Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

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by

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Supervisors
Professors Lars-Erik Persson and Peter Wall
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To my wife and children
Abstract

This PhD thesis in mathematics is focussed on some problems of great interest in applied mathematics. More precisely, we investigate some new questions in homogenization theory, which have been motivated by some concrete problems in tribology. From the mathematical point of view these questions are equipped with scales of Reynolds equations with rapidly oscillating coefficients. In particular, in this PhD thesis we derive the corresponding homogenized (averaged) equations. We consider the Reynolds equations in both the stationary and unstationary forms to analyze the effect of surface roughness on the hydrodynamic performance of bearings when a lubricant is flowing through it. In addition we have successfully developed a reiterated homogenization (with three scales) procedure which makes it possible to efficiently study problems connected to hydrodynamic lubrication including shape, texture and roughness.

Furthermore, we solve a linear parabolic initial-boundary value problem with singular coefficients in non-cylindrical domains. We accomplish this feat by developing a variant of Rothe’s method to prove the existence and uniqueness of a weak solution to the parabolic problem. By combining the Rothe’s method and the technique of two scale convergence we derive a homogenized equation for a linear parabolic problem with time dependent coefficients oscillating rapidly in the space variable. Moreover, we derive a concrete homogenization algorithm for giving a unique and computable approximation of the solution.

In Chapter 1 we describe some possible types of surfaces a bearing can take. Out of these we select two types and derive the appropriate Reynolds equations needed for their analysis.

Chapter 2 is devoted to the derivation of the homogenized equations associated with the stationary forms of the compressible and incompressible
Reynolds equations. We derive these homogenized equations by using the multiple scales expansion technique.

In Chapter 3 the homogenized equations for the unstationary forms of the Reynolds equations are considered and some numerical results based on the homogenized equations are presented.

In Chapter 4 we consider the equivalent minimization problem (variational principle) for the unstationary Reynolds equation and use it to derive a homogenized minimization problem. Moreover, we obtain both the lower and upper bounds for the derived homogenized problem.

Chapter 5 is devoted to studying the combined effect that arises due to shape, texture and surface roughness in hydrodynamic lubrication. This is accomplished by first studying a general class of problems that includes the incompressible Reynolds problem in both cartesian and cylindrical coordinate forms.

In Chapter 6 we prove a homogenization result for the nonlinear equation

\[ \text{div}(a(x, x/\epsilon, x/\epsilon^2, \nabla u_\epsilon) = \text{div} b(x, x/\epsilon, x/\epsilon^2), \]

where the coefficients are assumed to be periodic and \( a \) is monotone and continuous. This kind of problem has applications in hydrodynamic lubrication of surfaces with roughness on different length scales.

In Chapter 7 a variant of Rothe’s method is developed, discussed and used to prove existence and uniqueness result for linear parabolic problem with singular coefficients in non-cylindrical domains.

In Chapter 8 we combine the Rothe method with a homogenization technique (two-scale convergence) to handle a general time-dependent linear parabolic problem. In particular we prove that both the approximating sequence and the final approximate solution are unique. Finally, we derive a concrete homogenization algorithm on how to compute this homogenized solution.
This PhD thesis is written as a monograph. A brief description of the chapters are outlined in the abstract.

In particular, the author’s contributions in the following papers are included in this PhD thesis:


Also the following paper has influenced some results and ideas in this PhD thesis.

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# Contents

Abstract

Preface

Acknowledgements

1 Introduction

1.1 Reynolds type equations

1.1.1 Various forms of the Reynolds equations

1.1.2 Derivation of the linear forms (1.6) and (1.7)

1.1.3 Outline of the homogenization procedure

2 Multiple scale expansion for Reynolds equation (stationary case)

2.1 The stationary compressible (constant bulk modulus) case

2.2 The stationary incompressible case

3 Homogenization of the unstationary incompressible Reynolds equation

3.1 Introduction

3.2 The governing Reynolds type equations

3.3 Homogenization (constant bulk modulus)

3.4 Homogenization in the incompressible case

3.5 Numerical results

3.5.1 Incompressible case

3.5.2 Constant bulk modulus case

3.6 Concluding remarks
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Start Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Variational bounds applied to unstationary hydrodynamic lubrication</td>
<td>47</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>47</td>
</tr>
<tr>
<td>4.2</td>
<td>Homogenization of a variational principle</td>
<td>49</td>
</tr>
<tr>
<td>4.3</td>
<td>Bounds of arithmetic-harmonic type</td>
<td>53</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Upper bound</td>
<td>53</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Lower bound</td>
<td>55</td>
</tr>
<tr>
<td>4.4</td>
<td>Bounds of Reuss–Voigt type</td>
<td>58</td>
</tr>
<tr>
<td>4.5</td>
<td>Application to a problem in hydrodynamic lubrication</td>
<td>58</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Sinusoidal roughness</td>
<td>61</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Bisinusoidal roughness</td>
<td>65</td>
</tr>
<tr>
<td>4.5.3</td>
<td>A realistic surface roughness representation</td>
<td>68</td>
</tr>
<tr>
<td>4.6</td>
<td>Conclusions</td>
<td>71</td>
</tr>
<tr>
<td>4.7</td>
<td>Appendix (A dual variational principle)</td>
<td>72</td>
</tr>
<tr>
<td>5</td>
<td>Reiterated homogenization applied in hydrodynamic lubrication</td>
<td>75</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>75</td>
</tr>
<tr>
<td>5.2</td>
<td>The homogenization procedure</td>
<td>76</td>
</tr>
<tr>
<td>5.3</td>
<td>An additional result</td>
<td>78</td>
</tr>
<tr>
<td>5.4</td>
<td>Application to hydrodynamic lubrication</td>
<td>79</td>
</tr>
<tr>
<td>5.4.1</td>
<td>A numerical investigation of convergence</td>
<td>82</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Application to a thrust pad bearing problem</td>
<td>86</td>
</tr>
<tr>
<td>5.5</td>
<td>Conclusions</td>
<td>96</td>
</tr>
<tr>
<td>5.6</td>
<td>Appendix 1</td>
<td>98</td>
</tr>
<tr>
<td>5.7</td>
<td>Appendix 2</td>
<td>99</td>
</tr>
<tr>
<td>6</td>
<td>Reiterated homogenization of a nonlinear Reynolds-type equation</td>
<td>105</td>
</tr>
<tr>
<td>6.1</td>
<td>Introduction</td>
<td>105</td>
</tr>
<tr>
<td>6.2</td>
<td>Preliminaries and notation</td>
<td>106</td>
</tr>
<tr>
<td>6.3</td>
<td>Three-scale convergence</td>
<td>108</td>
</tr>
<tr>
<td>6.4</td>
<td>A three-scale homogenization procedure</td>
<td>114</td>
</tr>
<tr>
<td>6.5</td>
<td>The linear case</td>
<td>120</td>
</tr>
<tr>
<td>6.6</td>
<td>Application to hydrodynamic lubrication</td>
<td>122</td>
</tr>
<tr>
<td>6.7</td>
<td>A convergence result for periodic functions</td>
<td>124</td>
</tr>
<tr>
<td>7</td>
<td>Linear parabolic problems with singular coefficients in non-cylind-</td>
<td>129</td>
</tr>
<tr>
<td>7.1</td>
<td>Introduction</td>
<td>129</td>
</tr>
</tbody>
</table>
## Contents

7.2 Some preliminaries and auxiliary results ................................ 131
7.3 The main result ........................................................................... 143
7.4 Concluding examples and results ................................................. 148

8 Homogenization of linear parabolic problems by the method
of Rothe and two-scale convergence ............................... 153
8.1 Introduction .............................................................................. 153
8.2 Preliminaries ............................................................................ 154
8.3 Main results .............................................................................. 158
8.4 Proofs ...................................................................................... 160

Bibliography ............................................................................. 176
Chapter 1

Introduction

1.1 Reynolds type equations

Reynolds type equations are widely used in the field of Tribology. Tribology is a multidisciplinary field, which deals with the science, practice and technology of lubrication, wear prevention and friction control in machines. This enable lubrication engineers to minimize cost of moving parts. In this way machinery can be made more efficient, more reliable and more cost effective. In the field of hydrodynamic lubrication, the flow of fluid through machine elements such as bearings, gearboxes and hydraulic systems may be governed by the Reynolds equation. The Reynolds type equations are often used in analyzing the influence of surface roughness on the hydrodynamic performance of different machine elements when a lubricant is flowing through it.

The two surfaces through which a lubricant flows, may have any of the following characteristics:

(a) both surfaces are rough and moving,
(b) one surface is rough and stationary while the other is smooth and moving.

In Case (a), due to the motion of the rough surfaces, the coefficients in the governing Reynolds equation will be time dependent. As a result of this motion, the film thickness $h$ will be changing rapidly with respect to position $x$ and time $t$, thus giving rise to rapid variations (changes) in lubricant pressure within the machine element.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

In Case (b), the governing Reynolds type equation will be time independent. This is due to the fact that the film thickness at any position $x$ within the machine element remains the same at any time $t$. In both cases, due to the surface roughness, the coefficient $h$ in the Reynolds equation will be oscillating rapidly and therefore we may consider the possibility of solving the problem by using an averaging process, and here homogenization theory is a very useful method.

1.1.1 Various forms of the Reynolds equations

Figure 1.1 represents a cross section of two smooth bearing surfaces $s_1$ and $s_2$ with the governing Reynolds type equation given by

$$
\nabla \cdot \left[ \frac{\rho(p(x))h^3(x)}{12\eta} \nabla p(x) \right] = \frac{u_1 + u_2}{2} \frac{\partial}{\partial x_1} \left[ \rho(p(x))h(x) \right], \quad (1.1)
$$

where $u_1$ and $u_2$ are the velocities of $s_1$ and $s_2$, respectively, $\eta$ is the viscosity of the lubricant, which is assumed to be constant, and $\rho$ represents the density of the lubricant. Moreover, $h(x)$ is the film thickness between the two surfaces, while $p(x)$ is the pressure build up between the surfaces when the lubricant flows through it. The bearing domain is denoted by $\Omega$ and the space variable $x \in \Omega \subset \mathbb{R}^2$.

In general the density $\rho$ of a lubricant is a function of the pressure, so that with a converging film thickness, we expect the pressure to be changing.
Figure 1.2: One rough stationary surface and one smooth moving surface.

Figure 1.3: Both surfaces are rough and moving.
This change in pressure will cause the density of the lubricant to change. Figure 1.2 is a pictorial representation of case (b) above. Due to the periodic roughness on $s_2$, the film thickness will depend on the roughness wavelength $\varepsilon$, where $\varepsilon$ is a positive sequence converging to zero. This film thickness can now be described by introducing the following auxiliary function

$$ h = h(x, y) = h_0(x) + h_2(y), $$

where $h_2$ is assumed to be periodic. In this equation $h_0$ describes the global film thickness and the periodic function $h_2$ represent the roughness contribution of this surface to the overall film thickness. Without loss of generality it can also be assumed that for $h_2$ the cell of periodicity is represented by $Y = (0, 1) \times (0, 1)$, i.e. the unit cube in $\mathbb{R}^2$. As a result of this dependence of $h$ on $\varepsilon$ we can model the film thickness $h_\varepsilon$ by replacing $h(x)$ in (1.1) with $h_\varepsilon(x)$ to obtain the following equation:

$$ \nabla \cdot \left[ \frac{\rho(p_\varepsilon(x))h_\varepsilon^3(x)}{12\eta} \nabla p_\varepsilon(x) \right] = \frac{u_1 + u_2}{2} \frac{\partial}{\partial x_1} [\rho(p_\varepsilon(x))h_\varepsilon(x)], \quad (1.2) $$

where

$$ h_\varepsilon(x) = h(x, x/\varepsilon) = h_0(x) + h_2(x/\varepsilon), $$

$$ p_\varepsilon(x) = p(x, x/\varepsilon), $$

The variable $y = x/\varepsilon$ is called the local variable and $\varepsilon$ obviously describes how rapid the oscillations are. We will discuss this in detail later on in this PhD thesis and also study what happens when $\varepsilon \to 0^+$. Equation (1.2) is then the Reynolds equation, which takes into account the roughness contribution to the pressure build up in the bearing. If we assume that the rough surface is stationary, while the moving surface is smooth, then the film thickness $h_\varepsilon(x)$ at any position $x$ within the bearing will remain the same at any time $t$ and, hence, $h_\varepsilon(x)$ will be independent of time $t$. This explains why the Reynolds equation (1.2) does not involve time.

Figure 1.3 is a pictorial description of case (a) above. Here we consider the case where both surfaces are rough and moving. As a consequence of this motion, the film thickness will be changing rapidly, depending on the relative positions of the corresponding rough surfaces. In Figure 1.4, we see that the film thickness $h_\varepsilon$ at the position $x$ is different for the two time steps $t_1$ and $t_2$. This is due to the relative positions of the corresponding rough surfaces. This shows clearly that the film thickness $h$, which is dependent on $\varepsilon$, is a function of both $x$ and $t$ in case (a).
this case, the film thickness can be described by introducing the following auxiliary function

\[ h = h(x, t, y, \tau) = h_0(x, t) + h_2(y - \tau V_2) - h_1(y - \tau V_1), \]

where \( V_i = (u_i, 0) \) is the velocity of surface \( s_i, \ i = 1, 2 \) and \( u_i \) is constant, while \( h_1 \) and \( h_2 \) are assumed to be periodic. Here \( h_0 \) describes the global film thickness and the periodic functions \( h_1 \) and \( h_2 \) represents the roughness contribution of the two surfaces. By using this auxiliary function \( h \), we can model the film thickness \( h_\varepsilon \) by

\[ h_\varepsilon(x, t) = h(x, t, x/\varepsilon, t/\varepsilon) = h_0(x) + h_2\left(\frac{x - tV_2}{\varepsilon}\right) - h_1\left(\frac{x - tV_1}{\varepsilon}\right), \]

\[ p_\varepsilon(x, t) = p(x, t, x/\varepsilon, t/\varepsilon), \]

where \( y = x/\varepsilon \) and \( \tau = t/\varepsilon \). The Reynolds equation describing such a time dependent situation is given by

\[
\frac{\partial}{\partial t} \left[ \rho (p_\varepsilon(x,t)) h_\varepsilon(x,t) \right] = \nabla \cdot \left[ \frac{\rho (p_\varepsilon(x,t)) h_\varepsilon^2(x,t)}{12\eta} \nabla p_\varepsilon(x,t) \right] - \left( \frac{u_1 + u_2}{2} \right) \frac{\partial}{\partial x_1} \left[ \rho (p_\varepsilon(x,t)) h_\varepsilon(x,t) \right].
\]

In both the time independent and time dependent cases described above, we can expect the pressure to vary rapidly due to the rapidly changing nature of
the film thickness. As the roughness wavelength $\varepsilon$ tends to zero, we expect to have a rapidly oscillating pressure. This means that we will need such a fine mesh that it is impossible to solve it directly with any numerical method. This suggests some type of averaging. One rigorous way to do this is to use the general theory of homogenization, which we will describe, develop and use in later chapters. This theory facilitates the analysis of partial differential equations with rapidly oscillating coefficients, see e.g. Jikov et al. [41].

A more engineering oriented introduction can also be found in Persson et al. [69]. Homogenization has recently been applied to different problems connected to lubrication, see e.g. [6], [8], [10], [13], [15], [16], [20], [21], [22], [23], [28], [39], [40], [44], [45], [57] and [76] with much success. Some applications of homogenization have already been treated in the following thesis by other members of our research group in homogenization, see e.g. [25], [38], [56], [61], [73], [75], [77]. The main aim of this PhD thesis is to further develop and complement these results.

We remark that various kinds of inequalities are very important for the development of homogenization theory (e.g. those by Jensen, H"{o}lder, Minkowski, Poincare, Fredrich, Young, Hardy, Gronwall, etc). For example some new results concerning the close connection between inequalities and homogenization in domains with microinhomogeneous structure on the boundary are considered in the following papers, see Chechkin et. al [26], [27] and [29] (for further references see also the book [28].)

The Reynolds equation can be described as being compressible or incompressible depending on the functional dependence of $\rho$ on $p$ (i.e. $\rho(p_\varepsilon(x))$.)

If the lubricant is assumed to be incompressible, i.e. $\rho(p)$ is constant, then the equations (1.2) and (1.3) are reduced to

$$\nabla \cdot [h_3^3(x) \nabla w_\varepsilon(x)] = \Lambda \frac{\partial h_\varepsilon(x)}{\partial x_1},$$  

(1.4)

$$\Gamma \frac{\partial h_\varepsilon(x, t)}{\partial t} = \nabla \cdot [h_3^3(x, t) \nabla p_\varepsilon(x, t)] - \Lambda \frac{\partial h_\varepsilon(x, t)}{\partial x_1},$$  

(1.5)

where $\Gamma = 12\eta$, $\Lambda = 6\eta v$ and $v = u_1 + u_2$.

We note that the compressible equations (1.2) and (1.3) are non-linear. This means that in general it is much more difficult to analyze the compressible case. However, there is a relationship between the pressure and the density which will transform (1.2) and (1.3) respectively, into the linear forms below

$$\nabla \cdot (h_3^3(x) \nabla w_\varepsilon(x)) = \lambda \frac{\partial}{\partial x_1} (w_\varepsilon(x) h_\varepsilon(x)), $$  

(1.6)
\[
\gamma \frac{\partial}{\partial t} (w_\varepsilon(x, t)h_\varepsilon(x, t)) = \nabla \cdot (h_\varepsilon^3(x, t)\nabla w_\varepsilon(x, t)) - \lambda \frac{\partial}{\partial x_1} (w_\varepsilon(x, t)h_\varepsilon(x, t)), \tag{1.7}
\]

where \( \lambda = 6\eta v \beta^{-1}, \gamma = \frac{1}{2} \eta \beta^{-1}. \)

These linear forms of the compressible Reynolds equations are obtained under the assumption that the dependence of density on pressure obeys the relationship

\[
\rho(p_\varepsilon(x)) = \rho_a e^{(p_\varepsilon(x) - p_a)/\beta}, \tag{1.8}
\]

where \( \rho_a \) is the fluid’s density at the atmospheric pressure \( p_a \) and \( \beta \) is the bulk modulus of the fluid, which is assumed to be a positive constant. This assumption is valid for reasonably low pressures.

### 1.1.2 Derivation of the linear forms (1.6) and (1.7)

To further facilitate the transformation of (1.2) and (1.3) to the linear forms, we define a dimensionless density function \( w_\varepsilon(x) \) as

\[
w_\varepsilon(x) = \rho(p_\varepsilon(x))/\rho_a. \tag{1.9}
\]

Substituting (5.9) into (1.9), we get that

\[
w_\varepsilon(x) = e^{(p_\varepsilon(x) - p_a)/\beta}.
\]

Hence we have that

\[
\nabla w_\varepsilon(x) = e^{(p_\varepsilon(x) - p_a)/\beta} \frac{1}{\beta} \nabla p_\varepsilon(x)
\]

\[
= \frac{1}{\beta \rho_a e^{(p_\varepsilon(x) - p_a)/\beta}} \nabla p_\varepsilon(x)
\]

\[
= \beta^{-1} \rho_a^{-1} \rho(p_\varepsilon(x)) \nabla p_\varepsilon(x).
\]

This implies that

\[
\rho_a \beta \nabla w_\varepsilon(x) = \rho(p_\varepsilon(x)) \nabla p_\varepsilon(x). \tag{1.10}
\]

From (1.9) we see that

\[
ho(p_\varepsilon(x)) = \rho_a w_\varepsilon(x). \tag{1.11}
\]

By substituting (1.10) and (1.11) into (1.2) we obtain that

\[
\nabla \cdot (h_\varepsilon^3(x)\nabla w_\varepsilon(x)) = \lambda \frac{\partial}{\partial x_1} (w_\varepsilon h_\varepsilon) \text{ on } \Omega,
\]

where \( \lambda = 6\eta v \beta^{-1}, \) and (1.6) is derived.
Making similar substitutions of (1.11) and (1.10) into (1.3), we obtain the linear equation
\[
\gamma \frac{\partial}{\partial t} (w_\varepsilon(x,t)h_\varepsilon(x,t)) = \nabla \cdot (h_\varepsilon^2(x,t)\nabla w_\varepsilon(x,t)) - \lambda \frac{\partial}{\partial x_1} (w_\varepsilon(x,t)h_\varepsilon(x,t)),
\]
where \(\gamma = \frac{1}{2} \eta \beta - 1\), \(\lambda = 6\eta v \beta - 1\) and also (1.7) is derived.

1.1.3 Outline of the homogenization procedure

Homogenization is a branch within mathematics that involves the study of PDE’s with rapidly oscillating coefficients. The main purpose of homogenization of partial differential equations is to approximate p.d.e’s that have rapidly varying coefficients with equivalent homogenized p.d.e’s that, for example, more easily lend themselves to numerical treatment in a computer. The parameter \(\varepsilon\) is very important in homogenization, in that it describes how quickly the film thickness or material parameters vary and in the search for an equivalent homogenized p.d.e, one considers a sequence \(\{\varepsilon\} \to 0^+\). The smaller \(\varepsilon\) gets, the better the approximation becomes.

In deriving the homogenized Reynolds equation, we will model the lubricant film thickness in such a way that one part will describe the shape/geometry of the bearing, while the other part describes the surface roughness. The homogenized Reynolds equation is obtained by letting the wavelength of the modelled surface roughness tends to zero (i.e. \(\varepsilon \to 0^+\) in the modelling described above).

A first step to introduce and understand the homogenization of the equations (1.4) and (1.6) ) is to assume multiple scale expansions of the solutions in the following forms:
\[
p_\varepsilon(x) = p_0(x, \frac{x}{\varepsilon}) + \varepsilon p_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 p_2(x, \frac{x}{\varepsilon}) + ... \]
and
\[
w_\varepsilon(x) = w_0(x, \frac{x}{\varepsilon}) + \varepsilon w_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 w_2(x, \frac{x}{\varepsilon}) + ...
\]
where the functions \(p_i(x, y)\) and \(w_i(x, y)\), \((y = x/\varepsilon; \text{and } i = 0, 1, 2, ...\) are periodic in \(y\) for every \(x \in \Omega\). This means that \(y\) is a “local” variable, describing the behaviour of the solution on the unit cell scale. The “global” behaviour of the solution is expressed through the variable \(x\). The \(Y\)-periodicity means that the function is periodic in each coordinate with a period equal to the corresponding side length of \(Y\). In this way we arrive at an equation, which yields the approximation \(p_0\) of \(p_\varepsilon\) and \(w_0\) of \(w_\varepsilon\). For example by analysing
equation (1.4) using the formal method of multiple scale expansion we can prove that as \( \varepsilon \to 0 \), \( p_\varepsilon \to p_0 \), where \( p_0 \) solves a similar equation given by

\[
\nabla \cdot (B(x)\nabla p_0(x)) = \nabla \cdot (c(x)) \quad \text{on } \Omega \subset \mathbb{R}^2.
\]

This equation is the homogenized equation for (1.4). In particular \( B \) and \( c \) do not involve any rapid oscillations. This (more engineering oriented) approach is described in detail in Chapter 2.

Chapter 2 is devoted to the derivation of the homogenized equations associated with the stationary forms of the compressible and incompressible Reynolds equations. We derive these homogenized equations by using the multiple scales expansion technique.

In Chapter 3 the homogenized equations for the unstationary forms of the Reynolds equations are considered and some numerical results based on the homogenized equations are presented.

In Chapter 4 we consider the equivalent minimization problem (variational principle) for the unstationary Reynolds equation and homogenize it using multiple scale expansion. Finally, we obtain both the lower and upper bounds for the homogenized problem.

In Chapter 5, we study a class of problems with two oscillating scales. Homogenization of problems with two or more oscillating scales are referred to as reiterated homogenization, see e.g. [4], [19], [54] and [55]. Moreover we have successfully developed a reiterated homogenization procedure for this class of problems by using multiple scale expansion. In particular, by using this procedure we were able to study the combined effect that arises due to shape, texture and roughness in hydrodynamic lubrication. There are two steps involved in this type of homogenization process. First we homogenize the finer scale i.e. \( z = x/\varepsilon^2 \), whiles considering \( x \) and the other scale \( y = x/\varepsilon \) as parameters, and thereafter homogenize \( y \) to complete the process of obtaining the homogenized equation. In this process we still make use of the formal method of multiple scale expansion, but this time with \( p_\varepsilon \) assumed to be of the form

\[
p_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i p_i(x, x/\varepsilon, x/\varepsilon^2),
\]

where \( p_i = p_i(x, y, z) \) is both \( Y \) and \( Z \) periodic.

In Chapter 6 we prove a homogenization result for the nonlinear equation

\[
\text{div}(a(x, x/\varepsilon, x/\varepsilon^2, \nabla u_\varepsilon)) = \text{div} b(x, x/\varepsilon, x/\varepsilon^2),
\]
where the coefficients are assumed to be periodic and \( a \) is monotone and continuous. This kind of problem has applications in hydrodynamic lubrication of surfaces with roughness on different length scales. Some aspects of the theory concerning multi-scale convergence (three-scales) in Sobolev spaces \( W^{1,p}(\Omega) \) \((1 < p < \infty)\), needed to prove the homogenization result, are also developed.

In Chapter 7, a variant of Rothe’s method is developed, discussed and used to prove existence and uniqueness result for linear parabolic problem with singular coefficients in non-cylindrical domains. These results further extend and complement some recent results of this type in [32], [50], [51], [52] and [53].

In Chapter 8 we combine the Rothe method with a homogenization technique (two-scale convergence) to handle a general time-dependent linear parabolic problem. This two-scale convergence technique was introduced in 1989 by Nguetseng (see [65]) and later further developed by Allaire in [1]. Now the two-scale convergence technique is used frequently in the study of homogenization problems. We employ this technique to obtain a homogenized equation after using Rothe’s method, to prove existence and uniqueness of a parabolic problem. In particular we prove that both the approximating sequence and the final approximate solution are unique. Finally, we derive a concrete homogenization algorithm on how to compute this homogenized solution.

In Chapters 7 and 8 we develop and further extend some variants of the original Rothe method both in cylindrical and non-cylindrical domains. In order to put these results into a general frame we now give an overview of the Rothe method.

One approach in solving partial differential equations (e.g. evolution equations, reaction-diffusion equations, etc.) is by using the method suggested by E. Rothe in 1930. This method, known as the Rothe method (or method of lines or method of discretization in time or time discretization) makes it possible to convert parabolic partial differential equations into a set of elliptic differential equations. In particular, by using this approach it is possible to approximate the solution of a parabolic boundary value problem of the second order, in two variables \( x, t \) by the solution of a number of elliptic differential equations with the corresponding boundary conditions in the variable \( x \) and discrete values of \( t \). Finally, we can extend this approximative solution to all \( t \) in different ways e.g. as a piecewise linear functions or stepfunction in \( t \).

