*Finn Ouchterlony*¹

Extension of the Compliance and Stress Intensity Formulas for the Single Edge Crack Round Bar in Bending

REFERENCE: Ouchterlony, Finn, "Extension of the Compliance and Stress Intensity Formulas for the Single Edge Crack Round Bar in Bending," *Fracture Mechanics Methods for Ceramics, Rocks, and Concrete, ASTM STP 745*, S. W. Freiman and E. R. Fuller, Jr., Eds., American Society for Testing and Materials, 1981, pp. 237-256.

ABSTRACT: The compliance and stress intensity formulas for the solid single edge crack round bar in bending (SECRBB) specimen with a support span to diameter ratio $S/D = 3.33$ are extended to cover notch depth ratios $0 \leq a/D < 0.6$, an arbitrary Poisson's ratio, and a slight variation in S/D . The work is based on a precise and reliable measurement of the load point displacement on notched rock cores of Ekeberg marble. The measuring method virtually eliminates friction and twisting effects as well as nonlinear contact and crushing phenomena. It yields an accurate value of the Young's modulus.

The experimental results give the extension of the formulas which are used for a thorough fracture mechanics analysis of the SECRBB specimen. The matters considered are the crack growth stability, the secant offset at 2 percent apparent crack growth, and the logarithmic derivative relating the J-integral to work done. Hence a basis is laid for correct linear and nonlinear fracture toughness measurements with the SECRBB specimen.

KEY WORDS: fracture mechanics, stress intensity factor, J-integral, crack growth stability, compliance, load point displacement measurement, rock mechanics, core testing, rock material, crack propagation, composite materials

From a practical point of view, it will be extremely advantageous if the fracture resistance of geological materials can be determined from rock cores with a minimum of preparation. As such the "short rod" specimen is useful. This paper describes work done which hopefully will give the single edge crack round bar in bending, or SECRBB specimen, a wider use than it has today.

The SECRBB specimen has been studied by Bush [1],² who used aluminum bars. He analyzed both hollow and solid bars with two support span to

¹ Research scientist, Swedish Detonic Research Foundation, Stockholm, Sweden.

² The italic numbers in brackets refer to the list of references appended to this paper.

diameter ratios, $S/D = 3.33$ and 6.67 . My work [2] deals with the solid bar with $S/D = 3.33$ since it is the one best adapted to rock cores, Fig. 1. As a basis for future fracture mechanics work, the compliance and stress intensity formulas are extended to cover deeper notches, $0 \leq a/D < 0.6$, an arbitrary value of Poisson's ratio, and a slight variation in S/D .

The specimen halves obtained after the bend test may be used in ensuing "short rod" tests. In the future, it is expected that both specimen types will be used together with, for example, a round CT specimen to give the fracture resistance in three mutually orthogonal directions in a core.

Compliance and Stress Intensity Formulas

Assuming linear elastic material response, the compliance of the specimen is defined by

$$\lambda = \delta_F / F \quad (1)$$

For a 152 mm (6 in.) diameter solid aluminum bar with $S/D = 3.33$, Bush [1] yields

$$\lambda_{Al} = 1.661988 \{1 + 8.745143 (a/D)^{2.5}\} \quad (2)$$

- D = diameter of rock core specimen, appr. 41.4 mm,
- S = support span, appr. $3.33 \cdot 41.4 = 138$ mm,
- a = (maximum) notch depth, mm,
- A = reduced cross-sectional area,
- B = crack front length,
- F = load on specimen, kN,
- δ_F = (ideal) load point displacement, μm ,
- δ_{COD} = crack (mouth) opening displacement, μm ,
- N = notch tip, or front, and
- M_1, M_2 = saddle points of measuring base.

