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Epi-Convergence of Minimum Curvature Variation B-splines

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Abstract

We study the curvature variation functional, i.e., the integral over the square of arc-length derivative of curvature, along a planar curve. With no other constraints than prescribed position, slope angle, and curvature at the endpoints of the curve, the minimizer of this functional is known as a cubic spiral. It remains a challenge to effectively compute minimizers or approximations to minimizers of this functional subject to additional constraints such as, for example, for the curve to avoid obstacles such as other curves.

In this paper, we consider the set of smooth curves that can be written as graphs of three times continuously differentiable functions on an interval, and, in particular, we consider approximations using quartic uniform B-spline functions. We show that if quartic uniform B-spline minimizers of the curvature variation functional converge to a curve, as the number of B-spline basis functions tends to infinity, then this curve is in fact a minimizer of the curvature variation functional. In order to illustrate this result, we present an example of sequences of B-spline minimizers that converge to a cubic spiral.

1 Introduction

Let $\varphi = \varphi(s)$ be a curve in the plane, parameterized by its arc-length s, and let κ_φ and L_φ be the curvature and the length of φ, respectively. We study the *curvature variation functional*

$$F(\varphi) = \int_0^{L_\varphi} \dot{\kappa}_\varphi^2 \, ds, \tag{1}$$

where $\dot{\nu} = d\nu/ds$.

The litterature contains a number of studies of this functional and the problem of computing a curve that minimizes (1), subject to various constraints [13, 18, 14, 3]. Following Moreton [18], we call such a minimizer a *minimum variation curve (MVC)*. The computation of these curves is of great interest in path-planning [13, 8, 19, 3] as well as curve and surface design and reconstruction [18].

In the general case, when arbitrary constraints are imposed on the curve, to the best of the authors' knowledge, the representation of the MVC is not known. An example of such a case is when the curve is constrained not to intersect obstacles such as other curves [4]. Although, in the special case where the only constraints are second order constraints, i.e., prescribed position, slope angle, and curvature, at the endpoints of the curve, the representation of the MVC is known as a *cubic spiral*. It is also known as a *cubic* [13] and an *intrinsic spline* of degree 2 [8].

One drawback with the cubic is that its position can not be written as a closed form expression and that it is costly to compute. Therefore, both in the special and the general case, efficiently computed approximations, that can be constrained in various ways, of a cubic and a general MVC, is of great interest.

We consider approximations that belong to a certain class of smooth curves, namely the so called *B-spline functions*, which are also referred to as *B-splines* [6, 9]. One thing that makes B-splines attractive is the ease by which the shape of the resulting curve can be controlled. For this reason, B-splines are widely used in a variety of contexts such as data fitting, computer aided design (CAD), automated manufacturing (CAM), and computer graphics [11]. An advantage of B-splines compared to cubics is that they can be given in closed form and they are efficiently computed. It is also possible to bound them in the plane by piecewise linear envelopes in terms of the parameters describing the splines [15].

In this paper, we investigate whether B-splines can serve as good approximations to cubics in the special case where we only have second order constraints at the endpoints of the curve. We obtain convergence results for the curvature variation functional of (1) over a class of smooth curves containing B-spline functions. To this end, we use *epi-convergence*, or Γ-*convergence*, theory [12, 2]. Our approach follows the lines of Bruckstein et al. [5], who obtained convergence results for polygons with respect to the functional $\int_0^{L_\varphi} \kappa_\varphi^\alpha \, ds$, $1 \leq \alpha < \infty$.

In Section 2, we give some mathematical preliminaries regarding spaces, metrics, constraints, functionals, and Γ-convergence. In Section 3 we prove our convergence results. This is followed by an example of a converging sequence of B-spline minimizers in Section 4. Finally, in Section 5 we conclude the paper.

