $B_w(\ell^2)$ is not Schur algebra. Finally we collected in Section 4 all results concerning the Toeplitz matrices. For instance there is no difference between Toeplitz matrices from $B(\ell^2)$ and those from $B_w(\ell^2)$.

2. Preliminary results

**Proposition 2.** If $A$ is an upper triangular matrix then $\|A\|_{B_w(\ell^2)} \leq \|A\|_{A(\ell^2)}$. 

Proof. Let \( A \in A(\ell^2) \) be an upper triangular matrix. Then for every \((x_l)_{l \geq 0} \in \ell^2 \) with \(|x_l| \searrow 0 \) and \( \|x\|_2 \leq 1 \) we have

\[
\sum_{k=1}^{\infty} \left| \sum_{l=k-1}^{\infty} a_{l-k+1}^l x_l \right|^2 \leq \sum_{k=1}^{\infty} \left( \sum_{l=k-1}^{\infty} |a_{l-k+1}^l| |x_l| \right)^2
\]

\[
\leq \sum_{k=1}^{\infty} \left( \sum_{l=k-1}^{\infty} |a_{l-k+1}^l| \right)^2 \sup_{l \geq k-1} |x_l|^2
\]

\[
\leq \sum_{k=1}^{\infty} \left( \sup_k \sum_{l=k-1}^{\infty} |a_{l-k+1}^l| \right)^2 \sup_{l \geq k-1} |x_l|^2
\]

\[
\leq \|A\|_{A(\ell^2)}^2 \sum_{k=1}^{\infty} \sup_{l \geq k-1} |x_l|^2
\]

\[
= \|A\|_{A(\ell^2)}^2 \|x\|_2^2 \leq \|A\|_{A(\ell^2)}^2
\]

which implies that \( \|A\|_{B_w(\ell^2)} \leq \|A\|_{A(\ell^2)} \). \( \square \)

**Proposition 3.** Let \( A = A_0 \) given by \((a_k)_{k=1}^{\infty}\). Then \( A \in B_w(\ell^2) \) if and only if

\[
\sup_{n \in \mathbb{N}^*} \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}} < \infty.
\]

Moreover

\[
\|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}}.
\]

**Proof.** The sufficiency follows immediately from the factorization

\[
\ell^2 = d(1,2) \cdot g(1,2)
\]
where
\[ d(1,2) = \left\{ x : \sum_{n=1}^{\infty} \sup_{k \geq n} |x_k|^2 < \infty \right\} \]
and
\[ g(1,2) = \left\{ x : \sum_{k=1}^{n} |x_k|^2 = O(n) \right\} \] (see e.g. [5], p. 9).

For the necessity take \( x^{(n)} = \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}, 0, \ldots \right) \), \( \|x^{(n)}\|_2 = 1 \), \( |x_{k}^{(n)}| \searrow 0, n \geq 1 \)
\[ \|Ax^{(n)}\|_2 = \left( \frac{1}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \leq \|A\|_{B_w(\ell^2)} \]
and \( \sup_{n} \left( \frac{\sum_{k=1}^{n} |a_k|^2}{n} \right)^{\frac{1}{2}} < \infty. \]

If we translate this sequence \( (a_k)_{k=1}^{\infty} \) above or below of the main diagonal we obtain the following similar result.

**Corollary 4.**

a) Let \( k > 0 \) and \( A = A_k \) given by \( (a_l)_{l=1}^{\infty} \). Then \( A \in B_w(\ell^2) \) if and only if \( \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n+k} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}} < \infty \) and
\[ \|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n+k} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}}. \]

b) Let \( k < 0 \) and \( A = A_k \) given by \( (a_l)_{l=1}^{\infty} \). Then \( A \in B_w(\ell^2) \) if and only if
\[ \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}} < \infty \]
and \( \|A\|_{B_w(\ell^2)} = \sup_{n \in \mathbb{N}^*} \left( \frac{1}{n} \sum_{l=1}^{n} |a_l|^2 \right)^{\frac{1}{2}}. \)
Remark 5. 1. Clearly $B(\ell^2) \subseteq B_w(\ell^2)$ and using Proposition 3 for the matrix $A = A_0$ given by $(a_k)_{k=1}^\infty$,

$$a_k = \begin{cases} \sqrt{2^n} & \text{if } k = 2^n \\ 0 & \text{if } k \neq 2^n \end{cases} \quad n \geq 1$$

we can easily show that the inclusion is proper.

2. While $B(\ell^2)$ is closed under Schur multiplication $B_w(\ell^2)$ is not. For example it is easy to see that $A * A \notin B_w(\ell^2)$ where $A$ is the matrix defined previously.

3. The space $B_w(\ell^2)$ cannot be compared with $M(\ell^2)$ meaning that

$$(1) \quad B_w(\ell^2) \notin M(\ell^2) \quad \text{and} \quad (2) \quad M(\ell^2) \notin B_w(\ell^2).$$

For (1) there is $A = A_0$ given by $(a_k)_{k=1}^\infty$ such that $(a_k) \notin \ell^\infty$, $A \in B_w(\ell^2)$ and $A \notin M(\ell^2)$.

For (2) we take $A = \begin{pmatrix} 1 & 1 & \ldots & 1 & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \in M(\ell^2)$ but $A \notin B_w(\ell^2)$.

4. Let $A = A_0$ given by $(a_k)_{k=1}^\infty$. Then $A \in M(B_w(\ell^2), B_w(\ell^2)) = \{ M : M * B \in B_w(\ell^2) \text{ for every } B \in B_w(\ell^2) \}$ if and only if $(a_k) \in \ell^\infty$.

$A \in M(B_w(\ell^2), B_w(\ell^2))$ then $A * B \in B_w(\ell^2)$ for $B = B_0 \in B_w(\ell^2)$ given by $(b_k)_{k=1}^\infty$.

From Proposition 3,

$$(b_k)_{k=1}^\infty \in g(1,2), \|Ax\|_{\ell^2}^2 = \sum_{k=1}^\infty |a_kb_kx_k|^2 < \infty$$

for every $b = (b_k)_k \in g(1,2)$, and $x \in d(1,2)$.
Using the factorization $\ell^2 = d(1, 2) \cdot g(1, 2)$, provided in [6] we get that $(a_n)_n \in \ell^\infty$.

Now let $(a_n)_n \in \ell^\infty$. It is easy to see that

$$\|B_0\|_{B_w(\ell^2)} \leq \|B\|_{B_w(\ell^2)}.$$

3. Main results

Lemma 6.

\[
\sup_{|x_n| \searrow 0} \left| \sum_{n=1}^{\infty} a_n x_n \right| \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \approx \sup_{|x_n| \searrow 0} \left| \sum_{n=1}^{\infty} a_n x_n \right| \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} \approx \left( \sum_{n=1}^{\infty} |a_n| |x_n| \right) \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2},
\]

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers.

Proof. We denote $S = \sup_{|x_n| \searrow 0} \left| \sum_{n=1}^{\infty} a_n x_n \right| \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}$. From Theorem 1 we have

\[
S \approx \left( \int_{0}^{\infty} \left( \frac{V(t)}{W(t)} \right)^2 \hat{w}(t) dt \right)^{1/2} + \frac{\hat{V}(\infty)}{W^2(\infty)},
\]

where $v(n) = |a_n|$, $w(n) = 1$, $f(n) = |x_n|$ for every $n$ nonnegative integer.

In this case

\[
\tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1]} = \sum_{n=0}^{\infty} |a_n| \chi_{[n,n+1]}(s), \text{ where } a_0 = 0,
\]
therefore, for $t \in (j, j + 1)$, we have

\[
\tilde{V}(t) = \int_0^t \tilde{v}(s) \, ds = \int_0^j \tilde{v}(s) \, ds + \int_j^t \tilde{v}(s) \, ds
\]

\[
= \sum_{m=0}^{j-1} \int_m^{m+1} \tilde{v}(s) \, ds + \int_j^t \tilde{v}(s) \, ds
\]

\[
= \sum_{m=0}^{j-1} |a_m| + |a_j| (t - j),
\]

\[
\tilde{V}(\infty) = \int_0^\infty \tilde{v}(s) \, ds = \sum_{n=0}^\infty |a_n| \chi_{[n,n+1)}(s) \, ds = \sum_{n=1}^\infty |a_n| \text{ and}
\]

\[
\tilde{W}(\infty) = \int_0^\infty \tilde{w}(s) \, ds = \infty,
\]

since $\tilde{w}(s) = \sum_{n=0}^\infty \chi_{[n,n+1)}(s)$. Clearly, letting $\tilde{v}_M = \sum_{n=0}^M |a_n| \chi_{[n,n+1)}$ we have that

\[
\tilde{V}_M = \int_0^\infty \tilde{v}_M(s) \, ds
\]

\[
= \int_0^\infty \sum_{n=0}^M |a_n| \chi_{[n,n+1)}(s) \, ds
\]

\[
= \sum_{n=1}^M |a_n| < \infty,
\]
we get
\[ S = \sup_M \left( \sup_{|x_n|<\infty} \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} \right)^{\left( \sum_{n=1}^{\infty} |x_n|^2 \right)}^{\frac{1}{2}} \]
\[ \approx \sup_M \left[ \left( \int_0^\infty \left( \frac{\tilde{V}_M(t)}{t} \right)^2 dt \right)^{\frac{1}{2}} + \frac{\tilde{V}_M(\infty)}{W^{\frac{1}{2}}(\infty)} \right] \]
\[ \approx \left( \int_0^\infty \left( \frac{\tilde{V}(t)}{t} \right)^2 dt \right)^{\frac{1}{2}}. \]

But
\[ \int_0^\infty \left( \frac{\tilde{V}(t)}{t} \right)^2 dt = \sum_{j=1}^{j+1} \int_j^{j+1} \left( \sum_{m=0}^{j-1} |a_m| + |a_j| (t-j) \right)^2 dt \]
\[ \leq \sum_{j=1}^{\infty} \left( \int_j^{j+1} \frac{1}{t^2} dt \right) \left( \sum_{m=1}^{j} |a_m| \right)^2 \leq \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=1}^{j} |a_m| \right)^2. \]

On the other hand
\[ \int_0^\infty \left( \frac{\tilde{V}(t)}{t} \right)^2 dt \geq \sum_{j=1}^{j+1} \int_j^{j+1} \left( \sum_{m=1}^{j} \frac{|a_m|}{t} \right)^2 dt + \sum_{j=1}^{\infty} \int_j^{j+1} |a_j|^2 \frac{(t-j)^2}{t^2} dt \geq \]
\[ \geq \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=1}^{j} |a_m| \right)^2, \]
which implies that $S \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^2 \right)^{\frac{1}{2}}$.

**Theorem 7.** Let $M(B_w(\ell^2), B_w(\ell^2))$ the space of Schur multipliers from $B_w(\ell^2)$ to $B_w(\ell^2)$. Then

$$B(\ell^2) \subset M(B_w(\ell^2), B_w(\ell^2)).$$

**Proof.** Let us take $A \in B(\ell^2)$, $B \in B_w(\ell^2)$ and $x = (x_k)_{k \geq 1} \in \ell^2$ with $|x_k| \searrow 0$. Using Hölder inequality twice we have that

$$\sum_j \left| \sum_k a_{jk} b_{jk} x_k \right|^2 \leq \sum_j \left( \sum_k |a_{jk}| |b_{jk}| |x_k| \right)^2 \leq \sum_j \left( \sum_k |a_{jk}|^2 \right) \left( \sum_k |b_{jk}|^2 |x_k|^2 \right) \leq \sup_j \left( \sum_k |a_{jk}|^2 \right) \sum_j \left( \sum_k |b_{jk}|^2 |x_k|^2 \right).$$

Thus

$$(1) \quad \| (A \ast B)x \|_2 \leq \| A \|_{2,\infty} \left( \sum_j \left( \sum_k |b_{jk} x_k|^2 \right) \right)^{\frac{1}{2}} \leq \| A \|_{B(\ell^2)} \left( \sum_j \left( \sum_k |b_{jk} x_k|^2 \right) \right)^{\frac{1}{2}}$$
To estimate the second term in (1) we will use Rademacher functions on $[0, 1]$ denoted with $r_k$, $k \geq 1$ (see e.g. [9], p. 126).

$$\sum_j \left( \sum_k |b_{jk}x_k|^2 \right)$$

$$= \sum_j \int_0^1 \left( \sum_k b_{jk}x_k r_k(t) \right)^2 dt$$

$$\leq \operatorname{ess \ sup}_{t \in [0, 1]} \sum_j \left( \sum_k b_{jk}x_k r_k(t) \right)^2$$

$$\leq \|B\|_{B_w(\ell^2)}^2 \|x\|_{\ell^2}^2.$$  

The proof is complete.

We give now a necessary condition in order that an upper triangular Toeplitz matrix belong to $B(\ell^2)$:

**Proposition 8.** Let $A = 
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
0 & a_0 & a_1 & a_2 & a_3 & \ldots \\
0 & 0 & a_0 & a_1 & a_2 & \ldots \\
0 & 0 & 0 & a_0 & a_1 & \ldots \\
0 & 0 & 0 & 0 & a_0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$.

If $A \in B(\ell^2)$ then $\tilde{A} \in B_w(\ell^2)$, where $\tilde{A}$ is the diagonal matrix $\tilde{A} = \tilde{A}_0$ given by the sequence $(\tilde{a}_m)_{m=0}^{\infty}$ and $\tilde{a}_m = \sum_{j=0}^{m} a_j$.

**Proof.** Let $A \in B(\ell^2)$. Then

$$\sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_jx_{j+k} \right)^2 < \infty$$

for every $(x_j)_j \in \ell^2$.

But $\|e_1 + e_2 + \ldots + e_N\|_2^2 = N$, where $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$.  


Thus
\[ A \left( \frac{1}{\sqrt{N}} (e_1 + e_2 + ... + e_N) \right) = \frac{1}{\sqrt{N}} (\tilde{a}_N, \tilde{a}_{N-1}, ..., \tilde{a}_1, \tilde{a}_0, 0, ...) \]

which implies that \( \sup_N \frac{\sum_{m=0}^N |\tilde{a}_m|^2}{N} < \infty \) and, by Proposition 3 it follows that \( \tilde{A} \in B_w(\ell^2) \). 

We remark here that if we translate this result to the case of functions we shall get a necessary condition for a function to belong to \( H^\infty \) i.e. if \( f \in H^\infty \) then \( \sup_N \frac{\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} |\hat{f}(m)|^2}{N} < \infty \), where \( \hat{f}(k) \) is the Fourier coefficient of \( k \)-order.

**Theorem 9.** \( B_w (\ell^2) \cap \mathcal{T} = B (\ell^2) \cap \mathcal{T} \), where \( \mathcal{T} \) is the set of all Toeplitz matrices.

**Proof.** Let \( A \) be a Toeplitz matrix. Clearly, if \( A \in B (\ell^2) \) it follows that \( A \in B_w (\ell^2) \). It is well known that a Toeplitz matrix \( A = (a_{ij}) \), where \( a_{ij} = a_{i-j} \) for all \( i, j \in \mathbb{N}^* \), maps \( \ell^2 \) into \( \ell^2 \) precisely when there exists a measurable function essentially bounded on \([0, 2\pi] \) with Fourier coefficients \( \hat{f}(n) = a_n (n = 0, \pm 1, \pm 2, ...) \) and \( \|A\|_{B(\ell^2)} = \|f\|_\infty \) see e.g. [14].

Let \( f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikt} \), \( (x_n)_{n=0}^{\infty} \in \ell^2 \) with \( |x_n| \searrow 0 \) and \( h(t) = \sum_{k=0}^{\infty} x_k e^{2\pi ikt} \).

Then
\[
\|Ax\|_2 = \left( \sum_{k=0}^{\infty} \left| \sum_{j=-k}^{\infty} a_j x_{k+j} \right|^2 \right)^{\frac{1}{2}} = \left( \sum_{k=0}^{\infty} \left| \int_0^1 f(t) e^{2\pi ikt} h(-t) \ dt \right|^2 \right)^{\frac{1}{2}}
\]
where \( g(t) = \sum_{k=0}^{\infty} g_k e^{2\pi ikt} \).

Hence

\[
\|A\|_{B_w(\ell^2)} = \sup\{ \int_0^1 f(t)g(t)h(-t)dt : \|g\|_2 \leq 1, \|h\|_2 \leq 1, h \in L^2([0,1]) \}.
\]

If we take \( g(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{2\pi ij(t-t_0)} \) and \( h(-t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ik(t_0-t)} \) for \( n = 2p + 1, p \in \mathbb{N}^* \) and \( K_n \)-the Fejer kernel we obtain

\[
\|A\|_{B_w(\ell^2)} \geq \left| \int_0^1 f(t) \frac{1}{2p+1} \left| \sum_{j=-p}^{p} e^{2\pi ij(t-t_0)} \right|^2 dt \right| \geq \left| \int_0^1 f(t) K_{2p+1}(t-t_0) dt \right| \overset{a.e.}{\rightarrow} |f(t_0)|
\]

which implies that \( \|A\|_{B_w(\ell^2)} \geq \|f\|_{\infty} = \|A\|_{B(\ell^2)} \).  

**Theorem 10.** Let \( A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \) be a positive upper triangular Toeplitz matrix. Then \( A \in B(\ell^2) \) if and only if the sublinear operator \( T_A \) is bounded from \( \ell^2 \) into \( \ell^2 \) where

\[
T_A (b) (j) = \frac{1}{j} \sum_{m=0}^{j} |(a * b) (m)|, \quad T_A (b) (0) = |a_0 b_0|
\]

and \((a * b)(m) = \sum_{k+l=m} a_k b_l, a = (a_k)_{k \geq 0}, b = (b_k)_{k \geq 0} \in \ell^2\).
Proof. From Theorem 9, \( A \in B(\ell^2) \) if and only if \( \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right)_{k \geq 0} \in \ell^2 \) for any \( x \in \ell^2 \) with \( |x_j| \downarrow 0 \). This is equivalent with

\[
\sup_{\|b\|_{\ell^2} \leq 1} \left| \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k \right| < \infty \quad \text{for all} \quad x \in \ell^2, \ |x_j| \downarrow 0.
\]

\[
\sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k = \sum_{l=0}^{\infty} x_l c_l, \quad \text{where} \quad c_l = \sum_{k=0}^{l} a_{l-k} b_k. \quad \text{From lemma 6}
\]

\[
\sup_{|x_l| \downarrow 0} \frac{\left( \sum_{l=0}^{\infty} |x_l| |c_l| \right)}{\left( \sum_{l=0}^{\infty} |x_l|^2 \right)^{\frac{1}{2}}} = \sup_{|x_l| \downarrow 0} \frac{\sum_{l=0}^{\infty} |x_l| |c_l|}{\left( \sum_{l=0}^{\infty} |x_l|^2 \right)^{\frac{1}{2}}}
\]

\[
\approx |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} \left( \sum_{k=0}^{m} a_{m-k} b_k \right) \right)^2 \approx |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} |(a * b)(m)| \right)^2.
\]

Thus the proof is complete. \( \square \)

Now we show that \( M \left( B_w(\ell^2), B_w(\ell^2) \right) \cap \mathcal{T} \subseteq M(\ell^2) \cap \mathcal{T} \).

**Theorem 11.** If \( M \) is a Toeplitz matrix with \( M = (m_{j,k}) \), \( m_{j,k} = c_{j-k} \) where \( (j, k = 0, 1, 2, 3, \ldots) \) and \( M \in M \left( B_w(\ell^2), B_w(\ell^2) \right) \) there exists \( \mu \) a bounded, complex, Borel measure on \( \mathbb{T} \) with \( \hat{\mu}(n) = c_n \) for \( n = 0, \pm 1, \pm 2, \ldots. \) Moreover

\[
\|\mu\| \leq \|M\|_{M(B_w(\ell^2), B_w(\ell^2))}.
\]

**Proof.** We follow the standard method \([2]\) for solving the ”problem of moments”. Let \( p(t) = \sum_{n=-\infty}^{\infty} p_n e^{int} \) be an arbitrary trigonometric
polynomial and denote by $P$ the Toeplitz matrix generated by $p$. Then
\[ \left| \sum c_n p_n \right| = \lim_{N \to \infty} \left| \sum c_n p_n \left( 1 - \frac{|n|}{N} \right) \right| = \lim_{N \to \infty} \left| \langle (M \ast P)x^{(N)}, x^{(N)} \rangle \right| \]

where
\[ x^{(N)}(j) = \begin{cases} \frac{1}{\sqrt{N+1}} & \text{for } 0 \leq j \leq N \\ 0 & \text{for } j > N \end{cases} \]

We consider now the linear functional
\[ P \mapsto \sum c_n p_n \]
on the subspace of $C(\mathbb{T})$ generated by polynomials
\[ |\Lambda(p)| = \left| \sum c_n p_n \right| = \lim_{N \to \infty} \left| \langle (M \ast P)x^{(N)}, x^{(N)} \rangle \right| \leq \| (M \ast P)x^{(N)} \|_{\ell^2} \leq \| (M \ast P) \|_{B_w(\ell^2)} \leq \| M \|_{M(B_w(\ell^2), B_w(\ell^2))} \cdot \| P \|_{B_w(\ell^2)} = ( \text{by Theorem 9} ) = \| M \|_{M(B_w(\ell^2), B_w(\ell^2))} \cdot \| p \|_{\ell^2} \]

which implies that
\[ \| \Lambda \| \leq \| M \|_{M(B_w(\ell^2), B_w(\ell^2))} \cdot \| p \|_{\ell^2} \]

It is clear now that this map is well-defined and continuous and the existence of a measure satisfying all the requirements of the theorem follows easily from the Hahn-Banach and Riesz representation theorems.

Using an extension of the F. and M. Riesz theorem from [10] we have:

**Corollary 12.** If $M \in M(B_w(\ell^2), B_w(\ell^2))$ is a Toeplitz matrix $M = (m_{jk})$, $m_{jk} = c_{j-k}$ ($k, j = 0, 1, 2, \ldots$, and for $n < 0$, $c_n = 0$ with $n \notin E$, where $E$ is a set of type $\Lambda(1)$ (see [10]) then $c_n \to 0$ as $n \to \infty$.