FIG. 1—Schematic of experimental setup of SECRBB test. The load point displacement is measured indirectly as the displacement of the notch front (point N) relative to a measuring base riding on the specimen (points M_1 and M_2) directly over the support points.

in micrometres per kilonewton ($\mu\text{m/kN}$), valid when $0 \leq a/D < 0.35$. From λ_{AI} we may construct a dimensionless compliance function

$$g = \lambda ED \quad (3)$$

Within the limits of slender beam theory, this equation is valid for all scaled SECRBB specimens of linear elastic material. The unnotched SECRBB specimen with $S/D = 3.33$ is not exactly slender, however. Making the usual shear correction, assuming plane strain conditions at the crack front and using $\nu = 0.32$ and $E = 73.08$ GPa for aluminum leads to the generalization

$$g(a/D, \nu) = 15.6719 \{1 + 0.1372 (1 + \nu) + 11.5073 (1 - \nu^2) (a/D)^{2.5}\} \quad (4)$$

again valid when $0 \leq a/D < 0.35$. Alluding to the practical situation, small variations from the ideal $S/D = 3.33$ may be included in g by introducing a diameter ratio ρ which refers to a nominal test diameter D_s through $\rho = D_s/D = (S/D)/3.33 \approx 1$. The result is

$$g(a/D, \nu, \rho) = 15.6719 \{\rho^3 + 0.1372 (1 + \nu) \rho + 11.5073 (1 - \nu^2) \rho^2 (a/D)^{2.5}\} \quad (5)$$

This expression has been used to determine the Young's modulus of Ekeberg marble from compliance measurements on SECRBB specimens. For these purposes Bush's [1] stress intensity formula is written

$$K_I = 0.25 (S/D) Y' F/D^{1.5} = 0.8325 \rho Y' F/D^{1.5} \quad (6)$$

The dimensionless stress intensity factor Y' varies only slightly with S/D . It is given by

$$Y' = 2 (D/S) \left\{ \frac{dg(a/D)}{d(a/D)} / (1 - \nu^2) \right\}^{0.5} / \{(a/D) - (a/D)^2\}^{0.25} \quad (7)$$

or explicitly, with Eq 4 and $S/D = 3.33$, by

$$Y'(a/D) = 12.7527 (a/D)^{0.5} / \{1 - a/D\}^{0.25} \quad (8)$$

when $0 \leq a/D < 0.35$.

Y' takes a wide range of values in this interval. Bush [1] defines another dimensionless stress intensity factor which is better suited for accurate plotting, namely

$$Y = Y' \cdot \pi / \{32 (a/D)^{0.5}\} = 1.2520 / \{1 - a/D\}^{0.25} \quad (9)$$

(Curves of $Y'(a/D)$ and $Y(a/D)$ are given later in Figs. 7 and 8 as a complement to the extended versions of Eqs 17 and 18.)

Oddly enough Column B of Table 5 in Ref 1 gives $Y(0.0001) = 1.267$ where Eq 9 gives 1.252. Most likely Bush's results for Y' and Y in that column should be multiplied with the correction factor $1.252/1.267 = 0.9882$.

Experiments

Ekeberg marble was chosen as the testing material. It is fine to medium grained with a well recrystallized texture of mainly spherical carbonate grains. Petrographic and mechanical data are given in Appendix I. Its properties are fairly isotropic, and it has been used extensively for in-house fracture tests.

Twenty 0.5-m-long cores with $D = 41$ mm were taken in the same direction from one block of marble. Each could be notched and tested five times. The notch width was 1.6 mm. The first five cores were slotted in order to measure Young's modulus and fracture mechanics data in the notch depth interval $0 \leq a/D < 0.35$ where the previous compliance and stress intensity formulas are valid. Then the next five cores were slotted with the aim first of extending these formulas to $0 \leq a/D < 0.6$ and second of giving more fracture mechanics data. In all, 13 cores have been used at this stage. A total of 56 out of the 65 prepared tests were judged to give valid compliance data. The fracture mechanics part has not been concluded and will be reported later.

The tests have been performed in our Instron 1271 model servohydraulic testing machine. The setup is shown in Fig. 2. All tests were run in strain control with a clip gage, which measures the crack opening displacement δ_{COD} , giving the monitoring signal. The knife edges, between which δ_{COD} is measured, were affixed to the specimen with quick curing cyanoacrylate adhesive. During each test δ_{COD} increased linearly, typically at the rate of $0.1 \mu\text{m/s}$.

Load versus displacement curves $F(\delta_F)$ and $F(\delta_{\text{COD}})$ were recorded on a two-channel X-Y-recorder. A representative set of complete $F(\delta_F)$ -curves are shown in Fig. 3. The load point displacement at maximum load is about $25 \mu\text{m}$, the time consumed at this point roughly 2 to 3 min, and the time to complete failure about 15 min.