In the standard (cylindrical) version of this method, we consider a parabolic
Introduction

The equation of the form
\[ \frac{\partial u}{\partial t}(x, t) + A(x, t) u(x, t) = f(x, t), \quad (1.12) \]
defined in a cylindrical domain \( Q = \Omega \times (0, T) \), where \( \Omega \subset \mathbb{R}^N \) and \( A \) is an elliptic operator. The method of discretization of the time consists of

- dividing the time interval \( I = [0, T] \) for the variable \( t \) into \( p \) subintervals, each having a length \( h = T/p \).
- replacing the time derivative \( \frac{\partial u}{\partial t} \) in (1.12) by the difference quotient
  \[ \frac{z_j(x) - z_{j-1}(x)}{h} \approx \frac{\partial u}{\partial t}(x, t_j), \quad (1.13) \]
at each of the points of division \( t_j = jh \), (see Figure 1.5), \( j = 1, \ldots, p \), where \( h = t_j - t_{j-1} \). Here \( z_j(x) := u(x, t_j) \), \( j = 1, \ldots, p \).

Next we write equation (1.12) for \( t = t_j \) by substituting (1.13) into it to obtain the following system of \( p \) elliptic differential equations in \( x \) for the unknown functions \( z_j(x) \), \( j = 1, \ldots, p \):
\[ \frac{z_j(x) - z_{j-1}(x)}{h} + A(x, t_j) z_j(x) = f(x, t_j) \quad x \in \Omega. \quad (1.14) \]

Beginning with some initial condition \( z_0(x) = u_0(x) \) (and boundary conditions) we finally solve the following elliptic problems:
\[ \begin{cases} A z_j + \frac{1}{h} z_j = f_j + \frac{1}{h} z_{j-1} & \text{in } \Omega, \\ z_j = \frac{\partial z_j}{\partial n} = \ldots = \frac{\partial^{k-1} z_j}{\partial n^{k-1}} = 0 & \text{in } \partial \Omega, \end{cases} \quad (1.15) \]
where \( A = A(x, t_j) \), \( z_j = z_j(x) \) and \( f_j = f(x, t_j) \) to obtain approximate solutions of our original equation (1.12) at each of the points of divisions \( t = t_j \).

The weak formulation of the equation (6.45a) is as follows:
\[ z_j \in V \quad (A z_j, v) + \frac{1}{h} (z_j, v) = (f_j + \frac{1}{h} z_{j-1}, v) \quad v \in V. \]

Each of the functions \( z_j \), \( j = 1, \ldots, p \), can be taken as an approximation of the given problem at discrete values of the variable \( t \) only. To get an approximation in the whole domain \( Q = \Omega \times (0, T) \), for example, we construct the
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 1.5: Time discretization for \( p = 5 \) subintervals.

function, \( u_1(x, t) \) as a function continuous and piecewise linear in \( t \) for every fixed \( x \in \Omega \), assuming the values \( z_j(x) \) at the points \( t = t_j \) (see Figure 1.6).

Thus the first Rothe function \( u_1(x, t) \), in the \( j \)-th subinterval is defined by

\[
u_1(x, t) = z_{j-1}(x) + (t - t_{j-1}) \frac{z_j(x) - z_{j-1}(x)}{h} \quad \text{in } I_j = [t_{j-1}, t_j], \quad (1.16)
\]

\( j = 1, \ldots, p, \quad (t_0 = 0). \) We denote by \( d_1 \) the original division of \( I \) into \( p \) subintervals.

In constructing the second Rothe function \( u_2(x, t) \), we divide each of the previous subintervals by 2 and denote the points of divisions and the length of the new subintervals by \( t_{j}^2 = jh_2 \) and \( h_2 = T/2p \), respectively. For this second division of \( I \), denoted by \( d_2 \), we have \( 2p \) subintervals and the Rothe function \( u_2(x, t) \) defined for all \( t \in I \) is given by

\[
u_2(x, t) = z_{j-1}^2(x) + (t - t_{j-1}^2) \frac{z_{j}^2(x) - z_{j-1}^2(x)}{h_2} \quad \text{in } I_{j}^2 = [t_{j-1}^2, t_{j}^2],
\]

\( j = 1, \ldots, 2p. \)

Dividing the previous interval by 2 again, we obtain \( 2^2p \) subintervals, with \( t_{j}^3 = jh_3 \) and \( h_3 = T/2^2p \). We denote this third division of \( I \) by \( d_3 \) and the
corresponding Rothe function $u_5(x, t)$ defined for all $t \in I$ is given by

$$u_5(x, t) = z_2(x) + (t - t_4) \frac{z_3(x) - z_2(x)}{h_3} \text{ in } I_3^5 = [t_{j-1}, t_j],$$

for $j = 1, \ldots, 2^{n-1}p$.

Repeating this process we obtain a sequence of Rothe functions \(\{u_n(x, t)\}_{n=1}^{\infty}\) corresponding to the divisions $d_n$, $n = 2, 3, \ldots$ of the interval $I$, into $2^{n-1}p$ subintervals of length $h_n = T / (2^n - p)$. For the divisions on the time scale we define $t^n_j = jh_n$, for $j = 0, 1, \ldots, 2^{n-1}p$ and $I^n_j = [t_{j-1}^n, t_j^n]$, for $j = 1, \ldots, 2^{n-1}p$. Thus, in general by defining

$$z_0(x) = u_0(x) \quad n = 1, 2, \ldots$$

for a parabolic equation with some initial condition, we obtain a sequence of functions \(\{u_n(x, t)\}_{n=1}^{\infty}\) defined for all $t \in I$ by

$$u_n(t) = z_{j-1}^n(x) + \frac{z_j^n(x) - z_{j-1}^n(x)}{h_n} \left( t - t_{j-1}^n \right) \text{ in } I_j^n = [t_{j-1}^n, t_j^n],$$

(1.17)
in the domain $Q = \Omega \times (0, T)$ $(j = 1, \ldots, 2^{n-1}p)$. The sequence \(\{u_n(x, t)\}_{n=1}^{\infty}\) is called the Rothe sequence of approximate solutions of the given parabolic
14 Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

equation. It can be expected, intuitively, that this sequence will converge (in some appropriate sense) to a function $u(x, t)$ as $n \to \infty$, which will be a solution (also in in some appropriate sense) of the corresponding parabolic problem under consideration.

We note that the nature of the system of equations (6.45a), clearly, depends on the properties of the operator $A$ in (1.12). Rothe’s method is well developed when $A$ is an elliptic operator (linear or nonlinear).

Also the function $u(x, t)$ denotes the function of $x = (x_1, ..., x_N)$ and $t$ i.e. a function of the variables $x_1, ..., x_N, t$. Sometimes the function $u_n(t)$ when considered as a function of the variable $t \in I$ in $L_2(\Omega)$, or $V$, where $V$ is a subspace of the Sobolev space $W^{1,2}(\Omega)$.

The equation (1.17) tells us that to every $t \in I$ a certain function from the space $V$ is assigned. For example if $t = t^n_1$, then

$$u_n(t^n_1) = z^n_0 + \frac{z^n_1 - z^n_0}{h_n} (t^n_1 - t^n_0) = 0 + \frac{z^n_1}{h_n} \cdot h_n = z^n_1 \in V,$$

and if $t = \frac{3}{2}t^n_1$, then

$$u_n(\frac{3}{2}t^n_1) = z^n_1 + \frac{z^n_2 - z^n_1}{h_n} \left( t^n_2 - \frac{3}{2}t^n_1 \right) = z^n_1 + \frac{z^n_2 - z^n_1}{h_n} \cdot \frac{h_n}{2} = \frac{z^n_1 + z^n_2}{2} \in V,$$

too. (see Figure 1.7.)

Instead of considering this piecewise linear function in Figure 1.7 we can consider a corresponding step function $\tilde{u}_n(t)$ from $I$ into $V$ defined by

$$\left\{ \begin{array}{c}
\tilde{u}_n(0) = z^n_1 \\
\tilde{u}_n(t) = \tilde{z}^n_j \\
\end{array} \right., \quad \text{in } I^n_j = (t^n_{j-1}, t^n_j), \quad j = 1, ..., 2^{n-1}p, \quad (1.18)$$

(see Figure 1.8). Summing up:

- For a fixed $x$ the Rothe function $u_n(t)$ is a piecewise linear function in $t$ which takes the values $z^n_j$ at the points $t = t^n_j$. In particular, $u_n(t) \in V$. (see Figure 1.7.)

- For a fixed $x$ the Rothe function $\tilde{u}_n(t)$ is a piecewise constant function in $t$, which assumes the values $\tilde{z}^n_j \in V$ at the points $t = t^n_j$. In other words $\tilde{u}_n(t) \in V$. (see Figure 1.8).

- For any fixed $t \geq 0$ the function $u_n(t) = u_n(x, t)$ and $\tilde{u}_n(t) = \tilde{u}_n(x, t)$ may be regarded as approximative solution of the solution $u(x, t)$ we are looking for. (See Figure 1.6 for $u_n(x, t)$.)
Figure 1.7: Rothe function $u_n(t)$ as a piecewise linear function in $t$ in the interval $[0, T]$.

Figure 1.8: Rothe function as a step function from $I$ into $V$. 
Figure 1.9: Rothe functions $u_\ell(t)$ as a piecewise linear function of $t$ in the interval $[0,T]$ for a fixed $x$ in a non-cylindrical domain.

It is known that the Rothe sequence $\{u_\ell(x,t)\}$ is bounded in the space $L_2(I,V)$ and since this space is a Hilbert space a subsequence still denoted by $\{u_\ell(x,t)\}$ can be found to be weakly convergent in this space to a function $u \in L_2(I,V)$, (we write $u_\ell \rightharpoonup u$ in $L_2(I,V)$). It can also be proved that if $u_\ell \rightharpoonup u$ in $L_2(I,V)$, then also $\tilde{u}_\ell \rightharpoonup u$ in $L_2(I,V)$, where $\tilde{u}_\ell(t)$ is the sequence defined in (1.18). (See e.g. [72].)

In this PhD thesis we also consider the non-cylindrical case. This case was first considered and developed in the paper [32] by J. Dasht, J. Engström, A. Kufner and L. E. Persson and further developed in [53] and the PhD thesis of K. Kuliev [52]. In Chapter 7 of this PhD thesis we complement and further develop this fairly new research in various ways. In the non-cylindrical case the domain $Q$ is defined by $Q = \{(x,t); x \in \Omega_t, 0 < t < T\}$, and the time interval $I = [0,T]$ for the variable $t$ is still divided into $n$ subintervals $I_1, I_2, ..., I_n$ ($I_j = [t_{j-1}, t_j]$, $t_j = jh$, $j = 1, 2, ..., n$) each of length $h = \frac{T}{n}$. At each of the points of divisions $t_j$ on the time axis we replace the derivative $\frac{\partial u}{\partial t}$ by $\frac{z_j - z_{j-1}}{h}$ and put $z_{j-1} = 0$ on $\Omega_{t_{j-1}} \setminus \Omega_{t_{j-1}}$, $j = 1, 2, ..., n$ to obtain a sequence of elliptic problems on the different domains.
Ωt for \( t = t_j, \ j = 1, \ldots, n \). These problems are then solved in the following order: first we take for \( z_0(x) \) the initial value \( u_0(x) = 0 \), which is defined in \( \Omega_0 \), then we extend \( z_0(x) \) to the whole domain \( \Omega_T \) with zero. Next we solve the elliptic equation on \( \Omega_{t_1} \), and extend the solution obtained to the whole domain \( \Omega_T \) with zero. In a similar manner we solve the elliptic equation on \( \Omega_{t_2} \) and extend the solution obtained to the whole domain \( \Omega_T \) with zero. Going on in this way we get a sequence of functions, which are defined on the whole domain \( \Omega_T \), and construct the corresponding Rothe’s function. This function is then defined in the cylinder \( \Omega_T \times (0, T) \).

The significant difference between the domains in the cylindrical and non-cylindrical versions is that whereas in the cylindrical case the domain is the same for each time \( t_j \), we have different domains for each time division \( t_j \) in the non-cylindrical case. The Rothe function \( u_n(t) = u_n(x, t) \) is defined similarly as in the cylindrical domain case. For a fixed \( x \in \Omega_T \), \( u_n(t) = u_n(x, t) \) is a piecewise linear function in \( t \) on the interval \( I \) with values \( z_j(x) \) at the points \( t = t_j \) for \( n = 5 \) subintervals (see Figure 1.9). For a more detailed explanation see our chapter 7.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method
Chapter 2

Multiple scale expansion for Reynolds equation (stationary case)

In this chapter we will present the details concerning the multiple scale method (described in subsection 1.1.3) for deriving approximative solutions of the time independent equations (1.4) and (1.6). In each case we end up with concrete homogenization procedures, which can also be directly used by non experts in the area.

2.1 The stationary compressible (constant bulk modulus) case

The time independent compressible Reynolds equation given by (1.6), i.e.

\[ \nabla \cdot (h^3 \varepsilon \nabla w \varepsilon) = \lambda \frac{\partial}{\partial x_1} (w \varepsilon h \varepsilon) \quad \text{on } \Omega, \]

(2.1)

is used to describe the flow of thin films of fluid between two surfaces in relative motion. In this chapter we will use the method of multiple scale expansion to derive a "homogenized equation" for (2.1), which is a good approximation of (2.1) and which can be solved by using standard numerical methods. We will assume that only the stationary surface is rough.

To express the film thickness we introduce the following auxiliary function

\[ h(x, y) = h_0(x) + h_1(y), \]

19
where $h_1$ is assumed to be periodic. Without loss of generality it can also be assumed that for $h_1$ the cell of periodicity is $Y = (0,1) \times (0,1)$, i.e. the unit cube in $\mathbb{R}^2$. By using the auxiliary function $h$ we can model the film thickness $h_\varepsilon$ by

$$h_\varepsilon(x) = h(x, x/\varepsilon), \quad \varepsilon > 0.$$  

This means that $h_0$ describes the global film thickness, the periodic function $h_1$, represent the roughness contribution of the surface and that $\varepsilon$ is a parameter which describes the roughness wavelength. Further, since the coefficients $h_\varepsilon(x)$ of (2.1) are periodic functions of $x/\varepsilon$, it makes sense to expect that the solution is also a periodic function of its argument $x/\varepsilon$. Thus it is reasonable to assume a multiple scale expansion of the solution $w_\varepsilon(x)$ in the form

$$w_\varepsilon(x) = w_0(x, x/\varepsilon) + \varepsilon w_1(x, x/\varepsilon) + \varepsilon^2 w_2(x, x/\varepsilon) + ...$$  

(2.2)

where $w_i = w_i(x, y), i = 0, 1, ...$. If $y_j = \frac{x_j}{\varepsilon}$, then applying the chain rule on the smooth function

$$\psi_\varepsilon(x) = \psi(x, x/\varepsilon),$$

the partial derivatives with respect to $x_j$ becomes:

$$\frac{\partial \psi_\varepsilon}{\partial x_j}(x) = \left( \frac{\partial \psi}{\partial x_j} + \varepsilon^{-1} \frac{\partial \psi}{\partial y_j} \right) \left( \frac{x}{\varepsilon} \right), \quad j = 1, 2.$$  

(2.3)
Multiple scale expansion for Reynolds equation (stationary case)

Writing (2.3) in gradient form we have that
\[ \nabla_x \psi = \nabla_x \psi + \varepsilon^{-1} \nabla_y \psi. \tag{2.4} \]
Substituting (2.2) – (2.4) into (2.1) we obtain that
\[ \left( \nabla_x + \varepsilon^{-1} \nabla_y \right) \cdot \left[ h^3 \left( \nabla_x + \varepsilon^{-1} \nabla_y \right) w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \right] \tag{2.5} \]
\[ = \lambda \left( \frac{\partial}{\partial x_1} + \varepsilon^{-1} \frac{\partial}{\partial y_1} \right) \left( h w_0 + \varepsilon h w_1 + \varepsilon^2 h w_2 + \ldots \right). \]

To make the simplification more clear, we introduce the following notations:
\[ A_0 = \nabla_y \cdot (h^3 \nabla_y), \]
\[ A_1 = \nabla_y \cdot (h^3 \nabla_x) + \nabla_x \cdot (h^3 \nabla_y), \]
\[ A_2 = \nabla_x \cdot (h^3 \nabla_x). \]

Using these notations in (2.5) we obtain that
\[ (\varepsilon^{-2} A_0 + \varepsilon^{-1} A_1 + A_2) \left( w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \right) \]
\[ = +\varepsilon^{-1} \lambda \frac{\partial}{\partial y_1} (h w_0) + \lambda \left( \frac{\partial}{\partial x_1} (h w_0) + \frac{\partial}{\partial y_1} (h w_1) \right) \]
\[ + \varepsilon \lambda \left( \frac{\partial}{\partial y_1} (h w_2) + \frac{\partial}{\partial x_1} (h w_1) \right) + \varepsilon^2 \lambda \frac{\partial}{\partial x_1} (h w_2) + \ldots \]

Equating the three lowest powers of \( \varepsilon \), we obtain the following system of equations:
\[ A_0 w_0 = 0, \tag{2.6} \]
\[ A_1 w_0 + A_0 w_1 = \lambda \frac{\partial}{\partial y_1} (h w_0), \tag{2.7} \]
\[ A_0 w_2 + A_1 w_1 + A_2 w_0 = \lambda \left( \frac{\partial}{\partial x_1} (h w_0) + \frac{\partial}{\partial y_1} (h w_1) \right). \tag{2.8} \]

In order to solve (2.6)- (2.8), we need the following Lemma:

**Lemma 2.1.** Consider the boundary value problem
\[ A_0 \Phi = F \text{ in the unit cell } Y, \tag{2.9} \]
where \( F \in L^2(Y) \) and \( \Phi(y) \) is \( Y \)-periodic. Then the following holds true:
(i) There exists a weak \( Y \)-periodic solution \( \Phi \) of (2.9) if and only if
\[ \frac{1}{|Y|} \int_Y F dy = 0. \]
(ii) If there exists a weak \( Y \)-periodic solution of (2.9), then it is unique up to a constant, that is, if we find one solution \( \Phi_0(y) \), every solution is of the form \( \Phi(y) = \Phi_0(y) + c \), where \( c \) is a constant.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Proof. See [69, p. 39].

The operator $A_0$ involves only derivatives with respect to $y$ so $x$ is just a parameter in the solution of (2.6). One solution of (2.6) is $w_0(x, y) \equiv 0$. By Lemma 2.1, the general solution is $w_0(x, y) \equiv$ constant with respect to $y$, that is

$$w_0(x, y) = w_0(x).$$

(2.10)

In the sequel below we let

$$w_0 = w_0(x); \quad w_i = w_i(x, y) \text{ for } i = 1 \text{ and } 2.$$

From (2.7) it follows that

$$A_0 w_1 = \lambda \frac{\partial}{\partial y_1} (h w_0) - A_1 w_0,$$  

i.e.,

$$\nabla_y \cdot (h^3 \nabla_y w_1) = \lambda \frac{\partial}{\partial y_1} (h w_0) - \nabla_x \cdot (h^3 \nabla_y w_0) - \nabla_y \cdot (h^3 \nabla_x w_0).$$

According to (2.10), $w_0$ is a function of only $x$ and, hence, $\nabla_y w_0$ is equal to zero. Thus we have that

$$\nabla_y \cdot (h^3 \nabla_y w_1) = \lambda \frac{\partial}{\partial y_1} (h w_0) - \nabla_y \cdot (h^3 \nabla_x w_0).$$

(2.11)

Since the right hand side of (2.11) consists of three (by superposition) terms, we expect that $w_1(x, y)$ should be a linear function of three terms. Hence, we let

$$w_1(x, y) = \frac{\partial w_0}{\partial x_1} v_1(x, y) + \frac{\partial w_0}{\partial x_2} v_2(x, y) + w_0 v_3(x, y).$$

(2.12)

In the sequel we let $v_i = v_i(x, y)$ for $i = 1, 2$ and 3. Substituting (2.12) into (2.11) we get that

$$\nabla_y \cdot \left( h^3 \nabla_y \left( \frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 + w_0 v_3 \right) \right) = \lambda \frac{\partial}{\partial y_1} (h w_0) - \nabla_y \cdot (h^3 \nabla_x w_0).$$

(2.13)

But

$$\nabla_y \cdot (h^3 \nabla_x w_0) = \nabla_y \cdot \left( h^3 \frac{\partial w_0}{\partial x_1} e_1 + h^3 \frac{\partial w_0}{\partial x_2} e_2 \right),$$

(2.14)

where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$ and, hence, we can write (2.13) as

$$\nabla_y \left[ h^3 \nabla_y \left( \frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 + w_0 v_3 \right) \right]$$
Multiple scale expansion for Reynolds equation (stationary case)

\[\begin{align*}
= \lambda \frac{\partial}{\partial y_1} (h w_0) - \nabla_y \cdot \left( h^3 \frac{\partial w_0}{\partial x_1} e_1 + h^3 \frac{\partial w_0}{\partial x_2} e_2 \right).
\end{align*}\]

Comparing the corresponding terms we obtain the following three local (cell) problems

\[\begin{align*}
\nabla_y \cdot (h^3 \nabla_y v_3) &= \lambda \frac{\partial}{\partial y_1} (h), \\
\nabla_y \cdot (h^3 \nabla_y v_1) &= -\nabla_y \cdot (h^3 e_1), \\
\nabla_y \cdot (h^3 \nabla_y v_2) &= -\nabla_y \cdot (h^3 e_2).
\end{align*}\] (2.15)

Moreover, according to (2.8), we find that

\[\begin{align*}
A_0 w_2 + A_1 w_1 + A_2 w_0 &= \lambda \frac{\partial}{\partial x_1} (h w_0) + \lambda \frac{\partial}{\partial y_1} (h w_1).
\end{align*}\]

Averaging over the period \(Y\) we have that

\[\int_Y \left( A_0 w_2 + A_1 w_1 + A_2 w_0 - \lambda \frac{\partial}{\partial x_1} (h w_0) - \lambda \frac{\partial}{\partial y_1} (h w_1) \right) dy = 0.\]

By periodicity, \(\int_Y (A_0 w_2) dy = 0\) and, hence, we obtain that

\[\int_Y \left( A_1 w_1 + A_2 w_0 - \lambda \frac{\partial}{\partial x_1} (h w_0) - \lambda \frac{\partial}{\partial y_1} (h w_1) \right) dy = 0,\]

or

\[\int_Y \left( \nabla_x \cdot (h^3 \nabla_x w_1) + \nabla_y \cdot (h^3 \nabla_x w_1) + \nabla_x \cdot (h^3 \nabla_x w_0) \right) dy = \int_Y \lambda \frac{\partial}{\partial x_1} (h w_0) + \lambda \frac{\partial}{\partial y_1} (h w_1) dy.\]

But \(h^3 \nabla_x w_1\) and \(h w_1\) are periodic in \(Y\) so that \(\int_Y \nabla_y \cdot (h^3 \nabla_x w_1) dy = 0\), and \(\int_Y \frac{\partial}{\partial y_1} (h w_1) dy = 0\). Therefore, by Lemma 2.1 the last equation reduces to

\[\int_Y \left\{ \nabla_x \cdot \left[ h^3 \nabla_y \left( w_0 v_3 + \frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 \right) \right] + \nabla_x \cdot (h^3 \nabla_x w_0) - \lambda \frac{\partial}{\partial x_1} (h w_0) \right\} dy = 0.\] (2.16)

We note that

\[\begin{align*}
\nabla_x w_0 &= \frac{\partial w_0}{\partial x_1} e_1 + \frac{\partial w_0}{\partial x_2} e_2, \\
\lambda \frac{\partial}{\partial x_1} (h w_0) &= \nabla_x \cdot \begin{pmatrix} \lambda h w_0 \\ 0 \end{pmatrix}.
\end{align*}\] (2.17)
Substituting (2.17) in (2.16) and rearranging we get that

\[
\begin{align*}
\int_Y \nabla_x \cdot \left[ h^3 \nabla_y \left( \frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 \right) \right] dy \\
+ \int_Y \nabla_x \cdot \left( h^3 \frac{\partial w_0}{\partial x_1} e_1 + h^3 \frac{\partial w_0}{\partial x_2} e_2 \right) dy \\
= \int_Y \left( \nabla_x \cdot \left( \begin{array}{c}
\lambda h w_0 \\
0
\end{array} \right) - \nabla_x \cdot (h^3 \nabla_y w_0 v_3) \right) dy.
\end{align*}
\]

By simplifying we find that

\[
\nabla_x \cdot \left\{ \frac{\partial w_0}{\partial x_1} \int_Y (h^3 e_1 + h^3 \nabla_y v_1) \; dy \right\} \\
+ \frac{\partial w_0}{\partial x_2} \int_Y (h^3 e_2 + h^3 \nabla_y v_2) \; dy \right\} \\
= \nabla_x \cdot \int_Y \left[ \left( \begin{array}{c}
\lambda h w_0 \\
0
\end{array} \right) - \left( \begin{array}{c}
h^3 w_0 \frac{\partial v_3}{\partial y_1} \\
h^3 w_0 \frac{\partial v_3}{\partial y_2}
\end{array} \right) \right] \; dy,
\]

or

\[
\nabla_x \cdot \left\{ \frac{\partial w_0}{\partial x_1} \left( \begin{array}{c}
b_{11}(x) \\
b_{21}(x)
\end{array} \right) + \frac{\partial w_0}{\partial x_2} \left( \begin{array}{c}
b_{12}(x) \\
b_{22}(x)
\end{array} \right) \right\} \\
= \nabla_x \cdot w_0 \left( \begin{array}{c}
\int_Y \lambda h - h^3 \frac{\partial w_0}{\partial y_1} dy \\
\int_Y h^3 \frac{\partial w_0}{\partial y_2} dy
\end{array} \right),
\]

or

\[
\nabla_x \cdot \left\{ \left( \begin{array}{c}
b_{11}(x) \\
b_{21}(x)
\end{array} \right) \left( \begin{array}{c}
b_{12}(x) \\
b_{22}(x)
\end{array} \right) \left( \begin{array}{c}
\frac{\partial w_0}{\partial x_1} \\
\frac{\partial w_0}{\partial x_2}
\end{array} \right) \right\} = \nabla_x \cdot w_0 \left( \begin{array}{c}
c_1(x) \\
c_2(x)
\end{array} \right).
\]

We conclude that the homogenized equation for (2.1) is given by

\[
\nabla_x \cdot [B(x) \nabla w_0] = \nabla_x \cdot [w_0 C(x)], \quad (2.18)
\]

where \( B(x) \) is a matrix function defined by \( B(x) = (b_{ij}(x)) \), in terms of \( v_1 \) and \( v_2 \) by

\[
\begin{align*}
\left( \begin{array}{c}
b_{11}(x) \\
b_{21}(x)
\end{array} \right) &= \int_Y (h^3 e_1 + h^3 \nabla_y v_1) \; dy, \\
\left( \begin{array}{c}
b_{12}(x) \\
b_{22}(x)
\end{array} \right) &= \int_Y (h^3 e_2 + h^3 \nabla_y v_2) \; dy.
\end{align*}
\]
and $C(x) = (c_i(x))$ is a vector function defined in terms of $v_3$ by

\[
\begin{pmatrix}
  c_1(x) \\
  c_2(x)
\end{pmatrix} = \begin{pmatrix}
  \int_Y \lambda h - h^3 \frac{\partial p}{\partial x_1} dy \\
  \int_Y -h^3 \frac{\partial p}{\partial x_2} dy
\end{pmatrix}.
\] (2.20)

Note that the equation (2.18) describes the global behaviour of the solutions of (2.1) for small values of $\varepsilon$. Furthermore, the second term in (2.2), i.e. $\varepsilon w_1(x, x/\varepsilon)$ given by (2.7), yields important information about the local variations of the solutions, via the cell problems in (2.15) for $v_i(x, y), i = 1, 2, 3$, and the homogenized equation (2.18) for $w_0(x)$. We end this section by summing up our investigations so far in the form of an algorithm.