**Proof.** This follows immediately from Riemann-Lebesgue lemma. \(\square\)
References


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Paper 2
SCHUR MULTIPLIERS CHARACTERIZATION OF A CLASS OF INFINITE MATRICES

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ABSTRACT. Let $B_w(\ell^p)$ denote the space of infinite matrices $A$ for which $A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^\infty \in \ell^p$ with $|x_k| \searrow 0$. In this paper we characterize the upper triangular positive matrices from $B_w(\ell^p)$, $1 < p < \infty$, by using a special kind of Schur multipliers and the G. Bennett factorization technique. Also some related results are stated and discussed.

1. Introduction

In this paper we deal with infinite matrices $A$, whose entries $a_{kl}$, for $k \in \mathbb{Z}$ and $l \in \mathbb{Z}^+$, are indexed with respect to the $k$th diagonal and with the $l$th place on this diagonal. In what follows, sometimes we shall describe an infinite matrix by $A = (a_{kl})_{k \in \mathbb{Z}, l \in \mathbb{Z}^+}$ more precisely

$$ A = \begin{pmatrix} a^1_0 & a^1_1 & a^1_2 & a^1_3 & \cdots \\ a^2_{-1} & a^2_0 & a^2_1 & a^2_2 & \cdots \\ a^3_{-2} & a^3_{-1} & a^3_0 & a^3_1 & \cdots \\ a^4_{-3} & a^4_{-2} & a^4_{-1} & a^4_0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} $$

We started our study motivated by the paper [MM], where the first two authors introduced the space $B_w(\ell^2)$ of those infinite matrices $A$ for which $A(x) \in \ell^2$ for all $x = \{x_k\}_{k=1}^\infty \in \ell^2$ with $|x_k| \searrow 0$.

This space is of interest because the matrix version of the Wiener algebra $A(\mathbb{T})$, denoted by $A(\ell^2)$, which consists of all infinite matrices $A = (a^l_k)_{k \in \mathbb{Z}, l \in \mathbb{Z}^+}$ such that $\sup_{l \in \mathbb{Z}^+} \sum_{k \in \mathbb{Z}} |a^l_k| < \infty$, is not contained in the matrix version $C(\ell^2)$ of the space of all continuous functions $C(\mathbb{T})$ (see [BPP] for the definition and the properties of $C(\ell^2)$).

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Such an example is given by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

where on the \( \frac{n(n+1)}{2} \)-column there are \( n \) entries equals to 1 placed on the \( \frac{n(n-1)}{2} + 1, \ldots, \frac{n(n+1)}{2} \) rows and 0 otherwise. Clearly we have

\[
\sup_{k \in \mathbb{Z}} |a_k| = 1, \quad \text{hence } A \in A(\ell^2) \quad \text{and} \quad |Ae_n|_2 = n \quad \text{for all } e_n = (0, \ldots, 0, \frac{1}{n}, 0, \ldots).
\]

It yields that \( A(\ell^2) \subset B_w(\ell^2) \) (see Proposition 2 in [MM]), where \( B_w(\ell^2) \) is the Banach space with respect to the norm

\[
\|A\|_{B_w(\ell^2)} = \sup_{\|x\|_2 \leq 1, |x_k| < 0} \|A(x)\|_2.
\]

We remark that \( \ell^2_{\text{dec}} = \{x = (x_k) \searrow 0, \ x \in \ell^2\} \) is a cone and the solid hull of this cone, denoted by \( \text{so}(\ell^2_{\text{dec}}) \), coincides with the Banach space \( d(2) = \{x; \sum_{n=1}^{\infty} \sup_{k \geq n} |x_k|^2 < \infty\} \). The spaces \( d(p), \ p \geq 1 \) are introduced in [B], where it is described how they are connected to Hardy type inequalities (for historical information and results of this type we refer to the books [KMP] and [KP]). Here \( \text{so}(\ell^2_{\text{dec}}) = \{y = (y_k) \in \ell^2 \ \text{such that} \ |y_k| \leq x_k \ \text{for all} \ k \in \mathbb{N}, \ \text{where} \ x_k \searrow 0 \ \text{in} \ \ell^2\} \).

Let \( A \) be a positive matrix, that is such that all the elements of the sequence \( A(x) \) are positive whenever \( x = (x_j)_j \), is a sequence having only a finite number of nonzero positive elements. Clearly, if \( A \in B_w(\ell^2) \), then \( A \in B(d(2), \ell^2) \), that is \( A \) is a bounded linear operator from \( d(2) \) into \( \ell^2 \).

The next Lemma, which may be regarded as a discrete version of a special case of the Sawyer duality principle [Sa](see also [KP]) was obtained and applied in [MM].
Lemma 1.1. It yields that
\[
\sup_{|x_n| \searrow 0} \left| \sum_{n=1}^{\infty} a_n x_n \right| \approx \left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{1/2},
\]
where \((a_n)_n\) and \((x_n)_n\) are sequences of complex numbers.

For the investigations in this paper we need a corresponding (discrete Sawyer type) result for every \(p > 1\) and not only for \(p = 2\) as in Lemma 1.1 (see our Lemma 2.4).

In this paper we consider the space \(B_w(\ell^p)\) consisting of infinite matrices \(A\) for which \(A(x) \in \ell^p\) for all \(x = \{x_k\}_{k=1}^{\infty} \in \ell^p\) with \(|x_k| \searrow 0\) \((1 < p < \infty)\). In Theorem 2.1 we characterize the upper triangular positive matrices from \(B_w(\ell^p)\) by using a special kind of Schur multipliers. Also some related results are formulated in Section 2. The proofs can be found in Section 3. We pronounce that our proofs are heavily depending on various important factorization results by G. Bennett [B] and Lemma 2.4.

2. Main results

First let us recall the definition of Schur multipliers.

If \(A = (a_{jk})\) and \(B = (b_{jk})\) are matrices of the same size (finite or infinite) their Schur product (or Hadamard product) is defined to be the matrix of elementwise products
\[
A \ast B = (a_{jk} b_{jk}).
\]

There is, however, much justification for the term ”Schur product” and we refer the reader to [B1] and [St] for an historical discussion. This concept was first investigated by Schur in his paper [S] and has since arisen in several different areas of analysis: [Po], [SS1], [SS2](complex function theory); [B], [KwP](Banach spaces); [SW], [P], [BP](operator theory); [BPP], [BKP](matricial harmonic analysis) and [St](multivariate analysis).

If \(X\) and \(Y\) are two Banach spaces of matrices we define Schur multipliers from \(X\) to \(Y\) as the space \(M(X,Y) = \{M : M \ast A \in Y\text{ for every } A \in X\}\), equipped with the natural norm
\[
\|M\| = \sup_{\|A\|_{X} \leq 1} \|M \ast A\|_{Y}.
\]
We use a matrix operation introduced in [BLP], which extends to general matrices, the usual product of a Toeplitz matrix $A$ and a complex scalar $c$.

Namely, let $c = (c^1, c^2, \ldots)$ be a sequence of complex numbers. We denote by $[c]$ the matrix whose entries $[c]_{lk}$ are equal to $c^l$, for $l \geq 1$ and $k \in \mathbb{Z}$.

We observe that, for a Toeplitz matrix $A$ and for a constant sequence $c = (c^1, c^1, \ldots)$, the matrix $[c] \ast A$ coincides with the usual product between the complex number $c^1$ and the matrix $A$. Hence we denoted in [BLP] the product $[c] \ast A$ by $c \odot A$ and considered it as an external product between a matrix and a sequence of complex numbers.

In what follows, using the results about multipliers from [B], we will characterize the upper triangular positive matrices from $Bw(\ell^p) \cap \mathbb{R}^{n \times n}$ by studying the behaviour of the matrix $[c]$.

Here $Bw(\ell^p)$ denotes the space of those infinite matrices $A$ for which $A(x) \in \ell^p$ for all $x = \{x_k\}_{k=1}^\infty \in \ell^p$ with $|x_k| \downarrow 0$. It is clear that for $p > 1$ this is a Banach space with respect the norm

$$\|A\|_{Bw(\ell^p)} = \sup_{\|x\|_{\ell^p} \leq 1, |x_k| \downarrow 0} \|A(x)\|_p.$$ 

Here, as usual,

$$\ell^p = \{x = \{x_k\}_{k=1}^\infty : \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p} < \infty\}.$$

Moreover, let

$$d(p) = \{x = \{x_k\}_{k=1}^\infty : \left(\sum_{k=1}^\infty \sup_{n \geq k} |x_n|^p\right)^{1/p} < \infty\}.$$

Our first result reads:

**Theorem 2.1.** Let $B$ be an upper triangular matrix. Then $B \in B(\ell^p, ces(p))$, $1 < p < \infty$, if and only if $B * [c] \in B(\ell^p, \ell^1)$, for all $c \in d(p)$.

Here

$$ces(p) = \{x = \{x_k\}_{k=1}^\infty \text{ with } \sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p\right) < \infty\}$$

denotes the Banach space equipped with the norm

$$\|x\|_{ces(p)} = \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |x_k|\right)^p\right)^{1/p}.$$
Now we can state our main result concerning the characterization of the matrices belonging to $B_w(\ell^p)$.

**Theorem 2.2.** A lower triangular positive matrix $A$ belongs to $B_w(\ell^p)$, $1 < p < \infty$, if and only if $A^* [c] \in B(\ell^q, \ell^1)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for all $c \in d(p)$, where $A^*$ is the usual adjoint of the matrix $A$.

Besides the ces($p$)-spaces who have already attracted a fair deal of attention in the literature, an important role is played by $\ell^p, d(p)$ and also $g(p)$, defined by

$$g(p) = \left\{ x = \{x_k\}_{k=1}^\infty : \sup_{n \geq 1} \left( \frac{1}{n} \sum_{k=1}^n |x_k|^p \right)^{1/p} < \infty \right\}.$$ 

Therefore we also state the following result where ces($p$) in Theorem 2.2 is replaced by any of these spaces and $\ell^q \cdot d(p)$ is the sequence space of coordinative products (see [B] for further details).

**Theorem 2.3.** Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $B$ be an upper triangular matrix. Then

1. $B \in B(\ell^p, d(p))$ if and only if $B^* [c] \in B(\ell^q, \ell^1)$ for all $c \in$ ces($q$).
2. $B \in B(\ell^p, \ell^p)$ if and only if $B^* [c] \in B(\ell^q, \ell^1)$ for all $c \in \ell^q$.
3. $B \in B(\ell^p, g(p))$ if and only if $B^* [c] \in B(\ell^p, \ell^1)$ for all $c \in \ell^q \cdot d(p)$.

Our proof of Theorem 2.1 (and thus of Theorem 2.2) is heavily depending on the following extension of Lemma 1.1 of independent interest.

**Lemma 2.4.** If $p > 1$, then

$$\sup_{|x_n| \neq 0} \left( \sum_{n=1}^\infty |a_n x_n|^p \right)^{\frac{1}{p}} = \sup_{|x_n| \neq 0} \left( \sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}} \approx \left( \sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n |a_k| \right)^q \right)^{\frac{1}{q}} ,$$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers and $\frac{1}{p} + \frac{1}{q} = 1$.

**Remark 2.5.** If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$d(q)^\infty = ces(p).$$
Here $d(q)^\times$ is the associate space of $d(q)$, that is

$$d(q)^\times = \{ a = (a_n)_n; \text{ such that } \sum_{n=1}^\infty |a_n x_n| < \infty \text{ for all } (x_n)_n \in d(q) \}.$$  

This result which gives us the Köthe dual of $d(p)$ has been obtained also by G. Bennett in [B] using more technical methods, like factorization of some classical inequalities. This problem was first investigated by Jagers in 1974 in the paper [J].

Finally we note that

$$\text{so}(\ell^p_{\text{dec}}) = d(p)$$

($\ell^p_{\text{dec}}$ denotes the subspace of $\ell^p$ consisting of non-increasing sequences) and, hence, our results in particular implies Corollary 12.17 in paper [B] by G. Bennett.

3. Proofs

We first present a proof of the crucial Lemma 2.4, which is based on the following result of E. Sawyer [Sa]. For $p \leq 1$ a similar result has been proved by M. J. Carro and J. Soria in their paper [CS].

Lemma 3.1. Let $w = \{w(n)\}_{n=1}^\infty$, $v = \{v(n)\}_{n=1}^\infty$ be weights on $\mathbb{N}^*$, let

$$S = \sup_{f_n} \frac{\sum_{n=0}^\infty f(n) v(n)}{\left(\sum_{n=0}^\infty f(n)^p w(n)\right)^{\frac{1}{p}}}$$

and $\tilde{v} = \sum_{n=0}^\infty v(n) \chi_{[n,n+1)}$, $\tilde{w} = \sum_{n=0}^\infty w(n) \chi_{[n,n+1)}$ and $\tilde{V}(t) = \int_0^t \tilde{v}(s) ds$, $\tilde{W}(t) = \int_0^t \tilde{w}(s) ds$.

If $1 < p < \infty$, then

$$S \approx \left( \int_0^\infty \frac{\tilde{V}(t)^{q-1}}{\tilde{W}(t)} \tilde{v}(t) dt \right)^{\frac{1}{q}} \approx \left( \int_0^\infty \frac{\tilde{V}(t)^{q}}{\tilde{W}(t)} \tilde{w}(t) dt \right)^{\frac{1}{q}} + \frac{\tilde{V}(\infty)}{\tilde{W}(\infty)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Here, as usual, the relation $f \approx g$ means that there are two positive constants $C_0$ and $C_1$ so that $C_0 f(t) \leq g(t) \leq C_1 f(t)$, $t \in [0, \infty)$. 

Proof of Lemma 2.4. We denote
\[ S = \sup_{|x_n| > 0} \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}. \]

According to Lemma 3.1 we have that
\[ S \approx \left( \int_{0}^{\infty} \left( \frac{\tilde{V}(t)}{W(t)} \right)^{q} \tilde{w}(t) dt \right)^{\frac{1}{q}} + \frac{\tilde{V}(\infty)}{W(\tilde{\tau}(\infty))}, \]
where \( v(n) = |a_n|, \ w(n) = 1, \ f(n) = |x_n| \) for every nonnegative integer \( n \). In this case
\[ \tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1)} = \sum_{n=0}^{\infty} |a_n| \chi_{[n,n+1)}, \text{ where } a_0 = 0. \]

Therefore, for \( t \in (j, j + 1) \), it yields that
\[ \tilde{V}(t) = \int_{0}^{t} \tilde{v}(s) ds = \int_{0}^{j} \tilde{v}(s) ds + \int_{j}^{t} \tilde{v}(s) ds = \sum_{m=0}^{j-1} \int_{m}^{m+1} \tilde{v}(s) ds + \int_{j}^{t} \tilde{v}(s) ds = \sum_{m=0}^{j-1} |a_m| + |a_j|(t - j), \]
\[ \tilde{V}(\infty) = \int_{0}^{\infty} \tilde{v}(s) ds = \int_{0}^{\infty} \sum_{n=0}^{\infty} |a_n| \chi_{[n,n+1)}(s) ds = \sum_{n=1}^{\infty} |a_n| \]
and
\[ \tilde{W}(\infty) = \int_{0}^{\infty} \tilde{w}(s) ds = \infty, \]

since \( \tilde{w}(s) = \sum_{n=0}^{\infty} \chi_{[n,n+1)}(s) \).

Letting \( \tilde{v}_M = \sum_{n=0}^{M} |a_n| \chi_{[n,n+1)} \),
\[ \tilde{V}_M = \int_0^{\infty} \tilde{v}_M(s) \, ds = \int_0^M |a_n| \chi_{[n,n+1)}(s) \, ds = \sum_{n=1}^M |a_n| < \infty, \]

we get that

\[ S = \sup_{|x_n| \searrow 0} \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \]

\[ \approx \sup_M \left[ \left( \int_0^{\infty} \left( \frac{\tilde{V}_M(t)}{t} \right)^q \, dt \right)^{\frac{1}{q}} + \frac{\tilde{V}_M(\infty)}{W_{\frac{q}{2}}(\infty)} \right] \]

\[ \approx \left( \int_0^{\infty} \left( \frac{\tilde{V}(t)}{t} \right)^q \, dt \right)^{\frac{1}{q}}. \]

But

\[ \int_0^{\infty} \left( \frac{\tilde{V}(t)}{t} \right)^q \, dt = \sum_{j=1}^{\infty} \int_j^{j+1} \left( \frac{\sum_{m=0}^{j-1} |a_m| + |a_j|(t-j)}{t} \right)^q \]

\[ \approx \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=1}^{j} |a_m| \right)^q, \]

which implies that

\[ S \approx \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^q \right)^{\frac{1}{q}}. \]

The proof is complete. \(\square\)

**Proof of Theorem 2.1.** For clearness we first prove the theorem for the special case \(p = q = 2\). Note that \(A^*\) is an upper triangular matrix.

Let \(C \overset{\text{def}}{=} \{c\}\) be the upper triangular matrix obtained from \([c]\) taking the triangular projection \(P_T\), which acts as follows:

\[ P_T(A) = \begin{cases} 
 a_{ij} & \text{if } i \leq j \\
 0 & \text{otherwise.} 
\end{cases} \]

(See [BLP].)

Let \(B\) be an upper triangular matrix from \(B(\ell^2, ces(2))\). We have \(B(x) = \left( \sum_{j=1}^{\infty} b_{ij} x_j \right)_{i=1}^{\infty} \in ces(2)\), for all \(x = (x_j)_{j=1}^{\infty} \in \ell^2\). But \((B \ast \)
$C)(x) = \left( \sum_{j=1}^{\infty} b_{ij} x_j c_i \right)_{i=1}^{\infty}$ is the product of two sequences, one from $\text{ces}(2)$, and the other one completely arbitrary. By Proposition 15.4 in [B] we have that

\[ d(2) = I(2, 2) = \{ m : \sum_{k=1}^{\infty} |i_k - i_{k-1}|^2 < \infty; \text{for each sequence } i \text{ of integers with } i_0 = 0 < i_1 < i_2 < \ldots \} \]

Then, by using the table 29 on page 70 in [B], we get that $(B * C)(x) \in \ell^1$, where $c \in d(2) = I(2, 2)$ and $x \in \ell^2$. Hence $B * C \in B(\ell^2, \ell^1)$.

Conversely, let $B * C \in B(\ell^2, \ell^1)$ for each $c \in d(2)$. By Hölder’s inequality we have that $\ell^1 = \ell^2 \cdot \ell^1$, and, in view of Theorem 3.8 in [B], it follows that $\ell^2 = g(2) \cdot d(2)$, where

\[ g(2) = \left\{ x; \sup_n \frac{\sum_{k=1}^{n} |x_k|}{n} < \infty \right\} . \]

Hence $\ell^1 = (\ell^2 \cdot g(2)) \cdot d(2)$ and, according to Theorem 4.5 in [B], it yields that $\ell^1 = \text{ces}(2) \cdot d(2)$. On the other hand, by Proposition 14.5 in [B] $\text{ces}(2)$ has $d(2)$-cancellation property, that is the inclusion $y \cdot d(2) \subset \text{ces}(2) \cdot d(2)$ implies that $y \in \text{ces}(2)$.

Now, by hypotheses, for each $x \in \ell^2$, we have that

\[ (B * C)(x) = \left( \sum_{j=1}^{\infty} b_{ij} x_j c_i \right) \in \ell^1 = \text{ces}(2) \cdot d(2), \]

for all $c \in d(2)$. By the cancellation property it follows that

\[ B(x) = \left( \sum_{j=1}^{\infty} b_{ij} x_j \right) \in \text{ces}(2), \]

that is, by the closed graph theorem, $B \in B(\ell^2, \text{ces}(2))$.

Now we consider the case $p \neq 2$.

If $B \in B(\ell^p, \text{ces}(p))$, $c \in d(q)$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then as in the proof of the case $p = q = 2$ we have that $d(q) = I(q, q)$ and, in view of the table on page 70 in [B], it follows that

\[ (B * C)(x) \in \ell^1, \text{ for all } x \in \ell^p, \]

that is $B * C \in B(\ell^p, \ell^1)$.

Conversely, let $B * C \in B(\ell^p, \ell^1)$ for all $c \in d(q)$. Then, similarly as in the proof of the case $p = q = 2$ we find that

$\ell^1 = \ell^p \cdot \ell^q = \ell^p \cdot g(q) \cdot d(q) = (\text{by Theorem 4.5 in [B]}) = \text{ces}(p) \cdot d(q)$.

Since $\text{ces}(p)$ has $d(q)$-cancellation property (see Proposition 14.5 in [B]) it follows that $B \in B(\ell^p, \text{ces}(p))$.

The proof is complete. \hfill \Box
Proof of Theorem 2.2. First let us note that by using Lemma 2.4 it follows that $A \in B_w(\ell^p)$ if and only if $A^* \in B(\ell^q, \ces(q))$, for $\frac{1}{p} + \frac{1}{q} = 1$. It remains to apply Theorem 2.1. \hfill \Box

Proof of Theorem 2.3. (1). If $B \in B(\ell^p, d(p)), c \in \ces(q)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in \ell^p$ we have that

$$(B \ast [c])(x) = \left( \left( \sum_{j=1}^\infty b_{ij}x_j \right)^c \right)^{\infty}_{i=1} = (y_i c^i)_{i=1}^\infty \in d(p) \cdot \ces(q) = (\text{by Corollary 12.17 in } [B]) \cdot d(p) \cdot d(p)^* \subset \ell^1.$$ 

Hence, by the closed graph theorem, it yields that $B \ast [c] \in B(\ell^p, \ell^1)$.