Load and COD were measured with standard gages but not δ_F , since there are some extraneous factors which have to be avoided in order to measure it properly. Friction effects from the support rollers are taken care of routinely by making them outwardly mobile as required in the ASTM Test for Plane-Strain Fracture Toughness of Metallic Materials (E 399-78a). The use of a round bar specimen minimizes general twisting effects due to geometrical imperfections but aggravates nonlinear deformations and local crushing in roller to specimen contacts. Crushing marks were invariably found on the marble specimens after testing. However, the two local factors can be virtu-

ally eliminated by making differential displacement measurements in accordance with Bush [1].

The measuring arrangement may be seen in Figs. 2 and 4. Strictly seen it is the displacement of the notch tip relative to the support points which is measured. A rigid measuring base, through which the top roller protrudes, rides

FIG. 2—Experimental setup.

FIG. 3—Complete load displacement curves from SECRBB tests on notched rock cores of Ekeberg marble.

on the specimen directly over the support points. It is aligned to the base of the testing machine with screws and levels. An inverted yoke in the notch rests on two semicircular buttons. It is pulled into the notch by rubber bands fastened to the base.

The relative displacement is measured by two LVDTs with their housings attached to the base, symmetrically on each side of the top roller. Each measuring stylus rests on a shoulder of the yoke, which is in stable equilibrium. The two LVDTs are gaging transducers with linear ball bearings and an alleged reproducibility of approximately $0.1 \mu\text{m}$. The signals are summed to eliminate potential rotation effects, before being connected to the plotter.

FIG. 4—Measuring base with yoke and LVDTs.

The first part of the $F(\delta_F)$ curve is normally too steep to permit accurate reading of the initial slope $1/\lambda_{\text{exp}}$. Thus a low load cycle up to about 0.5 kN, with a slope between 1 and 2, was run with specimens with $a/D < 0.4$. For specimens with deeper notches the initial slope was read directly from the failure curves.

Determination of Young's Modulus

The relevant experimental data are shown in Table 1. The core diameter varied with up to 0.3 mm on each core and from 40.7 to 41.5 mm between cores. The support span was kept a constant $S = 3.33 \cdot 41.4 = 138$ mm during the test, that is $D_s = 41.4$ mm.

The Poisson's ratio of Ekeberg marble (E_m) is identical in tension and compression at low load levels. A representative value is $\nu_{E_m} = 0.22$, see Appendix I. Thus the generalized dimensionless compliance in Eq 5 reduces to

$$g_{E_m} = 15.6719 \cdot \left(\frac{41.4}{D} \right)^3 \left\{ 1 + 0.1674 \left(\frac{D}{41.4} \right)^2 + 10.9503 \left(\frac{D}{41.4} \right) (a/D)^{2.5} \right\} \quad (10)$$

TABLE 1—*Experimental test data and calculated values of Young's modulus.*

Test No.	Notch Depth Ratio	Diameter D , mm	Compliance λ_{exp} , $\mu\text{m}/\text{kN}$	Modulus E , GPa
E2	0.000	41.4	5.20	84.98
E3	0.154	41.5	5.70	83.56
E4	0.000	41.5	5.10	85.88
E5	0.300	41.4	7.55	85.60
mean number of tests $\pm$ SD				85.00 ₄ $\pm$ 1.03
F1	0.282	40.8	7.20	90.20
F3	0.130	40.7	5.70	87.27
F5	0.289	40.8	7.50	88.13
				88.53 ₃ $\pm$ 1.51
G1	0.287	40.8	7.90	83.25
G3	0.139	40.9	5.80	85.05
G4	0.000	40.9	5.25	88.06
G5	0.291	40.9	7.45	88.42
				86.19 ₄ $\pm$ 2.48
H1	0.284	40.9	7.55	85.71
H2	0.000	40.9	5.20	88.91
H3	0.139	40.9	5.80	85.05
H4	0.000	40.9	5.00	92.46
H5	0.287	40.8	7.50	87.69
				87.96 ₅ $\pm$ 2.95
I1	0.151	41.0	5.75	86.25
I3	0.142	40.9	5.70	86.84
I4	0.140	40.8	5.65	88.20
I5	0.140	40.8	5.80	85.92
				86.80 ₄ $\pm$ 1.01
J1	0.565	41.4	23.50	...
J2	0.408	41.4	10.90	...
J3	0.259	41.4	7.10	82.17
K1	0.573	41.4	24.50	...
K2	0.416	41.4	11.40	...
K3	0.143	41.4	5.90	80.33
K4	0.403	41.4	11.25	...
K5	0.565	41.4	24.25	...
L1	0.564	41.3	23.75	...
L2	0.407	41.0	11.25	...
L3	0.253	41.4	6.70	88.12
L4	0.406	41.4	10.75	...
L5	0.563	41.4	23.25	...
M1	0.571	41.3	24.00	...
M2	0.000	41.3	5.20	85.75
M3	0.259	41.4	7.15	81.60