2 Preliminaries

The canonical problem of finding a curve φ that minimizes $F(\varphi)$ of (1) subject to second order endpoint constraints, i.e., specified position, tangent, and curvature at the endpoints of the curve, is treated thoroughly by Kanayama and Hartman [13]. They show that there is one unique curve of finite length, having curvatures equal to zero at its endpoints, that minimizes $F(\varphi)$. This curve is either a so called symmetric cubic spiral or a pair of connected symmetric cubic spirals, and it is referred to as a cubic spiral, or cubic for short.

We do not address the problem of finding minimizers of $F(\varphi)$ as their representation is already given by the cubic. Instead, we are interested in the convergence and approximation properties of B-spline minimizers of $F(\varphi)$ for the case when we have symmetric endpoint constraints. These are the constraints that yield a symmetric cubic, which all cubics are concatenations of.

In this section, we first present the space and metrics of the smooth curves considered in this paper together with a short description of quartic uniform B-splines and the notion of symmetry. Second, we briefly present the method, relying on Γ-convergence theory, that we use to obtain convergence results.

2.1 Smooth curves, B-splines, symmetry, and metrics

We let x and y be Euclidian coordinates in the plane, and we consider a particular space of smooth curves, namely functions in $C^3[x_0, x_1]$, or more briefly C^3. Here, a *curve* φ is a vector $\varphi(x) = [x, f_\varphi(x)]^T$, where $f_\varphi \in C^3$. The curve φ can also be parameterized by its *arc-length* s, so as to be a unit-speed curve, through the one-to-one relation $ds = \sqrt{1 + f'_\varphi(x)^2}\, dx$, where $\nu' = d\nu/dx$. Throughout this paper, where appropriate, we use both of these parameterizations. We refer to $\|\varphi_1(s) - \varphi_2(s)\|$ as the *Euclidian distance* between the vectors $\varphi_1(s) = [x_{\varphi_1}(s), y_{\varphi_1}(s)]^T$ and $\varphi_2(s) = [x_{\varphi_2}(s), y_{\varphi_2}(s)]^T$. The *curvature* κ_φ of φ is defined as $\kappa_\varphi(s) = \|\ddot{\varphi}(s)\|$, where $\dot{\nu} = d\nu/ds$, and the *length* of φ is denoted L_φ. We consider the following *symmetric* second order endpoint constraints for $\varphi = \varphi(s)$:

$$\begin{aligned}
\varphi(0) &= [x_0, y_\varphi(0)]^T, \\
\varphi(L_\varphi) &= [x_1, y_\varphi(L_\varphi)]^T, \\
\frac{\dot{\varphi}(L_\varphi) + \dot{\varphi}(0)}{\|\dot{\varphi}(L_\varphi) + \dot{\varphi}(0)\|} &= \frac{\varphi(L_\varphi) - \varphi(0)}{\|\varphi(L_\varphi) - \varphi(0)\|}, \\
\ddot{\varphi}(0) &= 0, \\
\ddot{\varphi}(L_\varphi) &= 0.
\end{aligned}$$

For the approximation, we consider subsets $S_{4,n} \subset C^3$, $n \geq 6$, that are sets of uniform quartic B-spline functions built from n basis functions that are able to match the symmetric endpoint constraints. The reason for using quartic B-splines is that they are the B-splines of the lowest degree for which the derivative of curvature is continuous. A *uniform quartic B-spline* $y = B_n(b, x) = \sum_{i=1}^{n} b_i N_{i,4}(x)$ is a *piecewise polynomial* of degree 4 [6, 9]. It is defined by the *B-spline basis functions* $N_{i,4}(x)$, $i = 1, \ldots, n$, the *B-spline coefficient vector* $b = [b_1, \ldots, b_n]^T$, and a non-decreasing real number *knot sequence*. To handle the endpoint constraints, we use a *uniform* knot sequence with 5 *multiple* knots at the end points.