Conversely, if $B \ast [c] \in B(\ell^p, \ell^1)$ for all $c \in \ces(q)$, then, denoting by $(y_i)_i = \left( \sum_{j=1}^\infty b_{ij}x_j \right)_{i \in \mathbb{N}}$, we have that $(y_i c^i)_i \in \ell^1$ for all $c \in \ces(q)$. Thus

$$(y_i)_i \in \ces(q)^* = (\text{by Corollary 12.17 in } [B]) = d(p),$$

that is

$$B \in B(\ell^p, d(p)).$$

(2). If $B \in B(\ell^p, \ell^p), x \in \ell^p, c \in \ell^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder’s inequality, $B \ast [c] \in B(\ell^p, \ell^1)$.

Conversely, let $(y_i c^i)_i \in \ell^1$ for all $c \in \ell^q$. Then $(y_i)_i \in \ell^p$ and, consequently,

$$B \in B(\ell^p, \ell^p).$$

(3). If $B \in B(\ell^p, g(p))$ and $c \in \ell^q \cdot d(p)$, then, by using the previous notations, we find that

$$(y_i c^i)_i \in g(p) \cdot \ell^q \cdot d(p) = (\text{by Theorem 3.8 in } [B]) = \ell^p \cdot \ell^q \subset \ell^1.$$ 

Conversely, let $(y_i)_i \cdot \ell^q \cdot d(p) \in \ell_1 = (\text{by Theorem 3.8 in } [B]) = g(p) \cdot d(p) \cdot \ell^q$. Consequently we have to show that

$$(y_i)_i \in g(p).$$

This fact follows clearly if $g(p)$ has the $d(p) \cdot \ell^q$-cancellation property.

We note that by using Proposition 14.5 in [B], we get that $(y_i)_i \cdot d(p) \in g(p) \cdot d(p) \cdot \ell^p$.

Indeed, let $(z_i)_i \in d(p)$ be fixed. Then $(y_i z_i)_i \cdot \ell^1 \in \ell^p \cdot \ell^q$. Since $\ell^p$ has the $\ell^q$-cancellation property (see Proposition 14.5 in [B]) it follows that

$$(y_i z_i)_i \in \ell^p = g(p) \cdot d(p), \text{ for all } (z_i)_i \in d(p).$$
in other words \((y_i)_i \cdot d(p) \in g(p) \cdot d(p)\). Using now the fact that \(g(p)\) has \(d(p)\)-cancellation property, it follows that \((y_i)_i \in g(p)\). The proof is complete.

\[\square\]

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Paper 3
A NEW CHARACTERIZATION OF
BERGMAN-SCHATTEN SPACES AND A DUALITY
RESULT

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ABSTRACT. Let $B_0(D, \ell^2)$ denote the space of all upper triangular matrices $A$ such that $\lim_{r \to 1^-} (1 - r^2)^{1/2} \| (A \ast C(r))' \|_{B(\ell^2)} = 0$. We also denote by $B_{0,c}(D, \ell^2)$ the closed Banach subspace of $B_0(D, \ell^2)$ consisting of all upper triangular matrices whose diagonals are compact operators.

In this paper we give a duality result between $B_{0,c}(D, \ell^2)$ and the Bergman-Schatten spaces $L^1_a(D, \ell^2)$. We also give a characterization of the more general Bergman-Schatten spaces $L^p_a(D, \ell^2)$, $1 \leq p < \infty$, in terms of Taylor coefficients, which is similar to that of M. Mateljević and M. Pavlović [12] for classical Bergman spaces.

1. INTRODUCTION

The Bloch and Bergman spaces have been studied for a long time in complex analysis and in the last twenty years the interest concerning these spaces has increased. A direction of research was that to study vector valued analytic function, but considered from a Banach point of view. In this way appeared a series of papers e.g. by J. A. Arregui, O. Blasco [2] and [3] and O. Blasco [6]–[9]. In what follows we consider the Bloch and Bergman spaces in the framework of matrices e.g. infinite matrix valued functions. We use the powerful device of Schur multipliers and its characterizations in the case of Toeplitz matrices to prove the main theorems. The extension to the matriceal framework is based on the fact that there is a natural correspondence between Toeplitz matrices and formal Fourier series associated to $2\pi$-periodic functions (see e.g. [1], [4], [11] and [14]).

Let $A = (a_{jk})$ and $B = (b_{jk})$ be matrices of the same size (finite or infinite). Then their Schur product (or Hadamard product) is defined to be the matrix of elementwise products:

$$ A \ast B = (a_{jk}b_{jk}). $$

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Key words and phrases. Infinite matrices, Toeplitz matrices, Schur multipliers, Bergman-Schatten spaces, Bloch spaces, duality.
If \( X \) and \( Y \) are two Banach spaces of matrices we define Schur multipliers from \( X \) to \( Y \) as the space 
\[
M(X, Y) = \{ M : M * A \in Y \text{ for every } A \in X \},
\]
equipped with the natural norm 
\[
\| M \| = \sup_{\| A \|_X \leq 1} \| M * A \|_Y.
\]
In the case \( X = Y = B(\ell^2) \), where \( B(\ell^2) \) is the space of all linear and bounded operators on \( \ell^2 \), the space \( M(B(\ell^2), B(\ell^2)) \) will be denoted \( M(\ell^2) \) and a matrix \( A \in M(\ell^2) \) will be called Schur multiplier. We mention here an important result due to G. Bennett [5], which will be often used in this paper.

**Theorem 1.1.** The Toeplitz matrix 
\[
M = (c_{j-k})_{j,k},
\]
where \((c_n)_{n \in \mathbb{Z}}\) is a sequence of complex numbers, is a Schur multiplier if and only if there exists a bounded and complex Borel measure \( \mu \) on (the circle group) \( \mathbb{T} \) with 
\[
\hat{\mu}(n) = c_n, \text{ for } n = 0, \pm 1, \pm 2, \ldots.
\]
Moreover, we then have that 
\[
\| M \| = \| \mu \|.
\]

We will denote by \( A_k \), the \( k^{th} \)-diagonal matrix associated to \( A \) (see [4]). For an infinite matrix \( A = (a_{ij}) \) and an integer \( k \) we denote by \( A_k \) the matrix whose entries \( a'_{ij} \) are given by 
\[
a'_{ij} = \begin{cases} 
a_{ij} & \text{if } j - i = k \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( A = (a_{ij})_{i,j \geq 0} \), be an infinite matrix with complex entries and let \( n \geq 0 \). The matrix \( A \) is said to be of \( n \)-band type (see [4]) if \( a_{ij} = 0 \) for \( |i - j| > n \).

In what follows we will recall some definitions from [13], which we will use in this paper. We consider on the interval \([0, 1)\) the Lebesgue measurable infinite matrix-valued functions \( A(r) \). These functions may be regarded as infinite matrix-valued functions defined on the unit disc \( D \) using the correspondence \( A(r) \rightarrow f_A(r, t) = \sum_{k=-\infty}^{\infty} A_k(r) e^{ikt} \), where \( A_k(r) \) is the \( k^{th} \)-diagonal of the matrix \( A(r) \), the preceding sum is a formal one and \( t \) belongs to the torus \( \mathbb{T} \). This matrix \( A(r) \) is called analytic matrix if there exists an upper triangular infinite matrix \( A \) such that, for all \( r \in [0, 1) \), we have \( A_k(r) = A_k r^k \), for all \( k \in \mathbb{Z} \). In what follows we identify the analytic matrices \( A(r) \) with their corresponding upper triangular matrices \( A \) and we call them also analytic matrices.
We will denote by $C_p$, $0 < p < \infty$, the Schatten class operators (see e.g. [13]). We also recall the definition of Bergman-Schatten spaces (see e.g. [13]). Let $1 \leq p < \infty$. We denote

$$L^p(D, \ell^2) = \{r \to A(r)\text{ which are strong measurable } C_p - \text{valued functions defined on } [0, 1) \text{ such that}

\|A\|_{L^p(D, \ell^2)} := \left(2 \int_0^1 \|A(r)\|_{\ell^p}^p r \, dr\right)^{\frac{1}{p}} < \infty\}.$$ 

Let $\tilde{L}^p_0(D, \ell^2) = \{r \to A(r), \text{ where } A(r) = A \ast C(r) \text{ and } A \text{ are upper triangular matrices with } \|A\|_{\tilde{L}^p_0(D, \ell^2)} < \infty\}$. Here $C(r)$ denotes the Toeplitz matrix associated to the Cauchy kernel $\frac{1}{1 - r^2}$, for $0 \leq r < 1$. By $L^p_0(D, \ell^2)$ we mean the space of all upper triangular matrices such that $\|A\|_{L^p(D, \ell^2)} < \infty$. We identify $\tilde{L}^p_0(D, \ell^2)$ and $L^p_0(D, \ell^2)$ and call $L^p_0(D, \ell^2)$ Bergman-Schatten classes. For $p = \infty$ we denote $L^\infty(D, \ell^2) = \{r \to A(r) \text{ being a } w^* - \text{measurable function on } [0, 1) : \|A\|_{L^\infty(D, \ell^2)} := \text{ess sup}_{0 \leq r < 1} \|A(r)\|_{\ell^2} < \infty\}$ and $\tilde{L}^\infty(D, \ell^2)$ is the subspace of $L^\infty(D, \ell^2)$ consisting of all strong measurable functions on $[0, 1)$. We also consider

$$L^\infty_a(D, \ell^2) := \{A \text{ analytic matrix} : \|A\|_{L^\infty_a(D, \ell^2)} := \sup_{0 \leq r < 1} \|C(r) \ast A\|_{\ell^2} = \|A\|_{L^\infty(D, \ell^2)} < \infty\}.$$ 

An important tool in this paper is the Bergman projection. It is known (see e.g. [13]) that for all functions $A(r) \in L^2(D, \ell^2)$ defined on $[0, 1]$ and for all $i, j \in \mathbb{N}$ we have that

$$[P(A(\cdot))](r)(i, j) = \begin{cases} 2(j - i + 1)r^{j-i} \int_0^1 a_{ij}(s) \cdot s^{j-i+1} \, ds, & \text{if } i \leq j, \\ 0 & \text{otherwise}. \end{cases}$$

**Definition 1.2.** The matricial Bloch space $B(D, \ell^2)$ is the space of all analytic matrices $A$ with $A(r) \in B(\ell^2)$, $0 \leq r < 1$, such that

$$\|A\|_{B(D, \ell^2)} = \sup_{0 \leq r < 1} (1 - r^2)\|A'(r)\|_{B(\ell^2)} + \|A_0\|_{B(\ell^2)} < \infty$$

where $B(\ell^2)$ is the usual operator norm of the matrix $A$ on the sequence space $\ell^2$ and $A'(r) = \sum_{k=0}^\infty A_k kr^{k-1}$.

A matrix $A \in B(D, \ell^2)$ is called a Bloch matrix. It is clear that the Toeplitz matrices which belong to the set of analytic matrices $B(D, \ell^2)$ appears as an extension of the classical Bloch space of functions.
A very useful theorem is the following (see e.g. [13]):

**Theorem P** Both $P : L^\infty(D, \ell^2) \to \mathcal{B}(D, \ell^2)$ and $\tilde{P} : L^\infty(D, \ell^2) \to \mathcal{B}(D, \ell^2)$ are bounded surjective operators.

The main results in this paper are presented, proved and discussed in Sections 2 and 3 below.

In Section 2 we give a characterization of matrices in the little Bloch space $\mathcal{B}_0(D, \ell^2)$ using the Bergman projection. One of the main results in this section is a new duality result (see Theorem 2.9). Also some related results are formulated and proved. In Section 3, we begin to state and prove with three technical lemmas, which are necessary to prove the main result of this section namely a characterization of the Bergman-Schatten spaces $L^p_0(D, \ell^2)$, $1 \leq p \leq \infty$, in terms of Taylor coefficients (see Theorem 3.4).

2. A matrix version of the little Bloch space

Now we introduce another space of matrices, the so-called **little Bloch space** of matrices.

**Definition 2.1.** The space $\mathcal{B}_0(D, \ell^2)$ is the space of all upper triangular infinite matrices $A$ such that $\lim_{r \to 1^-} (1 - r^2)\|(A * C(r))'\|_{\mathcal{B}(\ell^2)} = 0$, where $C(r)$ is the Toeplitz matrix associated with the Cauchy kernel.

Clearly $\mathcal{B}_0(D, \ell^2)$ is a closed subspace of $\mathcal{B}(D, \ell^2)$ if the former is endowed with the norm of $\mathcal{B}(D, \ell^2)$.

We denote by $E$ the Toeplitz matrix having all its entries equal to 1. First we state the following Lemma of independent interest:

**Lemma 2.2.** Let $A \in \mathcal{B}(D, \ell^2)$ and $A_r(s) = A(rs) = A(r) * P(s)$ for all $0 \leq r < 1$ and $0 \leq s < 1$, where $P(s)$ is the Toeplitz matrix associated to the Poisson kernel, that is

$$P(s) = \left( \begin{array}{ccccccc} 1 & s & s^2 & s^3 & \cdots \\ s & 1 & s & r_0^2 & \cdots \\ s^2 & s & 1 & s & \cdots \\ s^3 & s^2 & s & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{array} \right)$$

Then it follows that $A_r$ is a matrix belonging to $\mathcal{B}_0(D, \ell^2)$ for all $0 \leq r < 1$.

**Proof.** First we note that

$$\lim_{s \to 1^-} (1 - s^2)\|A_r'(s)\|_{\mathcal{B}(\ell^2)} = \lim_{s \to 1^-} (1 - s^2)\|A'(rs)\|_{\mathcal{B}(\ell^2)}$$
and, by using well known facts about multipliers (see e.g. [5]) and elementary calculations, we find that
\[
\|A'(rs)\|_{B(\ell^2)} = \|\sum_{k=0}^{\infty} kA_k(r)k^{k-1}\|_{B(\ell^2)} = \|\sum_{k=0}^{\infty} kA_kr^k\|_{B(\ell^2)}
\]
\[
= \|A'(r)\|_{B(\ell^2)} \sum_{k=0}^{\infty} s^{k-1}E_k \leq \|A'(r)\|_{B(\ell^2)} \sum_{k=0}^{\infty} s^{k-1}E_k \|_{M(\ell^2)}
\]
\[
= \|A'(r)\|_{B(\ell^2)} \sum_{k=0}^{\infty} s^{k-1}e^{ik\eta} \|_{M(\ell^2)} = \|A'(r)\|_{B(\ell^2)} \cdot \frac{1}{s} \int_{-\pi}^{\pi} \frac{1}{1-se^{i\eta}} d\eta \leq
\]
\[
\leq \|A\|_{B(D,\ell^2)} \cdot \frac{1}{s(1-r^2)} \int_{-\pi}^{\pi} \frac{1}{1-se^{i\eta}} d\eta.
\]
Thus, by making some straightforward calculations, we find that
\[
\lim_{s \to 1} (1-s^2)\|A'_r(s)\|_{B(\ell^2)} \leq
\]
\[
\leq \|A\|_{B(D,\ell^2)} \cdot \frac{r}{1-r^2} \lim_{s \to 1} \frac{(1-s^2)}{s} \cdot \int_{-\pi}^{\pi} \frac{1}{1-se^{i\eta}} d\eta =
\]
\[
= \|A\|_{B(D,\ell^2)} \cdot \frac{r}{1-r^2} \lim_{s \to 1} (1-s^2) \ln \frac{1}{1-s} = 0.
\]
The proof is complete. □

Our first result in this Section reads:

**Theorem 2.3.** Let \(A \in B(D, \ell^2)\). Then \(A \in B_0(D, \ell^2)\) if and only if \(\lim_{r \to 1} -\|A_r - A\|_{B(D,\ell^2)} = 0\).

**Proof.** By Lemma 2.2 it follows that \(A_r \in B_0(D, \ell^2)\) and we use the fact that \(B_0(D, \ell^2)\) is a closed subspace of \(B(D, \ell^2)\) in order to conclude that the condition is sufficient.

Conversely, let \(A \in B_0(D, \ell^2)\). Then, for every \(\epsilon > 0\) there exists \(0 < \delta < 1\) such that \((1-s^2)\|A'(s)\|_{B(\ell^2)} < \epsilon\), for every \(\delta^2 < s < 1\). We remark that
\[
\|A_r - A\|_{B(D,\ell^2)} = \sup_{0 \leq s < 1} (1-s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)} \leq
\]
\[
\leq \sup_{\delta < s < 1} (1-s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)} + \sup_{0 \leq s \leq \delta} (1-s^2)\|A'_r(s) - A'(s)\|_{B(\ell^2)}.
\]
For \(\delta < r < 1\) the first term is smaller than
\[
(1-r^2s^2)\|A'(rs)\|_{B(\ell^2)} + (1-s^2)\|A'(s)\|_{B(\ell^2)} < 2\epsilon.
\]
The second term converges to 0 whenever \( r \to 1^- \). Indeed, for 
\[ 0 \leq s \leq \delta < \delta' < 1, \]
letting \( u = \frac{s}{\delta'} \) and making some straightforward calculations, we get that
\[
\| A'(s) - A'(s) \|_{B(\ell^2)} = \| r A'(rs) - A'(s) \|_{B(\ell^2)}
\]
\[
= \| r A'(s) \sum_{k=0}^{\infty} r^k E_k - A'(s) \|_{B(\ell^2)} = \| A'(s) \sum_{k=0}^{\infty} (r^k - 1) E_k \|_{B(\ell^2)}
\]
\[
= \| A'(u) \sum_{k=0}^{\infty} (r^k - 1) (\delta')^{k-1} E_k \|_{B(\ell^2)} \leq 
\]
\[
\leq \| A'(u) \|_{B(\ell^2)} \sum_{k=0}^{\infty} (r^k - 1) (\delta')^{k-1} E_k \|_{M(\ell^2)} \leq 
\]
\[
\leq \| A'(u) \|_{B(\ell^2)} \sum_{k=0}^{\infty} (r^k - 1) (\delta')^{k-1} E_k \|_{M(\ell^2)} \leq 
\]
\[
\leq \| A'(u) \|_{B(\ell^2)} \cdot (1 - r) \int_{-\pi}^{\pi} \frac{1}{|1 - r \delta' e^{i\theta}| \cdot |1 - \delta' e^{i\theta}|} \frac{d\theta}{2\pi} \leq 
\]
\[
\leq \| A'(u) \|_{B(\ell^2)} \cdot \frac{1 - r}{(1 - r \delta')(1 - \delta')} = \| A'(s) \|_{B(\ell^2)} \cdot \frac{(1 - r)}{(1 - \delta')(1 - r \delta')}. 
\]

Then
\[
\sup_{s \leq \delta} (1 - s^2) \| A'(s) - A'(s) \|_{B(\ell^2)} \leq 
\]
\[
\leq \sup_{s \leq \delta} \left[ \left(1 - \left(\frac{s}{\delta'}\right)^2\right) \| A' \|_{B(\ell^2)} \cdot \frac{1 - s^2}{1 - \frac{s^2}{\delta^2}} \right] \cdot \frac{(1 - r)}{(1 - \delta')(1 - r \delta')} \leq 
\]
\[
\leq \| A \|_{B(D,\ell^2)} \cdot \frac{1 - \delta^2}{1 - \frac{\delta^2}{\delta^2}} \cdot \frac{(1 - r)}{(1 - \delta')(1 - r \delta')}. 
\]

Consequently
\[
\lim_{r \to 1^-} (1 - s^2) \| A'(s) - A'(s) \|_{B(\ell^2)} = 0, \]
i.e.
\[
\lim_{r \to 1^-} \| A_r - A \|_{B(D,\ell^2)} = 0. 
\]

The proof is complete. \( \square \)

**Corollary 2.4.** \( B_0(D, \ell^2) \) is the closure of all matrices of finite band type in the Bloch norm. In particular, this implies that \( B_0(D, \ell^2) \) is a separable space.
Proof. Let $A \in \mathcal{B}_o(D, \ell^2)$ and $A^n = \sum_{k=0}^n A_k$. Then, by Theorem 2.3, it yields that for every $\epsilon > 0$ there is $r_0 < 1$ such that $\|A - A\|_{B(D, \ell^2)} < \epsilon/2$.

We note that $r \to A(r)$ for $r \in [0, 1)$ is a continuous $B(\ell^2)$-valued function on $[0, s]$ for $s < 1$.

Indeed, let $0 < s_n \leq s_0 < 1$ and $s_n \to s_0$. Then,
\[
\|A(s_n) - A(s_0)\|_{B(\ell^2)} = \left\| C \left( \frac{S_n}{s'} \right) - C \left( \frac{S_0}{s'} \right) * A(s') \right\|_{B(\ell^2)} \leq \|A(s')\|_{B(\ell^2)} \left\| C \left( \frac{S_n}{s'} \right) - C \left( \frac{S_0}{s'} \right) \right\|_{M(\ell^2)},
\]
where $s_n \to s_0 < s' < 1$. Hence, by putting $\delta = \frac{s_0}{s'} < 1$ and reasoning as in the proof of the previous theorem, we get that
\[
\|A(s_n) - A(s_0)\|_{B(\ell^2)} \leq \|A(s')\|_{B(\ell^2)} \left\| \sum_{k=0}^\infty \delta^k \left( \frac{S_n}{s_0} \right)^k \right\|_{M(\mathbb{T})} \to 0.
\]

Now, for a fixed $r_0 < 1$ we have that
\[
\sup_{0 \leq s \leq 1} \|A_{r_0}(s)\|_{B(\ell^2)} = M(r_0) = \|A(r_0)\|_{B(\ell^2)} < \infty
\]
for all analytic matrices.