TABLE 1—Continued

Test No.	Notch Depth Ratio	Diameter D , mm	Compliance λ_{exp} , $\mu\text{m/kN}$	Modulus E , GPa
M4	0.000	41.4	5.40	81.84
M5	0.568	41.4	23.75	...
				$83.06_3 \pm 2.33$
N1	0.408	41.4	11.25	...
N2	0.000	41.4	5.02	88.03
N3	0.254	41.4	6.83	84.44
N4	0.000	41.4	4.97	88.92
N5	0.403	41.4	10.20	...
				$87.13_3 \pm 2.37$
O1	0.279	40.9	7.70	83.00
O2	0.477	40.9	15.00	...
O3	0.167	40.8	5.70	90.51
O4	0.577	40.8	25.00	...
O5	0.375	40.8	9.95	...
P1	0.369	40.7	9.50	...
P4	0.474	40.7	15.00	...
P5	0.581	40.6	26.25	...
Q1	0.387	41.4	10.50	...
Q2	0.585	41.4	24.50	...
Q3	0.290	41.4	7.45	84.52
Q4	0.488	41.4	15.25	...
Q5	0.191	41.4	5.82	87.29

with D in millimetres. Given g_{Em} , D , and λ_{exp} in micrometres per kiloneutons, Eq 3 yields the Young's modulus

$$E = g_{Em} / \{\lambda_{exp} \cdot D\} \cdot 10^3 \quad (11)$$

in gigapascals.

The E -values from 33 valid tests, where $0 \leq a/D < 0.35$ and Eq 10 holds, are also given in Table 1.

An analysis of variances shows that there are no systematic differences between the E -values of the different cores. Neither is there any significant variation with respect to notch depth. Given as a mean value_{no tests} $\pm$ standard deviation, the results are

$$E = \begin{cases} 87.20_9 \pm 3.01 \text{ GPa when } a/D = 0 \\ 86.02_{11} \pm 2.63 \text{ GPa when } 0.1 < a/D \leq 0.2 \\ 85.60_{13} \pm 2.72 \text{ GPa when } 0.2 < a/D \leq 0.3 \end{cases} \quad (12)$$

One obvious systematic error in these values is a transverse contraction term, included because the yoke measures the deflection of the notch tip and not that of the neutral axis. It will make the specimen appear too stiff. An estimate based on simple beam theory yields a correction term $\Delta E \approx -1.7$ GPa when $a/D = 0$. When $a/D > 0$ this error is probably negligible.

The modulus values of Table 1 with this correction are plotted in Fig. 5. Adding an assessment of the maximum total error to the statistical limits of the 5 percent significance level, one obtains that for Ekeberg marble

$$E_{Em} = 85.7 \pm 3.3 \text{ GPa} \quad (13)$$

Appendix I gives the value 87 ± 5 GPa which is insignificantly higher. It was measured with strain gages.

As a result the measured E_{Em} -value and, as a consequence, also the adopted method of measuring the load point displacement appear both precise and reliable. The compliance formulas are thus validated, which is a prerequisite for the following.