Our convergence results are shown with respect to the norm in C^2. To simplify the notation in this paper, for the curves $\varphi_1 = [x, f_{\varphi_1}(x)]^T$ and $\varphi_2 = [x, f_{\varphi_2}(x)]^T$, where $f_{\varphi_1}, f_{\varphi_2} \in C^3$ and $x \in [x_0, x_1]$, the C^2 norm is written as

$$\begin{aligned}
D(\varphi_1, \varphi_2) &= \|f_{\varphi_2} - f_{\varphi_1}\|_{C^2} \\
&= \max_{x \in [x_0, x_1]} |f_{\varphi_2} - f_{\varphi_1}| + \\
&\quad \max_{x \in [x_0, x_1]} |f'_{\varphi_2} - f'_{\varphi_1}| + \\
&\quad \max_{x \in [x_0, x_1]} |f''_{\varphi_2} - f''_{\varphi_1}|.
\end{aligned} \tag{2}$$

In order to obtain results under the C^2 norm $D(\cdot, \cdot)$, we use a related metric $d(\cdot, \cdot)$ for smooth curves. These curves can be normalized and parameterized over $t \in [0, 1]$ (instead of in arc-length s). For the curves φ_1 and φ_2 that are parameterized as $\varphi_1(t) = [x_{\varphi_1}(t), y_{\varphi_1}(t)]^T$ and $\varphi_2(t) = [x_{\varphi_2}(t), y_{\varphi_2}(t)]^T$, this metric is

$$d(\varphi_1, \varphi_2) = \inf_{\psi:[0,1] \to [0,1]} \sup_{t \in [0,1]} \|\varphi_2(t) - \varphi_1(\psi(t))\|, \tag{3}$$

where ψ is a homeomorphism (reparametrization). Informally, the metric $d(\varphi_1, \varphi_2)$ can be seen as a measure of how far apart two pencils would have to separate if the curves φ_1 and φ_2 were drawn simultaneously. Furthermore, from the definition of distance between curves given by Alexandrov and Reshetnyak [1], it follows that the C^2 norm and $d(\cdot, \cdot)$ are related as

$$d(\varphi_1, \varphi_2) \leq D(\varphi_1, \varphi_2). \tag{4}$$

2.2 Epi-convergence and curvature variation functionals

Corresponding to the general curvature variation functional F, cf. (1), defined by

$$F(\gamma) = \int_0^{L_\gamma} \dot{\kappa}_\gamma^2 \, ds, \quad \gamma \in C^3, \tag{5}$$

we consider functionals of the form

$$F_n(\gamma) = \begin{cases} F(\gamma) & , \gamma \in S_{4,n} \subset C^3 \\ +\infty & , \gamma \in C^3 \setminus S_{4,n} \end{cases}, \tag{6}$$

where $S_{4,n} \subset C^3$, $n = 6, 7, \ldots$, are sets of uniform quartic B-spline functions B_n that are built from n basis functions. We prove that such functionals as in (6) approximate the functional in (5), with respect to the C^2 norm. In order to do this, we use epi-convergence theory, also referred to as Γ-convergence theory [12, 2, 16]. The main implication of this approximation is that minimizers of F_n converge to minimizers of F as $n \to \infty$.

The general structure of Γ-convergence is the following [16]. Let Ω be a separable metric space, where F_n, $n = 1, 2 \ldots$, and F are functionals defined over Ω. We say that F_n Γ-*converges to* F, which we denote $F_n \xrightarrow{\Gamma} F$, if

$$\begin{array}{llll} (L) & \forall \omega \in \Omega, & \omega_n \to \omega & : \liminf_{n\to\infty} F_n(\omega_n) \geq F(\omega), \\ (U) & \forall \omega \in \Omega, & \exists \tilde{\omega}_n \to \omega & : \limsup_{n\to\infty} F_n(\tilde{\omega}_n) \leq F(\omega), \end{array} \tag{7}$$

where $\omega_n, \tilde{\omega}_n \in \Omega$. We think of (L) and (U) as a lower and upper limit respectively. Proving Γ-convergence amounts to proving (L) and (U).