Thus, for $r_0 < r' < 1$ and by using the notation
\[
C^n \left( \frac{r_0}{r'} \right) = \sum_{k=0}^n C_k \left( \frac{r_0}{r'} \right),
\]
we find that
\[
\|A_{r_0}(\cdot) - (A_{r_0})^n(\cdot)\|_{L^\infty(D, \ell^2)} = \sup_{s<1} \|A - A^n)(r_0 s)\|_{B(\ell^2)} = \|(A - A^n)(r_0)\|_{B(\ell^2)} = \left\| C \left( \frac{r_0}{r'} \right) - C^n \left( \frac{r_0}{r'} \right) * A(r') \right\|_{B(\ell^2)} \leq \left\| C \left( \frac{r_0}{r'} \right) - C^n \left( \frac{r_0}{r'} \right) \right\|_{M(\ell^2)} \cdot \|A(r')\|_{B(\ell^2)} = \left\| \sum_{k=n+1}^\infty \left( \frac{r_0}{r'} \right)^k \cdot \|A(r')\|_{B(\ell^2)} \to 0.
\]
Consequently,
\[
\|A_{r_0}(\cdot) - (A_{r_0})^n(\cdot)\|_{L^\infty(D, \ell^2)} \to 0
\]
and
\[
\|A(\cdot) - (A_{r_0})^n(\cdot)\|_{B(D, \ell^2)} \leq \epsilon,
\]
whenever \( r_0 < 1 \) is fixed as before and \( n \) is sufficiently large. Since \( \epsilon > 0 \) is arbitrary and \( (A_{r_0})^n = \sum_{k=0}^n (A_{r_0})_k \) is a matrix of finite band type it follows that \( \mathcal{B}_0(D, \ell^2) \) is the closure of all matrices of finite band type in the Bloch norm. The proof is complete. \( \Box \)

The next theorem express a natural relation between the Bergman projection and the Bloch spaces. More exactly, our first main result in this Section is the following equivalence theorem:

**Theorem 2.5.** Let \( A \in \mathcal{B}(D, \ell^2) \). Then the following assertions are equivalent:

1) \( A \in \mathcal{B}_0(D, \ell^2) \).
2) There is a continuous \( B(\ell^2) \)-valued function \( r \to B(r) \) defined on \([0, 1] \) such that \( P(B(\cdot))(r) = A(r) \).
3) There is \( r \to B(r) \) which is a continuous \( B(\ell^2) \)-valued function such that \( \lim_{r \to 1} B(r) = 0 \) and satisfying \( P(B(\cdot))(r) = A(r) \).

**Proof.** To prove that 1) implies 3), let us take \( A \in \mathcal{B}_0(D, \ell^2) \). We define \( A_1(r) := \sum_{k=2}^\infty A_k r^k \), with \( r < 1 \). Thus \( A_1'(r) = \sum_{k=2}^\infty kA_k r^{k-1} \) and \( A(r) = A_0 + A_1 r + A_1(r) \). We take now

\[
B_2(r) = (1 - r^2)T * P(r) * A_1'(r),
\]

where \( T = (t_{ij})_{i,j} \) is a Schur multiplier which will be defined later on.

Thus, by the definition of the Bergman projection \( P \), we get that

\[
[PB_2(\cdot)](r)(i, j) = \begin{cases} 
 t_{ij} a_{ij} (j - i + 1)(j - i)r^{j-i} & \text{for } j - i \geq 2 \\
 0 & \text{otherwise}.
\end{cases}
\]

Consequently, by taking \( t_{ij} = \begin{cases} 
 \frac{3[3(j-i)+1]}{4(3-j)} & \text{for all } j \neq i, i, j \geq 1 \\
 \frac{9}{4} & \text{if } j = i, i \geq 1
\end{cases} \)

it follows that \( T \) is a Schur multiplier and \( [PB_2(\cdot)] = A_1(\cdot) \). Let now \( B(r) = 2(1 - r^2)A_0 + 3(1 - r^2)rA_1 + B_2(r) \). It is clear that \( [PB(\cdot)](r) = A(r) \).

Thus 3) holds.

It is obvious that 3) implies 2).

It remains to prove that 2) implies 1). Let 2) hold and choose \( B(r) \in B(\ell^2) \) be such that

\[
[PB(\cdot)](r) = A(r) \text{ for } r \in [0, 1].
\]

Assume that \( r \to B(r) \) is a continuous \( B(\ell^2) \)-valued function on \([0, 1] \) and let \( M = \sup_{0 \leq r \leq 1} \|B(r)\|_{B(\ell^2)} < \infty \).
Let $0 \leq r_0 < 1$ be fixed and let us consider $A_{r_0}(r)$ given by the formula

$$A_{r_0}(r)(i, j) = \begin{cases} \frac{(j - i + 1)(rr_0)^{j-i}}{2 \int_0^1 b_j(s)s^{j+i}\,ds} & \text{if } j - i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Consequently, according to Theorem P, we find that

$$A_{r_0}(\cdot) = P[P(r_0) \ast B(\cdot)] = P[B_{r_0}(\cdot)] \in \mathcal{B}(D, \ell^2),$$

where $P(r_0)$ is the Toeplitz matrix associated with the Poisson kernel.

Let $\mathcal{C}(D, \ell^2)$ denote the space of all continuous $\mathcal{B}(D, \ell^2)$-valued functions defined on $[0, 1]$. We will prove that the function $s \to P[P(r_0) \ast B(s)]$ belongs to $\mathcal{C}(D, \ell^2)$ if $B$ is a continuous $B(\ell^2)$-valued function. Moreover, we will show that

$$\lim_{r \to 1} \sup_{s \in [0, 1]} \| A_r(s) - A(s) \|_{\mathcal{B}(D, \ell^2)} = 0.$$  

This, in its turn, implies that $\lim_{r \to 1} A_r = A$ in $\mathcal{B}(D, \ell^2)$. Thus, by Theorem 2.3, it follows that $A \in \mathcal{B}_0(D, \ell^2)$.

Let $s, s_0 \in [0, 1]$. Then

$$\| P(r_0) \ast (B(s) - B(s_0)) \|_{\mathcal{B}(\ell^2)} \leq \| \sum_{k \in \mathbb{Z}} r_0^{[k]} e^{ik\theta} \|_{M(\mathbb{T})} \cdot \| B(s) - B(s_0) \|_{\mathcal{B}(\ell^2)} \to 0$$

for $s \to s_0$ and $B(s)$ is a continuous function on $[0, 1]$. Here we have used G. Bennett’s Theorem 1.1 and the fact that

$$\| \sum_{k \in \mathbb{Z}} r_0^{[k]} e^{ik\theta} \|_{M(\mathbb{T})} = \| \frac{1 - r_0^2}{1 - r_0 e^{i\theta}} \|_{M(\mathbb{T})} \leq 1.$$  

Thus, the function $s \to P[P(r_0) \ast B(s)]$ belongs to $\mathcal{C}(D, \ell^2)$.

Hence, it only remains to prove that (1) holds.

$$\| P(r) \ast B(s) - B(s) \|_{\mathcal{B}(\ell^2)} \leq \| \sum_{k \in \mathbb{Z}} (r^{[k]} - 1)e^{ik\theta} \|_{M(\mathbb{T})} \cdot \| B(s) \|_{\mathcal{B}(\ell^2)}$$

$$\leq M : \| \sum_{k \in \mathbb{Z}} (r^{[k]} - 1)e^{ik\theta} \|_{M(\mathbb{T})} \text{ for all } s \in [0, 1].$$

Denoting by $\mu_r(\theta)$ the measure $\sum_{k \in \mathbb{Z}} (r^{[k]} - 1)e^{ik\theta}$, then, for a trigonometric polynomial $\phi(\theta) = \sum_{n=-m}^{m} a_n e^{in\theta}$, we have that

$$\mu_r(\phi) = \sum_{n=-m}^{m} (r^{[n]} - 1)a_n \text{ and } |\mu_r(\phi)| \leq |\phi(r) - \phi(1)| \leq 2\|\phi\|,$$

where $\phi(r)$ is the value of the Poisson extension of $\phi$ in the point $r$.

Consequently $\mu_r$ is a measure with a norm smaller than 2. But $\lim_{r \to 1} \mu_r(\phi) = 0$ for all trigonometric polynomials $\phi$. Thus $w^* -
\[ \lim_{r \to 1} \mu_r = 0 \text{ in } M(T) \] and then is is clear that \( \lim_{r \to 1} \| \mu_r \| = 0 \) and by Theorem P the relation (1) is proved. Thus also the implication 2) \( \Rightarrow 1) \) is proved and the proof is complete. \( \square \)

By using similar ideas as in [13] we can obtain the following result:

**Theorem 2.6.** \( P_2 \) is a continuous operator (precisely a continuous projection) from \( L^1(D, \ell^2) \) onto \( L_a^1(D, \ell^2) \), where

\[
[P_2 A(\cdot)](r)(i, j) = \begin{cases} 
\frac{2\Gamma(j-i+4)}{j-i+3!} r^{j-i} \int_0^1 (1 - s^2)^2 a_{ij}(s)s^{j-i}(2sds) & \text{if } j \geq i, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** The topological dual of \( L^1(D, \ell^2) \) is \( L^\infty(D, \ell^2) \) with respect to the duality pair:

\[
<A(\cdot), B(\cdot)> = \int_0^1 tr(A(s)[B(s)]^*)(2sds),
\]

where \( A(\cdot) \in L^\infty(D, \ell^2), B(\cdot) \in L^1(D, \ell^2) \) (see e.g. Theorem 8.18.2 in [10]). Using a duality argument it is sufficient to prove that \( P_2^*: L^\infty(D, \ell^2) \to L^\infty(D, \ell^2) \) is bounded.

Now we are looking for the adjoint \( P_2^* \) of \( P_2 \). We note that

\[
<P_2^* A(\cdot), B(\cdot)> = \int_0^1 \sum_{i=1}^\infty \sum_{j=1}^\infty (P_2^* A(\cdot))(r)(i, j)b_{ij}(2rdr)
\]

\[
= \sum_{i=1}^\infty \sum_{j=1}^\infty \int_0^1 (P_2^* A(\cdot))(r)(i, j)b_{ij}(r)(2rdr).
\]

On the other hand it yields that

\[
<P_2^* A(\cdot), B(\cdot)> = <A(\cdot), P_2 B(\cdot)> = \int_0^1 tr A(r)(P_2 B)^*(r)(2rdr)
\]

\[
= \sum_{i=1}^\infty \sum_{j=1}^\infty \int_0^1 A(r)(i, j)(P_2 B)(r)(i, j)(2rdr)
\]

\[
= \sum_{i=1}^\infty \sum_{j=1}^{\infty} \Gamma(j - i + 4) (j - i)!\Gamma(3) \left( \int_0^1 [A(s)](i, j)s^{j-i}(2sds) \right) \times
\]

\[
\left( \int_0^1 b_{ij}(s)s^{j-i}(1 - s^2)^2(2sds) \right).
\]

Now let us consider \( \{I_k\} \), a sequence of intervals such that

\[
\lim_{k \to \infty} \mu(I_k) = 0, \ d\mu = 2sds \text{ and } r \in I_k.
\]
For every $k$ we take $B(s)(i,j) = \chi_{I_k}(s)/(\mu(I_k))$ and $B(s)(l,k) = 0$, $(l,k) \neq (i,j)$ for every $(i,j) \in \mathbb{N} \times \mathbb{N}$.

By Lebesgue’s differentiation theorem (see e.g. [15]) we have that

$$ (P^*_2 A(\cdot))(r)(i,j) = \begin{cases} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} r^{j-i}(1-r^2)^2 \int_0^r A(s)(i,j)s^{j-i}2ds & \text{if } j \geq i, \\ 0 & \text{if } j < i, \end{cases} \quad \text{a.e. for all } r \in [0,1). $$

We will now prove that $P^*_2 : L^\infty(D,\ell^2) \to L^\infty(D,\ell^2)$ is a bounded operator. In order to prove that we first note that

$$ \|A(r)\|_{L^\infty(D,\ell^2)}^2 = \text{ess sup}_{0 \leq r < 1} \|A(r)\|_{B(\ell^2)}^2 $$

Consequently,

$$ \|P^*_2 A(\cdot)\|_{L^\infty(D,\ell^2)}^2 = \text{ess sup}_{0 \leq r < 1} \sup_{\sum_{i=1}^\infty |h_i| \leq 1} \sum_{i=1}^\infty \sum_{j=1}^{\infty} h_i r^{j-i} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} (1-r^2)^2. $$

$$ \int_0^1 a_{ij}(s)s^{j-i}2ds = \text{ess sup}_{0 \leq r < 1} \sup_{\sum_{i=1}^\infty |h_i| \leq 1} \sum_{i=1}^\infty \sum_{j=1}^{\infty} a_{ij}(s) (1-r^2)^4 \int_0^1 \sum_{j=1}^{\infty} a_{ij}(s) (rs)^{j-i} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} h_j (2ds)^2. $$

Since the Toeplitz matrix $C(rs) = ((c_{ij}(rs)))_{i,j=1}^\infty$, where

$$ c_{ij}(rs) = c_{j-i}(rs) = \begin{cases} (rs)^{j-i}(j-i+3)(j-i+2)(j-i+1) & \text{if } j \geq i, \\ 0 & \text{otherwise}, \end{cases} $$

is a Schur multiplier with

$$ \|C(rs)\|_{L^1(\mathbb{T})} = \frac{6}{(1-rse^{i\theta})^4} \|L^1(\mathbb{T}) = 6 \sum_{n=0}^\infty (n+1)^2(rs)^{2n}, $$

$$ \int_0^1 (rs)^{j-i} \frac{\Gamma(j-i+4)}{(j-i)!\Gamma(3)} h_j (2ds)^2. $$
we get that
\[
\sup_{\sum_{j=1}^{\infty}|h_j|^2 \leq 1} \left( \sum_{i=1}^{\infty} \left| \sum_{j=i}^{\infty} a_{is}(s)(rs)^{j-i}(j-i+3)(j-i+2)(j-i+1)h_j \right|^2 \right)^{1/2}
\]
\[= \|A(s) \ast C(r)\|_{B(\ell^2)} \leq 6 \|A(s)\|_{B(\ell^2)} \cdot \sum_{n=0}^{\infty} (n+1)^2(rs)^{2n}.\]
Consequently,
\[
\|P_2^*A(\cdot)\|_{L^\infty(D, \ell^2)}^2 \leq 
\leq 9 \cdot \|A(\cdot)\|_{L^\infty(D, \ell^2)}^2 \left( \sum_{n=0}^{\infty} (n+1)^2 s^{2n} \right)^2 \sim 
\sim \|A(\cdot)\|_{L^\infty(D, \ell^2)}^2.
\]

which shows in turn that \(P_2^*: L^\infty(D, \ell^2) \rightarrow L^\infty(D, \ell^2)\) is bounded. The proof is complete. \(\square\)

Now let us denote by \(C_0(D, \ell^2)\) the space of all continuous \(B(\ell^2)\)-valued functions \(B(r)\) on \([0, 1]\) such that \(\lim_{r \to 1} B(r) = 0\) in the norm of \(B(\ell^2)\). Next we state the following Lemma of independent interest:

**Lemma 2.7.** Let \(V = (P_2)^*, \) i.e., \((P_2A(\cdot))^\ast(i, j) =
\[
\begin{cases}
\frac{1}{2} (j-i+3)(j-i+2)(j-i+1) & \text{if } j - i \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Then \(V\) is an isomorphic embedding of \(B_0(D, \ell^2)\) in \(C_0(D, \ell^2)\).

**Proof.** According to Theorem 2.5, for \(B \in B_0(D, \ell^2)\) we can find some \(A(\cdot) \in C_0(D, \ell^2)\) such that \([PA(\cdot)](r) = B(r)\). Clearly, we have that
\[
P_2^*B = P_2^*PA = T^1 \ast (P_2^*A_1),
\]
where \(A_1(r) = T \ast A(r)\), for \(T = (t_{j-i})_{i,j}\) with
\[
t_{j-i} = \begin{cases}
\frac{2(j-i+1)}{j-i+2} & \text{for } j - i \neq -2, \\
0 & \text{otherwise}.
\end{cases}
\]

\(T\) is a Schur multiplier and the same is true for \(T^1 = (t_{j-i}^1)_{i,j}\), where
\[
t_{j-i}^1 = \begin{cases}
\frac{j-i+2}{2(j-i+1)} & \text{for } j - i \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Thus, \( \|A_1(r)\|_{B(\ell^2)} \sim \|A(r)\|_{B(\ell^2)} \) for all \( r \in [0,1] \).

Hence, we obtain that

\[
\|P_2^* A_1(r)\|_{B(\ell^2)} = \sup_{\|h\|_{\ell^2} \leq 1} \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} h_i r^{j-i} \Gamma(j - i + 4)/(j - i)! \Gamma(3) (1 - r^2)^2 \int_0^1 a_{ij}(s) s^{j-i}(2sd\sigma)^2 \right|^2 \leq \sup_{\|h\|_{\ell^2} \leq 1} (1 - r^2)^4 \left[ \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij}(s) rs^{j-i} \Gamma(j - i + 4)/(2(j - i)! h_j)^2 (2sd\sigma)^2 \right)^2 \right]
\]

\[
\leq \sup_{\|h\|_{\ell^2} \leq 1} (1 - r^2)^4 \left[ \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij}(s) rs^{j-i} (j - i + 1)^2 (j - i + 3) h_j^2 \right)^2 \right].
\]

The Toeplitz matrix \( C(r, s) = (c_{j-i}(r, s))_{i,j} \), where

\[
c_{j-i}(r, s) = \begin{cases} (rs)^{j-i}(j - i + 3)(j - i + 1)^2 & \text{if } j \geq i, \\ 0 & \text{otherwise}, \end{cases}
\]

is a Schur multiplier, since \( \sum_{k=0}^{\infty} (rs)^k (k + 3) (k + 1)^2 e^{ik\theta} \sim \frac{1}{(1 - r^2e^{i\theta})^2} \)

and \( \int_{-\pi}^{\pi} \frac{1}{(1 - r^2e^{i\theta})^2} \frac{d\theta}{2\pi} \sim \sum_{n=0}^{\infty} n^2 (rs)^{2n} \).

Therefore, we have that

\[
\|P_2^* A_1(r)\|_{B(\ell^2)} \leq C(1 - r^2)^2 \int_0^1 \|A(s)\|_{B(\ell^2)} \cdot \sum_{n=0}^{\infty} n^2 (rs)^{2n}(2sd\sigma)^2.
\]

Since \( \lim_{s \to 1} \|A(s)\|_{B(\ell^2)} = 0 \), for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \|A(s)\| < \epsilon \) for all \( s \geq \delta \) and, consequently,

\[
\|P_2^* A_1(r)\|_{B(\ell^2)} \leq C(1 - r^2)^2 \delta^2 \cdot \frac{1 - r^2\delta^2}{(1 - r^2\delta^2)^2}.
\]

It follows that \( \lim_{r \to 1} \|P_2^* A_1(r)\|_{B(\ell^2)} = 0 \) and since \( T^1 \) is a Schur multiplier it follows that \( P_2^* B \in C_0(\ell^2) \) and \( \|P_2^* B\|_{B(\ell^2)} \leq C\|A(\cdot)\|_{C(D, \ell^2)} \).

Moreover, in view the proof of Theorem 2.5, we can find an \( A(\cdot) \in C_0(D, \ell^2) \) such that

\[
\|A(\cdot)\|_{C_0(D, \ell^2)} \leq C(\|B_0\|_{B(\ell^2)} + \|B(\cdot)\|_{B(D, \ell^2)}),
\]

where \( C > 0 \) is an absolute constant. By now also using the arguments in the proof of Theorem (2.6) it follows that \( P_2^* : B_0(D, \ell^2) \to C_0(D, \ell^2) \) is bounded.

On the other hand, if \( A \in B_0(D, \ell^2) \), then since \( A(r) \) is an analytic matrix it is obvious that \( A(r) = [PA(\cdot)](r) = (P[P_2^* A(\cdot)])(r) \) for all \( r \in [0,1] \). Thus, by using Theorem P, we conclude that there exists a constant \( C > 0 \) such that \( \|A(\cdot)\|_{B(D, \ell^2)} \leq C\|P_2^* A(\cdot)\|_{C(D, \ell^2)} \), which
implies that \( P_2^* : \mathcal{B}_0(D, \ell^2) \to \mathcal{C}_0(D, \ell^2) \) is an isomorphic embedding. The proof is complete.

**Remark 2.8.** From now on we identify \( \mathcal{B}_0(D, \ell^2) \) with the space \( \tilde{\mathcal{B}}_0(D, \ell^2) \) of all analytic matrices \( A \ast C(r) \) for \( A \in \mathcal{B}_0(D, \ell^2) \).

We denote now by \( \mathcal{B}_{0,c}(D, \ell^2) \) the closed Banach subspace of \( \mathcal{B}_0(D, \ell^2) \) consisting of all upper triangular matrices whose diagonals are compact operators. We are now ready to prove that the little Bloch space \( \mathcal{B}_{0,c}(D, \ell^2) \) in fact is the predual of the Bergman-Schatten space. More exactly, our last main result in this Section is the following duality result:

**Theorem 2.9.** It yields that \( \mathcal{B}_{0,c}(D, \ell^2)^* = L^1_a(D, \ell^2) \) with respect to the usual duality, whenever \( \mathcal{B}_0(D, \ell^2) \) is equipped with the norm induced by \( \mathcal{B}(D, \ell^2) \).

**Proof.** Let \( A \in L^1_a(D, \ell^2) \). Then \( B \to \int_0^1 tr[B(s)A^a(s)](2sd\sigma) \) defines a linear and bounded functional on \( \mathcal{B}_{0,c}(D, \ell^2) \) (see e.g. Theorem 22 in [13]). Conversely, let us assume that \( F \) is a bounded linear functional on \( \mathcal{B}_{0,c}(D, \ell^2) \). Then we shall prove that there is a matrix \( C \) from \( L^1_a(D, \ell^2) \) such that

\[
F(B) = \int_0^1 tr[B(r)C^a(r)](2rd\sigma),
\]

for \( B \) from a dense subset of \( \mathcal{B}_0(D, \ell^2) \).