Extension of Compliance and Stress Intensity Formulas

The λ_{exp} -values for the 56 valid tests in Table 1 cover the notch depth interval $0 \leq a/D < 0.6$. With the aid of Eqs 10, 11, and 13 they are converted

FIG. 5—Young's modulus of Ekeberg marble, E_{Em} . Experimental points and mean $\pm$ limits from Eq 13.

to equivalent compliance values λ_{eq} , relating to the nominal specimen diameter 41.4 mm. A transverse correction term of $\Delta\lambda = +0.10 \mu\text{m/kN}$ when $a/D = 0$ is included in this conversion. The λ_{eq} -values then form the basis for the curve fitting procedure.

Bush [1] discusses this procedure in some detail. His choice of form of the approximating function appears well founded, especially for shallow cracks, and it was used as a guide. Equations 3, 4, and 13 with $\nu_{Em} = 0.22$ yield the expected $\lambda(a/D)$ of a 41.4 mm diameter of a SECRBB specimen of Ekeberg marble

$$\lambda_{Em} = 5.1566 \{1 + 9.3800 (a/D)^{2.5}\} \quad (14)$$

in micrometres per kiloneuton, when $0 \leq a/D < 0.35$.

A suitable compliance function in the extended interval $0 \leq a/D < 0.6$ was found to be

$$\lambda_{Em} = 5.1566 \{1 + 9.3800 (a/D)^{2.5} + 65.815 (a/D)^7\} \quad (15)$$

in micrometres per kilonewton.

This result is achieved through use of a weighted least squares norm with a smoothness constraint. More than three terms in λ_{Em} tends to make K_I behave badly. The weight 1 was used when $a/D \leq 0.5$ and the weight 0.2 when $a/D > 0.5$ to compensate for the increase in variance of the λ_{eq} -values with notch depth. The unconstrained three term equation was found practically to coincide with the constrained one.

The λ_{eq} -values and $\lambda_{Em}(a/D)$ from both equations are plotted in Fig. 6. The closeness of fit appears satisfactory. The extended compliance function in Eq 15 yields values that are less than 1 percent higher than the old values when $a/D \leq 0.3$. Hence the feedback to the previous modulus determination is insignificant.

Converting back to an arbitrary Poisson's ratio leads to the dimensionless compliance function

$$g(a/D, \nu) = 15.6719 \{1 + 0.1372 (1 + \nu) + 11.5073 (1 - \nu^2) (a/D)^{2.5} \cdot [1 + 7.0165 (a/D)^{4.5}]\} \quad (16)$$

valid when $0 \leq a/D < 0.6$. A generalization which covers small variations from $S/D = 3.33$ is obvious from the analogy with Eqs 4 and 5.

The dimensionless stress intensity factors are then obtained from Eqs 7 and 9 as

$$Y'(a/D) = 12.7527 (a/D)^{0.5} \{1 + 19.646 (a/D)^{4.5}\}^{0.5} / \{1 - a/D\}^{0.25} \quad (17)$$

FIG. 6—Equivalent experimental compliance values λ_{eq} and fitted compliance curve λ_{Em} (a/D) from Eq 15. Broken curve depicts previous function valid when $0 \leq a/D < 0.35$.

and

$$Y(a/D) = 1.2520 \{1 + 19.646 (a/D)^{4.5}\}^{0.5} / \{1 - a/D\}^{0.25} \quad (18)$$

Both are plotted in Figs. 7 and 8 together with the old versions. The extended formulas yield somewhat higher values. The difference is from 2 to 9 percent when $0.25 < a/D < 0.35$ which could be significant.

Stability, Secant Offset, and J-Integral

The dimensionless compliance function contains more information relevant to fracture mechanics tests than just stress intensity factors. One such point is the matter of crack growth stability in the SECRBB specimen. In a soft testing machine the monotonically increasing $Y(a/D)$ curve in Fig. 7 implies that the crack growth is potentially unstable once it has been initiated. The point of instability in the $F(\delta_F)$ curve is at the maximum load.

In a stiff testing machine, where the displacement δ_F is the independent monitoring signal, the situation is different. The potential energy release rate of the system at incipient crack growth is

$$-\frac{\partial U}{\partial a} = G_1 B = \frac{\delta_F^2}{2\lambda^2} \cdot \frac{d\lambda}{da} \quad (19)$$

FIG. 7—Dimensionless stress intensity factor $Y'(a/D)$ of SECRBB specimen, extended and old versions. Extended version from Eq 17 shown as continuous curve.