The theory of Γ-convergence provides the following key result [16]:

Theorem 1 *Let $F_n \xrightarrow{\Gamma} F$ and let ω_n minimize F_n. If ω is a cluster point of $\{\omega_n\}$ then ω minimizes F.*

This means that all limit points of minimizers of F_n are minimizers of F which in turn implies that we can obtain approximations to minimizers of F without having explicit representations of them. In this paper, the separable metric space Ω is C^3, and the convergence is shown for $n = 6, 7 \ldots$, with respect to the C^2 norm.

3 Γ-convergence of curvature variation B-splines

In order to prove Γ-convergence of F_n to F, we are interested in showing that $F(\gamma)$ is lower semicontinuous, cf. (L) of (7). Then we first need the following property of a curve $\gamma \in C^3$ which comes from the fact that γ is an element of a much larger class of curves, namely the class of rectifiable curves with finite total absolute curvature (RFT-curves) [1]. This is a class of curves that admit an arc-length parametrization and that are endowed with the metric $d(\cdot,\cdot)$ as defined in (3). Such curves have the following property [1]:

Theorem 2 *Let c_i, $i = 1, 2, \ldots$, be a sequence of RFT curves converging to an RFT curve c with respect to $d(\cdot,\cdot)$. Then*

$$L_c \leq \liminf_{i\to\infty} L_{c_i}$$

We use this in order to state the following lemma regarding semicontinuity of F with respect to the C^2 norm.

Lemma 1 *If, for $\gamma_n, \gamma \in C^3$, $\lim_{n\to\infty} D(\gamma_n, \gamma) = 0$, then*

$$F(\gamma) \leq \liminf_{n\to\infty} F(\gamma_n).$$

Proof: We define the functional H as follows

$$H(\gamma) = \int_0^{L_\gamma} \| \dddot{\gamma} \|^2 \, ds \quad \gamma \in C^3.$$

Now, we use the Frenet formulas for a planar unit-speed curve [20]. By differentiating $\ddot{\gamma}$ we relate the curvature $\kappa_\gamma = \|\ddot{\gamma}\|$ by

$$\dddot{\gamma} = \dot{\kappa}_\gamma N_\gamma - \kappa_\gamma^2 T_\gamma,$$

where N_γ and T_γ are the *principal normal* and the *tangent* unit-vector fields on γ respectively. As N_γ and T_γ are orthonormal vectors, we conclude that

$$\| \dddot{\gamma} \|^2 = \dot{\kappa}_\gamma^2 + \kappa_\gamma^4.$$

This in turn means that,

$$H(\gamma) = F(\gamma) + G(\gamma), \tag{8}$$

where

$$F(\gamma) = \int_0^{L(\gamma)} \dot{\kappa}_\gamma^2 \, ds$$

is the same functional as in (5) and

$$G(\gamma) = \int_0^{L(\gamma)} \kappa_\gamma^4 \, ds = \int_0^{L(\gamma)} \|\ddot{\gamma}\|^4 \, ds.$$

As $d(\gamma_n, \gamma) \leq D(\gamma_n, \gamma)$, by (4), then $L_\gamma \leq \liminf_{n\to\infty} L_{\gamma_n}$, by Theorem 2. It follows easily, from standard lower semicontinuity of L^p norms, that

$$H(\gamma) \leq \liminf_{n\to\infty} H(\gamma_n).$$

This means that H is lower semicontinuous with respect to the C^2 norm. According to (8), $F(\gamma) = H(\gamma) - G(\gamma)$. It is evident that $G(\gamma) = \int_0^{L(\gamma)} \|\ddot{\gamma}\|^4 \, ds$ is continuous with respect to the C^2 norm. We treat G as a continuous perturbation of the lower semicontinuous functional H and conclude that F is also lower semicontinuous. □

Now, we are in position to prove the main result of this paper, namely the Γ-convergence of F_n to F with respect to the C^2 norm.

Theorem 3 *Let F_n and F be functionals defined over C^3 according to (5) and (6), respectively. Then, with respect to the C^2 norm,*

$$F_n \xrightarrow{\Gamma} F.$$

Proof: We prove the Γ-convergence by proving the lower and upper limits $(L)$and (U) as described in (7).