**Homogenization algorithm**: An approximate solution of the equation (2.1) can be obtained in the following way;

- **step 1**: Solve the local problem (2.15).
- **step 2**: Insert the solution of the local problem into (2.19) and (2.20) and compute the homogenized coefficient $B(x)$ and the vector function $C(x)$.
- **step 3**: Solve the homogenized equation (2.18), which corresponds to the approximative solution we are looking for.

We remark that all steps in this algorithm are easy to perform and, hence, we have a concrete algorithm which is easy to use in practice to solve an initially complicated problem.

### 2.2 The stationary incompressible case

In this section we consider multiple scale expansion of the incompressible Reynolds equation. According to (1.4) we have that

\[
\nabla \cdot (h^3(x) \nabla p_\varepsilon(x)) = \Lambda \frac{\partial}{\partial x_1} (h_\varepsilon(x)),
\] (2.21)

where $\Lambda = 6\eta v$. The parameters in the above equation have the same meanings as described in the previous section.

To express the film thickness we introduce the following auxiliary function

\[
h(x, y) = h_0(x) + h_1(y),
\]

where $h_1$ is assumed to be periodic. Without loss of generality it can also be assumed that for $h_1$ the cell of periodicity is $Y = (0, 1) \times (0, 1)$, i.e. the unit cube in $\mathbb{R}^2$. By using the auxiliary function $h$ we can model the film thickness $h_\varepsilon$ by

\[
h_\varepsilon(x) = h(x, x/\varepsilon), \quad \varepsilon > 0.
\]
This means that \( h_0 \) describes the global film thickness, the periodic function \( h_1 \), represent the roughness contribution of the surface and that \( \varepsilon \) is a parameter which describes the roughness wavelength.

We assume a multiple scale expansion of the solution \( p_\varepsilon(x) \) in the form

\[
p_\varepsilon(x) = p_0(x, x/\varepsilon) + \varepsilon p_1(x, x/\varepsilon) + \varepsilon^2 p_2(x, x/\varepsilon) + \ldots
\]

where \( p_i = p_i(x, y) \) for \( y = x/\varepsilon \), and \( i = 1, 2, \ldots \). Then the chain rule (see (2.3) and (2.4)) implies that (2.21) can be written as

\[
(\nabla_x + \varepsilon^{-1}\nabla_y) \cdot \left[ h_3 \left( \nabla_x + \varepsilon^{-1}\nabla_y \right) p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \right] = \Lambda \left( \frac{\partial}{\partial x_1} + \varepsilon^{-1} \frac{\partial}{\partial y_1} \right) h.
\]

(2.22)

For a simplification of (2.22), we introduce the following notations:

\[
A_0 = \nabla_y \cdot (h_3 \nabla_y),
A_1 = \nabla_y \cdot (h_3 \nabla_x) + \nabla_x (h_3 \nabla_y),
A_2 = \nabla_x \cdot (h_3 \nabla_x).
\]

Substituting the above notations in (2.22) we obtain that

\[
\left( A_2 + \varepsilon^{-1} A_1 + \varepsilon^{-2} A_0 \right) \left( p_0 + \varepsilon^{-1} p_1 + \varepsilon^{-2} p_2 \right) = \Lambda \left[ \frac{\partial}{\partial x_1} + \varepsilon^{-1} \frac{\partial}{\partial y_1} \right] h.
\]

Expanding we have that

\[
\varepsilon^{-2} A_0 p_0 + \varepsilon^{-1} (A_0 p_1 + A_1 p_0) + (A_0 p_2 + A_1 p_1 + A_2 p_0) + \varepsilon (A_2 p_1 + A_1 p_2) + \varepsilon^2 A_2 p_2
\]

\[
= \Lambda \left[ \frac{\partial}{\partial x_1} + \varepsilon^{-1} \frac{\partial}{\partial y_1} \right] h.
\]

By equating the three lowest powers of \( \varepsilon \) we get the following systems of equations:

\[
A_0 p_0 = 0, \quad (2.23)
A_0 p_1 + A_1 p_0 = \Lambda \frac{\partial h}{\partial y_1}, \quad (2.24)
A_0 p_2 + A_1 p_1 + A_2 p_0 = \Lambda \frac{\partial h}{\partial x_1}. \quad (2.25)
\]
Multiple scale expansion for Reynolds equation (stationary case)

The operator $A_0$ involves only derivatives with respect to $y$ and, thus, $x$ is just a parameter in the solution of (2.23). One solution of (2.23) is $p_0(x, y) \equiv 0$. By Lemma 2.1 the general solution $p_0(x, y) \equiv$ constant with respect to $y$, that is,

$$p_0(x, y) = p_0(x),$$

where $p_0(x)$ is sufficiently differentiable. In the sequel we let

$$p_0 = p_0(x); \quad p_i = p_i(x, y) \quad \text{for } i = 1 \text{ and } 2.$$

In view of (8.12) we see that

$$A_0 p_1 = \Lambda \frac{\partial h}{\partial y_1} - A_1 p_0, \text{ i.e.,}$$

$$\nabla_y \cdot (h^3 \nabla_y p_1) = \Lambda \frac{\partial h}{\partial y_1} - \nabla_x \cdot (h^3 \nabla_y p_0) - \nabla_y \cdot (h^3 \nabla_x p_0).$$

Moreover, $\nabla_y p_0$ is equal to zero since, according to (2.26), $p_0$ is a function of only $x$. Thus, we have that

$$\nabla_y \cdot (h^3 \nabla_y p_1) = \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot (h^3 \nabla_x p_0).$$

Since the right hand side consists of three linear terms we expect that $p_1(x, y)$ should be a linear function of three terms. By linearity we let

$$p_1(x, y) = \frac{\partial p_0}{\partial x_1} v_1(x, y) + \frac{\partial p_0}{\partial x_2} v_2(x, y) + v_3(x, y).$$

Substituting (2.28) into (2.27) we get that

$$\nabla_y \cdot \left( h^3 \nabla_y \left( \frac{\partial p_0}{\partial x_1} v_1 + \frac{\partial p_0}{\partial x_2} v_2 + v_3 \right) \right) = \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot \left( h^3 \nabla_x p_0 \right),$$

where $v_i = v_i(x, y) \text{ for } i = 1, 2 \text{ and } 3$. But

$$\nabla_y \cdot [h^3 \nabla_x p_0] = \nabla_y \cdot \left( h^3 \frac{\partial p_0}{\partial x_1} e_1 + h^3 \frac{\partial p_0}{\partial x_2} e_2 \right),$$

and, hence, we obtain that

$$\nabla_y \cdot \left( h^3 \nabla_y \left( \frac{\partial p_0}{\partial x_1} v_1 + \frac{\partial p_0}{\partial x_2} v_2 + v_3 \right) \right) = \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot \left( h^3 \frac{\partial p_0}{\partial x_1} e_1 + h^3 \frac{\partial p_0}{\partial x_2} e_2 \right).$$
Comparing the corresponding terms, we obtain the following periodic problems

\[
\begin{align*}
\nabla_y \cdot (h^3 \nabla_y v_3) &= \Lambda \frac{\partial h}{\partial y_1}, \\
\nabla_y \cdot (h^3 \nabla_y v_1 \frac{\partial p_0}{\partial x_1}) &= -\nabla_y \cdot (h^3 \frac{\partial p_0}{\partial x_1} e_1), \\
\nabla_y \cdot (h^3 \nabla_y v_2 \frac{\partial p_0}{\partial x_2}) &= -\nabla_y \cdot (h^3 \frac{\partial p_0}{\partial x_2} e_2),
\end{align*}
\]

(2.29)

where \( v_i = v_i(x, y) \) are their solutions.

Further, averaging over the period \( Y \) in (6.21) we obtain that

\[
\int_Y \left( A_0 p_2 + A_1 p_1 + A_2 p_0 - \Lambda \frac{\partial h}{\partial x_1} \right) dy = 0.
\]

By periodicity \( \int_Y (A_0 p_2) \, dy = 0 \), and, thus, we have that

\[
\int_Y \left( A_1 p_1 + A_2 p_0 - \Lambda \frac{\partial h}{\partial x_1} \right) dy = 0,
\]

or

\[
\int_Y \left[ \nabla_x \cdot (h^3 \nabla_y p_1) + \nabla_y \cdot (h^3 \nabla_x p_1) + \nabla_x \cdot (h^3 \nabla_x p_0) - \Lambda \frac{\partial h}{\partial x_1} \right] dy = 0.
\]

Since \( h^3 \nabla_x p_1 \) is periodic, it follows that \( \int_Y \nabla_y \cdot (h^3 \nabla_x p_1) \, dy = 0 \). Therefore the last equation reduces to

\[
\int_Y \nabla_x \cdot \left( h^3 \nabla_y \left( \frac{\partial p_0}{\partial x_1} v_1 + \frac{\partial p_0}{\partial x_2} v_2 + v_3 \right) \right) dy + \\
\nabla_x \cdot \left( h^3 \nabla_x p_0 \right) - \Lambda \frac{\partial h}{\partial x_1} dy = 0.
\]

Rearranging we get that

\[
\int_Y \nabla_x \cdot \left( h^3 \nabla_y \left( \frac{\partial p_0}{\partial x_1} v_1 + \frac{\partial p_0}{\partial x_2} v_2 \right) \right) dy + \\
\int_Y \nabla_x \cdot \left( h^3 \frac{\partial p_0}{\partial x_1} e_1 + h^3 \frac{\partial p_0}{\partial x_2} e_2 \right) dy \\
= \int_Y \left( \Lambda \frac{\partial h}{\partial x_1} - \nabla_x \cdot (h^3 \nabla_y v_3) \right) dy.
\]

Simplifying we obtain that

\[
\nabla_x \cdot \left( \frac{\partial p_0}{\partial x_1} \int_Y (h^3 e_1 + h^3 \nabla_y v_1) \right) dy
\]
Multiple scale expansion for Reynolds equation (stationary case)

\[ + \frac{\partial p_0}{\partial x_2} \int_Y \left( h^3 e_2 + h^3 \nabla_y v_2 \right) dy \]

\[ = \nabla_x \cdot \int_Y \left( \begin{array}{c} \Lambda h \\ 0 \end{array} \right) - \left( h^3 \frac{\partial p_0}{\partial y_1} \right) \right) dy, \]

or

\[ \nabla_x \cdot \left\{ \frac{\partial p_0}{\partial x_1} \left( \begin{array}{c} b_{11}(x) \\ b_{21}(x) \end{array} \right) + \frac{\partial p_0}{\partial x_2} \left( \begin{array}{c} b_{12}(x) \\ b_{22}(x) \end{array} \right) \right\} \]

\[ = \nabla_x \cdot \left( \begin{array}{c} \int_Y \Lambda h - h^3 \frac{\partial p_0}{\partial y_1} dy \\ \int_Y - h^3 \frac{\partial p_0}{\partial y_2} dy \end{array} \right), \]

or

\[ \nabla_x \cdot \left\{ \begin{array}{c} b_{11}(x) \\ b_{21}(x) \end{array} \right) \right( \begin{array}{c} \frac{\partial p_0}{\partial x_1} \\ \frac{\partial p_0}{\partial x_2} \end{array} \right) \right} = \nabla_x \cdot \left( \begin{array}{c} c_1(x) \\ c_2(x) \end{array} \right), \quad (2.30) \]

In a more compact form we can write the homogenized equation (2.30) as

\[ \nabla_x \cdot \left[ B(x) \nabla p_0 \right] = \nabla_x \cdot \left[ c(x) \right], \quad (2.31) \]

where \( B(x) \) is a matrix function defined by \( B(x) = b_{ij}(x) \) in terms of \( v_1 \) and \( v_2 \) as

\[ \left( \begin{array}{c} b_{11}(x) \\ b_{21}(x) \end{array} \right) = \int_Y \left( h^3 e_1 + h^3 \nabla_y v_1 \right) dy, \quad (2.32) \]

\[ \left( \begin{array}{c} b_{12}(x) \\ b_{22}(x) \end{array} \right) = \int_Y \left( h^3 e_2 + h^3 \nabla_y v_2 \right) dy, \]

and the vector function \( c(x) = \left( \begin{array}{c} c_1(x) \\ c_2(x) \end{array} \right) \) is defined in terms of \( v_3 \) as

\[ \left( \begin{array}{c} c_1(x) \\ c_2(x) \end{array} \right) = \left( \begin{array}{c} \int_Y \Lambda h - h^3 \frac{\partial p_0}{\partial y_1} dy \\ \int_Y - h^3 \frac{\partial p_0}{\partial y_2} dy \end{array} \right). \quad (2.33) \]

Summing up, in this section we have discussed the fact that it is possible to use the method of multiple scale expansion to derive a "homogenized equation" of (2.21), which easily can be solved numerically and which gives the approximative solution we are looking for. More exactly, we can use the following:

**Homogenization algorithm**: An approximate solution of the equation (2.21) can be obtained in the following way;
step 1: Solve the local problem (2.29).

step 2: Insert the solution of the local problem into (2.32) and (2.33) and compute the homogenized coefficient $B(x)$ and the vector function $C(x)$.

step 3: Solve the homogenized equation (2.31), which corresponds to the approximative solution we are looking for.

We remark that all steps in this algorithm are easy to perform and, hence, we have a concrete algorithm which is easy to use in practice to solve an initially complicated problem.

Remark 2.1. Also in this case it is possible to use two-scale convergence to rigorously verify that this homogenization algorithm gives the correct approximative solution we are looking for, for details see Wall [76].
Chapter 3

Homogenization of the unstationary incompressible Reynolds equation

3.1 Introduction

To increase the hydrodynamic performance in different machine elements during lubrication, e.g. journal bearings and thrust bearings, it is important to understand the influence of surface roughness. To consider the surface effects in the numerical analysis, a very fine mesh is needed to resolve the surface roughness, suggesting some type of averaging. A rigorous way to do this is to use the general theory of homogenization. This theory facilitates the analysis of partial differential equations with rapidly oscillating coefficients, see e.g. Jikov et al. [41]. Homogenization was recently applied to different problems connected to lubrication with much success, see e.g., [6], [10], [13], [15], [16], [20], [21], [22], [23], [39], [40], [44], [45], [57] and [76].

In general, the density of a lubricant is a function of the pressure. In this paper we will consider two special cases, where the density is assumed to be constant, i.e. an incompressible lubricant, and where the compressibility of the lubricant is modelled, assuming that the lubricant has a constant bulk modulus, see e.g. [34].

If only one of the two surfaces is rough and the rough surface is stationary, then the governing Reynolds type equation is stationary. When at least one of the moving surfaces is rough, then the governing Reynolds type equations will also involve time. Most of the previous studies on the effects of
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

surface roughness during lubrication are devoted to problems with no time dependency.

One technique within the homogenization theory is the formal method of multiple scale expansion, see e.g. [19] or [69]. Recently, the ideas in [10] were used to study the compressible unstationary Reynolds equation under the assumption of a constant bulk modulus. In this chapter, the method of multiple scale expansion is applied to derive a homogenization result for the incompressible unstationary Reynolds equation, see also [15]. In particular, the result shows a significant difference in the asymptotic behaviors between the incompressible case and the case with constant bulk modulus. More precisely, the homogenized equation contains a fast parameter in the incompressible case. Hence the pressure distribution oscillates rapidly in time, while it is almost smooth with respect to the space variable. This is contrary to the case of constant bulk modulus where the homogenized pressure solution does not contain any fast parameters, i.e. the pressure solution is smooth in both space and time. Moreover, it is clearly demonstrated by numerical examples that the homogenization result permits the surface effects in lubrication problems to be efficiently analyzed.

We want to point out that in the more mathematical oriented works in [15] and [20], Reynolds type equations modelling roughness on both surfaces were analyzed by using the method known as two-scale convergence. Concerning the concept of two-scale convergence, the reader is also referred to e.g. [1], [55] and [65]. However, in this work we use the more engineering oriented method of multiple scale expansions.

3.2 The governing Reynolds type equations

Let $\eta$ be the viscosity of the lubricant and assume that the velocity of surface $i$ is $V_i = (v_i, 0)$, where $i = 1, 2$ and $v_i$ is constant. Moreover, the bearing domain is denoted by $\Omega$, the space variable is represented by $x \in \Omega \subset \mathbb{R}^2$ and $t \in I \subset \mathbb{R}$ represents the time. To express the film thickness we introduce the following auxiliary function

$$ h(x, t, y, \tau) = h_0(x, t) + h_2(y - \tau V_2) - h_1(y - \tau V_1), $$

where $h_1$ and $h_2$ are assumed to be periodic. Without loss of generality it can also be assumed that for both $h_1$ and $h_2$ the cell of periodicity is $Y = (0, 1) \times (0, 1)$, i.e. the unit cube in $\mathbb{R}^2$. By using the auxiliary function $h$ we can model the film thickness $h_\varepsilon$ by

$$ h_\varepsilon(x, t) = h(x, t, x/\varepsilon, t/\varepsilon), \quad \varepsilon > 0. \quad (3.1) $$
Homogenization of the unstationary incompressible Reynolds equation

\[ h_0(x)+h_2(x/\varepsilon) \]

\[ h_1(x/\varepsilon) \]

Figure 3.1: Bearing geometry and surface roughness.

This means that \( h_0 \) describes the global film thickness, the periodic functions \( h_i, i = 1, 2 \), represent the roughness contribution of the two surfaces and \( \varepsilon \) is a parameter that describes the roughness wavelength, see Figure 3.1.

If the lubricant is compressible, i.e. the density \( \rho \) depends on the pressure, the pressure \( p(x, t) \) satisfies then the unstationary compressible Reynolds equation

\[
\frac{\partial}{\partial t} (\rho(p_\varepsilon)h_\varepsilon) = \nabla \cdot \left( \frac{h_3^3}{12\eta} \rho(p_\varepsilon) \nabla p_\varepsilon \right) - \frac{v}{2} \frac{\partial}{\partial x_1} (\rho(p_\varepsilon)h_\varepsilon), \quad \text{on} \ \Omega \times I, \quad (3.2)
\]

where \( v = v_1 + v_2 \). If the lubricant is incompressible, i.e. \( \rho \) is constant, the equation (3.2) is then reduced to the unstationary incompressible Reynolds equation

\[
\frac{\partial h_\varepsilon}{\partial t} = \nabla \cdot \left( \frac{h_3^3}{12\eta} \nabla p_\varepsilon \right) - \frac{v}{2} \frac{\partial h_\varepsilon}{\partial x_1}, \quad \text{on} \ \Omega \times I. \quad (3.3)
\]

Note that equation (3.2) is non-linear and equation (3.3) is linear. This means that in general it is much more difficult to analyze the compressible case. The situation is rather simplified if the relation between density and pressure is assumed to be of the form

\[
\rho(p_\varepsilon) = \rho_a e^{(p_\varepsilon-p_{\varepsilon})/\beta}, \quad (3.4)
\]
where the constant $\rho_a$ is the density at the atmospheric pressure $p_a$ and $\beta$ is a positive constant (bulk modulus). This relation is equivalent to the commonly used assumption that the lubricant has a constant bulk modulus $\beta$, see e.g. [34]. Note that this assumption is valid for reasonably low pressures. Due to the special form of the relation (3.4) it is possible to transform the nonlinear equation (3.2) into a linear equation. Indeed, if the function $w_\epsilon$ is defined as $w_\epsilon(x, t) = \rho(p_\epsilon(x, t))/\rho_a$, then
\[
\nabla w_\epsilon = \beta^{-1} e^{(p_\epsilon - p_a)/\beta} \nabla p_\epsilon = \beta^{-1} \rho_a^{-1} \rho(p_\epsilon) \nabla p_\epsilon
\]
and the equation (3.2) is converted into the linear equation
\[
\gamma \frac{\partial}{\partial t} (w_\epsilon h_\epsilon) = \nabla \cdot \left( h_\epsilon^3 \nabla w_\epsilon \right) - \lambda \frac{\partial}{\partial x_1} (w_\epsilon h_\epsilon), \quad \Omega \times I, \quad (3.5)
\]
where $\gamma = 12 \eta \beta^{-1}$ and $\lambda = 6 \eta v \beta^{-1}$.

For small values of $\epsilon$, the coefficients, including $h_\epsilon$, are rapidly oscillating functions. This implies that a direct numerical analysis of the deterministic problems (3.2), (3.3) and (3.5) becomes difficult for small values of $\epsilon$, because a very fine mesh is needed to resolve the surface roughness. This suggests some type of averaging. In this work, the multiple scale expansion method is used to homogenize the unstationary incompressible Reynolds equation (3.3), where $h_\epsilon$ is defined as in (3.1). These results will also be compared with known homogenization results for (3.5). A significant difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus will be seen.

Of note is that in the more mathematical oriented works [15] and [20] another method known as two-scale convergence was used to analyze Reynolds type equations modelling roughness on both surfaces. In particular, [20] considers air flow, where the air compressibility and slip-flow effects are considered. More precisely, the following non-linear equation is homogenized
\[
a \frac{\partial}{\partial t} (p_\epsilon h_\epsilon) = \nabla \cdot \left( \left( h_\epsilon^3 p_\epsilon + bh_\epsilon^2 \right) \nabla p_\epsilon \right) - c \cdot \nabla (p_\epsilon h_\epsilon), \quad \Omega \times I,
\]
where $a$ and $b$ are positive constants and $c \in \mathbb{R}^2$.

### 3.3 Homogenization (constant bulk modulus)

The focus of this work is the homogenization of the incompressible unstationary Reynolds equation. However, the results will be compared with the
Homogenization of the unstationary incompressible Reynolds equation

corresponding homogenization results for the unstationary equation corresponding to the constant bulk modulus case recently obtained in [10], see also [5]. Therefore, for the readers convenience, we review the main conclusions in [10].

Let $\chi_i$, $i = 1, 2, 3$ be the solutions of the local problems

$$\nabla_y \cdot (h^3 \nabla_y \chi_1) = \frac{\partial h^3}{\partial y_1}, \quad \text{on } Y$$

$$\nabla_y \cdot (h^3 \nabla_y \chi_2) = -\frac{\partial h^3}{\partial y_2}, \quad \text{on } Y$$

$$\nabla_y \cdot (h^3 \nabla_y \chi_3) = \gamma \frac{\partial h}{\partial \tau} + \lambda \frac{\partial h}{\partial y_1}, \quad \text{on } Y.$$ 

Moreover, let $h(x, t)$, the vector function $b(x, t)$ and the matrix function $A(x, t) = (a_{ij}(x, t))$ be defined as

$$h(x, t) = \int_T \int_Y h(x, t, y, \tau) \, dy \, d\tau,$$

$$b(x, t) = \int_T \int_Y (\lambda h e_1 - h^3 \nabla_y \chi_3) \, dy \, d\tau,$$

$$A(x, t) = \begin{pmatrix}
\int_T \int_Y h^3 \left(1 + \frac{\partial \chi_1}{\partial y_1}\right) \, dy \, d\tau & \int_T \int_Y h^3 \frac{\partial \chi_2}{\partial y_1} \, dy \, d\tau \\
\int_T \int_Y h^3 \frac{\partial \chi_1}{\partial y_2} \, dy \, d\tau & \int_T \int_Y h^3 \left(1 + \frac{\partial \chi_2}{\partial y_2}\right) \, dy \, d\tau
\end{pmatrix}.$$ 

The main result in [10] states that the deterministic solution $w_\varepsilon$ of (3.5) can be approximated with high accuracy by $w_0(x, t)$, where $w_0$ is the solution of the homogenized (averaged) equation

$$\gamma \frac{\partial}{\partial t} (h w_0) = -\nabla \cdot (b w_0) + \nabla \cdot (A \nabla w_0). \quad (3.6)$$

It was also clearly demonstrated that by using this homogenization result, an efficient method is obtained for analyzing the rough surface effects in problems where the lubricant has a constant bulk modulus and the governing equation is the time dependent compressible Reynolds equation (3.2).