FIG. 8—Dimensionless stress intensity factor $Y(a/D)$ of SECRBB specimen, extended and old versions. Extended version from Eq 18 shown as continuous curve.

Normalizing with respect to $E\delta_F^2$ like Broberg [3] we obtain a dimensionless stability function, or crack driving force

$$S = G_1 B / E \delta_F^2 = \frac{1}{2g} \cdot \frac{d \ln g}{d(a/D)} \quad (20)$$

$S(a/D)$ is plotted in Fig. 9 for $\nu = 0.32$ (aluminum), $\nu = 0.22$ (Ekeberg marble), and $\nu = 0.12$ (Bohus granite). The overall shape of the curve is nearly the same as for a rectangular specimen.

The stability curve for Ekeberg marble has a maximum at $a/D = 0.325$ beyond which unstable crack growth cannot take place. In the notch interval $0 < a/D < 0.325$, the crack driving force increases with crack length and crack growth tends to become unstable. This tendency may well show up as stable subcritical crack growth before maximum load when testing the SECRBB specimen in a servo-controlled machine.

In ASTM Method E 399, the interpretation of the load-displacement test record prescribes the drawing of a secant through the origin with a 5 percent decrease in slope relative to the initial tangent. In this way one finds the load which would correspond to 2 percent crack growth from a crack of given length, were it the only source of nonlinearity. The procedure goes on to establish a validity condition for the measured toughness value. It may be used as a guide for rock materials too.

FIG. 9—Dimensionless stability function, or crack driving force $S(a/D)$ of SECRBB specimen from Eq 20.

FIG. 10—Secant offset function $H(a/D)$ of SECRBB specimen from Eq 21. It correlates the relative change in load point displacement at constant load to the relative increase in crack length.

The secant corresponding to 2 percent apparent crack growth in a SECRBB specimen is obtained from a logarithmic differentiation of Eqs 1 and 3 as

$$\frac{\Delta \delta_F}{\delta_F} = H(a/D) \cdot \frac{\Delta a}{a} \text{ with } H(a/D) = (a/D) \cdot \frac{d \ln g}{d(a/D)} \quad (21)$$

Observe that this equation relates to the load point displacement δ_F rather than the COD.

Curves of $H(a/D)$ for $\nu = 0.12, 0.22$, and 0.32 are plotted in Fig. 10. If the same interval of acceptable notch depths is stipulated as in ASTM Method E 399 for the bend testing, that is, $0.45 \leq a/D \leq 0.55$, the curve for Ekeberg marble and $\Delta a/a = 0.02$ yields $\Delta \delta/\delta \approx 0.05$. This would imply the use of the same 5 percent offset secant procedure as in the ASTM standard to interpret the load-displacement record.

Although being accepted for metals, the use of J_{Ic} as a fracture parameter in the nonlinear range is new with rock materials. Some work has been reported by Wilkening [4], and Schmidt and Lutz [5]. In analogy with Eq 19, the J-integral may be defined from the nonlinear elastic potential energy release rate as

$$J_{Ic} = - \frac{\partial U}{\partial a} \quad (22)$$

When δ_F is being monitored this definition turns into

$$J_1 B = - \left. \frac{\partial W}{\partial a} \right|_{\delta_F} \quad (23)$$

where W is the work done from the $F(\delta_F)$ curve.

The crucial step comes when we wish to write

$$J_1 = \phi \cdot W/A \quad (24)$$

in order to avoid the differentiation procedure implied in Eq 23. A general expression for the logarithmic derivative ϕ for the bend specimen of ASTM Method E 399 was derived by Srawley [6]. He states that the significance of Eq 24 depends on the ability to estimate ϕ with practical accuracy for the ideal extremes of material behavior, that is, linear elastic and rigid perfectly plastic. If ϕ should happen to be practically the same for both extremes, then it is reasonable to assume that it might be the same for intermediate or real material behavior.