By Lemma 2.7 it follows that \( P_2^* : \mathcal{B}_0(D, \ell^2) \to \mathcal{C}_0(D, \ell^2) \) is an isomorphic embedding. Thus \( X = P_2^*(\mathcal{B}_{0,c}(D, \ell^2)) \) is a closed subspace in \( \mathcal{C}_0(D, C_\infty) \) and \( F \circ (P_2^*)^{-1} : X \to \mathbb{C} \) is a bounded linear functional on \( X \), where \( \mathcal{C}_0(D, C_\infty) \) is the subset in \( \mathcal{C}_0(D, \ell^2) \) whose elements are \( C_\infty \)-valued functions. By the Hahn-Banach theorem \( F \circ (P_2^*)^{-1} \) can be extended to a bounded linear functional on \( \mathcal{C}_0(D, C_\infty) \).

Let \( \Phi : \mathcal{C}_0(D, C_\infty) \to \mathbb{C} \) denote this functional. It follows that \( \mathcal{C}_0(D, C_\infty) = \mathcal{C}_0[0, 1] \otimes_{\varepsilon} C_\infty \) and, thus, \( \Phi \) is a bilinear integral map, that is there is a bounded Borel measure \( \mu \) on \([0, 1] \times U_{C_1}\), where \( U_{C_1} \) is the unit ball of the space \( C_1 \) with the topology \( \sigma(C_1, C_\infty) \), such that

\[
\Phi(f \otimes A) = \int_{[0, 1] \times U_{C_1}} f(r)tr(AB^a)d\mu(r, B)
\]

for every \( f \in \mathcal{C}_0[0, 1] \) and \( A \in C_\infty \).
Thus, for the matrix \( \sum_{k=0}^{n} A_k \in \mathcal{B}_{0,c}(D, \ell^2) \), identified with the analytic matrix \( \sum_{k=0}^{n} A_k r^k \), we have that

\[
F(\sum_{k=0}^{n} A_k) = F(\sum_{k=0}^{n} r^k A_k) = [F \circ (P_2^*)^{-1}][P_2^*(\sum_{k=0}^{n} r^k A_k)]
\]

\[
= \Phi \sum_{k=0}^{n} \frac{(k+3)(k+2)}{2} r^k (1-r^2)^2 A_k
\]

\[
= \int_{[0,1] \times U_{C_1}} \sum_{k=0}^{n} \text{tr}[(\frac{(k+3)(k+2)}{2} r^k (1-r^2)^2 A_k)B^*]d\mu(r, B)
\]

\[
def < \mu(r, B), \text{tr}(\sum_{k=0}^{n} \frac{(k+3)(k+2)}{2} r^k A_k)B^*(1-r^2)^2 >.
\]

On the other hand, we wish to have that

\[
F(\sum_{k=0}^{n} A_k) = \int_{0}^{1} \text{tr}(\sum_{k=0}^{n} s^k A_k)(C(s)^*)(2sd) = \int_{0}^{1} \text{tr}(\sum_{k=0}^{n} s^{2k} A_k C_k^*)(2sd) = \sum_{k=0}^{n} \text{tr}A_k \left( \frac{C_k^*}{k+1} \right).
\]

Therefore, letting \( A = e_{i+i+k} \), denote the matrix having 1 as the single nonzero entry on the \( i \)th-row and the \( (i+k) \)th-column, for \( i \geq 1 \) and \( j \geq 0 \), we have that

\[
C_k = < \overline{\nu}(r, B), \frac{(k+1)(k+2)(k+3)}{2} r^k (1-r^2)^2 B_k >, k = 0, 1, 2, \ldots .
\]

Then, it yields that

\[
\int_{0}^{1} \|C(s)\|_{C_1}(2sd) = \int_{0}^{1} \| \int_{[0,1] \times U_{C_1}} \sum_{k=0}^{n} \frac{(k+1)(k+2)(k+3)}{2} (sr)^k (1-r^2)^2 B_k \|_{C_1}(2sd) = \int_{[0,1] \times U_{C_1}} \sum_{k=0}^{n} \frac{(k+1)(k+2)}{2} (rs)^k.
\]

\[
d\overline{\nu}(r, B)\|_{C_1}(2sd) \leq \int_{[0,1] \times U_{C_1}} \| \int_{0}^{1} \| \sum_{k=0}^{n} \frac{(k+1)(k+2)(k+3)}{2} (rs)^k.
\]

\[
(1-r^2)^2 B_k \|_{C_1}(2sd) d\mu(r, B) \leq \int_{[0,1] \times U_{C_1}} \| \sum_{k=0}^{n} \frac{(k+1)(k+2)}{2} .
\]

\[
(k+3)(rs)^k (1-r^2)^2 e^{ik\theta} \|_{L^1(\mathbb{T})} \| B \|_{C_1}(2sd) d\mu(r, B) \leq \int_{[0,1] \times U_{C_1}} \frac{1}{1-rse^{i\theta}} \frac{d\theta}{2\pi} (2sd) d\mu(r, B)
\]
\[
\sim \int_{[0,1] \times U_{C_1}} \int_0^1 (1 - r^2)^2 \sum_{n=0}^{\infty} (n + 1)^2 (sr)^{2n} (2sd) d \mu |(r, B) \\
= \int_{[0,1] \times U_{C_1}} (1 - r^2)^2 \sum_{n=0}^{\infty} (n + 1)^2 r^{2n} d \mu |(r, B) = \| \mu \| < \infty.
\]

Consequently, \( C \in L^1(D, \ell^2) \) and we get the relation (3) by using the fact that the set of all matrices \( \sum_{k=0}^{n} A_k \) is dense in \( B_{0,\mathbb{C}}(D, \ell^2) \). The proof is complete. \( \square \)

3. A new characterization of Bergman-Schatten spaces

In this Section we give a characterization of the space \( L^p_a(D, \ell^2) \) in terms of Taylor coefficients which is similar to those obtained by M. Mateljevic and M. Pavlovic in [12]. For the proof of our main result (Theorem 3.4) we need the following three technical Lemmas.

**Lemma 3.1.** Let \( A = \sum_{k=m}^{n} A_k, \, 0 \leq m \leq n \). Then

\[
\| A \|_{C_p} r^n \leq \| A(r) \|_{C_p} \leq \| A \|_{C_p} r^m, \, (0 < r < 1)
\]

**Proof.** Let us take \( B[r] = \sum_{k=m}^{n} A_k r^{n-k} \).

Then

\[
\| A(r) \|_{C_p} = \| A(r)^* \|_{C_p} = \| \sum_{k=m}^{n} A_k r^{n-k} \|_{C_p} = \left\| B \left[ \frac{1}{r} \right] \right\|_{C_p} r^n.
\]

Since \( \frac{1}{p} > 1 \) and the function \( se^{it} \rightarrow \| B(se^{it}) \|_{C_p}, \, 0 < s < 1, \, t \in [0,2\pi] \) is a subharmonic function, we have that

\[
\left\| B \left[ \frac{1}{r} \right] \right\|_{C_p} \left( \int_0^{2\pi} \left\| B \left[ \frac{1}{r} \right] (e^{it}) \right\|_{C_p}^p \frac{dt}{2\pi} \right)^{\frac{1}{p}} \geq \left( \int_0^{2\pi} \| B[1]e^{it} \|_{C_p}^p \frac{dt}{2\pi} \right)^{\frac{1}{p}}
\]

\[
= \| B[1] \|_{C_p} = \| A^* \|_{C_p} = \| A \|_{C_p}.
\]

This proves the left hand side inequality. The proof of the right hand side inequality is similar so we omit the details. \( \square \)

**Lemma 3.2.** Let \( A \) be an upper triangular matrix,

\[
\sigma_k(A) = \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) A_k,
\]

the Cesaro mean of the order \( k \) and

\[
\| \sigma_n(A) \|_p = \sup_{0 < r < 1} \| \sigma_n(A)(r) \|_{C_p}, \, (n = 0, 1, 2, \ldots).
\]
Then
\[ \|σ_k(A)\|_p r^k \leq \|A(r)\|_{C_p} \leq (1 - r)^2 \sum_{n=0}^{\infty} \|σ_n(A)\|_p (n + 1) r^n. \]

**Proof.** First we observe that
\[ \|A(r)\|_{C_p} \geq \|σ_n(A)(r)\|_{C_p}, \text{ for every } r \in (0, 1), n \in \mathbb{N}. \]

Since
\[ \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt} \|F_n(t)\|_{L^1(T)} \leq 1, \]
\[ F_n \in M(\ell^2) = M(C_1) \subset M(C_p) \subset M(C_2) \]
for \(1 \leq p \leq 2\) and \(M(C_p) = M(C_{p'})\), \(\frac{1}{p} + \frac{1}{p'} = 1\), for \(2 \leq p < \infty\) it follows that \(\|σ_n(A)(r)\|_{C_p} = \|A(r) * F_n\|_{C_p} \leq \|A(r)\|_{C_p}\). By using this inequality and Lemma 3.1 we find that
\[ \|A(r)\|_{C_p} \geq \|σ_n(A)(r)\|_{C_p} \geq r^n \|σ_n(A)\|_{C_p} \geq r^n \|σ_n(A)\|_p, \]
and the left hand side of the inequality is proved. The proof of the right hand of the inequality follows by using the formula
\[ A(r) = (1 - r)^2 \sum_{n=0}^{\infty} σ_n(A)(n + 1) r^n \]
and Minkowski’s inequality. The proof is complete. \(\square\)

**Lemma 3.3.** Let \(A\) be an upper triangular matrix, \(0 \leq k < n\) and \(p \geq 1\). Then we have that
\[ (n - k + 1) \|σ_k(A)\|_p \leq (n + 1) \|σ_n(A)\|_p. \]

**Proof.** First we note that \(\|σ_n(A)\|_p = \sup_{0 < r < 1} \|σ_n(A)(r)\|_{C_p}\). Since
\[ \|σ_k(B)\|_{C_p} \leq \|B\|_{C_p} \]
it follows that
\[ \|σ_n(A)\|_p \geq \|σ_kσ_n(A)\|_p = \|σ_k(A) - \frac{r}{n + 1} σ_k' (A)\|_p \]
\[ \geq \|σ_k(A)\|_p - \frac{1}{n + 1} \|σ_k'(A)\|_p \geq \]
[ by Bernstein’s inequality ] \[ \geq \|σ_k(A)\|_p - \frac{k}{n + 1} \|σ_k(A)\|_p, \]
and the inequality (4) is proved. \(\square\)

Our main result in this Section reads:
Theorem 3.4. Let $A$ be an analytic matrix. Then $A \in L^p_a(D, \ell^2)$ if and only if
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \|\sigma_n(A)\|_p^p < \infty.
\]

Proof. First we prove that
\[
A \in L^p_a(D, \ell^2)
\]
if and only if
\[
(5) \quad \left( \int_0^1 \|A(r)\|_{C_p}^p \, dr \right)^{\frac{1}{p}} < \infty.
\]
If \( \left( \int_0^1 \|A(r)\|_{C_p}^p \, dr \right)^{\frac{1}{p}} < \infty \), then it is clear that $A \in L^p_a(D, \ell^2)$. Conversely if $A \in L^p_a(D, \ell^2)$, then $\|A(r)\|_{L^p_a(D, \ell^2)} < \infty$.

Let $E_{\theta}$ be the Toeplitz matrix corresponding to the Dirac measure $\delta_{\theta}$, i.e.,
\[
E_{\theta} = (\epsilon_{kj})_{k,j \geq 1}, \quad \epsilon_{kj} = e^{i(j-k)t}.
\]
Then $E_{\theta} \in M(\ell^2) = M(C_1) \subset M(C_\rho)$, $1 \leq p < \infty$.

Since $re^{i\theta} \to \|A(r) * E_{\theta}\|_{C_p}$ is subharmonic on $D$ it follows that
\[
\int_0^{2\pi} \|A(r) * E_{\theta}\|_{C_p}^p \, d\theta \leq \int_0^{2\pi} \|A(\sqrt{r}) * E_{\theta}\|_{C_p}^p \, d\theta.
\]

Therefore
\[
\int_0^1 \int_0^{2\pi} \|A(r) * E_{\theta}\|_{C_p}^p \, d\theta \, dr \leq \int_0^1 \int_0^{2\pi} \|A(\sqrt{r}) * E_{\theta}\|_{C_p}^p \, d\theta \, dr
\]
and
\[
\int_0^1 \|A(r)\|_{C_p}^p \, dr \leq \int_0^1 \|A(\sqrt{r})\|_{C_p}^p \, dr = 2 \int_0^1 \|A(s)\|_{C_p}^p \, ds < \infty,
\]
and (5) is proved.

Now let $A \in L^p_a(D, \ell^2)$. Then, by the first inequality in Lemma 3.2, we have that
\[
\|A(r)\|_{C_p}^p = (1-r) \sum_{n=0}^{\infty} \|A(r)\|_{C_p}^p \, r^n \geq (1-r) \sum_{n=0}^{\infty} \|\sigma_n(A)\|_{p^n}^{p^n(p+1)}.
\]

Now integration yields that
\[
\infty > \int_0^1 \|A(r)\|_{C_p}^p \, dr \geq \int_0^1 (1-r) \sum_{n=0}^{\infty} \|\sigma_n(A)\|_{p^n}^{p^n(p+1)} \, dr
\]
\[
\geq C^{-1} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \|\sigma_n(A)\|_p^p.
\]
Conversely, suppose that
\[ \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p^p \left( \frac{1}{n+1} \right)^2 < \infty. \]

Let \( x_n = \sum_{k=0}^{\infty} (k+1)(n-k+1) \| \sigma_k(A) \|_p. \) Then, by summing by parts as in (4.4) in [12], we find that
\[ \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1)x_n = (1-r)^2 \sum_{n=0}^{\infty} x_n r^n. \]

On the other hand, by using Lemma 3.3, we see that
\[ x_n \leq C(n+1)^3 \| \sigma_n(A) \|_p \]
and therefore,
\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3p+2}} x_n^p < \infty. \]

We use now Lemma 4.8 in [12] with \( q = p, \phi(r) = r^{\frac{1}{p}} \), \( r \in (0,1] \) and obtain that
\[ \int_0^1 \left( (1-r)^2 \sum_{n=0}^{\infty} x_n r^n \right)^p dr < \infty. \]

Moreover by using also (6) we arrive at
\[ \int_0^1 (1-r)^{4p} \left( \frac{1}{(1-r)^2} \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1)r^n \right)^p dr < \infty. \]

It follows that
\[ \int_0^1 (1-r)^{2p} \left( \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1)r^n \right)^p dr < \infty. \]

From the right hand side inequality in Lemma 3.2, we finally obtain that
\[ \int_0^1 \| A(r) \|_p^p dr \leq \int_0^1 (1-r)^{2p} \left( \sum_{n=0}^{\infty} \| \sigma_n(A) \|_p (n+1)r^n \right)^p dr < \infty. \]

The proof is complete. \( \square \)

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BERGMAN-SCHATTEN SPACES AND A DUALITY RESULT

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Paper 4
ABSTRACT. Let $B_p(\ell^2)$ denote the Besov-Schatten space of all upper triangular matrices $A$ such that

$$\|A\|_{B_p(\ell^2)} = \left[ \int_0^1 (1 - r^2)^{2p} \|A''(r)\|_{C_p} \, d\lambda(r) \right]^{\frac{1}{p}} < \infty.$$ 

In this paper we present and prove a duality result between $B_p(\ell^2)$ and $B_q(\ell^2)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and also a similar result for the limit cases $p = 1$ and $q = 1$. We also give a sufficient condition for a Hankel operator to be nuclear.

1. Introduction

Analytic Besov spaces first found a direct application in operator theory in Peller’s paper [9]. A comprehensive account of the theory of Besov spaces is given in Peetre’s book [7]. In what follows we consider the Besov-Schatten spaces in the framework of matrices e.g. infinite matrix valued functions. The extension to the matriceal framework is based on the fact that there is a natural correspondence between Toeplitz matrices and formal series associated to 2$\pi$-periodic functions (see e.g. [1], [3], [5], [14]). In particular the Besov-Schatten space $B_1(\ell^2)$ can be used to prove that the associated Hankel operator is nuclear. We use the powerful device Schur multipliers and its characterizations in the case of Toeplitz matrices to prove some of the main results.

The Schur product (or Hadamard product) of matrices $A = (a_{jk})_{j,k \geq 0}$ and $B = (b_{jk})_{j,k \geq 0}$ is defined as the matrix $A \ast B$ whose entries are the products of the entries of $A$ and $B$:

$$A \ast B = (a_{jk}b_{jk})_{j,k \geq 0}.$$ 

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If $X$ and $Y$ are two Banach spaces of matrices we define Schur multipliers from $X$ to $Y$ as the space $M(X,Y) = \{ M : M \ast A \in Y \text{ for every } A \in X \}$, equipped with the natural norm

$$
\|M\| := \sup_{\|A\|_X \leq 1} \|M \ast A\|_Y.
$$

In the case $X = Y = B(\ell^2)$, where $B(\ell^2)$ is the space of all linear and bounded operators on $\ell^2$, the space $M(\ell^2,\ell^2)$ will be denoted $M(\ell^2)$ and a matrix $A \in M(\ell^2)$ will be called Schur multiplier. We mention here the following important result due to G. Bennett [4], which will be often used in this paper:

**Theorem 1.1.** The Toeplitz matrix $M = (c_{j-k})_{j,k}$, where $(c_n)_{n \in \mathbb{Z}}$ is a sequence of complex numbers, is a Schur multiplier if and only if there exists a bounded and complex Borel measure $\mu$ on (the circle group) $T$ with

$$
\hat{\mu}(n) = c_n, \text{ for } n = 0, \pm 1, \pm 2, \ldots.
$$

Moreover, we then have that

$$
\|M\| = \|\mu\|.
$$

We will denote by $C_p$, $0 < p < \infty$, the Schatten class of operators (see e.g. [15]). Let us summarize briefly some well-known properties of the classes $M(C_p)$, which will be very often used in what follows. If $1 < p < \infty$, then $M(C_p) = M(C_p')$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $M(\ell^2) = M(C_1)$. Next, interpolating between the classes $C_p$, we can easily see that $M(C_{p_1}) \subset M(C_{p_2})$ if $0 < p_1 \leq p_2 \leq 2$ (see e.g. [2]). We will denote by $A_k$, the $k^{th}$-diagonal matrix associated to $A$ (see [3]). For an infinite matrix $A = (a_{ij})$ and an integer $k$ we denote by $A_k$ the matrix whose entries $a_{ij}'$ are given by

$$
a_{ij}' = \begin{cases} a_{ij} & \text{if } j - i = k \\ 0 & \text{otherwise} \end{cases}.
$$

In what follows we will recall some definitions from [11] (see also [6]), which we will use in this paper. We consider on the interval $[0,1)$ the Lebesgue measurable infinite matrix-valued functions $A(r)$. These functions may be regarded as infinite matrix-valued functions defined on the unit disc $D$ using the correspondence

$$
A(r) \to f_A(r,t) = \sum_{k=-\infty}^{\infty} A_k(r) e^{ikt},
$$
where $A_k(r)$ is the $k$-th diagonal of the matrix $A(r)$, the preceding sum is a formal one and $r$ belongs to the torus $\mathbb{T}$. This matrix $A(r)$ is called \textit{analytic matrix} if there exists an upper triangular infinite matrix $A$ such that, for all $r \in [0, 1)$, we have $A_k(r) = Akr^k$, for all $k \in \mathbb{Z}$. In what follows we identify the analytic matrices $A(r)$ with their corresponding upper triangular matrices $A$ and we call them also analytic matrices.

We also recall the definition of the matriceal Bloch space and the so-called little Bloch space of matrices (see [6]). The \textit{matriceal Bloch space} $B(D, \ell^2)$ is the space of all analytic matrices $A$ with $A(r) \in B(\ell^2)$, $0 \leq r < 1$, such that

$$\|A\|_{B(D, \ell^2)} = \sup_{0 \leq r < 1} (1 - r^2)\|A'(r)\|_{B(\ell^2)} + \|A_0\|_{B(\ell^2)} < \infty,$$

where $B(\ell^2)$ is the usual operator norm of the matrix $A$ on the sequence space $\ell^2$ and $A'(r) = \sum_{k=0}^{\infty} Ak r^{k-1}$.

The space $B_0(D, \ell^2)$ is the space of all upper triangular infinite matrices $A$ such that $\lim_{r \to 1-} (1 - r^2)\|(A*C(r))'\|_{B(\ell^2)} = 0$, where $C(r)$ is the Toeplitz matrix associated with the Cauchy kernel $\frac{1}{1-r}$, for $0 \leq r < 1$.

An important tool in this paper is the \textit{Bergman projection}. It is known (see e.g. [11]) that for all strong measurable $C_p$-valued functions $r \to A(r)$ defined on $[0, 1)$ with $\int_0^1 \|A(r)\|^p_{C_p} 2r dr < \infty$ and for all $i, j \in \mathbb{N}$ we have that

$$[P(A(\cdot))](r)(i, j) = \begin{cases} 2(j - i + 1)\int_0^1 a_{ij}(s) \cdot s^{j-i+1} ds, & \text{if } i \leq j, \\ 0 & \text{otherwise} \end{cases}$$

We recall now a Lemma from [6] that we will use in the following.

**Lemma 1.2.** Let $V = (P_2)^*$, i.e., $(P_2A(\cdot))^*(r)(i, j) = \begin{cases} \frac{(j-i+3)(j-i+2)(j-i+1)(j-i)(j-i-1)}{2} r^{j-i}(1 - r^2)^2 \int_0^1 a_{ij}(s)s^{j-i}(2sds) & \text{if } j - i \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Then $V$ is an isomorphic embedding of $B_0(D, \ell^2)$ in $C_0(D, \ell^2)$.

The paper is organized as follows: In Section 2 we give a characterization of matrices in the Besov-Schatten space $B_p(\ell^2)$ using the Bergman projection. The main result in Section 3 is a new duality result (see Theorem 3.2). Finally, in Section 4 we give a sufficient condition for a Hankel operator to be nuclear.