The lower limit (L) follows directly from the definition of F_n, cf. (6), together with the lower semicontinuity of F with respect to the C^2 norm, cf. (5) and Lemma 1. We have, for $\gamma_n, \gamma \in C^3$, $\lim_{n\to\infty} D(\gamma_n, \gamma) = 0$, that

$$\liminf_{n\to\infty} F_n(\gamma_n) \geq \liminf_{n\to\infty} F(\gamma_n) \geq F(\gamma),$$

which is also true for $\gamma_n = B_n$.

To prove the upper limit (U) we use an approximation scheme, applicable to uniform B-splines, introduced by de Boor and Fix [7], which is also mentioned by de Boor [6]. Using the scheme, it is possible to produce a uniform quartic B-spline function B_n, built from n basis functions, that is able to approximate curves with respect to the C^4 norm at the most. As we are dealing with curves in C^3, we can obtain approximations with respect to the C^3 norm. For $\gamma_1, \gamma_2 \in C^3$ the C^3 norm is given by $\tilde{D}(\gamma_1, \gamma_2) = D(\gamma_1, \gamma_2) + \max_{x \in [x_0, x_1]} |f'''_{\gamma_2} - f'''_{\gamma_1}|$. The scheme implies that, given a curve $\gamma \in C^3$, $\lim_{n\to\infty} \tilde{D}(B_n, \gamma) = 0$. As our functionals F and F_n are continuous with respect to the C^3 norm, $\lim_{n\to\infty} F(B_n) = F(\gamma)$, and as $0 \le D(\cdot, \cdot) \le \tilde{D}(\cdot, \cdot)$, we also have that $\lim_{n\to\infty} D(B_n, \gamma) = 0$, i.e. convergence in the C^2 norm.. Altogether, we know that, given a curve $\gamma \in C^3$ there exists a sequence of quartic uniform B-splines $B_n \in C^3$ for which

$$\lim_{n\to\infty} D(B_n, \gamma) = 0, \text{ such that } \lim_{n\to\infty} F(B_n) = F(\gamma).$$

This concludes the proof of (U) and also the proof of Γ-convergence. □

4 An example of convergent B-spline minimizers

In this section, we give an indication of the existence of a sequence of B-spline minimizers B_n of F_n, that converge to a minimizer γ of the curvature variation functional F. Here, we do this by a numerical study of one example of symmetric second order endpoint constraints. In this case, we know already that the unique minimizer $\gamma \in C^3$ of F, is a symmetric cubic spiral. Using standard nonlinear constrained programming software, we compute B-spline minimizers to the curvature variation functional over sets of B-spline functions. We investigate the convergence of the B-spline minimizers by comparing them with the already known cubic spiral solution.

In order to compute the B-spline minimizers, we use a B-spline implementation and a solver for constrained optimization problems `fmincon`, that are both provided by Matlab [17]. Involved integrals are computed by the routine `coteglob`, which is a globally doubly adaptive quadrature based on Newton-Cotes 5 and 9 points rules over a finite interval [10]. As an initial value for the solver, we use the straight line segment between the endpoints.

For B-splines built from a larger number of basis functions than 10, we get problems with the convergence when using the solver. The reason might be errors in the implementation of the solver itself or the accuracy of the quadrature routine. A feasible initial value, that is closer to the cubic than the one used here, would probably yield faster and more reliable convergence. Due to these problems mentioned, we study the convergence of B-spline minimizers for low numbers of basis functions only and take this as an indication of what happens for a larger number.

In our example, we consider the symmetric endpoint constraints $y(0) = 0$, $y'(0) = \tan(1)$, $y''(0) = 0$, and $y(1) = 0$, $y'(1) = \tan(-1)$, $y''(1) = 0$. The

n	$F_n(B_n) - F(\gamma)$	$D(\gamma, B_n)$
6	83.0496	3.4807
7	10.4786	2.3571
8	3.4175	1.7753
9	0.8430	1.2122
10	0.0811	0.8129

Table 1: Difference between cost functions and the distance between the symmetric cubic spiral γ and the B-spline minimizers B_n.

cubic spiral γ that minimizes F, and that can easily be derived [13], yield $F(\gamma) = 22.0615$.