In the linear range, Eqs 3, 19, 23, and 24 yield an expression for ϕ which has two drawbacks. First, since $g(a/D)$ in Eq 16 is valid only when $0 \leq a/D < 0.6$ so is $\phi(a/D)$ whereas the great practical interest presumably lies in specimens with deeper notches. Second, $\phi(a/D)$ does not increase in a monotonic fashion, see Fig. 6.3 in Ref 2. The latter problem may be avoided by relating J_1 to an equivalent rectangular uncracked area $B(D - a)$ through

$$J_1 = \phi_{eq} \cdot W/[B(D - a)] \quad (24a)$$

Then the equivalent logarithmic derivative in the elastic range becomes

$$\phi_{eq,el} = (1 - a/D) \cdot \frac{d \ln g}{d(a/D)} \quad (25)$$

This equation behaves much better.

An approximate analysis for deep cracks which is outlined in Appendix II shows that $\phi_{eq,el} \approx 2.5$ when $a/D \rightarrow 1$, see Eq 27. For a rigid perfectly plastic material there results the same limit, see Eq 28. $\phi_{eq,el}(a/D)$ is shown in Fig. 11 for $\nu = 0.12, 0.22$, and 0.32 . It approaches the limit in a monotonic fashion until $a/D \approx 0.53$ and then turns away, probably incorrectly. This small deviation may well result from not taking the asymptotic behavior of λ into account in the curve fitting procedure.

The following equation is suggested for practical purposes

$$\phi_{eq} = 2.5 \text{ when } 0.5 < a/D < 1.0 \quad (26)$$

FIG. 11—Equivalent logarithmic derivative $\phi_{eq}(a/D)$ of SECRBB specimen from Eqs 25 and 26. Horizontal line $\phi_{eq} = 2.5$ gives approximate elastic and plastic limits for deep cracks.

For shallow cracks in the short three-point bend specimen, Turner [7] in Fig. 2.30d finds that $\phi_{pl} > \phi_{el}$. Thus the aforementioned suggestion would be coupled to the use of $\phi_{eq,el}$ from Eq 25 as a tentative lower limit of ϕ_{eq} for real material behavior when $a/D < 0.5$. Before being put to widespread use, however, this needs further computational and experimental support.

Summary

1. The compliance and stress intensity factors of the solid SECRBB specimen with $S/D = 3.33$ are extended by way of an experimental compliance calibration. The extended formulas cover notch depth ratios $0 \leq a/D < 0.6$, an arbitrary Poisson's ratio, and slight variations in S/D . The analysis is based on the assumption of a linear elastic material behavior.

2. The compliance calibration is based on a precise and reliable method of measuring the load point displacement. This statement results from accurate Young's modulus value measurements on cores of Ekeberg marble with different notch depths, as compared with strain gage values.

3. The extended K_I -formula gives values that are up to 9 percent higher than what the old one gives in its range. The new formula, in turn, seems to underestimate the true K_I -value somewhat, roughly less than 3 percent, when $0.5 < a/D < 0.6$.

4. The stability curve for a displacement monitored SECRBB test is shown to agree qualitatively with that of the rectangular SENB test. The changeover from potentially unstable crack growth to stable crack growth occurs when $a/D \approx 0.30$ to 0.35 .

5. The secant offset function for the SECRBB specimen yields the same 5 percent offset procedure as in the ASTM Method E 399 for interpretation of the load displacement record of a SENB K_{Ic} -test, provided that the same crack lengths $0.45 < a/D < 0.55$ are stipulated. The procedure for the SECRBB test refers to the load-load point displacement curve, however.

6. The logarithmic derivative ϕ which relates the J-integral to work done on an elastic SECRBB specimen is calculated for $0 \leq a/D < 0.6$ and then extrapolated to $a/D \rightarrow 1$ to enhance its usefulness. This is based on a proper representation ϕ_{eq} and an approximate limit analysis.

7. Finally, the usefulness of ϕ_{eq} is extended to real nonlinear material behavior. This extension is based on circumstantial evidence and it is suggested that $\phi_{eq} \geq \phi_{eq,el}$ when $a/D < 0.5$ and $\phi_{eq} = 2.5$ when $0.5 < a/D < 1.0$ for real material behavior.

Acknowledgment

This work is a part of the rock mechanics research program of the Swedish Detonic Research Foundation, jointly supported by Swedish Industry and the Swedish Board for Technical Development (STU).