2. \textbf{Besov-Schatten spaces}

Now we introduce a new space of matrices namely the so called \textit{Besov-Schatten space}. [136x690]
Definition 2.1. Let $1 \leq p < \infty$ and a positive measure on $[0,1)$ given by

$$d\lambda(r) := \frac{2rdr}{(1-r^2)^2}.$$ 

The Besov-Schatten matrix space $B_p(\ell^2)$ is defined to be the space of all upper triangular infinite matrices $A$ such that

$$\|A\|_{B_p(\ell^2)} := \left[ \int_0^1 (1-r^2)^{2p}\|A''(r)\|_{C_p}^p d\lambda(r) \right]^{\frac{1}{p}} < \infty.$$ 

On $B_p(\ell^2)$ we introduce the norm

$$\|A\| := \|A_0\|_{C_p} + \|A\|_{B_p(\ell^2)}.$$ 

We also introduce the notation $L^p(D,d\lambda,\ell^2)$ for the space of all strongly measurable functions $r \rightarrow A(r)$ defined on the measurable space $([0,1),d\lambda)$ with $C_p$-values such that

$$\|A\|_{L^p(D,d\lambda,\ell^2)} := \left( \int_0^1 \|A(r)\|_{C_p}^p d\lambda(r) \right)^{\frac{1}{p}} < \infty.$$ 

We need the following useful Lemma in what follows (see [15], p.53):

Lemma 2.2. Let $z \in D$, $c$ is real, $t > -1$, and

$$I_{c,t} := \int_D \frac{(1-|w|^2)^t}{|1-z\overline{w}|^{2+t+c}} dA(w).$$

(1) If $c < 0$, then $I_{c,t}(z)$ is bounded in $z$;
(2) If $c > 0$, then

$$I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c} (|z| \rightarrow 1^-);$$
(3) If $c = 0$, then

$$I_{0,t}(z) \sim \log \frac{1}{1-|z|^2} (|z| \rightarrow 1^-).$$

In the next theorem we express a natural relation between the Bergman projection and the Besov-Schatten spaces. More precisely, our main result of this section is the following equivalence theorem:

Theorem 2.3. Let $1 \leq p < \infty$ and $A$ be an upper triangular matrix such that the $C_p$- valued function $r \rightarrow A''(r)$ is continuous on $[0,r_0)$ for some $r_0$, $0 < r_0 < 1$. Then the following assertions are equivalent:

(1) $A \in B_p(\ell^2)$;
(2) $(1-r^2)^2 A''(r) \in L^p(D,d\lambda,\ell^2)$;
(3) $A \in PL^p(D,d\lambda,\ell^2)$, where $P$ is the Bergman projection.
Proof. It is obvious that (1) is equivalent to (2). We observe that the Bergman projection may be described as follows:

\[ P(A(\cdot)) = \sum_{k=0}^{\infty} (k+1) \int_{0}^{1} [A(s)]_k s^k (2sds), \]

where \( A(\cdot) \in L^p(D, \ell^2) \). Then

\[ P \left( (1-r^2)^2 A_k r^k \right)(s) = \frac{2s^k A_k}{(k+2)(k+3)}, \]

for all \( k \geq 0 \), and all \( A_k \in C_p \).

It follows that each matriceal polynomial is in \( P L^p(D, d\lambda, \ell^2) \) for all \( 1 \leq p < \infty \).

Suppose that \( A \) is an upper triangular matrix with \( A_k \in C_p \) for all \( k \geq 0 \). We write

\[ A = \sum_{k=0}^{4} A_k + A^1, \]

where \( A^1 := \sum_{k=5}^{\infty} A_k \).

If \( (1-r^2)^2 A''(r) \in L^p(D, d\lambda, \ell^2) \), then we have that

\[ \Phi(r) := \sum_{k=0}^{4} \frac{(k+2)(k+3)}{2} (1-r^2)^2 A_k r^k + \frac{(1-r^2)^2 (A^1)''(r)}{2! r^2} \]

is in \( L^p(D, d\lambda, \ell^2) \) and, moreover, that \( A = P\Phi \).

Indeed, for \( 0 < r < r_0 \), \( r \to (A^1)''(r) \) is a continuous function. Therefore

\[ \int_{0}^{r_0} \frac{||(A^1)'(s)||_{C_p}}{s^{2p}} ds < \infty, \]

so that \( \Phi \in L^p(D, d\lambda, \ell^2) \).

Moreover \( A = P\Phi \) since

\[ \sum_{k=5}^{\infty} \int_{0}^{1} \frac{k(k-1) A_k s^{k-2}(1-s)^2}{s^2} (k+1) s^{k+1} ds = 0 \]

\[ = \sum_{k=5}^{\infty} (k-1)k(k+1) A_k \int_{0}^{1} s^{2k-3}(1-s^2)^2 ds = \sum_{k=5}^{\infty} A_k. \]

Thus we have proved that (2) implies (3).

It remains to prove that (3) implies (2). Suppose that (3) holds and let \( A = P\Phi \) for some \( \Phi(\cdot) \in L^p(D, d\lambda, \ell^2) \). Then we have that

\[ (1-r^2)^2 A''(r) = (1-r^2)^2 \int_{0}^{1} \left[ \Phi(s) * \frac{6s^2}{(1-rs)^4} \right] 2s ds. \]
Using Fubini’s Theorem and Lemma 2.2 we obtain that

\[
\int_0^1 (1 - r^2)^2 \| A''(r) \|_{C_1} \frac{2r dr}{(1 - r^2)^2} \leq
\]

\[
\leq \int_0^1 \left[ \int_0^1 \| \Phi(s) \|_{C_1} \int_0^{2\pi} \frac{6s^2 d\theta}{(1 - r se^{i\theta})^4} 2s ds \right] 2r dr =
\]

\[
= \int_0^1 6s^2 \| \Phi(s) \|_{C_1} \left[ \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - r se^{i\theta})^4} \right] 2r dr \sim
\]

\[
\sim \int_0^1 \| \Phi(s) \|_{C_1} \frac{12s^3}{(1 - s^2)^2} ds \leq 6 \int_0^1 \| \Phi(s) \|_{C_1} d\lambda(s) < \infty.
\]

Consequently, \( A \in L^1(D,d\lambda,\ell^2) \) and this proves that (3) implies (2) in the case \( p = 1 \). The proof in the case \( 1 < p < \infty \) is similar with the classical case of functions (see e.g. Theorem 5.3.3. in [15]). Let \( T(rs) = ((t_{ij}(rs))_{i,j=1}^\infty \) be the Toeplitz matrix with

\[
t_{ij}(rs) = t_{j-i}(rs) =
\]

\[
= \begin{cases} 
    s^2(rs)^{j-i}(j-i+3)(j-i+2)(j-i+1) & \text{if } j \geq i \\
    0 & \text{otherwise}.
\end{cases}
\]

Since \( T(rs) \) is a Schur multiplier with \( \| T(rs) \|_{M(C^2)} = \| T(rs) \|_{L^1(T)} = \| \frac{6s^2}{(1 - r se^{i\theta})^4} \|_{L^1(T)} \) and \( M(\ell^2) = M(C_1) \subset M(C_p), 1 \leq p < \infty \) we get that

\[
(1 - r^2)^2 \| A''(r) \|_{C_p} = (1 - r^2)^2 \| \int_0^1 [\phi(s) * \frac{6s^2}{(1 - r s)^4}] (2s ds) \|_{C_p} \leq
\]

\[
\leq (1 - r^2)^2 \int_0^1 \| \phi(s) \|_{C_p} \| \frac{6s^2}{(1 - r se^{i\theta})^4} \|_{L^1(T)} (2s ds) =
\]

\[
= (1 - r^2)^2 \int_0^1 \| \phi(s) \|_{C_p} (1 - s^2)^2 \| \frac{6s^2}{(1 - r se^{i\theta})^4} \|_{L^1(T)} d\lambda(s) := S_\phi(r).
\]

From Schur’s Theorem (see e.g. [15]) it follows that \( S_\phi(r) \) is bounded on \( L^p([0,1],d\lambda) \) which in its turn implies that

\[
(1 - r^2)^2 A''(r) \in L^p(D,d\lambda,\ell^2)
\]

for \( 1 < p < \infty \). Thus also the implication (3) \( \Rightarrow \) (2) is proved and the proof is complete.
3. The dual of Besov-Schatten spaces

Our aim in this Section is to characterize the Banach dual spaces of Besov-Schatten spaces.

First we prove the following lemma of independent interest:

**Lemma 3.1.** Let $V = (P_2)^*$, that is

$$[V(A(\cdot))] (r)(i,j) := \begin{cases} 
\frac{(j-i+3)(j-i+2)(j-i+1)}{2} r^{j-i}(1-r^2)^2 \int_0^1 a_{ij}(s)s^{j-i}(2ds) & \text{if } j - i \geq 0 \\
0 & \text{otherwise}.
\end{cases}$$

Then $V$ is an embedding from $B_p(\ell^2)$ into $L_p(D, d\lambda, \ell^2)$ for all $p \geq 1$, if $B_p(\ell^2) = PL_p(D, d\lambda, \ell^2)$ is equipped with the quotient norm.

**Proof.** Suppose that $A \in B_p(\ell^2)$ and $B(\cdot) \in L_p(D, d\lambda, \ell^2)$ with $A = PB(\cdot)$. Since

$$P(B(\cdot))(r)(i,j) = \begin{cases} 
2(j - i + 1)r^{j-i} \int_0^1 b_{ij}(s)s^{j-i+1}ds & \text{if } j - i \geq 0 \\
0 & \text{otherwise},
\end{cases}$$

it is easy to see that $PV = P$ and $VP = V$ on $L_p(D, d\lambda, \ell^2)$. Therefore $V(A) = V(B(\cdot))$ for all $A \in B_p(\ell^2)$ and $B(\cdot) \in L_p(D, d\lambda, \ell^2)$.

We will now prove that $V$ is a bounded operator on $L_p(D, d\lambda, \ell^2)$. We first prove this fact for $p = 1$. By Fubini’s theorem we have that

$$\|V(A(\cdot))\|_{L^1(D, d\lambda, \ell^2)} = \int_0^1 \|V(A(\cdot))\|_{C_1} d\lambda(r) =$$

$$\int_0^1 \|\sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k (1-r^2)^2 \int_0^1 A_k(s)s^k(2ds)\|_{C_1} d\lambda(r) \leq$$

$$\int_0^1 \int_0^1 \|\sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k (1-r^2)^2 A_k(s)s^k(2ds)\|_{C_1} (2rdr)d\lambda(s) =$$

$$\int_0^1 \int_0^1 \|\sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k A_k(s)s^k(1-s^2)^2\|_{C_1} (2rdr)d\lambda(s) =$$

$$= \int_0^1 \int_0^1 \|A(s) \ast C(rs)\|_{C_1} (2rdr)d\lambda(s),$$
where $C(rs) = (c_{ij}(rs))_{i,j=1}^\infty$ means the Toeplitz matrix given by

$$
c_{ij}(rs) = c_{j-i}(rs) = \begin{cases} 
  (rs)^{j-i} s^2 (1 - s^2)^2 \frac{(j-i+3)(j-i+2)(j-i+1)}{2} & \text{if } j \geq i \\
  0 & \text{otherwise.}
\end{cases}
$$

Since the Toeplitz matrix $C(rs)$ is a Schur multiplier with

$$
\|C(rs)\|_{M(\ell^2)} = \|6s^2(1 - s^2)^2\|_{L^1(T)},
$$

according to Lemma2.2 it follows that

$$
\int_1^0 \int_0^1 \|A(s) * C(rs)\|_{C_p(2rdr)} d\lambda(s) \leq \int_0^1 \|A(s)\|_{C_p} \int_0^1 \|C(rs)\|_{M(\ell^2)}(2rdr) d\lambda(s) \sim \int_0^1 \|A(s)\|_{C_p} d\lambda(s).
$$

Consequently, $V$ is bounded on $L^1(D, d\lambda, \ell^2)$. For $1 < p < \infty$ we have that

$$
\|VA(\cdot)(r)\|_{C_p} \leq \int_0^1 \left\| \sum_{k=0}^\infty \frac{(k+3)(k+2)(k+1)}{2} r^k s^k (1 - r^2)^2 (1 - s^2)^2 A_k(s) \right\|_{C_p} d\lambda(s) = \int_0^1 \|A(s) * T(rs)\|_{C_p} d\lambda(s),
$$

where $T(rs) = (t_{j-i}(rs))_{i,j}$ is a Toeplitz matrix and

$$
t_{j-i}(rs) = \begin{cases} 
  (rs)^{j-i} (1 - s^2)^2 (1 - r^2)^2 \frac{(j-i+3)(j-i+2)(j-i+1)}{2} & \text{if } j \geq i \\
  0 & \text{otherwise.}
\end{cases}
$$

$T(rs)$ is a Schur multiplier and therefore

$$
\int_0^1 \|A(s) * T(rs)\|_{C_p} d\lambda(s) \leq \int_0^1 \|A(s)\|_{C_p} (1 - r^2)^2 (1 - s^2)^2 \|6\|_{(1 - rs e^{i\theta})^4} \|_{L^1(T)} =: S_A(r).
$$

From Schur’s Theorem (see e.g., [15]) we obtain that $S_A(r)$ is bounded on $L^p([0,1), d\lambda)$. Hence $V$ is bounded on $L^p(D, d\lambda, \ell^2)$, $1 \leq p < \infty$, and there is a constant $C > 0$ such that

$$
\|V(A(\cdot))\|_{L^p(D, d\lambda, \ell^2)} \leq C \|B(\cdot)\|_{L^p(D, d\lambda, \ell^2)}.$$
for all $A = PB(\cdot)$. Taking the infimum over $B$, we get that
\[
\|V(A)\|_{L^p(D, d\lambda, \ell^2)} \leq C\|A\|_{B_p(\ell^2)}.
\]
Thus $V : B_p(\ell^2) \to L^p(D, d\lambda, \ell^2)$ is bounded. On the other hand, since $PV = P$ and $VP = V$ on $L^p(D, d\lambda, \ell^2)$ we get easily that $A = PV(A)$ for all $A \in B_p(\ell^2)$. Thus,
\[
\|A\|_{B_p(\ell^2)} = \inf \{\|B(\cdot)\|_{L^p(D, d\lambda, \ell^2)} : A = PB\} \leq \|V A\|_{L^p(D, d\lambda, \ell^2)}
\]
and, hence, $V : B_p(\ell^2) \to L^p(D, d\lambda, \ell^2)$ is an embedding. The proof is complete. \qed

Now we can formulate and prove the duality of Besov-Schatten spaces.

**Theorem 3.2.** Under the pairing
\[
< A, B > = \int_0^1 tr(V(A)[V(B)]^*)d\lambda(r)
\]
we have the following dualities:
(1) $B_p(\ell^2)^* \approx B_q(\ell^2)$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;
(2) $B_{0,c}(D, \ell^2)^* \approx B_1(\ell^2)$ and $B_1(\ell^2)^* \approx B(D, \ell^2)$.

**Proof.** Since $V$ is an embedding from $B_p(\ell^2)$ into $L^p(D, d\lambda, \ell^2)$ for all $1 \leq p < \infty$, Hölder’s inequality shows that $B_q(\ell^2) \subset B_p(\ell^2)^*$ for $1 \leq p < \infty$ and $B_1(\ell^2) \subset B^*_{0,c}(D, \ell^2)$.

Suppose that $F$ is a bounded linear functional on the Besov-Schatten space $B_p(\ell^2)$ with $1 \leq p < \infty$. Then $F \circ V^{-1} : VB_p(\ell^2) \to \mathbb{C}$ can be extended to a bounded linear functional on $L^p(D, d\lambda, \ell^2)$. Thus there exists $C(\cdot) \in L^p(D, d\lambda, \ell^2)$ such that $\|C(\cdot)\|_{L^p(D, d\lambda, \ell^2)} = \|F \circ V^{-1}\|$ and
\[
(F \circ V^{-1})(B) = \int_0^1 tr(B(r))[C(r)]^*d\lambda(r), \quad B(\cdot) \in L^p(D, d\lambda, \ell^2).
\]
In particular, if $B(\cdot) = V(A)$ with $A \in B_p(\ell^2)$, then
\[
F(A) = \int_0^1 tr((VA)(r))[C(r)]^*d\lambda(r).
\]
Let $B = P(C)$; then $B \in B_q(\ell^2)$ and it is easy to check that
\[
F(A) = \int_0^1 tr((VA)(r)(VB)(r))^*d\lambda(r), \quad A \in B_p(\ell^2),
\]
with $\|B\|_{B_q(\ell^2)} \leq \|C(\cdot)\|_{L^p(D, d\lambda, \ell^2)} = \|F \circ V^{-1}\| \leq \|V^{-1}\| \|F\|$. This proves the duality $B_p(\ell^2)^* \approx B_q(\ell^2)$ for $1 \leq p < \infty$.

It remains to prove the duality $B^*_{0,c}(D, \ell^2) \approx B_1(\ell^2)$.\[\]
Let us assume that \( F \) is a bounded linear functional on \( B_{0,c}(D, \ell^2) \). Then we shall prove that there is a matrix \( C \) from \( B_1(\ell^2) \) such that

\[
F(B) = \int_0^1 \text{tr}[VB(r)(VC)^*(r)]d\lambda(r),
\]

for \( B \) from a dense subset of \( B_0(D, \ell^2) \). By Lemma 1.2 it follows that \( V : B_0(D, \ell^2) \to C_0(D, \ell^2) \) is an isomorphic embedding. Thus \( X = V(B_{0,c}(D, \ell^2)) \) is a closed subspace in \( C_0(D, C_\infty) \) and \( F \circ (V)^{-1} : X \to \mathbb{C} \) is a bounded linear functional on \( X \), where \( C_0(D, C_\infty) \) is the subset in \( C_0(D, \ell^2) \) whose elements are \( C_\infty \)-valued functions. By the Hahn-Banach theorem \( F \circ (V)^{-1} \) can be extended to a bounded linear functional on \( C_0(D, C_\infty) \).

Let \( \Phi : C_0(D, C_\infty) \to \mathbb{C} \) denote this functional. It follows that \( C_0(D, C_\infty) = C_0[0, 1] \otimes C_\infty \) and, thus, \( \Phi \) is a bilinear integral map, that is there is a bounded Borel measure \( \mu \) on \( [0, 1] \times U_{C_1} \), where \( U_{C_1} \) is the unit ball of the space \( C_1 \) with the topology \( \sigma(C_1, C_\infty) \), such that

\[
\Phi(f \otimes A) = \int_{[0, 1] \times U_{C_1}} f(r)\text{tr}(AB^*)d\mu(r, B)
\]

for every \( f \in C_0[0, 1] \) and \( A \in C_\infty \).

Thus, for the matrix \( \sum_{k=0}^n A_k \in B_{0,c}(D, \ell^2) \), identified with the analytic matrix \( \sum_{k=0}^n A_k r^k \), we have that

\[
F(\sum_{k=0}^n A_k) = F(\sum_{k=0}^n r^k A_k) = [F \circ (V)^{-1}][V(\sum_{k=0}^n r^k A_k)] = \\
= \Phi(\sum_{k=0}^n \frac{(k+3)(k+2)}{2} r^k (1-r^2)^2 A_k) = \\
= \int_{[0, 1] \times U_{C_1}} \sum_{k=0}^n \text{tr}[(\frac{(k+3)(k+2)}{2} r^k (1-r^2)^2 A_k)B^*]d\mu(r, B) \\
\text{def} = < \mu(r, B), \text{tr}(\sum_{k=0}^n \frac{(k+3)(k+2)}{2} r^k A_k)B^* (1-r^2)^2 >.
\]
On the other hand, we wish to have that

\[ F(A) = \int_0^1 \text{tr} V(A)(V(C))^* d\lambda(s) = \]

\[ = \int_0^1 \text{tr} \left( \sum_{k=0}^n \frac{(k+3)(k+2)}{2} s^k A_k \right) (V(C))^* (2s ds) = \]

\[ = \int_0^1 \text{tr} \left( \sum_{k=0}^n \frac{s^{2k}(k+3)^2(k+2)^2}{4}(1 - s^2)^2 A_k C_k^* \right) (2s ds) = \]

\[ = \sum_{k=0}^n \text{tr} A_k \left( \frac{(k+3)(k+2)}{2(k+1)} C_k^* \right). \]

Therefore, letting \( A = e_{i,i+k} \), denote the matrix having 1 as the single nonzero entry on the \( i \)th-row and the \((i+k)\)th-column, for \( i \geq 1 \) and \( j \geq 0 \), we have that

\[ C_k = < \mu(r, B); (k+1)r^k(1 - r^2)^2 B_k >, k = 0, 1, 2, \ldots \]

Then, it yields that

\[ \int_0^1 \|C''(s)\|_{C_1} 2s ds = \]

\[ = \int_0^1 \left\| \sum_{k=2}^n \frac{(k+1)!(k-2)!}{r^k(1 - r^2)^2 B_k} d\mu(r, B) \right\|_{C_1}(2s ds) \leq \]

\[ \leq \int_{U_{C_1}} \left[ \int_0^1 \left\| \sum_{k=2}^n \frac{(k+1)!(k-2)!}{r^k(1 - r^2)^2 B_k} \|_{C_1}(2s ds) \right\| d|\mu|(r, B) \]

\[ \leq \int_{U_{C_1}} \left[ \int_0^1 \left\| \sum_{k=2}^n \frac{(k+1)!(k-2)!}{r^k(1 - r^2)^2 B_k} \|_{C_1}(2s ds) \right\| \right] d|\mu|(r, B) \]

\[ \leq \int_{U_{C_1}} \left( \int_0^1 \int_0^{2\pi} \frac{r^2(1 - r^2)^2 d\theta}{|1 - re^{i\theta}|^2} (2s ds) \right) d|\mu|(r, B) \sim \]

\[ \sim \int_{U_{C_1}} r^2(1 - r^2)^2 \frac{1}{(1 - r^2)^2} d|\mu|(r, B) \leq \|\mu\| < \infty. \]

Consequently, \( C \in B_1(\ell^2) \) and we get the relation (1) by using the fact that the set of all matrices \( \sum_{k=0}^n A_k \) is dense in \( \mathcal{B}_{0,c}(D, \ell^2) \). The proof is complete.