In Table 4, for an increasing number, n, of basis functions, we present the relation of the B-spline minimizers B_n to γ, with respect to difference in cost function and C^2 norm. Figure 1 shows a plot of the cubic spiral solution together with the B-spline minimizers. Both numerical and visual inspection indicates that there is a sequence of B-spline minimizers that converge to the optimal curve, which is a symmetric cubic.

5 Conclusions and future work

In this paper we have studied the problem of computing a smooth planar curve $\gamma \in C^3$ that, with symmetric second order endpoint constraints, minimizes the curvature variation functional F over C^3, i.e., the integral over the square of arc-length derivative of curvature, along the curve. The curve γ that solves this problem is already known as a symmetric cubic spiral.

We have considered approximations in C^3 represented by efficiently computed uniform quartic B-splines $B_n \in S_{4,n} \subset C^3$ built from n B-spline basis functions. The functional F_n, which is the same as F over the space $S_{4,n}$ of B-splines and $+\infty$ otherwise, has been introduced. We have proved that F_n Γ-converges, or epi-converges, to F with respect to the C^2 norm. The main implication of this is that, if there is a converging sequence of minimizers B_n of F_n, then B_n converges, in C^2 norm, to a minimizer γ of F as $n \to \infty$.

Through an example with symmetric endpoint constraints, for which we already know the cubic spiral minimizer γ of F, we gave an indication of the existence of such a converging sequence of B-spline minimizers B_n of F_n. It is in fact possible to prove that such a sequence exists, using the Sobolev embedding theorem and assuming that curve lengths are uniformly

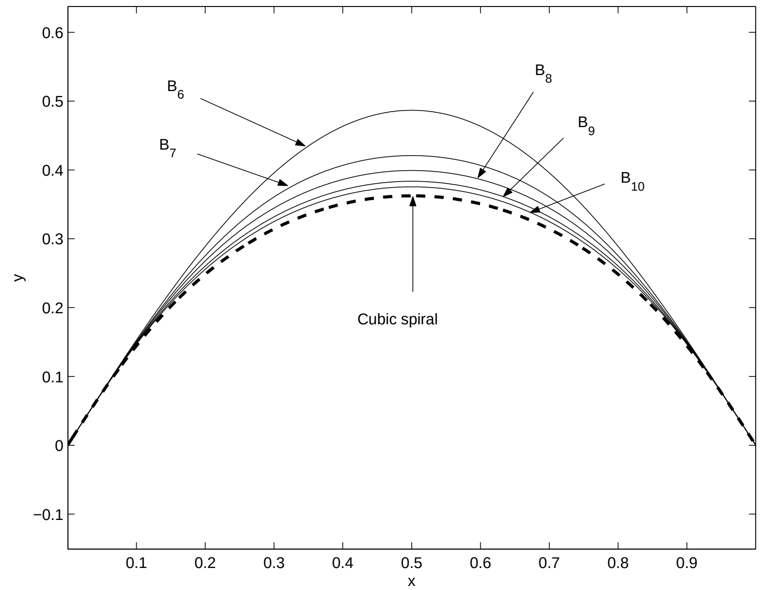

Figure 1: Plots of B-spline minimizers B_n of F_n, $n = 6, \dots, 10$, together with the symmetric cubic spiral γ (dashed) that minimizes F.

bounded. We will treat this issue in forthcoming work.

Other future work includes the convergence of quartic uniform B-spline approximations for general second order endpoint constraints as well as for various other constraints, e.g., for the curve to not intersect other prescribed curves [4]. Is the minimizer B_n of F_n unique even for these more general constraints? Another interesting issue is the convergence rate, i.e., the decrease in F_n with respect to n. How many B-spline basis functions are needed in order to obtain a given accuracy of the approximation?

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