References

- [1] Bush, A. J., *Experimental Mechanics*, Vol. 16, 1976, pp. 249-257.
- [2] Ouchterlony, F., "Compliance Measurements on Notched Rock Cores in Bending," Report DS 1980:2, Swedish Detonic Research Foundation, Stockholm, Sweden.
- [3] Broberg, K. B., *Journal of Mechanics and Physics of Solids*, Vol. 23, 1975, pp. 215-237.
- [4] Wilkening, W. W. in *Proceedings*, 19th U.S. Symposium on Rock Mechanics, Y. S. Kim Ed., University of Nevada, 1978, pp. 254-258.
- [5] Schmidt, R. A. and Lutz, T. J. in *Fracture Mechanics Applied to Brittle Materials*, ASTM STP 678, American Society for Testing and Materials, Philadelphia, 1979, pp. 166-182.
- [6] Srawley, J. E., *International Journal of Fracture*, Vol. 12, 1976, pp. 470-474.
- [7] Turner, C. E. in *Post-Yield Fracture Mechanics*, D. G. H. Latzko Ed., Applied Science Publishers, Barking, England, 1979, pp. 23-210.

APPENDIX I

Petrographic and Mechanical Data of Ekeberg Marble

Ekeberg marble from the Närke province in Sweden is quarried for building purposes. The variety used in these tests is yellowish white, sometimes with faint veins. It is fine to medium grained with a well recrystallized texture. The grains are comparatively spherical and equidimensional.

A thin section is shown in Fig. 12. The average grain diameter is 0.5 mm. The contents are roughly 90 percent carbonate (calcite + dolomite) and 10 percent muscovite.

The density of Ekeberg marble is $\approx 2.9 \cdot 10^3 \text{ kg/m}^3$ and its porosity ≈ 0.25 percent. The mechanical properties are comparatively isotropic. A compilation is made in Ref 2 Appendix B. Summing up

Young's modulus in tension/compression: $87 \pm 5 \text{ GPa}$
 Poisson's ratio in tension/compression: 0.15 to 0.25, typically 0.22
 Uniaxial tensile strength: 7 to 10 MPa
 Tensile strength index from point load test: 5 to 12 MPa
 Tensile strength from Brazilian disk test: 12 to 16 MPa
 Tensile strength from bend test: 15 to 21 MPa
 Uniaxial compressive strength: 120 to 210 MPa

Details about specimen size, number of tests, and standard deviations are given in Ref 2.

APPENDIX II

Derivation of Approximate Limit for ϕ_{eq}

The derivation of an approximate limiting value for deep cracks of ϕ_{eq} , as defined in Eq 24a, is outlined next for linear elastic and rigid perfectly plastic material behavior. When the remaining ligament becomes small, $h = D - a \ll D$, then $B \approx 2(Dh)^{0.5}$. The moment of inertia $I \propto Bh^3$.

FIG. 12—Thin section of Ekeberg marble in plane polarized light. Average grain diameter 0.5 mm.

The strain energy of flexure is roughly concentrated to the ligament within an approximate volume $\propto Ah$. For an elastic material simple beam theory with the acting moment $M \propto F$ yields

$$W \propto \frac{M^2}{2EI} \cdot h \propto F^2 h / I \propto F^2 / B h^2 \propto F^2 / h^{2.5}$$

Since $W = F^2 \lambda / 2$ and $g = \lambda E D$ it follows that $g \sim (1 - a/D)^{2.5}$ when $a/D \rightarrow 1$ and hence from Eq 25 that

$$\phi_{eq,el} \approx 2.5 \text{ when } a/D \rightarrow 1 \quad (27)$$

For rigid perfectly plastic material behavior the work done in a displacement monitored test $W \approx F_l \cdot \delta_F$, where F_l is the limit load. Equations 23 and 24a now yield

$$\phi_{eq,pl} = -(1 - a/D) \cdot \frac{d \ln F_l}{d(a/D)}$$

Converting F_l to limit moment M_l and to yield stress σ_s gives

$$F_l = 4M_l/S \propto \sigma_s I/h \propto h^{2.5}$$

Hence, just as in the elastic case

$$\phi_{eq,pl} \approx 2.5 \text{ when } a/D \rightarrow 1 \quad (28)$$