**Remark 3.1.** If $h$ is independent of $t$, i.e. $h = h(x, y, \tau)$, then the homogenized equation (3.6) has the form

$$0 = -\nabla \cdot (b w_0) + \nabla \cdot (A \nabla w_0). \quad (3.7)$$
3.4 Homogenization in the incompressible case

Consider the incompressible transient Reynolds equation

\[
\Gamma \frac{\partial h_\varepsilon}{\partial t} + \Lambda \frac{\partial h_\varepsilon}{\partial x_1} - \nabla \cdot (h_\varepsilon^3 \nabla p_\varepsilon) = 0,
\]

(3.8)

where \( \Gamma = 12\eta \) and \( \Lambda = 6\eta v \). Assume the following multiple scale expansion of the solution \( p_\varepsilon \)

\[
p_\varepsilon = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots
\]

(3.9)

where \( p_i = p_i(x, y, t, \tau) \). The chain rule then implies that

\[
\Gamma \left( \frac{\partial}{\partial t} + \frac{1}{\varepsilon} \frac{\partial}{\partial \tau} \right) h + \Lambda \left( \frac{\partial}{\partial x_1} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_1} \right) h
\]

\[ - \left( \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \right) \left[ h_3 \left( \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \right) \left( p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \right) \right] = 0.
\]

Let \( A_0, A_1 \) and \( A_2 \) be defined as

\[
A_0 = \frac{\partial}{\partial y_i} \left( h_3 \frac{\partial}{\partial y_i} \right) = \nabla_y \cdot (h_3 \nabla_y),
\]

\[
A_1 = \frac{\partial}{\partial x_i} \left( h_3 \frac{\partial}{\partial y_i} \right) + \frac{\partial}{\partial y_i} \left( h_3 \frac{\partial}{\partial x_i} \right) = \nabla_x \cdot (h_3 \nabla_y) + \nabla_y \cdot (h_3 \nabla_x),
\]

\[
A_2 = \frac{\partial}{\partial x_i} \left( h_3 \frac{\partial}{\partial x_i} \right) = \nabla_x \cdot (h_3 \nabla_x).
\]

Then (3.8) may be written as

\[
\Gamma \left( \frac{\partial}{\partial t} + \frac{1}{\varepsilon^1} \frac{\partial}{\partial \tau} \right) h + \Lambda \left( \frac{\partial}{\partial x_1} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_1} \right) h
\]

\[ - (\varepsilon^{-2} A_0 + \varepsilon^{-1} A_1 + A_2) \left( p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \right) = 0.
\]

The idea is now to collect terms of the same order of \( \varepsilon \). For the homogenization it is sufficient to consider the orders \(-2, -1\) and \(0\).

\[
-A_0 p_0 = 0,
\]

(3.10)

\[
\Gamma \frac{\partial h}{\partial \tau} + \Lambda \frac{\partial h}{\partial y_1} - A_0 p_1 - A_1 p_0 = 0,
\]

(3.11)

\[
\Gamma \frac{\partial h}{\partial t} + \Lambda \frac{\partial h}{\partial x_1} - A_0 p_2 - A_1 p_1 - A_2 p_0 = 0.
\]

(3.12)
Homogenization of the unstationary incompressible Reynolds equation

It is well-known that equations of the form $A_0 u = f$ has a unique solution up to an additive constant, if and only if the average over $Y$ of the right hand side is 0, see e.g. page 93 in [3]. Hence, it is clear from (3.10) that $p_0$ does not depend on $y$, i.e. $p_0 = p_0(x, t, \tau)$. Using this fact and averaging (3.11) with respect to $y$ gives

$$\int_Y \left( \frac{\partial h}{\partial \tau} + \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot (h^3 \nabla_y p_1) - \nabla_y \cdot (h^3 \nabla_x p_0) \right) dy = 0.$$ 

By considering $Y$-periodicity, this is reduced to

$$\int_Y \frac{\partial h}{\partial \tau} dy = 0. \quad (3.13)$$

Hence, the assumption that $p_\varepsilon$ may be expanded as in (3.9) requires $h$ to satisfy (3.13). We observe that $h$ fulfills this condition in our case. Physically this means that the surface-to-surface volume does not depend on the relative position of the surface roughness. The fact that $p_0 = p_0(x, t, \tau)$ implies that the equation (3.11) is

$$\nabla_y \cdot (h^3 \nabla_y p_1) = \frac{\partial h}{\partial \tau} + \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot (h^3 \nabla_x p_0),$$

where $x, t$ and $\tau$ are parameters. By linearity, $p_1$ is of the form

$$p_1(x, y, t, \tau) = v_1(x, y, t, \tau) + \frac{\partial p_0}{\partial x_1} v_2(x, y, t, \tau) + \frac{\partial p_0}{\partial x_2} v_3(x, y, t, \tau),$$

where $v_i$ is the solutions of the following local problems

$$\nabla_y \cdot (\Lambda e_1 - h^3 \nabla_y v_1) = -\frac{\partial h}{\partial \tau},$$

$$\nabla_y \cdot (h^3 (e_1 + \nabla_y v_2)) = 0,$$

$$\nabla_y \cdot (h^3 (e_2 + \nabla_y v_3)) = 0,$$

and $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$.

Averaging the equation (3.12) with respect to $y$ gives the equation

$$\Gamma \frac{\partial}{\partial t} \int_Y h dy + \nabla_x \cdot \int_Y (\Lambda e_1 - h^3 \nabla_y v_1) dy -$$

$$\nabla_x \cdot \left( \frac{\partial p_0}{\partial x_1} \int_Y h^3 [e_1 + \nabla_y v_2] dy + \frac{\partial p_0}{\partial x_2} \int_Y h^3 [e_2 + \nabla_y v_3] dy \right) = 0. \quad (3.14)$$
If we introduce the notation \( \overline{h}(x, t, \tau) = \int_Y h \, dy \) and define the homogenized vector \( b(x, t, \tau) \) and the homogenized matrix \( A(x, t, \tau) = (a_{ij}(x, t, \tau)) \) as

\[
\begin{align*}
  b &= \int_Y (\Delta h e_1 - h^3 \nabla_y v_1) \, dy, \\
  \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} &= \int_Y h^3 (e_1 + \nabla_y v_2) \, dy \quad \text{and} \quad \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \int_Y h^3 (e_2 + \nabla_y v_3) \, dy,
\end{align*}
\]

then (3.14) takes the following form:

\[
\Gamma \frac{\partial \overline{h}}{\partial t}(x, t, \tau) + \nabla_x \cdot b(x, t, \tau) - \nabla_x \cdot (A(x, t, \tau) \nabla p_0) = 0. \tag{3.15}
\]

Note that \( t \) and \( \tau \) are just parameters. The appearance of the fast parameter \( \tau \) in the homogenized equation (3.15) means that for small wavelengths the pressure will oscillate rapidly in time. This should be compared with the case of liquid flow with a constant bulk modulus, see (3.6), where the pressure is almost smooth with respect to time, i.e. the amplitude of the oscillations in time, in the deterministic pressure solution \( p_\epsilon \), are very small for small wavelengths. In both cases, the pressure is almost smooth in the space variable.

It should be noted that if \( h \) is independent of \( t \), i.e. \( h = h(x, y, \tau) \), then the homogenized equation (3.15) has the form

\[
\nabla_x \cdot (A(x, \tau) \nabla x p_0(x, \tau)) = \nabla_x \cdot b(x, \tau). \tag{3.16}
\]

It should also be noted that if only one of the surfaces is rough (either the moving or the stationary), i.e. \( h \) is of the form \( h(x, y, t, \tau) = h_0(x, t) + h_i(y - \tau V_i) \), where \( i = 1 \) or \( i = 2 \), then \( \overline{h} \), \( b \) and \( A \) are independent of \( \tau \). This means that the solution \( p_0 \) of the homogenized problem (3.15) is independent of \( \tau \) and this simplifies the problem (3.15).

### 3.5 Numerical results

In this section we present some numerical results based on the homogenized equations obtained in the previous sections. To perform the numerical analysis, the algorithms presented in [10] and [6] are used. In all examples the solution domain \( \Omega \) is a subset of \( \mathbb{R}^2 \) such that \( 0 \leq x_1 \leq L \) and \( -L/2 \leq x_2 \leq L/2 \). For simplicity, the global film thickness \( h_0 \) is assumed to be time independent. More precisely,

\[
h_0(x) = \begin{cases} 
  h_{\text{min}} (1 + k), & x_1 < L/2, \\
  h_{\text{min}}, & x_1 > L/2,
\end{cases}
\]
Homogenization of the unstationary incompressible Reynolds equation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
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<td>$h_{\text{min}}$</td>
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<td>m</td>
</tr>
<tr>
<td>$k$</td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>$c_1 = c_2$</td>
<td>$1/8$</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>$1 \cdot 10^{-1}$</td>
<td>m</td>
</tr>
<tr>
<td>$v_1$</td>
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<td>$ms^{-1}$</td>
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<tr>
<td>$v_2$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$\eta$</td>
<td>$0.14$</td>
<td>$Pa \cdot s$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$1 \cdot 10^{11}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Common problem specific parameters.

and the roughness contribution is represented by

$$h_i(y - \tau v_i) = c_i h_{\text{min}} \sin (2\pi (y - \tau v_i)).$$

This means that a step bearing with surface roughness is considered (in the numerical simulations the discontinuity has been smoothened). The specific parameters, common to all the numerical computations, may be found in Table 3.1.

3.5.1 Incompressible case

Figure 3.2 depicts the deterministic solutions $p_\varepsilon$ of (3.8) for a fixed $\varepsilon$ and time $t$. In Figure 3.3 the corresponding homogenized solution $p_0$ of (3.16) is plotted. It should be noted that the deterministic solution $p_\varepsilon$ oscillates rapidly, while the homogenized solution is smooth (fixed time $t$ and $\varepsilon$).

The convergence of the deterministic pressure towards the homogenized pressure $p_0$, as $\varepsilon \to 0$, was analyzed above by multiple scale expansions. This convergence will now be illustrated by means of numerical solutions. Indeed, Figure 3.4 represent part of the pressure distribution between the two rough surfaces along the $x_2 = 0$ line at a particular point in time for different values of $\varepsilon$. As seen in the figure, the pressure distribution $p_\varepsilon$ approaches that of the homogenized pressure as $\varepsilon$ tends to zero. Figure 3.5 represents an enlargement of a portion of Figure 3.4, showing clearly the decrease in the amplitude of the pressure distribution towards the homogenized pressure solution as the roughness wavelength $\varepsilon$ tends to zero.

As mentioned before in the analysis by multiple scale expansions, the appearance of the fast parameter $\tau$ in the homogenized equations (3.15) and (3.16) means that for small wavelengths the pressure will oscillate rapidly in time. This fact is illustrated in Figure 3.6, which depicts the pressure
Figure 3.2: Pressure distribution in the incompressible case for a fixed $\varepsilon$. 
Homogenization of the unstationary incompressible Reynolds equation

Figure 3.3: Homogenized pressure distribution for the incompressible case.

Figure 3.4: Pressure solutions at $x_2 = 0$ for various $\varepsilon$ as well as the corresponding homogenized solution at time $t = 0$ in the incompressible case.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 3.5: Zoomed portion of Figure 3.4.

distribution at some different times (within a period) for a fixed $\varepsilon$ and the corresponding homogenized solutions.

In addition to the visual illustration of the convergence of $p_\varepsilon$ to $p_0$, a more quantitative convergence analysis is considered here. For this purpose we consider what happens with the load carrying capacity as $\varepsilon$ tends to 0. The load carrying capacity $l_\varepsilon$ corresponding to $p_\varepsilon$ and $l_0$ corresponding to $p_0$, are defined as

$$
l_\varepsilon(t) = \int_\Omega p_\varepsilon(x,t) \, dx \quad \text{and} \quad l_0(\tau) = \int_\Omega p_0(x, \tau) \, dx.
$$

(3.17)

In Figure 3.7 we see that $l_\varepsilon \to l_0$ as $\varepsilon$ approaches zero. The difference in load carrying capacity at $t = \tau = 0$, which is the worst case scenario, is approximately 1%. It is also noted that, in the case with perfectly sinusoidal surface roughness descriptions, for a specific value of $\varepsilon$ between $1/64$ and $1/32$, a seemingly small variation of the load carrying capacity in time is obtained, i.e. it is possible to optimize the surfaces to reduce vibrations.

3.5.2 Constant bulk modulus case

In the analysis by multiple scale expansions we observed a significant difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus. No fast parameter $\tau$ is found in the homogenized equation of the constant bulk modulus case. This implies that
Homogenization of the unstationary incompressible Reynolds equation

Figure 3.6: Pressure solutions at $x_2 = 0$ for three different $\varepsilon$ as well as the corresponding homogenized solution.

Figure 3.7: Convergence of the load carrying capacity in the incompressible case.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 3.8: The pressure solutions for a fixed $\varepsilon$ at three different time steps and the homogenized solution in the compressible case.

we only have one homogenized solution in our example where $h_0 = h_0(x)$ contrary to the incompressible case where we have different homogenized solutions for different times $t$ within a period. This fact is illustrated in Figure 3.8, which corresponds to the Figure 3.6 in the incompressible situation.

In Figure 3.9 we observe that, for perfectly smooth surfaces, as $\beta$ is increased, the pressure distribution in the constant bulk modulus case, approaches that in the incompressible case. However, this does not seem to be the case for rough surfaces, due to the different asymptotic behavior between the constant bulk modulus case and the incompressible case.

3.6 Concluding remarks

We have clearly demonstrated that homogenization may be used to efficiently analyze the effects of surface roughness in incompressible thin film unstationary lubrication flow. This has been done using the method of asymptotic expansions and numerical examples where we visualize the convergence and give a quantitative convergence analysis of the load capacity. One important observation is that there is a difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus. When the lubricant is assumed to be incompressible, the homogenized (averaged)
Homogenization of the unstationary incompressible Reynolds equation

Figure 3.9: Comparison of the pressure distribution between the incompressible and compressible case when the bearing surfaces are smooth for different values of $\beta$.

equation contains a fast parameter that is connected to the time. This means that for small wavelengths the pressure distribution oscillates rapidly in time, while it is almost smooth with respect to the space variable. For liquid flow of a lubricant with a constant bulk modulus, the pressure solution of the homogenized equation does not contain any of the fast parameters. Thus, for small wavelengths the pressure is almost smooth in both the space and time variables. There are many interesting directions to deepen our study of hydrodynamic lubrication, where both surfaces are assumed to be rough. For example, to include a model that regards cavitation, another would be to consider non-Newtonian lubricants.
Homogenization of Reynolds equations and of some parabolic problems via Rothe's method
Chapter 4

Variational bounds applied to unstationary hydrodynamic lubrication

4.1 Introduction

It is well known that the surface micro topography is an important parameter in determining the performance of moving machine parts operating in the hydrodynamic lubrication regime. An important problem in thin-film lubrication theory is therefore to estimate the effects of surface roughness on the pressure solution.

In Reynolds lubrication model, the effects of surface roughness are solely determined by the induced variations in film thickness, both in space and time. Due to these oscillations, a direct computation of the pressure solution may not be possible in practice since that would require a high resolution mesh. A remedy may be to consider some sort of averaging. Surface roughness in hydrodynamic lubrication has been considered by many authors and several averaging techniques have been proposed in the literature. A rigorous form of averaging is accomplished by homogenization. Reynolds type differential equations have been analyzed by homogenization techniques in e.g. [9], [11], [15], [20], [21], [22], [23], [33], [40] and [57] and by other averaging techniques in e.g. [30], [36], [63] and [68].
The procedure for solving the homogenized (averaged) equation can be described as follows: First one solves a number of local problems by some numerical method. Then these local solutions are used to compute the coefficients in the homogenized equation. Finally, the homogenized equation is solved. This means that the homogenized solution, although not as complex as the deterministic solution, may still be very demanding to compute because of all the local problems that need to be solved. In the recent works [9] and [57], an alternative approach was introduced for analyzing the stationary Reynolds equation (one rough surface). The advantage of this approach is that no local problems need to be computed. The main idea is to obtain bounds on the homogenized "energy density" appearing in the variational formulation corresponding to the homogenized equation. For a summary of the state of the art concerning bounds in general the reader may consult e.g. [41]. The conclusion in [9] and [57] were that when both precision and computational time are important, bounds are a very cost-effective method of estimating the effects of surface roughness in stationary hydrodynamic lubrication.

The main purpose of this work is to develop the ideas in [9] and [57] to include the situation where both surfaces are rough. More precisely, in Section 4.2 we study the homogenization of a class of variational problems by multiple scale expansion. In particular, this class of variational problems includes the variational formulation associated with the stationary and the unstationary Reynolds equation formulated in either Cartesian or polar coordinates. In Section 4.3 and 4.4 we derive bounds that apply to the homogenized energy functional by using the techniques described in [9] and [57]. The bounds of arithmetic-harmonic mean type are optimal, i.e., there exists surface roughness descriptions where the lower and upper bound coincide. In Section 4.5 we show that our general results may be applied to analyze the effects of rough surface in connection with the unstationary Reynolds equation. Moreover, we present several numerical experiments. In particular, the pressure solutions obtained by using the new bounds is compared with both the homogenized and the deterministic solutions for different types of surface roughness. The experiments clearly show that bounds may be used to accurately estimate the effects of surface roughness in hydrodynamic lubrication.
4.2 Homogenization of a variational principle

In this section we consider the homogenization of a class of variational problems by multiple scale expansion. In particular, this class of variational problems includes the variational formulation associated with the unstationary Reynolds equation, which appears in hydrodynamic lubrication, see Section 4.4. The homogenized variational problem will thereafter serve as the starting point for deriving lower and upper bounds.

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^2 \) and \( T > 0 \). In the remainder of this paper \( a_i, b_i \) (\( i = 1, 2 \)) and \( \delta \) denote functions from \( \Omega \times [0, T] \times \mathbb{R}^2 \times \mathbb{R} \) to \( \mathbb{R} \) that are \( Y \)-periodic in the third argument and \( Z \)-periodic in the fourth argument. For simplicity we assume \( Y = (0, 1)^2 \) and \( Z = (0, 1) \). Moreover, define

\[
\mathbf{A} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \mathbf{b} = (b_1, b_2)
\]

and assume that there exists a constant \( \alpha \) such that \( a_1, a_2 \geq \alpha > 0 \).

For each \( \varepsilon > 0 \) consider the variational principle

\[
\begin{cases}
\min_p I_\varepsilon(p) \\
I_\varepsilon(p) \overset{\text{def}}{=} \int_0^T \int_\Omega \frac{1}{2} \nabla_x p \cdot \mathbf{A}_\varepsilon \nabla_x p + \mathbf{b}_\varepsilon \cdot \nabla_x p + \delta_\varepsilon p \, dx \, dt
\end{cases}, \quad (4.1)
\]

where

\[
\mathbf{A}_\varepsilon(x,t) = \mathbf{A}(x,t,x/\varepsilon,t/\varepsilon), \quad \mathbf{b}_\varepsilon(x,t) = \mathbf{b}(x,t,x/\varepsilon,t/\varepsilon), \quad \delta_\varepsilon(x,t) = \delta(x,t,x/\varepsilon,t/\varepsilon)
\]

A smooth functions \( p = p(x,t) \) satisfying \( p = 0 \) on the boundary of \( \Omega \) is called an admissible function and the minimum in (4.1) is understood to be taken over all such functions \( p \). Moreover, it is assumed that the minimization problem is well posed and we denote by \( p_\varepsilon \) the minimizer in the class of admissible functions.

As \( \varepsilon \to 0 \) the coefficients in (4.1) will oscillate rapidly, which suggest some type of averaging. The homogenization problem consists of analyzing the asymptotic behavior as \( \varepsilon \to 0 \). The main question is: does \( \lim_{\varepsilon \to 0} I_\varepsilon(p_\varepsilon) \) exist? We shall answer this question in the affirmative by showing that there exist a homogenized variational problem \( \min_p I_0(p) \), such that

\[
\lim_{\varepsilon \to 0} \min_p I_\varepsilon(p) = \min_p I_0(p).
\]

We start the homogenization by first postulating that the sequence of real numbers \( \{I_\varepsilon(p_\varepsilon)\}_{\varepsilon > 0} \) is bounded. Moreover, we assume that \( p_\varepsilon \) admits
the following multiple-scale expansion

\[ p_\varepsilon(x,t) = \sum_{i=0}^{\infty} \varepsilon^i p_i(x,t,x/\varepsilon,t/\varepsilon), \quad (4.2) \]

where the functions \( p_i = p_i(x,t,y,\tau) \) \((i = 0,1,2,\ldots)\) are assumed to be smooth functions that are \(Y\)-periodic in \(y\) and \(Z\)-periodic in \(\tau\). The idea is to apply the chain rule and formally insert (4.2) into (4.1) and to determine thereby the limiting behavior of \( I_\varepsilon(p_\varepsilon) \) as \(\varepsilon \to 0\). After some calculations we obtain the following expansion for \( I_\varepsilon(p_\varepsilon) \):

\[ I_\varepsilon(p_\varepsilon) = \varepsilon^{-2} \int_0^T \int_\Omega \frac{1}{2} \nabla_y p_0 \cdot A_\varepsilon \nabla_y p_0 \, dx \, dt \]

\[ + \varepsilon^{-1} \int_\Omega \int_0^T \nabla_y p_0 \cdot A_\varepsilon (\nabla_x p_0 + \nabla_y p_1) + b_\varepsilon \cdot \nabla_y p_0 \, dx \, dt \]

\[ + \varepsilon^0 \int_0^T \int_\Omega \left[ \frac{1}{2} (\nabla_x p_0 + \nabla_y p_1) \cdot A_\varepsilon (\nabla_x p_0 + \nabla_y p_1) + \nabla_y p_0 \cdot A_\varepsilon (\nabla_x p_1 + \nabla_y p_2) + b_\varepsilon \cdot (\nabla_x p_0 + \nabla_y p_1) + \delta p_0 \right] \, dx \, dt + \varepsilon(...) \quad (4.3) \]

Consider the first term in the expansion (4.3), i.e.

\[ \varepsilon^{-2} \int_0^T \int_\Omega \frac{1}{2} \nabla_y p_0 \cdot A_\varepsilon \nabla_y p_0 \, dx \, dt. \]

In view of the conditions on \(a_i\) we have that

\[ \int_0^T \int_\Omega \frac{1}{2} \nabla_y p_0 \cdot A_\varepsilon \nabla_y p_0 \, dx \, dt \geq \frac{\alpha}{2} \int_0^T \int_\Omega |\nabla y p_0|^2 \, dx \, dt \]

for some \(\alpha > 0\). Furthermore, according to the “mean value property” (see for example [65])

\[ \lim_{\varepsilon \to 0} \int_0^T \int_\Omega |\nabla y p_0(x,t,x/\varepsilon,t/\varepsilon)|^2 \, dx \, dt \]

\[ = \int_0^T \int_\Omega \int_Z \int_Y |\nabla y p_0(x,t,y,\tau)|^2 \, dy \, d\tau \, dx \, dt. \]

Thus, for the boundedness assumption to hold it is necessary that

\[ \int_0^T \int_\Omega \int_Z \int_Y |\nabla y p_0|^2 \, dy \, d\tau \, dx \, dt = 0. \]
This implies \( \nabla_y p_0 = 0 \), hence \( p_0(x, t, y, \tau) = p_0(x, t, \tau) \), and so (4.3) can be reduced to

\[
I_\varepsilon(p_\varepsilon) = \int_0^T \int_\Omega \left\{ \frac{1}{2} (\nabla_x p_0 + \nabla_y p_1) \cdot A_\varepsilon (\nabla_x p_0 + \nabla_y p_1) \\
+ b_\varepsilon \cdot (\nabla_x p_0 + \nabla_y p_1) + \delta_\varepsilon p_0 \right\} \, dx \, dt + O(\varepsilon). \tag{4.4}
\]

According to the mean value property we obtain

\[
\lim_{\varepsilon \to 0} I_\varepsilon(p_\varepsilon) = \int_0^T \int_\Omega \int_Z \int_Y \frac{1}{2} (\nabla_x p_0 + \nabla_y w) \cdot A (\nabla_x p_0 + \nabla_y w) \\
+ b \cdot (\nabla_x p_0 + \nabla_y w) + \delta p_0 \, dy \, d\tau \, dx \, dt. \tag{4.5}
\]

Thus, the limiting behavior of \( I_\varepsilon(p_\varepsilon) \) is governed by the functions \( p_0 \) and \( p_1 \) alone. This suggests that \( p_0 \) and \( p_1 \) are determined by the variational principle

\[
\begin{cases}
\min_{p, w} J_0(p, w) \\
J_0(p, w) \overset{\text{def}}{=} \int_0^T \int_\Omega \int_Z \int_Y \frac{1}{2} (\nabla_x p + \nabla_y w) \cdot A (\nabla_x p + \nabla_y w) \\
+ b \cdot (\nabla_x p + \nabla_y w) + \delta p_0 \, dy \, d\tau \, dx \, dt.
\end{cases} \tag{4.6}
\]

The minimum in (4.6) is taken over all smooth \( p = p(x, t, \tau) \) that vanish on the boundary and smooth \( w = w(x, t, y, \tau) \) that are \( Y \)-periodic in \( y \). In other words, we claim

\[
\lim_{\varepsilon \to 0} I_\varepsilon(p_\varepsilon) = J_0(p_0, p_1) = \min_{p, w} J_0(p, w).
\]

Let \( W_{\text{per}} \) denote the set of all \( Y \)-periodic smooth functions \( w = w(y) \). Then we can write (4.6) as

\[
\begin{cases}
\min_{p} I_0(p) \\
I_0(p) \overset{\text{def}}{=} \int_0^T \int_\Omega \int_Z \int_Y f_0(x, t, \tau, \nabla_x p) + \delta_0(x, t, \tau)p \, d\tau \, dx \, dt,
\end{cases} \tag{4.7}
\]

where \( \delta_0(x, t, \tau) = \int_Y \delta(x, t, y, \tau) \, dy \) and

\[
f_0(x, t, \tau, \xi) = \min_{w \in W_{\text{per}}} \int_Y \frac{1}{2} (\xi + \nabla_y w) \cdot A(x, t, y, \tau)(\xi + \nabla_y w) \\
+ b(x, t, y, \tau) \cdot (\xi + \nabla_y w) \, dy. \tag{4.8}
\]
for any $\xi \in \mathbb{R}^2$.

Summing up we have derived the following homogenization result:

$$\lim_{\varepsilon \to 0} \min_p I_\varepsilon(p) = \lim_{\varepsilon \to 0} I_\varepsilon(p_\varepsilon) = I_0(p_0) = \min_p I_0(p),$$

(4.9)

where $I_0$ is given by (4.7).