\( \square \)
4. Nuclear Hankel operators and the space $\mathcal{M}_1$

Let us denote with $T_2$ the subspace of $C_2$ consisting of all upper triangular matrices. We denote by $P_+$ the triangular projection, $P_+ A = \sum_{k \in \mathbb{Z}} A_k$ and $P_- = I - P_+$, where $I$ is the identity on $T_2$. Let $\Phi$ be an infinite matrix such that for all matrices $A = \sum_{k=0}^{n} A_k$, $n \in \mathbb{N}$, we have $P_-(\Phi A) \in C_2$.

Examples of such matrices are either all matrices representing linear bounded operators on $\ell^2$, or matrices $\Phi$ such that $P_- \Phi = 0$.

We define the matrix version of Hankel operator to be $H_\Phi: T_2 \to (T_2)_- := C_2 \ominus T_2$, on the dense subspace in $T_2$ of all matrices $A = \sum_{k=0}^{n} A_k$, $n \in \mathbb{N}$ such that $\Phi A \in C_2$ by $H_\Phi(A) = P_- (\Phi A)$.

The matrix $\Phi$ is called the symbol operator of Hankel operator $H_\Phi$.

We briefly indicate the connection between the matrix version of Hankel operator and the vector Hankel operator in the sense described by V. Peller in [10].

Let $f$ be a function on the torus $T$ such that for any function $g \in H^2(T)$, $f g$ belongs to $L^2(T)$. Examples of such functions are analytic functions from the classical Besov space $B_1$ (see e.g. [15]).

If $\Phi$ is a matrix as above, then the Hankel operator in the sense described by V. Peller, $H_{f \otimes \Phi} : H^2(T_2) \to (H^2)_-(T_2_-)$ coincides with the tensor product $H_f \otimes H_\Phi$ between the classical Hankel operator $H_f$ and the matrix version of Hankel operator $H_\Phi$.

We mention here that the matrix version of Hankel operator was first considered by S. C. Power in 1986 [13] and recently studied in more detail in [12].

We give now a sufficient condition in order that the Hankel operator $H_A$ to be nuclear.

We use the space $\mathcal{M}_1$ considered by A. Pelczynski and F. Sukochev in [8], that is $\mathcal{M}_1 = \{ A \text{ such that } \| A \|_{\mathcal{M}_1} := \sum_{k \in \mathbb{Z}+} \| A_k \|_{C_1} < \infty \}$.

This space was investigated in [8] from the viewpoint of Schur multipliers with range consisting of matrices with absolutely summable entries.

We intend to show that for an upper triangular matrix $A$ such that $A' = \sum_{k=0}^{\infty} k A_k \in \mathcal{M}_1$, the operator $H_A$ is nuclear. It is easy to see that
an upper triangular matrix $A$ as above belongs in fact to the Besov-Schatten matrix space $B_1(ℓ^2)$. Thus, our next Theorem gives a partial answer to the following conjecture.

**Conjecture 4.1.** If $A \in B_1(ℓ^2)$ then $H_{A^*}$ is a nuclear operator.

Let $1 \leq p < \infty$. Then let us denote by $S_p$ the Schatten class of all operators from $T_2$ into $T_{2-}$.

Then we have the following result:

**Theorem 4.2.** Let $A$ be an upper triangular infinite matrix such that $A' \in \mathcal{M}_1$. Then it follows that the Hankel operator $H_{A^*} : T_2 \rightarrow T_{2-}$ is nuclear. Moreover, there is a constant $C > 0$, such that

$$
\|H_{A^*}\|_{S_1} \leq \|A'\|_{\mathcal{M}_1}.
$$

**Proof.** The proof uses induction with respect to $n$. Of course if $A = A_0$, then $H_{A^*} = 0$, and $A' = 0$. Now let $A = A_1 = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots \\ 0 & 0 & a_1^2 & 0 & \cdots \\ 0 & 0 & 0 & a_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Then $H_{A^*}(B) = (I - P)(A^*B)$, where $B \in T_2$, $P$ is the upper triangular projection and $I$ is the identity matrix. Then

$$(I - P) \begin{pmatrix} 0 & 0 & 0 & \cdots \\ a_1^2 & 0 & 0 & \cdots \\ 0 & a_1^2 & 0 & \cdots \\ 0 & 0 & a_1^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots \\ 0 & b_0^2 & b_1^2 & \cdots \\ 0 & 0 & b_0^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} =
$$

$$
\begin{pmatrix} 0 & 0 & 0 & \cdots \\ \overline{a_1^2} b_0 & 0 & 0 & \cdots \\ 0 & \overline{a_1^2} b_0^2 & 0 & \cdots \\ 0 & 0 & \overline{a_1^2} b_0^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$
Consequently, \( H_{A^*}(B) = A_1^* \tau_{-1} B_0 \), for all \( B \in T_2 \), where

\[
\tau_{-1} \begin{pmatrix}
    b_0^1 & b_1^1 & b_2^1 & \cdots \\
    0 & b_0^2 & b_1^2 & \cdots \\
    0 & 0 & b_0^3 & \cdots \\
    \vdots & \cdots & \cdots & \ddots 
\end{pmatrix} = \begin{pmatrix}
    b_0^1 & b_1^1 & b_2^1 & \cdots \\
    b_0^2 & b_1^2 & b_2^2 & \cdots \\
    0 & b_0^3 & b_1^3 & \cdots \\
    \vdots & \cdots & \cdots & \ddots 
\end{pmatrix}.
\]

Denoting by \( E_k^j \), where \( j \in \mathbb{Z} \), and \( k \geq 1 \), the infinite matrix having as entries 1 on the \( k \)th place on the \( j \)th-diagonal, and 0 otherwise, we get easily that

\[
H_{A^*}(B) = \sum_{k=1}^{\infty} a_k^1 \langle E_0^k, B \rangle E_{k-1}^k = \sum_{k=1}^{\infty} a_k^1 \langle E_0^k \otimes E_{k-1}^k, B \rangle,
\]

where \( \sum_{k=1}^{\infty} |a_k^1| = \|A\|_{\mathcal{M}_1} < \infty \).

It follows that \( H_{A^*} \) is a nuclear operator for \( A = A_1 \in \mathcal{M}_1 \) and \( \|H_{A^*}\|_{\mathcal{S}_1} \leq \|A\|_{\mathcal{M}_1} \).

We consider now the case \( n = 2 \), that is \( A = A_1 + A_2 \). We have that

\[
H_{A^*}(B) = (I - P) \begin{pmatrix}
    \begin{pmatrix}
        0 & 0 & 0 & \cdots \\
        a_1^1 & 0 & 0 & \cdots \\
        a_1^2 & a_1^2 & 0 & \cdots \\
        0 & a_1^2 & a_1^2 & \cdots \\
        \vdots & \cdots & \cdots & \ddots 
    \end{pmatrix} & \begin{pmatrix}
        b_0^1 & b_1^1 & b_2^1 & \cdots \\
        b_0^2 & b_1^2 & b_2^2 & \cdots \\
        0 & b_0^3 & b_1^3 & \cdots \\
        \vdots & \cdots & \cdots & \ddots 
    \end{pmatrix} \\
    \begin{pmatrix}
        \bar{a}_1^1 b_0^1 & 0 & 0 & \cdots \\
        0 & 0 & 0 & \cdots \\
        \bar{a}_2^2 b_0^1 + a_1^2 b_0^2 & 0 & 0 & \cdots \\
        a_2^2 b_0^1 & \bar{a}_1^2 b_0^2 + a_1^2 b_0^2 & 0 & \cdots \\
        \vdots & \cdots & \cdots & \ddots 
    \end{pmatrix} \end{pmatrix} =
\]

\[
\begin{pmatrix}
    0 & 0 & 0 & \cdots \\
    a_1^1 b_0^1 & 0 & 0 & \cdots \\
    \bar{a}_2^2 b_0^1 & a_1^2 b_0^1 + \bar{a}_1^2 b_0^2 & 0 & \cdots \\
    0 & \bar{a}_2^2 b_0^1 & \bar{a}_1^2 b_0^2 + a_1^2 b_0^2 & 0 & \cdots \\
    \vdots & \cdots & \cdots & \ddots 
\end{pmatrix}.
\]

Then we obtain that

\[
H_{A^*} = \sum_{k=1}^{\infty} \frac{a_2^k}{a_2^2} E_0^k \otimes E_{-2}^k + \sum_{k=1}^{\infty} \frac{a_1^k}{a_1^2} E_0^k \otimes E_{-1}^k + \sum_{k=1}^{\infty} \frac{a_2^k}{a_2^2} E_0^k \otimes E_{-1}^k,
\]

which implies that \( H_{A^*} \) is a nuclear operator and

\[
\|H_{A^*}\|_{\mathcal{S}_1} \leq \|A_1\|_{\mathcal{C}_1} + 2 \|A_2\|_{\mathcal{C}_1} = \|A\|_{\mathcal{M}_1}.
\]

Let us consider now the case \( n = 3 \) which contains all the difficulties of the general case.
If $A = A_1 + A_2 + A_3$, then

$H_{A^*}(B) = (I - P)\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
\overline{a_1} & 0 & 0 & 0 & \cdots \\
\overline{a_2} & \overline{a_1} & 0 & 0 & \cdots \\
\overline{a_3} & \overline{a_2} & \overline{a_1} & 0 & \cdots \\
0 & \overline{a_3} & \overline{a_2} & \overline{a_1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix}
b_0^1 & b_1^1 & b_2^1 & b_3^1 & \cdots \\
0 & b_0^2 & b_1^2 & b_2^2 & \cdots \\
0 & 0 & b_0^3 & b_1^3 & \cdots \\
0 & 0 & 0 & b_0^4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$. 

Consequently

$H_{A^*} = \sum_{k=1}^{\infty} \overline{a_1} E_0^k \otimes E_{-1}^k + \sum_{k=1}^{\infty} \overline{a_2} (E_0^k \otimes E_{-2}^k + E_{-1}^k \otimes E_{-1}^{k+1}) + \sum_{k=1}^{\infty} \overline{a_3} (E_0^k \otimes E_{-3}^k + E_{-2}^k \otimes E_{-2}^{k+1} + E_{-1}^k \otimes E_{-1}^{k+2})$.

Hence

$\|H_{A^*}\|_{S1} \leq \|A_1\|_{C_1} + 2\|A_2\|_{C_1} + 3\|A_3\|_{C_1} = \|A^*\|_{M1}$

and the theorem can be proved in general by using an induction procedure. □

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Paper 5
A CLASS OF LINEAR OPERATORS ON QUASI-MONOTONE SEQUENCES IN $\ell^2$ AND ITS SCHUR MULTIPLIERS

LIVIU GABRIEL MARCOCI

Abstract. In this paper we introduce and investigate a new class of linear operators. In particular, we give concrete characterizations for some cases and prove some new results concerning Schur multipliers.

1. Introduction

We started our study motivated by the papers [3], [7], [16] and [17]. In the paper [16], the authors introduced the space $B_w(\ell^2)$ of those infinite matrices $A$ for which $A(x) \in \ell^2$ for all $x = (x_n)_n \in \ell^2$ with $|x_n| \searrow 0$. This Banach space of infinite matrices can be regarded as a weaker version of the space $B(\ell^2)$, the space of all bounded operators from $\ell^2$ into $\ell^2$. It is a weaker version because it consists of those operators which maps the sequences which are decreasing in modulus from $\ell^2$ into $\ell^2$. This space actually appeared in the study of matricial analogues of classical function spaces like $C(\mathbb{T})$ (the continuous functions on the torus), the Wiener algebra $A(\mathbb{T})$ and the Lebesgue space $L^1(\mathbb{T})$.

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It is easy to see that $B(\ell^2) \subset B_w(\ell^2)$ and that the inclusion is proper. It is interesting that these two spaces coincides on the subspace consisting of Toeplitz matrices. Let

$$B_w^α(\ell^2) = \{ A \text{ infinite matrix } ; Ax \in \ell^2 \text{ for every } \}
\begin{align*}
x = (x_n)_n \in \ell^2 \text{ with } \frac{|x_n|}{n^α} \searrow 0, \quad α \geq 0\end{align*}.$$  

Thus $B_w^0(\ell^2) = B_w(\ell^2)$.

The Schur product of two matrices is defined by

$$A \ast B = (a_{ij} \cdot b_{ij})_{i,j \geq 1},$$

where $A = (a_{ij})_{i,j \geq 1}, B = (b_{ij})_{i,j \geq 1}$. We denote by

$$M(\ell^2) = \{ M : M \ast A \in B(\ell^2) \text{ for every } A \in B(\ell^2) \}$$

the space of all Schur multipliers equipped with the following norm

$$\| M \| = \sup_{\| A \|_{B(\ell^2)} \leq 1} \| M \ast A \|_{B(\ell^2)}.$$

For an infinite matrix $A = (a_{ij})$ and an integer $k$ we denote by $A_k = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. we have that

$$A_k = \begin{pmatrix} 0 & 0 & \ldots & a_{1k} & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \ldots & 0 & a_{2k+1} & \ldots & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & a_{k2k-1} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}. $$
A k is called the Fourier coefficient of k:th order of the matrix A (see e.g. [3]). In particular, if \( a = (a_k)_{k \geq 1} \) is a sequence we write that
\[
A_0 = 
\begin{pmatrix} 
  a_1 & 0 & 0 & 0 & \cdots \\
  0 & a_2 & 0 & 0 & \cdots \\
  0 & 0 & a_3 & 0 & \cdots \\
  0 & 0 & 0 & a_4 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]
and we sometimes say that the diagonal matrix \( A_0 \) is given by the sequence \( a = (a_k)_{k \geq 1} \). The definition of the matrix \( A_k \) given by the sequence \( a = (a_k)_{k \geq 1} \) is similar.

The paper is organized as follows: In Section 2 we give some preliminaries. One of the main tools in this paper is to use Sawyer’s duality Theorem in the discrete case. We include also some additional definitions and results that we need in the sequel. In Section 3 we present the main results of this paper. In Section 4 we present the proofs of the main results. Finally, for the readers convenience, we include an appendix with proofs of some Theorems from Section 2.

2. Preliminaries

The following standard notations will be used very often: Letters such as \( X \) or \( Y \) are always \( \sigma \)-finite measure spaces and \( \mathcal{M}(X) (\mathcal{M}^+(X)) \) denotes the space of measurable (resp. measurable and nonnegative) functions on \( X \). We denote by \( \mathbb{R}_+ = (0, \infty) \). The letters \( w, \tilde{w}, w_0, ... \) are used for weight functions in \( \mathbb{R}_+ \) (nonnegative locally integrable functions in \( \mathbb{R}_+ \)). For a given weight \( w \) we write \( W(r) = \int_0^r w(t)dt < \infty, 0 \leq r < \infty \). If \( f \) is a positive nonincreasing function we will write \( f \searrow \). The following Lemma (see e.g. [9]) will be very useful.

Lemma 2.1. Let \( 0 < p < \infty \) and \( v \geq 0 \) be a measurable function in \( \mathbb{R}_+ \). Let \( V(r) = \int_0^r v(s)ds, 0 \leq r < \infty \). Then, for every decreasing function \( f \) we have that
\[
\int_0^\infty f^p(s)v(s)ds = \int_0^\infty pt^{p-1}V(\mu_f(t))dt.
\]
Next we recall an important result due to E. Sawyer which will be used in what follows. The original proof can be seen in [21] and other proofs in [8] and [25]. For completeness we present in our appendix a proof, which is a combination of the original proof and the one from [8].

**Theorem 2.2** (Sawyer’s Theorem). Suppose that $1 < p < \infty$ and that $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbb{R}_+$, with $v$ locally integrable. Then

$$\sup \frac{\int_0^\infty f(x)g(x)dx}{(\int_0^\infty f(x)^p v(x)dx)^{1/p}} \approx \left( \int_0^\infty \left( \int_0^x g(t)dt \right)^{p'} v(x) \left( \int_0^x v(t)dt \right)^{-p'} dx \right)^{1/p'} + \frac{\int_0^\infty g(t)dt}{(\int_0^\infty v(t)dt)^{1/p}},$$

where the supremum is taken over all nonnegative and nonincreasing functions $f$. Moreover, the right hand side of (1) can be replaced with the integral

$$\left( \int_0^\infty \left( \int_0^x g(t)dt \right)^{p'-1} \left( \int_0^x v(t)dt \right)^{1-p'} g(x)dx \right)^{1/p}.$$

The expression $f \approx g$ will indicate the existence of two positive constants $a$ and $b$ such that $af \leq g \leq bf$.

For $p \leq 1$, we also have (see [8]) the following result.

**Theorem 2.3.** Suppose $p \leq 1$ and that $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbb{R}_+$ with $v$ locally integrable. Then

$$\sup \frac{\int_0^\infty f(x)g(x)dx}{(\int_0^\infty f(x)^p v(x)dx)^{1/p}} \approx \sup_{r > 0} \left( \int_0^r g(x)dx \left( \int_0^r v(x)dx \right)^{-1/p} \right),$$

where the supremum is taken over all nonnegative and nonincreasing functions $f$. 
The discrete version of Sawyer’s formula is given by the next theorem (see [10]). We first need some preliminaries. In the paper [10] the authors considered the problem of characterizing the boundedness of

\[ T : (L^{p_0}(X) \cap L, \| \cdot \|_{p_0}) \to L^{p_1}(Y), \]

that is

\[ \| Tf \|_{L^{p_1}(Y)} \leq C \| f \|_{L^{p_0}(X)}, \quad f \in L, \]

where \( L \) is a subclass of \( \mathcal{M}(X) \) and \( T \) is an operator (usually linear or sublinear). In some cases inequality (2) holds for every \( f \in L \) if and only if it holds for the characteristic functions \( f = \chi_A \in L \). This happens (see [8] and [24]) in the case \( L = L^{p_0}(w_0), X = (\mathbb{R}_+, w_0(t)dt) \) and \( Y = (\mathbb{R}_+, w_1(t)dt) \) in the range of indices \( 0 < p_0 \leq 1, p_0 \leq p_1 < \infty \) and operators of the type

\[ Tf(r) = \int_0^\infty k(r,t)f(t)dt, \quad r > 0, \]

for positive kernels \( k \). In [10], they obtained a more general principle that can be applied to a more general class of operators than those of integral type and for a bigger class of functions that includes the monotone functions. We recall now some definitions from [10].

**Definition 2.4.** We say that \( \emptyset \neq L \subset \mathcal{M}(X) \) is a regular class in \( X \) if, for every \( f \in L \),

(i) \( |\alpha f| \in L \), for every \( \alpha \in \mathbb{R} \),

(ii) \( \chi_{\{|f|>t\}} \in L \), for every \( t > 0 \), and

(iii) there exists a sequence of simple functions \( (f_n)_n \subset L \) such that

\[ 0 \leq f_n(x) \leq f_{n+1}(x) \to |f(x)| \text{ a.e. } x \in X. \]

**Example 2.5.** (i) If \( X \) is an arbitrary measure space, every functional lattice in \( X \) (i.e., a vector space \( L \subset \mathcal{M}(X) \) with \( g \in L \) if \( |g| \leq |f|, f \in L \) with the Fatou property (see [5]), is a regular class. In particular the Lebesgue space \( L^p(X), 0 < p \leq \infty \) is a regular class.

(ii) In \( \mathbb{R}_+ \) the set of positive decreasing functions is a regular class and the same holds for the increasing functions.
Definition 2.6. Let $Y$ be a measure space, $L$ a regular class and $T : L \to \mathcal{M}(Y)$ an operator.

(i) $T$ is sublinear if

$$|T(\alpha_1 f_1 + \cdots + \alpha_n f_n)(y)| \leq |\alpha_1 T f_1(y)| + \cdots + |\alpha_n T f_n(y)|$$

a.e. $y \in Y$, for every $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and every $f_1, \ldots, f_n \in L$ such that $\alpha_1 f_1 + \cdots + \alpha_n f_n \in L$.

(ii) $T$ is monotone if

$$|Tf(y)| \leq |Tg(y)|$$

a.e. $y \in Y$, if $|f(x)| \leq |g(x)|$ a.e. $x \in X$, $f, g \in L$.

(iii) We say that $T$ is order continuous if it is monotone and if, for every sequence $(f_n)_n \subset L$ with $0 \leq f_n(x) \leq f_{n+1}(x) \to f(x)$ a.e. $x \in X$, we have that

$$\lim_n |T f_n(y)| = |T f(y)|$$

a.e. $y \in Y$.

Remark 2.7. The identity operator $f \mapsto f$ is obviously order continuous.

We recall an important result which will be used in what follows and where the boundedness of certain operators is characterized by their restriction to the set of characteristic functions.

Theorem 2.8 (see e.g. [10]). Let $L \subset \mathcal{M}(X)$ be a regular class and $T : L \to \mathcal{M}(Y)$ be an order continuous sublinear operator. If $0 < p_0 \leq 1$, $p_0 \leq p_1 < \infty$ then we have that

$$\sup_{f \in L} \frac{\|Tf\|_{L^{p_1}(Y)}}{\|f\|_{L^{p_0}(X)}} = \sup_{\chi_B \in L} \frac{\|T\chi_B\|_{L^{p_1}(Y)}}{\|\chi_B\|_{L^{p_0}(X)}}.$$

This result is actually a Corollary of a more general Theorem (see Theorem 1.2.11 in [10]).