We proceed by giving an alternative formula for $f_0$ that is more suitable for computation. From the definition of $f_0$ we obtain

$$f_0(x, t, \tau, \xi) = \min_{w \in W_{\text{per}}} \int_Y \frac{1}{2} \nabla_y w \cdot A \nabla_y w + A \xi \cdot \nabla_y w + \frac{1}{2} \xi \cdot A \xi + b \cdot (\xi + \nabla_y w) \, dy. \quad (4.10)$$

Omitting all constant terms it is clear that a minimizer $w_\xi$ of (4.10) is also a solution of

$$\min_{w \in W_{\text{per}}} \int_Y \frac{1}{2} \nabla_y w \cdot A \nabla_y w + (A \xi + b) \cdot \nabla_y w \, dy. \quad (4.11)$$

The Euler–Lagrange equation corresponding to (4.11) is

$$\nabla_y \cdot (A \nabla_y w + (A \xi + b)) = 0.$$

By linearity $w$ can be written $w = w_\xi + v_0$, where $v_\xi$ and $v_0$ are solutions of the local problems

$$\nabla_y \cdot A(\xi + \nabla_y v_\xi) = 0 \quad \text{in } Y \quad (4.12)$$

$$\nabla_y \cdot (A \nabla_y v_0 + b) = 0 \quad \text{in } Y. \quad (4.13)$$

Inserting $w = v_\xi + v_0$ into (4.10) yields

$$f_0(x, t, \tau, \xi) = \int_Y \frac{1}{2} (\xi + \nabla_y w) \cdot A(\xi + \nabla_y v_\xi) + \frac{1}{2} (\xi + \nabla_y w) \cdot (A \nabla_y v_0 + b) \, dy.$$

Integration by parts together with (4.12) and (4.13) further gives

$$f_0(x, t, \tau, \xi) = \int_Y \frac{1}{2} \xi \cdot A_0 \xi + b_0 \cdot \xi + \frac{1}{2} \int_Y b \cdot \nabla_y v_0 \, dy, \quad (4.14)$$
Variational bounds applied to unstationary hydrodynamic lubrication

where

\[ A_0 \xi = \int_Y A(\xi + \nabla_y v_\xi) \, dy, \quad b_0 = \int_Y (A \nabla_y v_0 + b) \, dy. \]

It can be noted from the homogenization result above that it is possible to deduce a homogenization result for the Euler equation to (4.1), i.e.

\[-\nabla_x \cdot (A_x \nabla_x p_x + b_x) + \delta_x = 0 \quad \text{in } \Omega \times (0, T). \quad (4.15)\]

By taking the alternative formula (4.14) and (4.7) into account it follows that the corresponding homogenized equation is

\[-\nabla_x \cdot (A_0 \nabla_x p_0 + b_0) + \delta_0 = 0 \quad \text{in } \Omega \times (0, T). \quad (4.16)\]

We remark that the homogenization of (4.15) was studied in [15] by a different method namely two-scale convergence and that their results are in agreement with ours.

### 4.3 Bounds of arithmetic-harmonic type

The aim of this section is to extend the ideas in [9] and [57] in order to obtain upper and lower bounds on the function \( f_0 \). First we note that \( x, t \) and \( \tau \) may be regarded as parameters. To emphasize this and to avoid lengthy notation we temporarily write \( A = A(y) \) and \( b = b(y) \).

More precisely, we prove an estimate of the type

\[ \frac{1}{2} \xi \cdot A^- \xi + b^- \cdot \xi + c^- \leq f(\xi) \leq \frac{1}{2} \xi \cdot A^+ \xi + b^+ \cdot \xi + c^+, \]

where \( A^\pm \) (diagonal matrices), \( b^\pm \) (vectors) and \( c^\pm \) (scalars) are constants, on the function \( f: \mathbb{R}^2 \to \mathbb{R} \) defined by

\[ f(\xi) = \min_{w \in W_{\text{per}}} \int_Y \frac{1}{2} (\xi + \nabla_y w) \cdot A(\xi + \nabla_y w) + b \cdot (\xi + \nabla_y w) \, dy. \quad (4.17)\]

The proof of the bounds now follows by using the same arguments as in [57], but is included for the readers convenience.

#### 4.3.1 Upper bound

Let \( V = \{ \phi \in W_{\text{per}} : \phi(y) = \phi_1(y_1) + \phi_2(y_2) \} \). Then clearly

\[ f(\xi) \leq \min_{w \in V} \int_Y \frac{1}{2} (\xi + \nabla_y w) \cdot A(\xi + \nabla_y w) + b \cdot (\xi + \nabla_y w) \, dy \underset{\text{def}}{=} f^+(\xi). \quad (4.18)\]
We compute $f^+(\xi)$ explicitly by solving the corresponding weak Euler–Lagrange equation:

$$
\int_Y (A(\xi + \nabla_y w) + b) \cdot \nabla_y \phi \, dy = 0 \quad \forall \phi \in V. \tag{4.19}
$$

Upon inserting $w(y) = w_1(y_1) + w_2(y_2)$ and $\phi(y) = \phi_1(y_1) + \phi_2(y_2)$ into (4.19), we obtain

$$
\begin{align*}
\int_0^1 \left\{ \int_0^1 (a_1(\xi_1 + w_1') + b_1) \, dy_2 \right\} \phi_1'(y_1) \, dy_1 \\
+ \int_0^1 \left\{ \int_0^1 (a_2(\xi_2 + w_2') + b_2) \, dy_1 \right\} \phi_2'(y_2) \, dy_2 = 0.
\end{align*}
$$

We conclude that the expressions within curly brackets must be constant.

**Notation.** We assume in the sequel that the indices $i$ and $j$ are complementary in $\{1, 2\}$, i.e. either $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$.

Thus, $k_i$ ($i = 1, 2$) defined by

$$
k_i = \int_0^1 a_i(y) \left( \xi_i + w_i'(y_i) \right) + b_i(y) \, dy_j. \tag{4.20}
$$

are constants. But (4.20) implies that

$$
\xi_i + w_i' = \frac{k_i - B_i}{A_i}, \tag{4.21}
$$

where

$$
A_i(y_i) = \int_0^1 a_i(y) \, dy_j \quad \text{and} \quad B_i(y_i) = \int_0^1 b_i(y) \, dy_j.
$$

We compute $k_i$ by integrating (4.21) on $(0, 1)$ with respect to $y_j$ and then solve for $k_i$. This gives

$$
k_i = \frac{\xi_i + \int_0^1 \frac{B_i}{A_i} \, dy_i}{\int_0^1 A_i^{-1} \, dy_i}.
$$

After some straightforward computations we obtain

$$
\begin{align*}
f^+(\xi) = \sum_{i=1}^2 \frac{1}{2} \left( \xi_i + \int_0^1 \frac{B_i}{A_i} \, dy_i \right)^2 - \frac{1}{2} \int_0^1 \frac{B_i^2}{A_i} \, dy_i. \tag{4.22}
\end{align*}
$$
Define the matrix $A^+$ and the vector $\beta^+$ as
\[
A^+(y) = \begin{pmatrix}
\int_0^1 a_1(y_2) \, dy_2 \\
\int_0^1 a_2(y_1) \, dy_1
\end{pmatrix}
\quad \text{and} \quad
\beta^+(y) = \begin{pmatrix}
\int_0^1 b_1(y_2) \\
\int_0^1 b_2(y_1)
\end{pmatrix}.
\]
Then $f^+$ can be written as
\[
f^+(\xi) = \frac{1}{2} \xi \cdot A^+ \xi + b^+ \cdot \xi + c^+,
\]
where $A^+ = \langle (A^+)^{-1} \rangle^{-1}$, $b^+ = \langle (A^+)^{-1} \rangle^{-1} \langle (A^+)^{-1} \beta^+ \rangle$ and
\[
c^+ = \frac{1}{2} \langle (A^+)^{-1} \beta^+ \rangle \cdot \langle (A^+)^{-1} \rangle^{-1} \langle (A^+)^{-1} \beta^+ \rangle - \frac{1}{2} \langle (A^+)^{-1} \beta^+ \rangle \cdot (A^+)^{-1} \beta^+.
\]

4.3.2 Lower bound

The upper bound $f^+$ was straightforward to derive but a lower bound calls for more sophisticated techniques. The derivation of lower bounds relies on the dual variational principle (see Appendix)
\[
f^*(\eta) = \min_{\sigma \in S} \int_Y \frac{1}{2} (\sigma + \eta - b) \cdot A^{-1} (\sigma + \eta - b) \, dy,
\]
where $f^*: \mathbb{R}^2 \to \mathbb{R}$ denotes the conjugate function of $f$, or the Legendre transformation of $f$, defined by
\[
f^*(\eta) = \max_{\xi \in \mathbb{R}^2} \{ \eta \cdot \xi - f(\xi) \}
\]
and $S = \{ \sigma: Y \to \mathbb{R}^2 : \nabla \cdot \sigma = 0 \text{ and } \int_Y \sigma \, dy = 0 \}$ is the space of solenoidal vector fields on $Y$ with mean value zero. As before, the idea is to look for a solution of the minimization problem (4.26) in a smaller subspace.

Define the vector field space $S^* \subset S$ by
\[
S^* = \{ \sigma \in S : \sigma = (\sigma_1(y_2), \sigma_2(y_1)) \}.
\]
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Obviously

\[ f^*(\eta) \leq \min_{\sigma \in \mathcal{S}} \int_Y \frac{1}{2} (\sigma + \eta - \mathbf{b}) \cdot \mathbf{A}^{-1}(\sigma + \eta - \mathbf{b}) \, dy \overset{\text{def}}{=} (f^*)^+(\eta). \quad (4.27) \]

We compute \((f^*)^+(\eta)\) by solving the corresponding Euler–Lagrange equations: find \(\sigma \in \mathcal{S}\) such that

\[ \int_Y \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \mathbf{A}^{-1}(\sigma + \eta - \mathbf{b}) \, dy = 0 \quad (4.28) \]

for all \(\phi_1, \phi_2\) with mean value zero. From (4.28) we see that

\[ \int_0^1 \left\{ \int_0^1 a_i^{-1}(\sigma_i(y_j) + \eta_i - b_i) \, dy_i \right\} \phi_i(y_j) \, dy_j = 0. \]

By choosing \(\phi_i = \psi'\), such that \(\psi\) is smooth and has compact support in \((0, 1)\), we can conclude that the expression between the curly brackets does not depend on \(y\). Thus, \(m_i\) defined by

\[ m_i = \int_0^1 a_i^{-1}(\sigma_i(y_j) + \eta_i - b_i) \, dy_i \quad (4.29) \]

is constant, where \(i\) and \(j\) are complementary. However, (4.29) implies

\[ \sigma_i + \eta_i = H_i(m_i + G_i), \quad (4.30) \]

where

\[ H_i = \left( \int_0^1 a_i^{-1} \, dy_i \right)^{-1} \quad G_i = \int_0^1 \frac{b_i}{a_i} \, dy_i. \]

Integration of (4.30) w.r.t. \(y_j\) yields

\[ m_i = \frac{\eta_i - \int_0^1 H_i G_i \, dy_j}{\int_0^1 H_i \, dy_j}. \quad (4.31) \]

By the fundamental theorem of calculus

\[ \frac{(\sigma_i + \eta_i - b_i)^2}{2a_i} - \frac{b_i^2}{2a_i} = \int_0^{\sigma_i + \eta_i} \frac{\theta - b_i}{a_i} \, d\theta. \]

We compute

\[ \int_Y \int_0^{\sigma_i + \eta_i} \frac{\theta - b_i}{a_i} \, d\theta \, dy = \frac{\left( \eta_i - \int_0^1 H_i G_i \, dy_j \right)^2}{2 \int_0^1 H_i \, dy_j} - \frac{1}{2} \int_0^1 H_i G_i^2 \, dy_j. \]
Variational bounds applied to unstationary hydrodynamic lubrication

to find that

\[
\int_Y \frac{(\sigma_i + \eta_i - b_i)^2}{2a_i} \, dy = \left( \frac{\eta_i - \int_0^1 H_i G_j \, dy_j}{2 \int_0^1 H_i \, dy_j} \right)^2 - \frac{1}{2} \int_0^1 H_i G_j^2 \, dy_j + \int_Y \frac{b_i^2}{2a_i} \, dy
\]

and consequently

\[
(f^+)(\eta) = \sum_{i=1}^{2} \frac{1}{2} \left( \frac{\eta_i - \int_0^1 H_i G_j \, dy_j}{2 \int_0^1 H_i \, dy_j} \right)^2 + \int_Y \frac{b_i^2}{2a_i} \, dy - \frac{1}{2} \int_0^1 H_i G_j^2 \, dy_j.
\]

Now define

\[
A^-(y) = \begin{pmatrix} f_0^1 & 0 \\ 0 & f_0^1 \end{pmatrix},
\]

and

\[
\beta^-(y) = A^-(y) \begin{pmatrix} f_0^1 \\ f_0^1 \end{pmatrix},
\]

Then

\[
(f^+)^+(\eta) = \frac{1}{2} (\eta - \langle \beta^- \rangle) \cdot \langle A^- \rangle^{-1} (\eta - \langle \beta^- \rangle) + \frac{1}{2} (\beta^- \cdot A^- \beta^- - \frac{1}{2} \langle \beta^- \cdot A^- \beta^- \rangle).
\]

Because of the involutive property of the Legendre transformation the conjugate function of \((f^+)^+, ((f^+)^+)^+,\) yields a lower bound on \(f\). We compute

\[
((f^+)^+)^*(\xi) = \frac{1}{2} \xi \cdot \langle A^- \rangle \xi + \langle \beta^- \rangle \cdot \xi + \frac{1}{2} (\beta^- \cdot \langle A^- \rangle^{-1} \beta^-) - \frac{1}{2} \langle b \cdot A^{-1} b \rangle.
\]

Hence, \(f\) is bounded from below by the function

\[
f^- (\xi) = \frac{1}{2} \xi \cdot A^- \xi + b^- \cdot \xi + c^-,
\]

where

\[
A^- = \langle A^- \rangle \quad b^- = \langle \beta^- \rangle \quad c^- = \frac{1}{2} (\beta^- \cdot \langle A^- \rangle^{-1} \beta^-) - \frac{1}{2} \langle b \cdot A^{-1} b \rangle.
\]

It should be noted that the bounds of arithmetic-harmonic mean type are optimal, in the sense that there exists matrices \(A\) for which \(f^-_0\) and \(f^+_0\) coincide with \(f_0\).
4.4 Bounds of Reuss–Voigt type

A trivial bound on $f$ is obtained by taking $w = 0$ in the right hand side of (4.17). Indeed

$$f(\xi) \leq \int_{Y} \frac{1}{2} \xi \cdot A \xi + b \cdot \xi \, dy = \frac{1}{2} \xi \cdot \langle A \rangle \xi + \langle b \rangle \cdot \xi \overset{\text{def}}{=} f_{\text{RV}}^+(\xi).$$

Similarly, by taking $\sigma = 0$ in the dual variational principle (4.26), it is clear that

$$f^*(\eta) \leq \int_{Y} \frac{1}{2} \eta \cdot (\eta - b) \cdot A^{-1} (\eta - b) \, dy = \frac{1}{2} \eta \cdot \langle A^{-1} \eta \rangle - \frac{1}{2} \eta \cdot \langle A^{-1} b \rangle + \frac{1}{2} \langle b \cdot A^{-1} b \rangle.$$

The Legendre transformation of the last expression is

$$f_{\text{RV}}^{-}(\xi) \overset{\text{def}}{=} \frac{1}{2} (\xi + \langle A^{-1} b \rangle) \cdot \langle A^{-1} \rangle^{-1} (\xi + \langle A^{-1} b \rangle) - \frac{1}{2} \langle b \cdot A^{-1} b \rangle,$$

which yields the lower Reuss–Voigt bound on $f$. This proves the pointwise estimate $f_{\text{RV}}^- \leq f \leq f_{\text{RV}}^+$. This proves the pointwise estimate $f_{\text{RV}}^- \leq f \leq f_{\text{RV}}^+$ where

$$A_{\text{RV}}^+ = \langle A \rangle \quad A_{\text{RV}}^- = \langle A^{-1} \rangle^{-1} \quad b_{\text{RV}}^+ = \langle b \rangle \quad b_{\text{RV}}^- = \langle A^{-1} \rangle^{-1} \langle A^{-1} b \rangle \quad c_{\text{RV}}^+ = 0 \quad c_{\text{RV}}^- = \frac{1}{2} \langle A^{-1} b \rangle \cdot \langle A^{-1} \rangle^{-1} \langle A^{-1} b \rangle - \frac{1}{2} \langle b \cdot A^{-1} b \rangle. \quad (4.34)$$

4.5 Application to a problem in hydrodynamic lubrication

We apply the preceding general results to a thrust pad bearing problem where both the pad and the shaft surfaces exhibit periodic roughness and the lubricant is assumed to be an incompressible Newtonian fluid with viscosity $\mu$ and density $\rho$. For a schematic description of the model problem see Figure 4.1. A point on the reference plane is identified by its polar coordinates $x = (x_1, x_2)$, $x_1$ denoting the angular coordinate and $x_2$ the radial coordinate. The gap between the two smooth surfaces is given by $h_0$ and the lower surface rotates uniformly at the angular speed $\omega$.

To model surface roughness we introduce the auxiliary film thickness function

$$h(x, y, \tau) = h_0(x) + h_U(y) - h_L(y - \tau(\omega, 0)),$$

where
Variational bounds applied to unstationary hydrodynamic lubrication

Figure 4.1: Schematic of a thrust pad bearing.

- $h_0$ is a function of the global variable $x$ (space variable) that describes the global shape of the film thickness,

- $h_U$ and $h_L$ are $Y$-periodic functions of the local variables $y = x/\varepsilon$ and $\tau = t/\varepsilon$ that represent the roughness contribution of the upper and lower surfaces respectively.

Note that the results obtained in Section 4.2 are easily extended to allow for an arbitrary $\omega$ by defining $Z = (0, 1/\omega)$.

For a given wavelength $\varepsilon$, the corresponding film thickness $h_\varepsilon$ is given by $h_\varepsilon(x, t) = h(x, x/\varepsilon, t/\varepsilon)$. The pressure $p_\varepsilon = p_\varepsilon(x, t)$ that builds up in the film is assumed to be governed by Reynolds variational principle, i.e. $I_\varepsilon(p_\varepsilon) = \min_p I_\varepsilon(p)$ where

$$I_\varepsilon(p) = \int_0^T \int_\Omega h_\varepsilon^3 \left( \frac{\partial p}{\partial x_1} \right)^2 + \frac{x_3 h_\varepsilon^3}{2} \left( \frac{\partial p}{\partial x_2} \right)^2 - \lambda x_2 \tilde{h} \frac{\partial p}{\partial x_1} \, dx \, dt,$$

$$\tilde{h} \text{ is defined by } \tilde{h} = h_0 + h_U + h_L \text{ and } \gamma = 12\mu \text{ and } \lambda = \gamma \omega/2 \text{ are constants.}$$

Moreover, the bearing domain is assumed to be annulus-shaped, i.e. there exists $R_1$ and $R_2$ such that for each $x \in \Omega$ it holds that $0 < R_1 \leq x_2 \leq R_2$. 

To see that (4.36) is of the general form (4.1), take
\[ A(x, y, \tau) = \begin{pmatrix} a_1(x, y, \tau) & 0 \\ 0 & a_2(x, y, \tau) \end{pmatrix}, \]
\[ b(x, y, \tau) = (b_1(x, y, \tau), 0), \]
\[ \delta_0(x, t) = 0. \]

Since \( h \) is of special form (4.35), it is readily checked that
\[
\lambda \frac{\partial h_\varepsilon}{\partial x_1}(x, t) + \gamma \frac{\partial h_\varepsilon}{\partial t}(x, t) = \lambda \frac{\partial}{\partial x_1} \tilde{h}(x, x/\varepsilon, t/\varepsilon).
\]

Using this relation we see that the Euler–Lagrange equation corresponding to (4.36) is the unstationary Reynolds equation in polar coordinates, i.e.
\[
\frac{\partial}{\partial x_1} \left( h_\varepsilon^3 \frac{\partial p_\varepsilon}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( x_2 h_\varepsilon^3 \frac{\partial p_\varepsilon}{\partial x_2} \right) = \lambda x_2 \frac{\partial h_\varepsilon}{\partial x_1} + \gamma x_2 \frac{\partial h_\varepsilon}{\partial t}.
\]

The coefficients in the arithmetic-harmonic and Reuss–Voigt bounds are easily computed according to the formulae (4.25), (4.33) and (4.34) respectively. By using the bounds (4.23) and (4.32) we obtain
\[
I_0^- (p^-) = \min_p I_0^- (p) \leq \min_p I_0^0 (p) \leq \min_p I_0^+ (p) = I_0^+ (p^+),
\]

where
\[
I_0^\pm (p) = \int_0^T \int_\Omega \int_Z \frac{1}{2} \nabla_x p \cdot A^\pm \nabla_x p + b^\pm \cdot \nabla_x p + c^\pm d\tau dx dt. \quad (4.37)
\]

The Euler–Lagrange equations corresponding to \( I_0^\pm \) are
\[
-\nabla_x \cdot \left( A^\pm \nabla_x p^\pm + b^\pm \right) = 0 \quad \text{in} \ \Omega \times (0, T), \quad (4.38)
\]

which allows us to compute the bounds solutions \( p^+ \) and \( p^- \).

We now proceed with some numerical examples. All results are presented in dimensionless form, i.e. the following dimensionless variables are introduced:
\[
X_2 = \frac{x_2}{R_1}, \quad H = \frac{h}{h_0}, \quad P = \lambda \frac{h_0^2}{R_1^2} p,
\]

where the inner pad radius \( R_1 \) and \( h_0 \), the film thickness at the trailing edge (of \( h_0 \)), are used as scaling parameters. In this case, the dimensionless global film thickness \( H_0 \) is given by
\[
H_0 (X) = 1 - K X_2 \frac{R_1}{R_2} \sin X_1 - \sin \theta_2
\]
where \( R_1/R_2 = 3/7 \), \( \theta_2 = -\theta_1 = 27.5^\circ \) and \( K = 1/4 \). These values are used in the simulation of a single pad in a thrust pad bearing, which is assumed to consist of a total of 6 pads separated by an angle of 5\(^\circ\) and operating at 1/4 inclination with \( R_1/R_2 = 3/7 \). This leads to the dimensionless Reynolds equation:

\[
\nabla X \cdot \left( H^3 \begin{pmatrix} 1/X_2 & 0 \\ 0 & X_2 \end{pmatrix} \nabla X P \right) = X_2 \frac{\partial \tilde{H}_e}{\partial X_1}.
\]

(4.39)

The advantage of this form is that it does not contain any reference or input parameters. By solving this equation once, for a given \( R_1/R_2 \) ratio, a given constant \( K \) and for a specific surface roughness representation (\( h_U \) and \( h_L \)), we simulate a 6 pad bearing, with pads separated by an angle of 5\(^\circ\), given any choice of the parameters \( \mu, h_00, R_1 \) and \( \omega \) (if the generalized \( Z = (0, 1/\omega) \) is considered). In the subsequent sections we present some illustrative cases of the numerical simulations performed.

### 4.5.1 Sinusoidal roughness

First, roughness due to a one-dimensional transversal sinusoidal perturbation of the global film thickness is considered. More precisely, the roughness is described by

\[
H_U(y, \tau) = -\frac{c}{2} \left( \sin (2\pi y_1) - 1 \right),
\]

\[
H_L(y, \tau) = \frac{c}{2} \left( \sin (2\pi (y_1 - \omega \tau)) - 1 \right),
\]

(4.40)

where \( c = 1/8 \). Figure 4.2 is a plot of the dimensionless deterministic pressure distribution corresponding to the perturbations (4.40) of the global film \( H_0 \). As pointed out before, the bounds coincide for transversal roughness, hence \( P^- = P_0 = P^+ \) for sinusoidal roughness. In general, the computation of the homogenized solution is a fairly complex task and several local problems need to be solved in the process. Even if it is not needed in this example, we use this general approach to compute \( P_0 \). To quantify the numerical accuracy, the following measure of the relative difference between the bounds solutions and the homogenized solution is introduced

\[
\int_{\Omega} |P^\pm (x, \tau_k) - P_0 (x, \tau_k)| \, dx / \int_{\Omega} P^\pm (x, \tau_k) \, dx,
\]

Figure 4.3 represents a plot of this measure against \( \tau_k \) for an eight-step overtaking cycle. Recall that the solutions \( P_0 \) and \( P^\pm \) are functions of \( x, \tau \) and periodic in \( \tau \), whereas the deterministic solution \( P_\varepsilon \) is a function of
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

$x$, $t$ and periodic in $t$. In this simulation a single period of the sinusoids was discretely represented by 64 spatial nodes. According to Figure 4.3 the maximum relative difference did not exceed $0.02\%$, which verifies the feasibility, regarding the numerical accuracy, of our method for computing $P_0$ by the general procedure.

The deterministic load carrying capacity $lcc_\varepsilon(t) = \int_\Omega P_\varepsilon(x,t) \, dx$ and the load carrying capacity $lcc_0(\tau) = \int_\Omega P_0(x,\tau) \, dx$ associated with the homogenized solution are depicted in Figure 4.4. It is clear that for $\varepsilon = 1/2^6$ the difference between the load carrying capacities associated with the deterministic solution $P_\varepsilon$ and the homogenized solution $P_0$ is small. In fact, the difference attains a maximum value which is less than $2.5\%$, half way through the cycle ($\tau = \tau_4$).

In Figure 4.5 three pressure solutions $P_\varepsilon$, $\varepsilon = 1/2^4$, $1/2^5$ and $1/2^6$ - at an intermediate time step and $x_2 = R_m$, where $R_m = (1 + R_2/R_1)/2$ - are shown. For clarity an enlarged portion of the central region is also displayed in 4.6. In Figure 4.7 an enlarged portion of the deterministic pressure solution $P_\varepsilon$, for $\varepsilon = 1/2^6$ and the homogenized solution $P_0$ at three consecutive time steps is shown. In particular, this figure illustrates the unstationary behavior of Reynolds equation. Finally, the Reuss–Voigt bounds are compared to the homogenized solution. Figure 4.8 shows a snapshot of the Reuss–
Variational bounds applied to unstationary hydrodynamic lubrication

Figure 4.3: Sinusoidal roughness: A $\tau$-cycle of the relative difference in load carrying capacity between the arithmetic-harmonic bounds solution and the homogenized solution.

Figure 4.4: Sinusoidal roughness: A cycle of the deterministic load carrying capacity and homogenized dito computed at each time step for three different $\varepsilon$. 
Figure 4.5: Sinusoidal roughness: The pressure solution at $x_2 = R_m$, for three different choices of $\varepsilon$.

Figure 4.6: An enlargement showing the central region. The homogenized solution is indicated by the dashed line.
Variational bounds applied to unstationary hydrodynamic lubrication

Figure 4.7: Sinusoidal roughness: The deterministic pressure solution $P_\varepsilon$, $\varepsilon = 1/2^6$, and the homogenized solution $P_0$ at three consecutive time steps.

Voigt bounds pressure solutions $P_{RV}^{\pm}$ and the homogenized solution $P_0$ at an intermediate time step and Figure 4.9 the measure of the relative differences

$$\int_{\Omega} \left| P_{RV}^{\pm} (x, \tau_k) - P_0 (x, \tau_k) \right| \, dx / \int_{\Omega} P_0 (x, \tau_k) \, dx,$$

for the complete $\tau$-cycle.

Of course, the preciseness of the Reuss–Voigt mean type bounds is not comparable to that of the coinciding arithmetic-harmonic bounds, however, as can be observed from Figure 4.9 the maximum relative difference between the upper and the lower bound is less than 5%.

4.5.2 Bisinusoidal roughness

The second simulation is devoted to bisinusoidal surface roughness, i.e., roughness in both directions and we define $H_U$ and $H_L$$\text{as}$$

$$H_U (y, \tau) = -\frac{c}{2} \left( \cos (2\pi y_1) \cos (2\pi y_2) - 1 \right),$$

$$H_L (y, \tau) = \frac{c}{2} \left( \cos (2\pi (y_1 - \omega \tau)) \cos (2\pi y_2) - 1 \right).$$

(4.41)

As before, $c = 1/8$. The pressure distribution $P_\varepsilon$, $\varepsilon = 1/2^4$, is shown in Figure 4.10 and the convergence of the deterministic solution is investigated in Figure 4.11 where the load carrying capacity is plotted as a function of $\tau$. We note that each period of the deterministic, bisinusoidal, roughness
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 4.8: Sinusoidal roughness: The Reuss–Voigt bounds pressure solutions $P_{RV}^{\pm}$ and the homogenized solution $P_0$ at $\tau = \tau_2$.

Figure 4.9: Sinusoidal roughness: A $\tau$-cycle of the relative difference in load carrying capacity between the Reuss–Voigt bounds solutions and the homogenized solution. The upper bound solution is indicated by filled squares, the lower bound solution is indicated by filled circles.
Variational bounds applied to unstationary hydrodynamic lubrication

Figure 4.10: Bisinusoidal roughness: The pressure distribution $P_{\varepsilon, \varepsilon = 1/2^4}$, at an intermediate time step.

Figure 4.11: Bisinusoidal roughness: $\tau$-cycles of homogenized and deterministic load carrying capacities, for three choices of $\varepsilon$. 
Figure 4.12: Bisinusoidal roughness: A $\tau$-cycle of the relative difference in load carrying capacity between the arithmetic-harmonic bounds solutions. The upper bound solution is indicated by filled squares, lower bound is indicated by filled circles.

representation is resolved with only $8 \times 8$ discrete grid nodes - meaning a total number of $513 \times 513$ grid nodes for $\varepsilon = \frac{1}{2^6}$. For this type of roughness the arithmetic-harmonic bounds pressure solutions are not equal to the homogenized one but Figure 4.12 reveals that the difference between the load carrying capacity corresponding to the upper and the lower bound is small. In the bisinusoidal case, we also observe that the Reuss–Voigt bounds are close - the difference between the upper and the lower bound is almost as small as for the arithmetic-harmonic bounds - with the maximum difference being approximately 2.40% as compared to $(\approx 1.03 + 1.35 =) 2.38\%$ for the arithmetic-harmonic bounds. For details see [11].

4.5.3 A realistic surface roughness representation

Figure 4.13 shows a surface originating from an optical interference measurement. The original grounded surface was coarsened by resampling on a $17 \times 33$ grid - to reduce the discretization errors by enabling successive linear interpolation - and also normalized. For the results presented here each period was interpolated onto $65 \times 129$ discrete nodes. Let the roughness of the measured specimen be represented by the function $f_r$, which is extended
Variational bounds applied to unstationary hydrodynamic lubrication

Figure 4.13: A surface roughness representation originating from a real surface measurement.

by periodicity. As we want to state the governing equation in dimensionless form, let

\[ H_U(y, \tau) = -c(f(y) - 1), \]
\[ H_L(y, \tau) = c(f(y_1 - \omega \tau, y_2) - 1), \]  

(4.42)

where, as before, \( c = 1/8 \). Solving the deterministic problem, corresponding to the surface roughness representation (4.42) with \( f_r \) as in Figure 4.13 yields the pressure distribution, at an intermediate time step \( t_k \) as shown in Figure 4.14.

Figure 4.15 displays the deterministic and the Reuss–Voigt bounds pressure solutions, for the same time step as in Figure 4.14, at \( x_2 = R_m \). Figure 4.16 is a magnification of the central region, where also the arithmetic-harmonic bounds have been added. This figure highlights the preciseness of the bounds as it visualizes the very small differences between the upper and the lower bounds pressure solutions. To be more precise, the relative difference between the upper and lower arithmetic-harmonic bound is only about 0.1%. A small investigation was carried out showing that when the amplitude of the surface roughness increase by a factor of 2 the difference between the upper and lower bound solution increase by a factor slightly less than 4 at all \( \tau \)-steps. Note that this also holds for the Reuss–Voigt bounds as well.
Figure 4.14: Realistic roughness: The pressure distribution, $P_\varepsilon$, $\varepsilon = 1/2^3$, at an intermediate time step.

Figure 4.15: Realistic roughness: The deterministic and the Reuss–Voigt pressure solutions at $x_2 = R_m$, same time step as in Figure 4.14
Figure 4.16: Realistic roughness: An enlargement of the central region of Figure 4.15 with also the arithmetic-harmonic bounds added.

as for the previously considered sinusoidal and bisinusoidal problems. For example, considering the arithmetic-harmonic bounds, $c = 1/4, 1/2$ and $1$ yields relative differences of approximately (and certainly less than) 0.4%, 1.6% and 3.2%, for the realistic surface representation (4.42) with $f_r$ as in Figure 4.13.

4.6 Conclusions

It is well-known that homogenization is very useful in analyzing the effects of surface roughness on the performance of moving machine parts operating in the hydrodynamic lubrication regime. The results presented here confirm this once again. However, the homogenized solution has a severe limitation in that it may be very demanding to compute because of all the local problems that need to be solved. In the recent works [11] and [57], an alternative approach was introduced in connection with problems which are modeled by the stationary Reynolds equation (one rough surface). The main result of this work is that these ideas have been extended to include the situation where both surfaces are rough. In order to do so we first derived a homogenization result for a class of variational problems, using the method of multiple scale expansion. Thereafter, we proved bounds on the corresponding homogenized local “energy density”. We showed that the class of bounds includes
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

e.g. those associated with problems which are modeled by the unstationary Reynolds equation (both surfaces rough) formulated in either Cartesian or polar coordinates. The bounds are optimal, i.e. there exists surface roughness descriptions where the lower and upper bound coincide. The advantage of using bounds instead of the explicit formulas obtained by homogenization is that the complexity of the numerical analysis is significantly reduced because one must solve only two smooth problems, i.e. no local scale has to be considered. To demonstrate that the bounds may be used to accurately estimate the effects of surface roughness in hydrodynamic lubrication, we have presented several numerical examples in connection with problems that are modeled by the unstationary Reynolds equation.

4.7 Appendix (A dual variational principle)

Suppose \( Q \) is a symmetric invertible \( 2 \times 2 \) matrix. As a special case of Young–Fenchel’s inequality the following holds:

\[
\frac{1}{2} u \cdot Qu + \frac{1}{2} v \cdot Q^{-1}v \geq u \cdot v \quad \forall u, v \in \mathbb{R}^2.
\] (4.43)

Using (8.1) pointwise with \( Q = A(y) \), \( u = \xi + \nabla w(y) \) and \( v = \sigma(y) - b(y) \), where \( \sigma \) is any vector field on \( Y \), we obtain

\[
f(\xi) \geq \min_{w \in W_{\text{per}}} \int_Y \sigma \cdot (\xi + \nabla w) - \frac{1}{2} (\sigma - b) \cdot A^{-1}(\sigma - b) \, dy.
\] (4.44)

Let \( V_{\text{sol}} \) consist of the \( Y \)-periodic vector fields that have zero divergence. Then it follows from (8.2) that

\[
f(\xi) \geq \max_{\sigma \in V_{\text{sol}}} \int_Y \sigma \cdot \xi - \frac{1}{2} (\sigma - b) \cdot A^{-1}(\sigma - b) \, dy.
\] (4.45)

It turns out that the inequality (4.45) is actually an equality. Indeed, the solution \( w_\xi \) of

\[
\min_{w \in W_{\text{per}}} \int_Y \frac{1}{2} (\xi + \nabla w) \cdot A(\xi + \nabla w) + b \cdot (\xi + \nabla w) \, dy
\]
is also the unique \( w_\xi \) in \( W_{\text{per}} \) such that

\[
\int_Y (A(\xi + \nabla w_\xi) + b) \cdot \nabla \phi \, dy = 0 \quad \forall \phi \in W_{\text{per}}.
\] (4.46)
Let \( \sigma^* = A(\xi + \nabla w\xi) + b \). From (4.46) it is clear that \( \sigma^* \) belongs to \( V_{\text{sol}} \).

By choosing \( \sigma = \sigma^* \) in (4.45) we obtain equality in (4.45), i.e.

\[
 f(\xi) = \max_{\sigma \in V_{\text{sol}}} \int_Y \sigma \cdot \xi - \frac{1}{2}(\sigma - b) \cdot A^{-1}(\sigma - b) \, dy.
\]

From the (orthogonal) decomposition \( V_{\text{sol}} = \mathbb{R}^2 \oplus S \), where \( S \) denotes the vector fields in \( V_{\text{sol}} \) with mean value zero we have

\[
 f(\xi) = \max_{\sigma \in S} \int_Y \sigma \cdot \xi - \frac{1}{2}(\sigma - b) \cdot A^{-1}(\sigma - b) \, dy
 = \max_{\eta \in \mathbb{R}^2} \int_Y \eta \cdot \xi - \frac{1}{2}(\sigma + \eta - b) \cdot A^{-1}(\sigma + \eta - b) \, dy
 = \max_{\eta \in \mathbb{R}^2} \left\{ \eta \cdot \xi - \min_{\sigma \in S} \int_Y \frac{1}{2}(\sigma + \eta - b) \cdot A^{-1}(\sigma + \eta - b) \, dy \right\},
\]

which shows that \( f \) is the Legendre transformation (w.r.t. the variable \( \eta \)) of the function

\[
 F(\eta) = \min_{\sigma \in S} \int_Y \frac{1}{2}(\sigma + \eta - b) \cdot A^{-1}(\sigma + \eta - b) \, dy.
\]

In other words \( f = F^* \). Since \( F \) is convex and lower semicontinuous, \( (\cdot)^{**} \) acts as the identity. Thus \( f^* = F^{**} = F \).

This proves the dual variational principle

\[
 f^*(\eta) = \min_{\sigma \in S} \int_Y \frac{1}{2}(\sigma + \eta - b) \cdot A^{-1}(\sigma + \eta - b) \, dy. \tag{4.47}
\]
HOMOGENIZATION OF REYNOLDS EQUATIONS AND OF SOME
PARABOLIC PROBLEMS VIA ROTE’S METHOD
Chapter 5

Reiterated homogenization applied in hydrodynamic lubrication

5.1 Introduction

Throughout the years, the general theory of homogenization has been successfully applied to different problems connected to hydrodynamic lubrication, see e.g. [6], [9], [16], [17], [24], [44], and [45]. In these works it was shown that the rapid oscillations (in the coefficients of the Reynolds type equation under consideration) induced by the surface roughness, could efficiently be averaged by the homogenization method employed. In these previous results, it is assumed that the lubrication problem exhibits two separable scales, i.e. a global scale describing the geometric shape of the application and a local scale describing the surface roughness.

In the present work it is assumed that the problem of interest, exhibits three separable scales, i.e. one global scale describing geometry, one oscillating local scale describing the surface texture and a faster oscillating local scale describing the surface roughness. Homogenization of problems with two or more oscillating scales are referred to as reiterated homogenization, see e.g. [4], [19], and [55]. In this paper, a generalized form of the Reynolds problem is considered, governing incompressible and Newtonian flow, with the advantage to unify both the Cartesian and the cylindrical coordinate formulations. In particular, the aim is to obtain a general homogenized problem that corresponds to a class of problems modelled by (5.1). One technique
within the homogenization theory is the formal method of multiple scale expansion, see e.g. [19] and [69]. To accomplish this aim the formal method of multiple scale expansion is employed to obtain a homogenized problem (5.8) for (5.1). For other problems connected to the incompressible Reynolds that have been studied by multiple scale expansion see [9], [16], and [24].

By means of numerical analysis, the convergence, of the direct numerical solution toward the homogenized counterpart, in terms of load carrying capacity and hydrodynamically induced friction is quantified. The results show that the combined effect due to texture and roughness on a modelled bearing can be effectively analyzed through reiterated homogenization. More specifically, the discrepancies, between the proposed method and the direct numerical approach, in terms of predicted load carrying capacity and friction force are tolerably small; $O(1\%)$ for textures as well as roughness of wavelengths likely to be found in a real application. That is, wavelengths within the ranges $1/100 - 1/10$ of the length bearing for the texture and $1/10000 - 1/100$ for the roughness.

5.2 The homogenization procedure

In this section a class of equations that includes the Reynolds equation governing incompressible Newtonian flow is considered. It can be seen that the generalized form (5.1), makes it possible to study the Reynolds problem in its Cartesian, and cylindrical coordinate forms. (See Section 5.4)

Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$, $Y = (0, 1)^2$ and $Z = (0, 1)^2$. Introduce the auxiliary matrix $A = (a_{ij})$, where $a_{ij} = a_{ij}(x, y, z)$, and $i = 1, 2$, and $j = 1, 2$ are smooth functions that are $Y$-periodic in $y$ and $Z$-periodic in $z$. It is also assumed that a constant $\alpha > 0$ exists such that

$$\sum_{i,j=1}^{2} a_{ij}(x, y, z)\xi_i\xi_j \geq \alpha |\xi|^2$$

for every $\xi \in \mathbb{R}^2$.

Moreover, we introduce the auxiliary vector $b = (b_i)$, where $b_i = b_i(x, y, z)$ and $i = 1, 2$, are smooth functions that are $Y$-periodic in $y$ and $Z$-periodic in $z$. Let $\varepsilon > 0$ and define the matrix $A_{\varepsilon}$ and the vector $b_{\varepsilon}$ as

$$A_{\varepsilon}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} = A(x, x/\varepsilon, x/\varepsilon^2),$$

$$b_{\varepsilon}(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix} = b(x, x/\varepsilon, x/\varepsilon^2).$$
Consider the following boundary value problem

\[ \nabla_x \cdot (A_\varepsilon(x) \nabla_x p_\varepsilon(x)) = \nabla_x \cdot b_\varepsilon(x) \quad \text{in } \Omega, \]  
\[ p_\varepsilon(x) = 0 \quad \text{on } \partial \Omega. \]  
(5.1)

For small values of the parameter \( \varepsilon \) the coefficients in (5.1) are rapidly oscillating. This suggests some type of asymptotic analysis. We will see that \( p_\varepsilon \to p_0 \) as \( \varepsilon \to 0 \) and that \( p_0 \) can be found by solving a so called homogenized equation (5.8), which does not contain any rapid oscillations. This means that \( p_0 \) may be used as an approximation of the solution \( p_\varepsilon \) for small values of \( \varepsilon \).

We will use the method of multiple scale expansion developed in the homogenization theory to derive a homogenization result connected to (5.1). For general information concerning this method in connection to homogenization, see e.g. [19] and [69]. The homogenization of Reynolds type equations involving only one local scale have been studied by multiple scale expansion in [6], [10] [16], and [45]. Assume that \( p_\varepsilon \) is of the form

\[ p_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i p_i(x, x/\varepsilon, x/\varepsilon^2), \]  
(5.2)

where \( p_i = p_i(x, y, z) \) is \( Y \)-periodic in \( y \) and \( Z \)-periodic in \( z \). The main idea is to insert the expansion (5.2) into (5.1), and then collect terms of the same order of \( \varepsilon \) and analyze the system of equations obtained. A comprehensive analysis can be found in Appendix 5.7. The main result is that the leading term \( p_0 \) in the expansion (5.2) is of the form \( p_0 = p_0(x) \) and is found by the following homogenization algorithm:

**Step 1:** Solve the local problems (on the \( z \)-scale)

\[ \nabla_z \cdot (A (\nabla_z u_i + e_i)) = 0 \quad \text{in } Z, \ (i = 1, 2), \]  
(5.3)

\[ \nabla_z \cdot (A \nabla_z u_0 - b) = 0 \quad \text{in } Z. \]  
(5.4)

Here \( u_i = u_i(x, y, z), \ i = 0, 1, 2, \) is \( Y \)-periodic in \( y \), \( Z \)-periodic in \( z \) and \{\( e_1, e_2 \)\} is the canonical basis in \( \mathbb{R}^2 \). Use these local solutions to define the matrix

\[ A = A(x, y, z) = \begin{pmatrix} 1 + \frac{\partial u_1}{\partial z_1} & \frac{\partial u_2}{\partial z_1} \\ \frac{\partial u_1}{\partial z_2} & 1 + \frac{\partial u_2}{\partial z_2} \end{pmatrix} \].

**Step 2:** Solve the local problems (on the \( y \)-scale)

\[ \nabla_y \cdot (A A^z (\nabla_y v_i + e_i)) = 0 \quad \text{in } Y, \ (i = 1, 2), \]  
(5.5)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

\[ \nabla_y \cdot \left( \overline{A}_z \overline{A}^z \nabla_y v_0 - \left( b - \overline{A} \overline{w}_0 \right) \right) = 0 \quad \text{in } Y. \quad (5.6) \]

Here \( v_i = v_i(x, y), \ i = 0, 1, 2, \) is \( Y \)-periodic in \( y \) and \( \overline{A}_z \overline{A}^z \) is the average with respect to \( Z \). Use these local solutions to define the matrix

\[ B = B(x, y) = \begin{pmatrix} 1 + \frac{\partial v_1}{\partial y_1} & \frac{\partial v_2}{\partial y_1} \\ \frac{\partial v_1}{\partial y_2} & 1 + \frac{\partial v_2}{\partial y_2} \end{pmatrix}. \]

**Step 3:** Compute the homogenized matrix \( A_0 \) and the homogenized vector \( b_0 \) by the following formulas

\[ A_0(x) = \overline{A} \overline{B}^{xy} \quad \text{and} \quad b_0(x) = b - \overline{A} \overline{w}_0 - \overline{A} \overline{w}_0 \overline{y}_0. \quad (5.7) \]

**Step 4:** Find \( p_0 \) by solving the so called homogenized problem

\[ \nabla_x \cdot (A_0(x) \nabla_x p_0(x)) = \nabla_x \cdot b_0(x) \quad \text{in } \Omega, \quad (5.8) \]
\[ p_0(x) = 0 \quad \text{on } \partial \Omega. \]

The main advantage of the above algorithm is that the scales are treated separately, i.e. first one "averages" with respect to the \( z \)-scale, then with respect to the \( y \)-scale and finally one solves the homogenized equation. We note that the homogenized equation does not contain any oscillating coefficients, nevertheless, it takes into account the effects of the local scales, see (5.7). The fact that the scales can be separated in this way, significantly simplifies the numerical analysis of the problem.

**5.3 An additional result**

In this section, the convergence of \( \nabla p_\varepsilon \) is investigated. The functions \( p_i, \ i = 0, 1, 2, \) in the expansion is of the form,

\[ p_0 = p_0(x), \quad p_1 = p_1(x, y), \quad p_2 = p_2(x, y, z), \]

see Appendix 5.7. When inserted into (5.2) we see that

\[ \nabla_x p_\varepsilon (x) = \nabla_x p_0(x) + \nabla_y p_1(x, y) + \nabla_z p_2(x, y, z) + \varepsilon [\ldots], \]

which means that

\[ \nabla_x p_\varepsilon (x) \approx \nabla_x p_0(x) + \nabla_y p_1(x, y) + \nabla_z p_2(x, y, z) \]
for small values of $\varepsilon$. According to the analysis in Appendix 5.7, $p_1$ and $p_2$ can be expressed in terms of the solutions $u_i$ and $v_i$ of the local problems (5.3), (5.4), (5.5) and (5.6), respectively. Making use of (5.49) and (5.53) in addition to (5.45) and (5.58) yields,

$$\nabla_x p_\varepsilon (x) \approx \nabla_z u_0 (x, y, z) + \mathcal{A} (x, y, z) \nabla_y v_0 (x, y) + \mathcal{A} (x, y, z) \mathcal{B} (x, y) \nabla_x p_0 (x),$$

(5.9)

after some straightforward calculations. According to [4], [55], and [65], the following convergence holds

$$\int_\Omega \nabla_x p_\varepsilon (x) \varphi(x/\varepsilon, x/\varepsilon^2) \, dx \longrightarrow$$

$$\int_\Omega \int_Y \int_Z [\nabla_z u_0 + \mathcal{A} \nabla_y v_0 + \mathcal{A} \mathcal{B} \nabla_x p_0] \varphi(x, y, z) \, dz \, dy \, dx,$$

(5.10)

for any smooth function $\varphi$ that is $Y$-periodic in $y$ and $Z$-periodic in $z$.

### 5.4 Application to hydrodynamic lubrication

In this section we study how our general reiterated homogenization result can be applied to analyze the effects of texture and surface roughness in hydrodynamic lubrication governed by the Reynolds equation. For this purpose we introduce an auxiliary function that may be used to represent the lubricant film thickness

$$h(x, y, z) = h_0(x) + h_T(x, y) + h_R(x, y, z),$$

(5.11)

where

- $h_0(x)$ describes the geometry of the bearing,
- $h_T(x, y)$ is a $Y$-periodic function in $y$, representing surface texture,
- $h_R(x, y, z)$ is a $Y$-periodic function in $y$ and a $Z$-periodic in $z$, representing the roughness contribution.

Note that this formulation admits studying problem where the texture and the roughness changes with position at the tribological interface. For example, this enables studying the effects of a texture only on a part of the surface, which in turn may exhibit different surface roughness patterns at different parts of the texture itself. However, here the numerical examples are restricted to consider the case where the texture and the roughness
representation does not change with the position, i.e., \( h_T = h_T(y) \) and \( h_R = h_R(z) \). By making use of the auxiliary function \( h \), it is possible to model the deterministic film thickness \( h_\varepsilon \) as

\[
h_\varepsilon(x) = h(x, x/\varepsilon, x/\varepsilon^2) = h_0(x) + h_T(x/\varepsilon) + h_R(x/\varepsilon^2), \tag{5.12}
\]

where \( \varepsilon \) is a parameter that describes the texture and roughness wavelength.

Now, by choosing

\[
A_\varepsilon(x) = \begin{pmatrix}
 h_\varepsilon^3(x) & 0 \\
 0 & h_\varepsilon^3(x)
\end{pmatrix}, \tag{5.13a}
\]

\[
b_\varepsilon(x) = 6\mu U h_\varepsilon(x) e_1, \tag{5.13b}
\]

in (5.1), we obtain the Reynolds equation describing incompressible Newtonian flow in Cartesian coordinates, i.e.,

\[
\nabla_x \cdot \left( \begin{pmatrix}
 h_\varepsilon^3(x)/x_2 & 0 \\
 0 & x_2 h_\varepsilon^3(x)
\end{pmatrix} \nabla_x p_\varepsilon \right) = 6\mu U \nabla_x \cdot (h_\varepsilon e_1) \quad \text{in } \Omega, \tag{5.14}
\]

\[
p_\varepsilon(x) = 0 \quad \text{on } \partial \Omega.
\]

Here, \( p_\varepsilon \) is the hydrodynamically induced pressure distribution, \( \mu \) is the (constant) viscosity of the Newtonian lubricant and \( U \) is the linear speed of the moving smooth surface.

We also observe that by choosing

\[
A_\varepsilon(x) = \begin{pmatrix}
 h_\varepsilon^3(x)/x_2 & 0 \\
 0 & x_2 h_\varepsilon^3(x)
\end{pmatrix}, \tag{5.15a}
\]

\[
b_\varepsilon(x) = 6\mu \omega x_2 h_\varepsilon(x) e_1, \tag{5.15b}
\]

in (5.1), where \( \omega \) is the angular speed of the smooth rotating surface and \((x_1, x_2)\) are the angular and the radial coordinates we obtain the Reynolds equation describing incompressible Newtonian flow in cylindrical coordinates

\[
\nabla_x \cdot \left( \begin{pmatrix}
 h_\varepsilon^3(x)/x_2 & 0 \\
 0 & x_2 h_\varepsilon^3(x)
\end{pmatrix} \nabla_x p_\varepsilon \right) = 6\mu \omega x_2 \nabla_x \cdot (h_\varepsilon e_1) \quad \text{in } \Omega, \tag{5.16}
\]

\[
p_\varepsilon(x) = 0 \quad \text{on } \partial \Omega.
\]

It should be noted that our homogenization result, which is that \( p_\varepsilon \to p_0 \) as \( \varepsilon \to 0 \), do not require any restrictions on the geometry, neither on the texture (\( y \)-scale) nor on the roughness (\( z \)-scale). The only limitation is that \( \varepsilon \) should be sufficiently small in order to approximate the hydrodynamic
Reiterated homogenization applied in hydrodynamic lubrication

pressure $p_e$ with $p_0$. As will be seen this is actually no limitation since $\varepsilon$ is very small in realistic examples.