We present now the discrete version of Sawyer’s formula as it was obtained in [10].
Theorem 2.9. Let $w = (w(n))_n$, $v = (v(n))_n$ be weights in $\mathbb{N}^*$ and let

$$S = \sup_{f \geq 0} \frac{\sum_{n=0}^{\infty} f(n)v(n)}{(\sum_{n=0}^{\infty} f(n)^p w(n))^{\frac{1}{p}}}.$$

(i) If $0 < p \leq 1$, then

$$S = \sup_{n \geq 0} \frac{V(n)}{W^{\frac{1}{p'}}(n)},$$

with $W$ defined by $W(n) = \sum_{k=0}^{n} w(k)$, $n = 0, 1, 2, ...$ and $V$ analogously.

(ii) If $1 < p < \infty$, then

$$S \approx \left( \int_0^{\infty} \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{\frac{p'-1}{p'}} \tilde{v}(t) \, dt \right)^{\frac{1}{p'}}$$

$$\approx \left( \int_0^{\infty} \left( \frac{\tilde{V}(t)}{\tilde{W}(t)} \right)^{\frac{p'}{p}} \tilde{w}(t) \, dt \right)^{\frac{1}{p'}} + \frac{\tilde{V}(\infty)}{\tilde{W}^{\frac{1}{p'}}(\infty)},$$

where $\tilde{v}$ is the weight in $\mathbb{R}_+$ defined by

$$\tilde{v} = \sum_{n=0}^{\infty} v(n) \chi_{[n,n+1]}$$

and $\tilde{V}(t) = \int_0^t \tilde{v}(s) \, ds$ and analogously for $\tilde{w}$ and $\tilde{W}$.

Moreover, the implicit constants in the symbol $\approx$ only depend on $p$.

For the reader's convenience we include a proof also of this theorem in our appendix (see [10]).

3. Main results

We begin with a characterization of main diagonal matrices.
Theorem 3.1. Let $\alpha \geq 0$, $A = A_0$ be given by the sequence $a = (a_n)_n$. Then $A \in B_\alpha^w(\ell^2)$ if and only if

$$\sup_{n \geq 1} \left( \frac{\sum_{k=1}^n |a_k|^2 k^{2\alpha}}{\sum_{k=1}^n k^{2\alpha}} \right)^{\frac{1}{2}} < \infty.$$  

Moreover, the norm

$$\|A\|_{B_\alpha^w(\ell^2)} = \sup_{n \geq 1} \left( \frac{\sum_{k=1}^n |a_k|^2 k^{2\alpha}}{\sum_{k=1}^n k^{2\alpha}} \right)^{\frac{1}{2}}.$$  

Example 3.2. It is clear that $B(\ell^2) \subset B_\alpha^w(\ell^2)$ and using the previous theorem for the matrix $A = A_0$ given by the sequence $a = (a_k)_{k=1}^n$, where

$$a_k = \begin{cases} 2^{\frac{n-1}{2} - \alpha}, & \text{if } k = 2^p \\ 0, & \text{otherwise}, \end{cases}$$

we can easily show that the inclusion is proper. An easy computation show us that $\|A\|_{B_\alpha^w(\ell^2)} < \infty$ but the sequence is not bounded. This fact implies that $A \notin B(\ell^2)$.

Remark 3.3. Using the previous example we can conclude that:

$$B(\ell^2) \subset B_\alpha^w(\ell^2)$$

and the inclusion is proper. It is known that $B(\ell^2)$ is closed under Schur multiplication (see e.g. [7]). Thus it is a natural question if $B_\alpha^w(\ell^2)$ is closed under Schur multiplication. The answer is negative: we can take for example the sequence defined in the Example 3.2 and we can see that $A \ast A \notin B_\alpha^w(\ell^2)$.

If $\alpha = 0$ it is known that the space of all bounded operators on $\ell^2$ is included in the space of all Schur multipliers from $B_0^w(\ell^2)$ to $B_0^w(\ell^2)$ (see e.g. [16]). The following Theorem shows that the result in fact is true for $\alpha \geq 0$. Our next result reads:

Theorem 3.4. The space $B(\ell^2)$ is included in the space of all Schur multipliers from $B_\alpha^w(\ell^2)$ to $B_\alpha^w(\ell^2)$ where $\alpha \geq 0$.  

Our next result of this Section gives us the coincidence of the spaces $B(\ell^2)$ and $B^\alpha_w(\ell^2)$ in the case of Toeplitz matrices.

**Theorem 3.5.** Let us denote $\mathcal{T}$ the set of all Toeplitz matrices. Then we have that

$$B^\alpha_w(\ell^2) \cap \mathcal{T} = B(\ell^2) \cap \mathcal{T}.$$ 

For the next result we need the following Lemma from [16]:

**Lemma 3.6.**

$$\sup_{|x_n| \to 0} \left| \sum_{n=1}^\infty a_n x_n \right| \left( \sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}} \approx \sup_{|x_n| \to 0} \left| \sum_{n=1}^\infty |a_n x_n| \right| \left( \sum_{n=1}^\infty |x_n|^2 \right)^{\frac{1}{2}},$$

where $(a_n)_n$ and $(x_n)_n$ are sequences of complex numbers.

**Theorem 3.7.** Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ be an upper triangular positive Toeplitz matrix. Then $A \in B(\ell^2)$ if and only if the sublinear operator $T_A$ is bounded from $\ell^2$ into $\ell^2,$ where

$$T_A (b) (j) = \frac{1}{j} \sum_{m=0}^{j} |(a * b) (m)|, \quad T_A (b) (0) = |a_0 b_0|$$

and $(a * b) (m) = \sum_{k+l=m} a_k b_l,$ $a = (a_k)_{k \geq 0},$ $b = (b_k)_{k \geq 0} \in \ell^2.$

Our next Theorem is a characterization of diagonal matrices, which are Schur multipliers from $B^\alpha_w(\ell^2)$ to $B^\alpha_w(\ell^2).$
**Theorem 3.8.** For $M = M_0$ given by the sequence $m = (m_k)_{k \geq 1}$ we have that

$$M \in M(B_w^a(\ell^2), B_w^a(\ell^2))$$

if and only if $m = (m_k)_{k \geq 1} \in \ell^\infty$.

Moreover, in the case of Toeplitz matrices we have the following embeddings:

**Theorem 3.9.** It yields that

$$M(B_w^a(\ell^2), B_w^a(\ell^2)) \cap T \subseteq M(\ell^2) \cap T.$$
It follows that
\[ S = \sup_{n \geq 1} \left( \frac{\sum_{k=1}^{n} k^{2\alpha} |a_k|^2}{\sum_{k=1}^{n} k^{2\alpha}} \right)^{\frac{1}{2}}. \]
and
\[ \|A\|_{B_\alpha^w(\ell^2)} = \sup_{n \geq 1} \left( \frac{\sum_{k=1}^{n} k^{2\alpha} |a_k|^2}{\sum_{k=1}^{n} k^{2\alpha}} \right)^{\frac{1}{2}}. \]

The proof is complete. \( \square \)

**Proof of Theorem 3.4.** Let us take \( A \in B(\ell^2) \) and \( B \in B_\alpha^w(\ell^2) \). Then we have that
\[ \sum_j |\sum_k a_{jk} b_{jk} x_k|^2 \leq \sum_j \left( \sum_k |a_{jk}| \|b_{jk}\| \|x_k\| \right)^2 \]
\[ \leq \sum_j \left( \sum_k |a_{jk}| \right) \sum_k \left( |b_{jk}| \|x_k\| \right)^2 \]
\[ \leq \sup_j \left( \sum_k |a_{jk}| \right) \sum_j \left( \sum_k |b_{jk}| \|x_k\| \right)^2. \]

To estimate the last term in (4) we consider the Rademacher functions \( r_k(t) = sgn \sin(2^n \pi t) \) on \( [0, 1] \), for \( k \geq 1 \) and we use the equality (see [12] p. 126)
\[ \sum_k |y_k|^2 = \int_0^1 |\sum_k y_k r_k(t)|^2 dt. \]

It follows that
\[ \sum_j \left( \sum_k |b_{jk}| \|x_k\|^2 \right) = \sum_j \int_0^1 |\sum_k b_{jk} x_k r_k(t)|^2 dt \]
\[ \leq \text{esssup}_{t \in [0,1]} \sum_j \int_0^1 \sum_k |b_{jk} x_k r_k(t)|^2 dt \]
\[ \leq \|B\|_{B_\alpha^w(\ell^2)} \|x\|_2. \]

Thus, we obtain that
\[ \| A \ast B \|_{B_2(\ell^2)} \leq \| A \|_{2,\infty} \cdot \| B \|_{B_2(\ell^2)}. \]

Since \( \| A \|_{2,\infty} \leq \| A \|_{B(\ell^2)} \) the proof is complete. \( \square \)

**Proof of Theorem 3.5.** It follows from the definition that

\[ B(\ell^2) \subset B_{w_1}^0(\ell^2) \subset B_w(\ell^2). \]

From Theorem 9 in [16] we have that the Toeplitz matrices from \( B(\ell^2) \) coincides with Toeplitz matrices from \( B_w(\ell^2) \). Thus, the Toeplitz matrices from \( B(\ell^2) \) coincides with those from \( B_{w_1}^0(\ell^2) \). The proof is complete. \( \square \)

**Proof.** From Theorem 3.5 we have that \( A \in B(\ell^2) \) if and only if \( A \in B_{w_1}^0(\ell^2) \) i.e. that \( \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right)_{k \geq 0} \in \ell^2 \) for any \( x \in \ell^2 \) with \( \frac{|x_j|}{n^\alpha} \downarrow 0 \).

This, in its turn, is equivalent to that

\[ \sup_{\| b \|_{\ell^2} \leq 1} \left| \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k \right| < \infty \]

for all \( x \in \ell^2, \frac{|x_j|}{n^\alpha} \downarrow 0 \). We define the convolution \( c_l = \sum_{k=0}^{l} a_{l-k} b_k \) and it yields that

\[ \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j x_{j+k} \right) b_k = \sum_{l=0}^{\infty} x_l c_l. \]
From Lemma 3.6 it follows that

\[
\sup_{|x| \in \ell_2^0} \left( \sum_{l=0}^{\infty} |x_l c_l| \right)^{1/2} \approx \sup_{|x| \in \ell_2^0} \left( \sum_{l=0}^{\infty} |x_l| |c_l| \right)^{1/2} \\
\approx |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} \left| \sum_{k=0}^{m} a_{m-k} b_k \right| \right)^2 \\
= |a_0 b_0|^2 + \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{m=0}^{j} |(a \ast b)(m)| \right)^2.
\]

Thus the proof is complete. \( \square \)

**Proof of Theorem 3.8.** Suppose that \( M \in M(B_w^\alpha(\ell^2), B_w^\alpha(\ell^2)) \). Then \( M \ast A \in B_w^\alpha(\ell^2) \), for every \( A \in B_w^\alpha(\ell^2) \). In particular,

\[
M \ast A_0 \in B_w^\alpha(\ell^2),
\]

for \( A_0 = diag(a) \), with \( a = (a_k)_{k \geq 1} \). We have that \( A_0 x \in \ell^2 \) for every \( x \in \ell^2 \) with \( \frac{|x_k|}{k^\alpha} \searrow 0 \).

Since \( (M \ast A_0)x \in \ell^2 \) it implies that \( m = (m_k)_{k \geq 1} \in \ell^\infty \).

Conversely, assume that \( m = (m_k)_{k \geq 1} \in \ell^\infty \). Observe first that if \( A = (a_{jk})_{j,k} \) is an infinite matrix and \( A_0 \) is the matrix defined as \( A_0 = diag(a_{kk}) \), then

\[
\|A_0\|_{B_w^\alpha(\ell^2)} \leq \|A\|_{B_w^\alpha(\ell^2)}.
\]
Indeed, using the same arguments as in proof of Theorem 3.4, where \( r_k, k \geq 1 \), are the Rademacher functions we have that

\[
\|A_0 x\|^2 = \sum_k |a_{kk} x_k|^2 \\
\leq \sum_j \sum_k |a_{jk} x_k|^2 \\
= \sum_j \int_0^1 \sum_k a_{jk} x_k r_k(t) \, dt \\
\leq \text{ess sup}_{t \in [0,1]} \sum_j \sum_k a_{jk} x_k r_k(t) \\
\leq \|A\|^2_{B_w^\alpha(\ell^2)} \|x\|_2^2,
\]

for every \( x = (x_k)_{k \geq 1} \in \ell^2 \) with \( \frac{|x_k|}{k^\alpha} \searrow 0 \).

Let \( A \) be an infinite matrix such that \( A \in B_w^\alpha(\ell^2) \). Then

\[
\|(M \ast A)x\|^2 = \|(M \ast A_0)x\|^2 \\
= \|MA_0 x\|^2 \\
\leq \|m\|^2_\infty \cdot \|A_0 x\|^2 \\
\leq \|m\|^2_\infty \cdot \|A_0\|^2_{B_w^\alpha(\ell^2)} \cdot \|x\|^2 \\
\leq \|m\|^2_\infty \cdot \|A\|^2_{B_w^\alpha(\ell^2)} \cdot \|x\|^2,
\]

for every \( x = (x_k)_{k \geq 1} \in \ell^2 \) such that \( \frac{|x_k|}{k^\alpha} \searrow 0 \). The proof is complete. \( \square \)

**Proof of Theorem 3.9.** Let \( M = (m_{jk})_{jk} \) be a Toeplitz matrix of the form

\[
m_{jk} = c_{j-k}, \ j, k = 0, 1, 2, \ldots
\]

such that \( M \in M(B_w^\alpha(\ell^2), B_w^\alpha(\ell^2)) \). Using the same techniques as in [16], applying Theorem 3.5 we have that there exists a bounded,
complex, Borel measure $\mu$ on the circle group $T$ with
$$\hat{\mu}(n) = c_n \text{ for } n = 0, \pm 1, \pm 2, \cdots.$$ 

Moreover,
$$\|\mu\| \leq \|M\|_{M(B_2^2(\ell^2)_*,B_2^2(\ell^2))}.$$

Applying now Theorem 8.1 in [7] we have that the Toeplitz matrix (5) is in $M(\ell^2)$. The proof is complete. \qed

REFERENCES

Proof of Theorem 2.2. By virtue of the monotone convergence theorem, we may assume that $g$ is supported in a compact subset of $(0, \infty)$ with $\int_0^\infty g(x)dx = 1$ and that $0 < \int_0^\infty v(t)dt < \infty$, for all $x > 0$. Set

$$f(x) = \left( \int_x^\infty \frac{g(t)}{\int_0^t v(s)ds} dt \right)^{p'-1}, \quad 0 < x < \infty.$$ 

Then $f$ is bounded and nonincreasing on $(0, \infty)$, and an integration by parts shows that

$$\int_0^\infty f(x)^p v(x)dx = f(x)^p \left[ \int_0^x v(t)dt \right]_0^\infty -$$

$$- \int_0^\infty pf(x)^{p-1}f'(x) \left( \int_0^x v(t)dt \right) dx$$

$$= p' \int_0^\infty f(x)g(x)dx.$$ 

Thus the supremum on the left side of (1) is at least

$$C \left( \int_0^\infty f(x)g(x)dx \right)^{\frac{1}{p'}}.$$
Now let \( x_j \) satisfy \( \int_0^{x_j} g(x) \, dx = 2^j, \quad -\infty < j \leq 0 \). Then

\[
\int_0^\infty f(x)g(x) \, dx = \sum_{j=-\infty}^{0} \int_{x_{j-1}}^{x_j} \left( \int_x^\infty \frac{g(t)}{\int_0^t v(s) \, ds} \, dt \right)^{p'-1} g(x) \, dx \geq
\]

\[
\geq \sum_{j=-\infty}^{-1} \left( \int_{x_j}^{x_{j+1}} g(t) \, dt \right)^{p'-1} \left( \int_0^{x_{j+1}} v(t) \, dt \right)^{1-p'} \left( \int_{x_{j-1}}^{x_j} g(t) \, dt \right) \geq
\]

\[
\geq C \sum_{j=-\infty}^{-2} \left( \int_{x_j}^{x_{j+1}} g(t) \, dt \right)^{p'-1} \left( \int_0^{x_{j+1}} v(t) \, dt \right)^{1-p'} \left( \int_{x_{j-1}}^{x_j} g(t) \, dt \right) \geq
\]

\[
\geq C \int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'-1} \left( \int_0^x v(t) \, dt \right)^{1-p'} g(x) \, dx
\]

since \( \int_0^{x_j} g(t) \, dt = 2^j \). Using this inequality in (6), we conclude that

\[
\left( \sup_{f \geq 0} \int_0^\infty f(x)g(x) \, dx \right) \left( \int_0^\infty f(x) \, dx \right)^{\frac{p'}{2}} \geq
\]

\[
\geq C \left[ \int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'-1} \left( \int_0^x v(t) \, dt \right)^{1-p'} g(x) \, dx \right]^{\frac{1}{p}}.
\]

However, integration by parts yields

\[
\int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'-1} \left( \int_0^x v(t) \, dt \right)^{1-p'} g(x) \, dx =
\]

\[
= \frac{1}{p'} \left( \int_0^x g(t) \, dt \right)^{p'-1} \left( \int_0^x v(t) \, dt \right)^{1-p'} \bigg|_0^\infty +
\]

\[
\frac{1}{p} \int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'} \left( \int_0^x v(t) \, dt \right)^{-p'} v(x) \, dx =
\]

\[
= \frac{1}{p} \int_0^\infty \left( \int_0^x g(t) \, dt \right)^{p'} \frac{v(x)}{\left( \int_0^x v(t) \, dt \right)^{p'}} \, dx +
\]
+ \frac{1}{p'} \left( \int_0^\infty g(t)dt \right)^{p'} \left( \int_0^\infty v(t)dt \right)^{1-p'}

where the second term above is zero if \( \int_0^\infty v(t)dt = \infty \) (recall that \( \int_0^\infty g(t)dt = 1 \) for large \( x \)). Now, the inequality “\( \geq \)" from (1) can be obtained from (7) and (8). For the reverse inequality we give a proof from [8]. Set

\[ h(t) = \left( \int_t^\infty \left( \int_0^x g(s)ds \right)^{p-1} \left( \int_0^x v(s)ds \right)^{-p} v(x)dx + \right) \left( \int_0^\infty \frac{g(s)ds}{v(s)ds} \right)^{\frac{1}{p'}} . \]

Then

\[ \int_0^\infty f(x)g(x)dx = \int_0^\infty f(x)g(x)h(x)h(x)^{-1}dx \leq \]

\[ \leq \left( \int_0^\infty f''(x)h^{-p}(x)g(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty h''(x)g(x)dx \right)^{\frac{1}{p'}} . \]

Applying the Fubini theorem to the last factor in the previous inequality we obtain the right hand side of (1). For the first factor, we observe that

\[ h(x)^{-p} \leq \left( \int_0^x g(t)dt \right)^{-1} \left( \int_0^x v(t)dt \right) \]

and, thus, by Lemma 2.1 we have that

\[ \int_0^\infty f''(x)h^{-p}(x)g(x)dx = p \int_0^\infty y^{p-1} \int_0^{\mu_f(y)} h^{-p}(x)g(x)dx dy. \]

Integrating by parts the inner integral and erasing the negative terms one has that the previous expression can be bounded, up to multiplicative constants, by

\[ \int_0^\infty y^{p-1} \left( \int_0^{\mu_f(y)} g(x)dx \right) h^{-p}(\mu_f(y))dy \leq \]
\[ \leq \int_0^\infty y^{p-1} \left( \int_0^\mu(y) v(x) dx \right) dy \approx \int_0^\infty f^p(x) v(x) dx. \]

The proof is complete. \(\square\)

Proof of Theorem 2.9. (i) is obtained applying Theorem 2.8 with \(p_1 = 1, p_0 = p, X = Y = \mathbb{N}^*, T = Id\) to the regular class \(L\) of decreasing sequences in \(\mathbb{N}^*\).

(ii) can be deduced from Theorem 2.2 by observing that

\[ S = \sup \frac{\int_0^\infty \tilde{f}(t) \tilde{\nu}(t) dt}{\int_0^\infty \tilde{f}^p(t) \tilde{\omega}(t) dt} \tag{9} \]

To see this, note that if \(f = (f(n))_n\) is a decreasing sequence in \(\mathbb{N}^*\) and we define

\[ \tilde{f} = \sum_{n=0}^\infty f(n) \chi_{[n,n+1)} \in \mathcal{M}_{dec}(\mathbb{R}^+). \]

It is obvious that

\[ \frac{\sum_{n=0}^\infty f(n)v(n)}{(\sum_{n=0}^\infty f(n)^p w(n))^{\frac{1}{p}}} = \frac{\int_0^\infty \tilde{f}(t) \tilde{\nu}(t) dt}{(\int_0^\infty \tilde{f}^p(t) \tilde{\omega}(t) dt)^{\frac{1}{p}}}. \]

Therefore \(S\) is less than or equal to the second member in (9). On the other hand, if \(g \geq 0\) is a decreasing function in \(\mathbb{R}^+\) and we define the decreasing sequence \(f(n) = (\int_n^{n+1} g^p(s) ds)^{\frac{1}{p}}, n = 0, 1, \ldots\) then we obtain that

\[ \int_0^\infty g^p(t) \tilde{\omega}(t) dt = \sum_n f(n)^p w(n), \]
while, by Hölder’s inequality, we have that
\[
\int_0^\infty g(t)\tilde{v}(t)dt = \sum_n v(n) \int_n^{n+1} g(t)dt \\
\leq \sum_n v(n) \left( \int_n^{n+1} g^p(t)dt \right)^{\frac{1}{p}} \\
= \sum_n v(n)f(n).
\]
Hence,
\[
\frac{\int_0^\infty g(t)\tilde{v}(t)dt}{\left( \int_0^\infty g^p(t)\tilde{w}(t)dt \right)^{\frac{1}{p}}} \leq \frac{\sum_{n=0}^\infty f(n)v(n)}{\left( \sum_{n=0}^\infty f(n)^p w(n) \right)^{\frac{1}{p}}} \leq S.
\]
Thus (9) is proved and (ii) follows by applying Theorem 2.2. The proof is complete. \[\square\]