From the homogenization result, convergence of load carrying capacity $I_\varepsilon$ automatically follows, i.e.

$$I_\varepsilon = \int_\Omega p_\varepsilon (x) \ dx \longrightarrow \int_\Omega p_0 (x) \ dx = I_0.$$  

Moreover, we studied the convergence of hydrodynamically induced friction force, $F_\varepsilon$, and frictional torque, $T_\varepsilon$, are connected to the derivative $\partial p_e / \partial x_1$, and by making use of (8.1) we obtain the following expressions

$$F_\varepsilon = \int_\Omega \left( \frac{\mu U}{h_\varepsilon(x)} + \frac{h_\varepsilon(x) \partial p_e}{2 \partial x_1} \right) dx \longrightarrow \tag{5.17}$$

$$F_0 = \int_\Omega \int_Y \int_Z \left( \frac{\mu U}{h(x,y,z)} + \frac{h(x,y,z) \partial p_0}{2 \partial x_1} \right) dzdydx$$

$$+ \int_\Omega \int_Y \int_Z \frac{h(x,y,z)}{2} \left[ \frac{\partial u_0}{\partial z_1} + \frac{\partial u_1 \partial p_0}{\partial z_1 \partial x_1} + \frac{\partial u_2 \partial p_0}{\partial z_2 \partial x_2} \right] dzdydx$$

$$+ \int_\Omega \int_Y \int_Z \frac{h(x,y,z)}{2} \left[ \frac{\partial v_0}{\partial y_1} + \frac{\partial v_1 \partial p_0}{\partial y_1 \partial x_1} + \frac{\partial v_2 \partial p_0}{\partial y_2 \partial x_2} \right] dzdydx$$

$$+ \int_\Omega \int_Y \int_Z \frac{h(x,y,z)}{2} \left[ \left( \frac{\partial u_1}{\partial z_1} \right) \cdot \left( \frac{\partial v_0}{\partial y_1} \right) + \left( \frac{\partial v_1}{\partial y_1} \right) \cdot \left( \frac{\partial v_2}{\partial y_2} \right) \right] dzdydx$$

for friction force and

$$T_\varepsilon = \int_\Omega x_2 \left( \frac{\mu \varepsilon x_2}{h_\varepsilon(x)} + \frac{h_\varepsilon(x) \partial p_e}{2 \partial x_2} \right) \ dx_2 \ dx_1 dx_2 \longrightarrow \tag{5.18}$$

$$T_0 = \int_\Omega \left\{ \int_Y \left[ \int_Z \frac{h(x,y,z)}{2} \left[ \frac{\partial u_0}{\partial z_1} + \frac{\partial u_1 \partial p_0}{\partial z_1 \partial x_1} + \frac{\partial u_2 \partial p_0}{\partial z_2 \partial x_2} \right] dzdy \right] x_2 \ dx_1 dx_2 \right\}$$

$$+ \int_\Omega \left\{ \int_Y \left[ \int_Z \frac{h(x,y,z)}{2} \left[ \frac{\partial v_0}{\partial y_1} + \frac{\partial v_1 \partial p_0}{\partial y_1 \partial x_1} + \frac{\partial v_2 \partial p_0}{\partial y_2 \partial x_2} \right] dzdy \right] x_2 \ dx_1 dx_2 \right\}$$

$$+ \int_\Omega \left\{ \int_Y \left[ \int_Z \frac{h(x,y,z)}{2} \left[ \left( \frac{\partial u_1}{\partial z_1} \right) \cdot \left( \frac{\partial v_0}{\partial y_1} \right) + \left( \frac{\partial v_1}{\partial y_1} \right) \cdot \left( \frac{\partial v_2}{\partial y_2} \right) \right] dzdy \right] \right\} x_2 \ dx_1 dx_2$$
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

for frictional torque. To clarify, from the equations above, the resulting homogenized quantity is made up of friction force/torque due to the smooth (averaged) film thickness plus a corrector term identified by three separate contributions, i.e. due to roughness or texture acting alone or roughness and texture acting together.

In the following, we conduct numerical investigations into the convergence associated with load carrying capacity and the hydrodynamically induced friction force by employing a second order finite difference scheme. The results of these investigations, justify the applicability of the homogenization process presented in this paper. Subsequently, we study the effects of periodic texture and surface roughness by considering a thrust pad bearing problem. It is observed that for one-dimensional texture and roughness representation only very small differences exist between the homogenized numerical solution (HNS) and the direct numerical solution (DNS). We point out that it is only possible to find the DNS in the case of transversal and longitudinal (i.e. one-dimensional) texture and roughness due to the enormous number of discretization points that is required in the general case. From the general analysis, it is clear that it is always possible to obtain an approximate solution $p_0$ of $p_\varepsilon$, with very high accuracy by solving the homogenized equation. From an application point of view this means that for arbitrary (i.e. also two-dimensional) yet physically relevant, texture and roughness, a highly accurate approximation $p_0$ of the pressure solution $p_\varepsilon$, can be obtained by solving the homogenized equation. This is one of the benefits with our method.

5.4.1 A numerical investigation of convergence

Computationally, it is extremely demanding to retrieve the DNS for short wavelength roughness (and texture). Therefore, to assess and quantify convergence, the one-dimensional problem was first revisited. This elementary problem constitutes an excellent benchmark for the implemented numerics, since it is possible to obtain closed form expressions for the coefficients in the homogenized equation. Specifically, we obtain a one-dimensional representation of the Reynolds equation, for incompressible and Newtonian flow, in Cartesian coordinates by considering (5.14), i.e.

\[
\frac{d}{dx} \left( h_\varepsilon^3(x) \frac{dp_\varepsilon}{dx}(x) \right) = 6\mu U \frac{dh_\varepsilon}{dx}(x) \quad \text{in} \quad 0 \leq x \leq L, \quad (5.19)
\]

\[
p_\varepsilon(0) = p_\varepsilon(L) = 0.
\]
Reiterated homogenization applied in hydrodynamic lubrication

Here, \( L \) is the length of the stationary surface exhibiting texture and roughness.

Through (5.11), the film thickness function of the modeled linear slider bearing, is described with

\[
h_0 (x) = h_{\text{min}} + \frac{h_{\text{min}}}{4} \left( 1 - \frac{x}{L} \right),
\]

\[
h_T (s) = 2 h_R (s) = \frac{h_{\text{min}}}{4} \left( \frac{1}{2} (1 - \cos (2 \pi s)) \right),
\]

where \( h_{\text{min}} \) denotes the fixed minimum film thickness of the corresponding smooth problem, i.e. the problem with a smooth stationary surface as well as a smooth moving surface. To generalize the results, the dimensionless variables \( X = x/L, H = h/h_{\text{min}} \) and \( P_\varepsilon = P_\varepsilon / (6 \mu UL/h_{\text{min}}^2) \) were introduced to obtain a dimensionless Reynolds problem,

\[
\frac{d}{dX} \left( H^3 \left( X \right) \frac{dP_\varepsilon}{dX} \left( X \right) \right) = \frac{dH}{dX} \left( X \right), \quad \text{in} \quad 0 \leq X \leq 1, \quad (5.20)
\]

\[
P_\varepsilon (0) = P_\varepsilon (1) = 0.
\]

We also present the dimensionless representation of the auxiliary film thickness function, in terms of these dimensionless variables, i.e.

\[
H (X, y, z) = 1 + \frac{1}{4} (1 - X) + \frac{1}{4} \left( \frac{1}{2} (1 - \cos (2 \pi y)) \right) + \frac{1}{8} \left( \frac{1}{2} (1 - \cos (2 \pi z)) \right).
\]

The homogenized problem corresponding to (5.20) reads as

\[
\frac{d}{dX} \left( \frac{1}{H^{-3} (X, y, z)} \frac{dP_0}{dX} \right) = \frac{d}{dX} \left( \frac{H^{-2} (X, y, z)}{H^{-3} (X, y, z)} \right), \quad \text{in} \quad 0 \leq X \leq 1,
\]

\[
P_0 (0) = P_0 (1) = 0.
\]

Figure 5.1 illustrates the convergence of load carrying capacity \( I_\varepsilon \) toward \( I_0 \) with decreasing \( \varepsilon \). In fact, it is the measure

\[
|I_\varepsilon - I_0| / I_0,
\]

(5.22)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 5.1: Convergence of load carrying capacity $I_\varepsilon$ towards $I_0$ with decreasing $\varepsilon$.

If $P_\varepsilon, P_0 \geq 0$ that is considered as being a function of $\varepsilon$ in the figure. When computing the DNS, $2^5$ discrete nodes were used to represent a single wavelength of the texture, e.g., for $\varepsilon = 2^{-7}$, a total number of $(1/2^{-7})^2 2^5 = 2^{19}$ grid nodes were used. As deduced from the figure, the rate of convergence is very close to linear, with the goodness of fit equaling 0.99. To further elaborate on the convergence of $P_\varepsilon$ towards $P_0$, we illustrate a set of DNS ($P_\varepsilon$) and the HNS ($P_0$) in Figure 5.2.

To facilitate the derivation of the specific version of (5.17) that corresponds to the one-dimensional dimensionless form of (5.1), we first choose $A_\varepsilon = H_\varepsilon^3(x_1)$ and $b_\varepsilon = H_\varepsilon(x_1)$. Then, owing to (5.17), we have the following convergence, in terms of dimensionless friction force $F_\varepsilon = \mathcal{F}_\varepsilon / (\mu UL/h_{\min})$,

$$F_\varepsilon = \int_0^1 \frac{1}{H_\varepsilon(X)} + 3H_\varepsilon(X) \frac{dP_\varepsilon}{dX}(X) \, dX \longrightarrow$$

(5.23)

$$F_0 = \int_0^1 \int_0^1 \int_0^1 \left( \frac{1}{H} + 3H \frac{dp_0}{dX} \right) \, dz \, dy \, dX$$

$$+ \int_0^1 \int_0^1 \int_0^1 3H \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} \frac{dp_0}{dX} \right) \, dz \, dy \, dX$$
Reiterated homogenization applied in hydrodynamic lubrication

Figure 5.2: A set of DNS ($P_\varepsilon$) and the HNS ($P_0$).

\[
+F_0 = \int_0^1 \int_0^1 \int_0^1 \frac{1}{H(X,y,z)} + 3H(X,y,z) \left[ H^{-2}(X,y,z) - \frac{H^{-2}(X,y,z)H^{-3}(X,y,z)}{H^{-3}(X,y,z)} \right] dP_0 dX.
\]

For the one-dimensional problem we can solve the cell problems (5.3), (5.4), (5.5) and (5.6) explicitly. Inserting the solutions

\[
\frac{\partial u_0}{\partial z} = H^{-2}(X,y,z) - \frac{H^{-2}(X,y,z)}{H^{-3}(X,y,z)} H^{-3}(X,y,z), \quad \frac{\partial u_1}{\partial y} = -1 + \frac{H^{-3}(X,y,z)}{H^{-3}(X,y,z)}
\]

into $F_0$ we find that

\[
F_0 = \int_0^1 \int_0^1 \int_0^1 \frac{1}{H(X,y,z)} + 3H(X,y,z) \left[ H^{-2}(X,y,z) - \frac{H^{-2}(X,y,z)H^{-3}(X,y,z)}{H^{-3}(X,y,z)} \right] dP_0 dX.
\]
Figure 5.3: Convergence of friction force $F_\varepsilon$ toward $F_0$ with decreasing $\varepsilon$

Figure 5.3 displays the convergence of $F_\varepsilon$. Actually, Figure 5.3 visualizes the variation with $\varepsilon$ in the expression

$$\frac{|F_\varepsilon - F_0|}{F_0}$$

In comparison to the (almost) linear convergence for $I_\varepsilon$, the rate of convergence of $F_\varepsilon$ is lower than linear, according to the figure. However, the results presented above, particularly those shown in Figures 5.1 and 5.3, clearly serve as justification of the applicability of the proposed reiterated homogenization result. More specifically, the discrepancies in terms of predicted load carrying capacity and friction force are tolerably small; $O(1\%)$ for textures as well as roughness of wavelengths likely to be found in a real application. That is, wavelengths within the ranges $1/100 - 1/10$ for the texture and $1/10000 - 1/100$ for the roughness.

### 5.4.2 Application to a thrust pad bearing problem

The effects of periodic texture and surface roughness are here exemplified by considering a thrust pad bearing problem. The flow is assumed to be modelled through the cylindrical coordinate formulation of the Reynolds problem, i.e. (5.16). A point $x$ in the bearing is identified by its cylindrical coordinates...
Reiterated homogenization applied in hydrodynamic lubrication

\[ x = (x_1, x_2) \in \Omega = [-\theta_0/2, \theta_0/2] \times [R, 2R] \] (with \( x_1 \) denoting the angular and \( x_2 \) the radial coordinate). In this case convergence of frictional torque, \( T_\varepsilon \), is given by (5.18).

There are two ways of approaching the lubrication problem. In the preceding section, we regarded the separation \( h_{\text{min}} \) between the surfaces on the global scale as an input parameter and retrieved the solution in terms of the single dependent parameter, i.e. dimensionless hydrodynamic pressure \( P_\varepsilon \), by solving the Reynolds equation (5.20). Observe that due to the specific dimensionless formulation chosen we obtained the solution \( p_\varepsilon \) for arbitrary \( h_{\text{min}} > 0 \).

In approaching the present thrust pad bearing problem, we employ a force-balance equation

\[ W - \int_\Omega p_\varepsilon (x) x_2 dx_2 dx_1 = 0, \] (5.26)

where the applied load \( W \) appears as an input parameter. We then solve the Reynolds equation (5.16) and the force-balance equation (5.26) to retrieve the solution in terms of the two dependent parameters, namely the separation between the surfaces on the global scale \( h_{00} \) and the hydrodynamic pressure \( p_\varepsilon \). Again, we use (5.11) to represent the film thickness and define

\[ h_0 (x) = h_{00} - \frac{x_2 (\sin(x_1) - \sin(\theta_0))}{R \sin(\theta_0)} \theta_0 R \tan \alpha, \] (5.27)

to model a single bearing segment. Note that \( h_{00} \) exactly defines the height of the parallel gap between the trailing edge and the rotating shaft surface and that its value depends on the applied load \( W \). (For the smooth problem \((h_T = h_R \equiv 0)\), \( h_{00} \) represents the minimum film thickness.) In (5.27), \( R \) denotes the inner radius, \( \theta_0 \) defines the size of the pad in radians and \( \alpha \) controls the pad inclination. See Figure 5.4 for a schematic description of a bearing segment within the bearing.

By considering the homogenized correspondence (5.8) to (5.16), we can proceed to examine the effects of periodic texture and surface roughness. The case of transversal sinusoidal surface texture as well as surface roughness is addressed first,

\[ h_\varepsilon (x) = h_{00} - \frac{x_2 (\sin(x_1) - \sin(\theta_0))}{R \sin(\theta_0)} \theta_0 R \tan \alpha + h_\varepsilon^T (x) + h_\varepsilon^R (x), \] (5.28)

where

\[ h_\varepsilon^T (x) := h_T (x/\varepsilon) \quad \text{and} \quad h_\varepsilon^R (x) := h_R (x/\varepsilon^2). \]
Explicitly, the auxiliary functions are

\[ h_T(y) = \frac{a_T}{2} (1 - \cos (2\pi y_1)) \quad (5.29) \]

and

\[ h_R(z) = \frac{a_R}{2} (1 - \cos (2\pi z_1)) . \quad (5.30) \]

The separation \( h_{00}^\varepsilon \) is regarded as a parameter that is parameterized in \( \varepsilon \) and dependent on \( W \), and we can therefore solve the Reynolds equation (5.16) and the force-balance criterion (5.26) for \( h_{00}^\varepsilon \) and \( p_\varepsilon \) (or \( h_{00}^0 \) and \( p_0 \) for the corresponding homogenized system of equations). The input parameters chosen for this specific problem are found in Table 5.1.

Table 5.1: Input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_0 )</td>
<td>Pad size (in degrees)</td>
<td>25</td>
<td>deg</td>
</tr>
<tr>
<td>R</td>
<td>Pad inner radius</td>
<td>10 \cdot 10^{-3}</td>
<td>m</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Fluid viscosity</td>
<td>0.3</td>
<td>Pa s</td>
</tr>
<tr>
<td>( \omega )</td>
<td>Smooth surf. angular speed</td>
<td>2.5</td>
<td>rad/s</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Pad inclination</td>
<td>1.6 \cdot 10^{-4}</td>
<td>rad</td>
</tr>
<tr>
<td>W</td>
<td>Applied load</td>
<td>10</td>
<td>N</td>
</tr>
<tr>
<td>a</td>
<td>Roughness amp. scaling param.</td>
<td>0.5 \cdot 10^{-6}</td>
<td>m</td>
</tr>
</tbody>
</table>

Figure 5.4: A schematic descriptions of a single pad.
Reiterated homogenization applied in hydrodynamic lubrication

Table 5.2: Normalized homogenized property \( h_{00}^0/h_{00}^s \), transversal sinusoidal texture and roughness.

<table>
<thead>
<tr>
<th>( a_T )</th>
<th>( a_R )</th>
<th>0</th>
<th>a</th>
<th>2a</th>
<th>4a</th>
<th>8a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1.0000</td>
<td>0.9639</td>
<td>0.9298</td>
<td>0.8677</td>
<td>0.7636</td>
</tr>
<tr>
<td>a</td>
<td></td>
<td>0.9639</td>
<td>0.9278</td>
<td>0.8937</td>
<td>0.8317</td>
<td>0.7277</td>
</tr>
<tr>
<td>2a</td>
<td></td>
<td>0.9298</td>
<td>0.8937</td>
<td>0.8597</td>
<td>0.7978</td>
<td>0.6942</td>
</tr>
<tr>
<td>4a</td>
<td></td>
<td>0.8677</td>
<td>0.8317</td>
<td>0.7979</td>
<td>0.7363</td>
<td>0.6341</td>
</tr>
<tr>
<td>8a</td>
<td></td>
<td>0.7636</td>
<td>0.7277</td>
<td>0.6942</td>
<td>0.6341</td>
<td>0.5364</td>
</tr>
</tbody>
</table>

To resolve the direct numerical solution (DNS) properly, each roughness wavelength is resolved with \( 2^5 \) discrete nodes. For the results presented here, this means a total number of uniformly distributed nodes of \( 2^6 (2^4)^2 = 2^{13} \) in the \( x_1 \)-direction for \( \varepsilon = 2^{-4} \), while \( 2^6 \) nodes were considered sufficient for the discretization in the \( x_2 \)-direction. The coefficients in the homogenized matrix and vector, both given in (5.7), were obtained by solving the (one-dimensional) periodic \( Y \) or \( Z \) cell problems with \( 2^6 \) nodes in the \( y_1 \)- and the \( z_1 \)-directions.

Table 5.2 displays normalized homogenized separation \( h_{00}^0/h_{00}^s \), where \( h_{00}^s = 6.72 \cdot 10^{-6} \) m denotes the minimum film thickness for the correspondingly smooth problem. In the table, texture amplitude \( a_T \) increases vertically downwards, while roughness amplitude \( a_R \) increases horizontally to the right, as indicated by \( a_T \backslash a_R \).

For \( \varepsilon = 2^{-4} \) the maximum relative difference between \( h_{00}^0 \) and \( h_{00}^s \) was found to be less than 0.01. More specifically, for \( \varepsilon = 2^{-4} \), corresponding to a texture wavelength \( \theta_0 2R/2^4 \approx 0.5 \cdot 10^{-3} \) m measured at the outer radius \( (x_2 = 2R) \) and for \( (a_T, a_R) = (8a, 8a) \), we obtained \( |h_{00}^s - h_{00}^0|/h_{00}^0 = 0.0097 \). The fact that the maximum difference occurs for \( (a_T, a_R) = (8a, 8a) \) is to be expected, as an increase in texture amplitude or roughness amplitude also increases the discretization errors. Since, in theory, it makes sense to distinguish between roughness and texture only when both of them are present, the first row and column in Table 5.2 could be used as a benchmark of the numerical routine employed. Although the figures in the table seem to indicate that the rigid body separation is symmetrical with respect to texture and roughness amplitude, no theoretical evidence supporting this is reported here.

Table 5.3 presents the variation in the normalized homogenized frictional torque, \( T_0/T_s \). The numerical values of \( T_0 \) are computed from (5.18) and the
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Table 5.3: Normalized homogenized frictional torque $T_0/T_s$, transversal sinusoidal texture and roughness.

<table>
<thead>
<tr>
<th>$a_T \setminus a_R$</th>
<th>0</th>
<th>$a$</th>
<th>$2a$</th>
<th>$4a$</th>
<th>$8a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0005</td>
<td>1.0021</td>
<td>1.0083</td>
<td>1.0312</td>
</tr>
<tr>
<td>$a$</td>
<td>1.0005</td>
<td>1.0011</td>
<td>1.0026</td>
<td>1.0087</td>
<td>1.0314</td>
</tr>
<tr>
<td>$2a$</td>
<td>1.0021</td>
<td>1.0027</td>
<td>1.0043</td>
<td>1.0106</td>
<td>1.0333</td>
</tr>
<tr>
<td>$4a$</td>
<td>1.0083</td>
<td>1.0089</td>
<td>1.0106</td>
<td>1.0170</td>
<td>1.0404</td>
</tr>
<tr>
<td>$8a$</td>
<td>1.0312</td>
<td>1.0318</td>
<td>1.0336</td>
<td>1.0403</td>
<td>1.0641</td>
</tr>
</tbody>
</table>

The frictional torque exhibited for a set of perfectly smooth surfaces, is found to be $T_s = 1.16 \cdot 10^{-3}$ Nm. Also, for $\varepsilon = 2^{-4}$ and $(a_T, a_R) = (8a, 8a)$ we find that $|T_s - T_0|/T_0 = 0.0037$.

According to Table 5.3, the previously remarked symmetry observed in Table 5.2, with respect to texture and roughness amplitude also to hold true for the homogenized frictional torque. For example, a texture of amplitude $2a$ combined with roughness of amplitude $0$, i.e. $(a_T, a_R) = (2a, 0)$, and a texture of amplitude $0$ combined with roughness of amplitude $2a$, i.e. $(a_T, a_R) = (0, 2a)$, yields approximately the same $h_{00}$ or $T_0$ according to the tables, i.e. $h_{00} = 0.9298$ and $T_0 = 1.0021$, whereas $(a_T, a_R) = (a, a)$ results in $h_{00} = 0.9278$ and $T_0 = 1.0011$. However, superpositioning the effects resulting from $(a_T, a_R) = (a, 0)$ and $(a_T, a_R) = (0, a)$ gives, with 4 decimal places, $h_{00} = 0.9278$ and $T_0 = 1.0010$. The relative discrepancies between the superpositioned results and the directly computed results were found to be $3.15 \cdot 10^{-6}$ for $h_{00}$ and $1.35 \cdot 10^{-5}$ for $T_0$. For the frictional torque, we suggest that this relative difference is attributed to the last term in (5.18), i.e. the term for the combined effect of texture and roughness.

Next, we consider the textured pad from the preceding case, i.e. (5.28) and (5.29), but with a longitudinal instead of a transversal sinusoidally shaped surface roughness,

$$h_R(z) = \frac{a_R}{2} (1 - \cos (2\pi z)) . \tag{5.31}$$

The results are compiled in Tables 5.4 and 5.5.

These tables illustrate how a longitudinally shaped roughness (or texture, interpreting the data in the first row as being induced by a surface texture instead of surface roughness) influences film formation to a higher degree than the transversal correspondence. When considering the induced
Reiterated homogenization applied in hydrodynamic lubrication

Table 5.4: Normalized homogenized property $h^{0}_{00}/h^{s}_{00}$, transversal sinusoidal texture and longitudinal sinusoidal roughness.

<table>
<thead>
<tr>
<th>$a_{T}$ \ $a_{R}$</th>
<th>0</th>
<th>$a$</th>
<th>2$a$</th>
<th>4$a$</th>
<th>8$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>0.9627</td>
<td>0.9251</td>
<td>0.8493</td>
<td>0.6947</td>
</tr>
<tr>
<td>$a$</td>
<td>0.9639</td>
<td>0.9266</td>
<td>0.8890</td>
<td>0.8132</td>
<td>0.6585</td>
</tr>
<tr>
<td>2$a$</td>
<td>0.9298</td>
<td>0.8925</td>
<td>0.8549</td>
<td>0.7790</td>
<td>0.6242</td>
</tr>
<tr>
<td>4$a$</td>
<td>0.8677</td>
<td>0.8304</td>
<td>0.7928</td>
<td>0.7167</td>
<td>0.5610</td>
</tr>
<tr>
<td>8$a$</td>
<td>0.7636</td>
<td>0.7262</td>
<td>0.6884</td>
<td>0.6115</td>
<td>0.4532</td>
</tr>
</tbody>
</table>

Table 5.5: Normalized homogenized frictional torque $T_{0}/T_{s}$, transversal sinusoidal texture and longitudinal sinusoidal roughness.

<table>
<thead>
<tr>
<th>$a_{T}$ \ $a_{R}$</th>
<th>0</th>
<th>$a$</th>
<th>2$a$</th>
<th>4$a$</th>
<th>8$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0003</td>
<td>1.0013</td>
<td>1.0053</td>
<td>1.0219</td>
</tr>
<tr>
<td>$a$</td>
<td>1.0005</td>
<td>1.0009</td>
<td>1.0019</td>
<td>1.0059</td>
<td>1.0225</td>
</tr>
<tr>
<td>2$a$</td>
<td>1.0021</td>
<td>1.0025</td>
<td>1.0035</td>
<td>1.0075</td>
<td>1.0242</td>
</tr>
<tr>
<td>4$a$</td>
<td>1.0033</td>
<td>1.0087</td>
<td>1.0098</td>
<td>1.0138</td>
<td>1.0305</td>
</tr>
<tr>
<td>8$a$</td>
<td>1.0312</td>
<td>1.0316</td>
<td>1.0327</td>
<td>1.0369</td>
<td>1.0541</td>
</tr>
</tbody>
</table>
Table 5.6: Normalized homogenized property $h^{0}_{00}/h^{*}_{00}$, different textures and roughnesses.

<table>
<thead>
<tr>
<th>Type \ R Type</th>
<th>Smooth</th>
<th>Eq.(5.30)</th>
<th>Eq.(5.31)</th>
<th>Fig. 5.7</th>
<th>Fig. 5.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth</td>
<td>1.0000</td>
<td>0.8054</td>
<td>0.7615</td>
<td>0.5641</td>
<td>0.7781</td>
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<tr>
<td>Eq.(5.30)</td>
<td>0.7636</td>
<td>0.5752</td>
<td>0.5218</td>
<td>0.3355</td>
<td>0.5464</td>
</tr>
<tr>
<td>Eq.(5.31)</td>
<td>0.6947</td>
<td>0.5002</td>
<td>0.4573</td>
<td>0.2596</td>
<td>0.4723</td>
</tr>
<tr>
<td>Eq.(5.32)</td>
<td>0.9346</td>
<td>0.7420</td>
<td>0.6951</td>
<td>0.5012</td>
<td>0.7142</td>
</tr>
<tr>
<td>Eq.(5.33)</td>
<td>0.9808</td>
<td>0.7866</td>
<td>0.7422</td>
<td>0.5454</td>
<td>0.7592</td>
</tr>
</tbody>
</table>

Table 5.7: Normalized homogenized property $T_{0}/T_{s}$, different textures and roughnesses.

<table>
<thead>
<tr>
<th>Type \ R Type</th>
<th>Smooth</th>
<th>Eq.(5.30)</th>
<th>Eq.(5.31)</th>
<th>Fig. 5.7</th>
<th>Fig. 5.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth</td>
<td>1.0000</td>
<td>1.0199</td>
<td>1.0133</td>
<td>1.0245</td>
<td>1.0197</td>
</tr>
<tr>
<td>Eq.(5.30)</td>
<td>1.0312</td>
<td>1.0524</td>
<td>1.0451</td>
<td>1.0573</td>
<td>1.0521</td>
</tr>
<tr>
<td>Eq.(5.31)</td>
<td>1.0219</td>
<td>1.0471</td>
<td>1.0370</td>
<td>1.0534</td>
<td>1.0458</td>
</tr>
<tr>
<td>Eq.(5.32)</td>
<td>1.0143</td>
<td>1.0349</td>
<td>1.0281</td>
<td>1.0396</td>
<td>1.0347</td>
</tr>
<tr>
<td>Eq.(5.33)</td>
<td>1.0027</td>
<td>1.0228</td>
<td>1.0161</td>
<td>1.0274</td>
<td>1.0226</td>
</tr>
</tbody>
</table>

frictional torque, the effects caused by the longitudinally shaped roughness (or texture) shows a less pronounced effect than that of the corresponding transversal case. This corresponds well with what would be intuitively expected and confirms what is already well-known within the field.

Tables 5.6 and 5.7 compare the two more realistic surface roughness representations found in Figures 5.7 and 5.8 with the previously considered sinusoidal representations as well as the smooth case. In addition to the transversal and longitudinal sinusoidal textures, the textures given by (5.32) (displayed in Figure 5.5) and (5.33) (displayed in Figure 5.6) were also considered. All four roughness representations were scaled to exhibit an average roughness value $R_a = \int_Z |h_R(z)| dz$ of $1\mu$m. This means that the corresponding amplitude of the sinusoidal representations (both the transversal and the longitudinal) become $a_R = R_z/2 = \pi \mu$m, i.e., $R_z = 2\pi \mu$m $\approx 6.28\mu$m. The rough surface in the Figure 5.7 has $R_z = 6.10\mu$m and the one in Figure 5.8 has $R_z = 11.00\mu$m. In all simulations (except for the case without any texture) the texture amplitude was
Figure 5.5: Artificially ground surface texture \( h_T(y) \).

held fixed, i.e. \( a_T = 4 \mu m \).

Figure 5.5 presents the mathematical description of the surface texture given by

\[
h_T(y) = 10^{-50(y_1-1/2)^2} \cos (2\pi (y_1 - 1/2)),
\]

while the surface representation presented in Figure 5.6 is modelled mathematically by

\[
h_T(y) = 10^{-25((y_1-1/2)^2+(y_2-1/2)^2)} \cos (2\pi (y_1 - 1/2)) \cos (2\pi (y_2 - 1/2)).
\]

Figure 5.7 displays a surface roughness representation \( h_R(z) \), exhibiting an almost unskewed striated pattern, while Figure 5.8 displays a negatively skewed surface roughness representation \( h_R(z) \) that exhibits a reasonably random pattern. Both of these roughnesses originate from measurements but have been re-sampled and normalized for the assessments conducted here. Normalized to an average roughness value, \( R_a = 1 \mu m \), these roughness representations have \( R_z = 6.10 \mu m \) and \( R_z = 11.00 \mu m \) (as previously mentioned) and their corresponding skewness values, \( R_{SK} = -0.0061 \) and \( R_{SK} = -1.7284 \). In studying Table 5.6 one notices that the longitudinal texture deteriorates film formation most, i.e. produces the smallest values of the ratio \( h_{100}^0/h_{00}^0 \), and the artificially dimpled texture (5.33) the least without considering the perfectly smooth surface. The surface roughness representation shown in Figure 5.7 is by far the most detrimental in terms of film formation. This surface roughness representation exhibits exactly the same \( R_a \) (ensured by the scaling) and approximately the same \( R_z \) and
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 5.6: Artificially dimpled surface texture $h_T(y)$.

Figure 5.7: A surface roughness $h_R(z)$, exhibiting a striated pattern, originating from a surface measurement.
Reiterated homogenization applied in hydrodynamic lubrication

$R_{Sk}$ values as those corresponding to the transversal sinusoidal representation. The same table clearly shows that after the perfectly smooth surfaces, it is the transversal sinusoidal roughness representation that generates the thickest film. Hence we conclude that for a prediction to be reliable it must consider more information than the three abovementioned surface roughness parameters.

Addressing frictional torque, it is - according to Table 5.7 - the artificially dimpled texture (5.33) is again the texture inducing the smallest effect. However, it is the transversal and not the longitudinal sinusoidal texture that influences frictional torque the most. In optimizing the performance in terms of film formation and induced frictional torque, it is clear that perfectly smooth surfaces are preferred, this was also previously confirmed, see e.g. [24]. However, disregarding the unrealistic perfectly smooth bearing, it is the artificially dimpled surface texture (5.33) that yields the thickest films and induces the smallest frictional torque. As well, it is the grounded surface roughness representation displayed in Figure 5.7 that clearly has the most severe influence on film formation and frictional torque. Thus, from a manufacturing point-of-view, in choosing from the selection of textures and roughnesses found in Tables 5.6 and 5.7, it would probably be most convenient to use a laser dimpling technique to achieve the 4$\mu$m deep texture and then radially grind to a 1$\mu$m $R_a$-value. This would be a rather successful combination according to the present findings. However, if the surface is further processed from its grounded state, e.g. also chemically de-burred, it might display a surface finish similar to that presented in Figure 5.8. In turn,
this should facilitate film formation as well as lower the induced frictional torque, according to the results presented here.

5.5 Conclusions

Our main result is that we have successfully developed a reiterated homogenization procedure for a class of problems by using multiple scale expansion. In particular, the Reynolds problem, which governs incompressible and Newtonian flow in Cartesian and cylindrical coordinates, belongs to this class. This made it possible to efficiently study problems connected to hydrodynamic lubrication including shape, texture and roughness. Herein lies the novelty of our results, whereas only two scales, i.e. shape and roughness, have been considered previously we can consider a third scale, i.e., the texture.

In addition, we have analyzed the convergence of the pressure gradient. This enabled us to study the limiting behavior of hydrodynamically induced friction force and frictional torque, as the wavelengths of the local scales tend to zero.

To demonstrate the applicability and effectiveness of our method, several numerical results are presented, which clearly show the convergence of the deterministic solutions toward the homogenized solution. The quantification of convergence was given in terms of load carrying capacity and friction force. In these convergence illustrations only transversal and longitudinal roughness and texture were considered. The reason for this was that it is impossible to obtain the full numerical solution for two dimensional roughness and texture, due to enormous amount of discretization points which are required to resolve the surface. However, by using our homogenization result it is possible to study the effects of arbitrary roughness with very high accuracy by solving the derived smooth homogenized equation. This was demonstrated in an example connected to a realistic thrust pad bearing problem, where the effects of texture and roughness on film formation and frictional torque were investigated.

Based on the general convergence result for the pressure gradient, we were able to deduce the limit of the deterministic expression for the friction force. The resulting homogenized quantity is made up of friction force due to the smooth (averaged) film thickness plus a corrector term. Moreover, in this corrector term, one can identify three separate contributions, i.e. due to either roughness or texture acting alone or texture and roughness acting together. The presence of terms of the latter kind implies that roughness could enhance (or diminish) certain effects that are essentially due to tex-
Reiterated homogenization applied in hydrodynamic lubrication (and vice versa). Our numerical results indicate that the combined effect due to texture and roughness on the modelled hydrodynamic bearings can be efficiently analyzed using reiterated homogenization. The resulting discrepancies in terms of predicted load carrying capacity and friction force are small; $\mathcal{O}(1\%)$ for textures as well as roughnesses of wavelengths likely to be found in a real application. That is, wavelengths within the ranges $1/100 - 1/10$ of the length bearing for the texture and $1/10000 - 1/100$ for roughness.

From the assessment of the combined effects of texture and roughness - that arise in the modelled thrust pad bearing - we adhere to the conclusion that reiterated homogenization is a feasible tool, and for any prediction to be reliable it must consider more information regarding the surface than the three surface roughness parameters, $R_a$, $R_z$ and $R_{SK}$. 
5.6 Appendix 1

\( \alpha_{ij} \) Elements of matrix \( A \)

\( A_{\varepsilon} \) Deterministic matrix

\( A_0 \) Homogenized matrix

\( A_i \) Differential operator, \( i = 0, \ldots, 4 \)

\( b_{\varepsilon} \) Deterministic vector

\( b_0 \) Homogenized vector

\( e_i \) Canonical basis in \( \mathbb{R}^2 \), \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \)

\( f^z \) Average of \( f \) with respect to \( Z \) \( (= \int_Z f dz) \)

\( \overline{f}^{zy} \) Average of \( f \) with respect to \( Z \) and \( Y \) \( (= \int_Y \int_Z f dzdy) \)

\( F_{\varepsilon} \) Deterministic frictional force

\( F_0 \) Homogenized frictional force

\( F_{\varepsilon} \) Dimensionless friction force \( = F_{\varepsilon} / (\mu U L / h_{\text{min}}) \)

\( F_0 \) Dimensionless homogenized friction force

\( h \) Auxiliary function used to model film thickness

\( h_0 \) Function describing global geometry of bearing

\( h_R \) Function describing the roughness part of film thickness

\( h_{\varepsilon} \) Deterministic film thickness

\( h_T \) Function describing the texture part of film thickness

\( h_{\text{min}} \) Fixed minimum film thickness \( = \min \) of \( h_0 \)

\( H \) Dimensionless film thickness \( = h / h_{\text{min}} \)

\( I_{\varepsilon} \) Deterministic load carrying capacity

\( I_0 \) Homogenized load carrying capacity

\( L \) Length of stationary surface exhibiting texture and roughness

\( p_{\varepsilon} \) Deterministic pressure solution

\( p_i \) The \( i \)th term in the expansion of the pressure \( p_{\varepsilon} \)

\( p_0 \) Homogenized pressure solution

\( P_{\varepsilon} \) Dimensionless deterministic pressure \( = p_{\varepsilon} / (6 \mu U L / h_{\text{min}}^2) \)

\( T_{\varepsilon} \) Deterministic frictional torque

\( T_0 \) Homogenized frictional torque

\( T_s \) Frictional torque for a perfectly smooth surface \( (= 1.16 \cdot 10^{-3} \text{Nm}) \)

\( u_i \) \( Z \)-periodic solution of the local problems, \( i = 0, 1, 2 \)

\( U \) Linear speed of moving surface

\( v_i \) \( Y \)-periodic solution of the local problems, \( i = 0, 1, 2 \)

\( x \) Local spatial coordinate, \( x = (x_1, x_2) \)

\( X \) Dimensionless spatial coordinate \( = x / L \)

\( y \) Local spatial coordinate, \( y = (y_1, y_2) = (x_1 / \varepsilon, x_2 / \varepsilon) \)

\( Y \) \( Y \)-cell \( = [0, 1]^2 \)

\( z \) Local spatial coordinate, \( z = (z_1, z_2) = (x_1 / \varepsilon^2, x_2 / \varepsilon^2) \)

\( Z \) \( Z \)-cell \( = [0, 1]^2 \)

\( \partial \Omega \) Boundary of \( \Omega \)

\( \varepsilon \) Parameter describing the roughness and texture scale \( (\varepsilon > 0) \)

\( \nabla_x \) Gradient operator, \( \nabla_x = \nabla \)

\( \nabla_y \) Gradient operator, \( \nabla_y = (\partial / \partial y_1, \partial / \partial y_2) \)

\( \nabla_z \) Gradient operator, \( \nabla_z = (\partial / \partial z_1, \partial / \partial z_2) \)

\( \Omega \) Open bounded subset of \( \mathbb{R}^2 \)
Reiterated homogenization applied in hydrodynamic lubrication

5.7 Appendix 2

In this appendix the analysis leading to our homogenization result is presented, by deriving the homogenized equation (5.8) corresponding to (5.1). The method we use is known as multiple scale expansion. For more information concerning this method in connection with homogenization see e.g. [19].

Let us first observe that the chain rule applied to a smooth function of the form $\psi_\varepsilon(x) = \psi(x, y, z)$, where $y = x/\varepsilon$ and $z = x/\varepsilon^2$ gives that

$$\nabla_x \psi_\varepsilon(x) = \left( \nabla_x + \frac{1}{\varepsilon} \nabla_y + \frac{1}{\varepsilon^2} \nabla_z \right) \psi(x, y, z). \tag{5.34}$$

Inserting the expansion (5.2) (of $p_\varepsilon$) into (5.1) and making use of (5.34) we obtain

$$\left( \nabla_x + \frac{1}{\varepsilon} \nabla_y + \frac{1}{\varepsilon^2} \nabla_z \right) \cdot \left[ A_0 \left( \nabla_x + \frac{1}{\varepsilon} \nabla_y + \frac{1}{\varepsilon^2} \nabla_z \right) \sum_{i=0}^{\infty} \varepsilon^i p_i \right]$$

$$= \left( \nabla_x + \frac{1}{\varepsilon} \nabla_y + \frac{1}{\varepsilon^2} \nabla_z \right) \cdot \mathbf{b}. \tag{5.35}$$

Let the differential operators $A_i$, $i = 0, ..., 4$ be defined as

$$A_0 = \nabla_z \cdot (A \nabla_z),$$
$$A_1 = \nabla_z \cdot (A \nabla_y) + \nabla_y \cdot (A \nabla_z),$$
$$A_2 = \nabla_x \cdot (A \nabla_z) + \nabla_y \cdot (A \nabla y) + \nabla_z \cdot (A \nabla x),$$
$$A_3 = \nabla_x \cdot (A \nabla y) + \nabla_y \cdot (A \nabla x),$$
$$A_4 = \nabla_x \cdot (A \nabla x).$$

Using the above notation (5.35) may be written as

$$(\varepsilon^{-4} A_0 + \varepsilon^{-3} A_1 + \varepsilon^{-2} A_2 + \varepsilon^{-1} A_3 + A_4) \left( p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \right)$$

$$= (\varepsilon^{-2} \nabla_z + \varepsilon^{-1} \nabla_y + \nabla_x) \cdot \mathbf{b}.$$ 

By comparing terms with the same order of $\varepsilon$ (from $-4$ to $0$), we obtain the following system of equations

$$A_0 p_0 = 0, \tag{5.36a}$$
$$A_0 p_1 + A_1 p_0 = 0, \tag{5.36b}$$
$$A_0 p_2 + A_1 p_1 + A_2 p_0 = \nabla_z \cdot \mathbf{b}. \tag{5.36c}$$
\[ A_0 p_3 + A_1 p_2 + A_2 p_1 + A_3 p_0 = \nabla_y \cdot b, \quad (5.36d) \]
\[ A_0 p_4 + A_1 p_3 + A_2 p_2 + A_3 p_1 + A_4 p_0 = \nabla_x \cdot b. \quad (5.36e) \]

In the following, we make frequent use of the following well-known result:

\[ \Gamma u = f \text{ has a solution if and only if } \int_Z f dz = 0. \quad (5.37) \]

In this case \( u \) is unique up to an additive constant.

Here, \( \Gamma \) is any of the operators \( A_0, A_1, A_2, \ldots \) and \( Z \) may be replaced with \( Y \). See for example [69, p. 39] for a proof of (5.37). According to (5.37), it is clear that \( p_0 \) in (5.36a) does not depend on \( z \), i.e.

\[ p_0 = p_0(x, y), \quad (5.38) \]

and this simplifies (5.36b) to

\[ A_0 p_1(x, y, z) = -\nabla_z \cdot (A (x, y, z) \nabla y p_0(x, y)). \quad (5.39) \]

By linearity

\[ p_1(x, y, z) = u_1(x, y, z) \frac{\partial p_0}{\partial y_1}(x, y) + u_2(x, y, z) \frac{\partial p_0}{\partial y_2}(x, y) + \tilde{p}_1(x, y), \quad (5.40) \]

where the \( Z \)-periodic function \( u_i = u_i(x, y, z) \), \( i = 1, 2 \) is a solution (unique up to a constant) to the following local problem

\[ \nabla_z \cdot (A (\nabla_z u_i + e_i)) = 0 \text{ in } Z. \quad (5.41) \]

According to (5.37) we can solve (5.36c) for \( p_2 \) if and only if

\[ \int_Z (A_1 p_1 + A_2 p_0) \ dz = 0. \quad (5.42) \]

Substituting (1.6) into (5.42) and considering \( Z \)-periodicity we find that

\[ \nabla_y \cdot \left( \int_Z \left[ A \left( \nabla_y p_0 + \nabla_z \left( u_1 \frac{\partial p_0}{\partial y_1} + u_2 \frac{\partial p_0}{\partial y_2} \right) \right) \right] dz \right) = 0. \quad (5.43) \]

This is identical to

\[ \nabla_y \cdot \left( A (x, y, z) A (x, y, z) \nabla y p_0(x, y) \right) = 0, \quad (5.44) \]
Reiterated homogenization applied in hydrodynamic lubrication

where \( \mathcal{F} = \int_Z f \, dz \) and

\[
\mathcal{A} = \mathcal{A}(x, y, z) = \begin{pmatrix}
1 + \frac{\partial u_1}{\partial z_1} & \frac{\partial u_2}{\partial z_1} \\
\frac{\partial u_1}{\partial z_2} & 1 + \frac{\partial u_2}{\partial z_2}
\end{pmatrix}.
\] (5.45)

We remark that the equation (5.44) is the homogenized equation after the first reiteration.

Equation (5.44) implies that

\[
p_0(x, y) = p_0(x).
\] (5.46)

Thus, by virtue of (1.6),

\[
p_1 = p_1(x, y).
\] (5.47)

Using (5.46) and (5.47) in (5.36c) and simplifying, we have

\[
A_0 p_2 = \nabla_z \cdot b - \nabla_z \cdot (\mathcal{A} (\nabla_y p_1 + \nabla_x p_0)).
\] (5.48)

By linearity we find that \( p_2 \) is of the form

\[
p_2(x, y, z) = u_0(x, y, z) + u_1(x, y, z) \left( \frac{\partial p_0}{\partial x_1}(x) + \frac{\partial p_1}{\partial y_1}(x, y) \right) + u_2(x, y, z) \left( \frac{\partial p_0}{\partial x_2}(x) + \frac{\partial p_1}{\partial y_2}(x, y) \right) + \tilde{p}_2(x, y),
\] (5.49)

where \( u_0 \) is a solution (unique up to an additive constant) to the local problem

\[
\nabla_z \cdot (\mathcal{A} \nabla_z u_0 - b) = 0 \quad \text{in } Z.
\] (5.50)

Recall that even though \( y \) is a parameter in this context, \( u_0 \) in (5.50) and \( u_1 \) and \( u_2 \) in (5.41) are not only \( Z \)-periodic, but also \( Y \)-periodic functions.

To solve (5.36d) for \( p_3 \) it must hold that

\[
\int_Z (A_1 p_2 + A_2 p_1 + A_3 p_0 - \nabla_y \cdot b) \, dz = 0.
\]

Expansion yields

\[
\int_Z (\nabla_y \cdot (\mathcal{A} \nabla_z p_2) + \nabla_y \cdot (\mathcal{A} \nabla_y p_1) + \nabla_y \cdot (\mathcal{A} \nabla_x p_0) - \nabla_y \cdot b) \, dz = 0.
\] (5.51)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Inserting (5.49) in (5.51) and rearranging the terms we obtain that
\[
\int_Z \nabla_y \cdot \left[ A \left( \nabla_x p_0 + \nabla_z \left( u_1 \frac{\partial p_0}{\partial x_1} + u_2 \frac{\partial p_0}{\partial x_2} \right) \right) + \right.
\]
\[
+ (A \nabla_z u_0 - b) + A \left( \nabla_y p_1 + \nabla_z \left( u_1 \frac{\partial p_1}{\partial y_1} + u_2 \frac{\partial p_1}{\partial y_2} \right) \right) \right] \, dz
\]
\[
= 0,
\]
and by virtue of (5.45), this reduces to
\[
\nabla_y \cdot \left( \overline{A} \overline{A}^T \nabla_y p_1 \right) = -\nabla_y \cdot \left( \overline{A} \overline{A}^T \nabla_x p_0 \right) + \nabla_y \cdot (b - A \nabla_z u_0)^x. \tag{5.52}
\]

By linearity, the equation (5.52) is satisfied if
\[
p_1(x, y) = v_0(x, y) + v_1(x, y) \frac{\partial p_0}{\partial x_1}(x) + v_2(x, y) \frac{\partial p_0}{\partial x_2}(x) + \tilde{p}_1(x). \tag{5.53}
\]
where the Y-periodic functions \( v_i(x, y) \) \((i = 0, 1, 2)\) are the solutions of the following local problems involving y
\[
\begin{cases}
\nabla_y \cdot \left( \overline{A} \overline{A}^T \nabla_y v_0 - \left( b - A \nabla_z u_0 \right)^x \right) = 0 \text{ in } Y, \\
\nabla_y \cdot \left( \overline{A} \overline{A}^T (\nabla_y v_i + e_i) \right) = 0 \text{ on } Y, \ (i = 1, 2).
\end{cases} \tag{5.54}
\]
Here \( x, \) is regarded as a parameter.

A necessary condition for solving (5.36e) for \( p_4 \) is that
\[
\int_Z A_1 p_3 + A_2 p_2 + A_3 p_1 + A_4 p_0 - \nabla_x \cdot b \, dz = 0. \tag{5.55}
\]

By integrating (5.55) over \( Y, \) expanding the differential operators \( A_i \) and making use of the \( Y \) and \( Z \) periodicity yields
\[
\int_Y \int_Z \nabla_x \cdot \left( A (\nabla_z p_2 + \nabla_y p_1 + \nabla_x p_0) - b \right) \, dydz = 0. \tag{5.56}
\]

Next, we show that the condition (8.31) leads to the homogenized equation. By inserting (5.49) and (5.53) in (8.31) we obtain
\[
\nabla_x \cdot \left( \overline{A} \nabla_z \left( u_0 + u_1 \frac{\partial p_0}{\partial x_1} + u_2 \frac{\partial p_0}{\partial x_2} \right) + u_1 \frac{\partial p_1}{\partial y_1} + u_2 \frac{\partial p_1}{\partial y_2} + \tilde{p}_2(x, y) \right)^y \right) +
\]
\[
\nabla_x \cdot \left( \overline{A} \nabla_y \left( v_0 + v_1 \frac{\partial p_0}{\partial x_1} + v_2 \frac{\partial p_0}{\partial x_2} + \tilde{p}_1(x) \right)^x \right) +
\]
Reiterated homogenization applied in hydrodynamic lubrication

\[ \nabla_x \cdot \left( \mathbb{A}^{xy} \nabla_x p_0 \right) \]

\[ = \nabla_x \cdot \left( \mathbb{B}^{xy} \right). \]

By simplifying and rearranging the following, we have that

\[ \nabla_x \cdot \left( \mathbb{A} \left( \nabla_x p_0 + \nabla_y \left( v_1 \frac{\partial p_0}{\partial x_1} + v_2 \frac{\partial p_0}{\partial x_2} \right) \right) \right) + \]

\[ \nabla_x \cdot \left( \mathbb{A} \nabla_z \left( u_1 \frac{\partial}{\partial y_1} \left( v_1 \frac{\partial p_0}{\partial x_1} + v_2 \frac{\partial p_0}{\partial x_2} \right) \right) \right) + \]

\[ \nabla_x \cdot \left( \mathbb{A} \nabla_z \left( u_2 \frac{\partial}{\partial y_2} \left( v_1 \frac{\partial p_0}{\partial x_1} + v_2 \frac{\partial p_0}{\partial x_2} \right) \right) \right) = \]

\[ \nabla_x \cdot \left( \mathbb{B} - \mathbb{A} \left( \nabla_z u_0 + \nabla_y v_0 + \nabla_z \left( u_1 \frac{\partial v_0}{\partial y_1} \right) + \nabla_z \left( u_2 \frac{\partial v_0}{\partial y_2} \right) \right) \right). \] (5.57)

By defining

\[ \mathbb{B} = \mathbb{B} (x, y) = \begin{pmatrix} 1 + \frac{\partial v_1}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \\ \frac{\partial v_1}{\partial y_1} & 1 + \frac{\partial v_2}{\partial y_2} \end{pmatrix}, \] (5.58)

we see that the compressed form of

\[ \nabla_x \cdot \left( \mathbb{A} \left( \nabla_x p_0 + \nabla_y \left( v_1 \frac{\partial p_0}{\partial x_1} + v_2 \frac{\partial p_0}{\partial x_2} \right) \right) \right) = \nabla_x \cdot (\mathbb{A} \nabla_x p_0). \] (5.59)

Inserting (8.40) into (8.36) and rearranging the terms, we find that

\[ \nabla_x \cdot \left( \mathbb{A} \mathbb{B}^{xy} \nabla_x p_0 \right) + \]

\[ \nabla_x \cdot \left( \begin{pmatrix} \frac{\partial u_1}{\partial z_1} & \frac{\partial u_1}{\partial z_1} \\ \frac{\partial u_1}{\partial z_1} & \frac{\partial u_2}{\partial z_2} \end{pmatrix} \nabla_x p_0 \right) + \]

\[ \nabla_x \cdot \left( \begin{pmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_1} \\ \frac{\partial u_1}{\partial y_1} & \frac{\partial u_2}{\partial y_1} \end{pmatrix} \nabla_x p_0 \right) + \]

\[ \nabla_x \cdot \left( \begin{pmatrix} \frac{\partial u_1}{\partial z_1} & \frac{\partial u_1}{\partial y_1} \\ \frac{\partial u_1}{\partial z_1} & \frac{\partial u_2}{\partial y_1} \end{pmatrix} \nabla_x p_0 \right) + \]

\[ \nabla_x \cdot \left( \begin{pmatrix} \frac{\partial u_2}{\partial z_1} & \frac{\partial u_2}{\partial z_2} \\ \frac{\partial u_1}{\partial z_1} & \frac{\partial u_2}{\partial y_1} \end{pmatrix} \nabla_x p_0 \right). \]
\[
\n\nabla_x \cdot \left( A \begin{pmatrix}
\frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \\
\frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2}
\end{pmatrix} \begin{pmatrix}
\nabla_x p_0 \\
\nabla_x p_0
\end{pmatrix}
\right)
\]

\[
= \nabla_x \cdot \left( b - A \left( \nabla_z u_0 + \nabla_y v_0 + \nabla_z \left( u_1 \frac{\partial v_0}{\partial y_1} \right) + \nabla_z \left( u_2 \frac{\partial v_0}{\partial y_2} \right) \right) \right). \tag{5.60}
\]

By adding the corresponding components of the matrices in the inner brackets and simplifying, we obtain that

\[
\nabla_x \cdot \left( \overline{A \overline{AB}^{-1}} \nabla_x p_0 \right)
\]

\[
= \nabla_x \cdot \left( b - A \nabla_z u_0 - A \left( \nabla_y v_0 + \nabla_z \left( u_1 \frac{\partial v_0}{\partial y_1} \right) + \nabla_z \left( u_2 \frac{\partial v_0}{\partial y_2} \right) \right) \right). \tag{5.62}
\]

Moreover,

\[
\nabla_y v_0 + \nabla_z u_1 \frac{\partial v_0}{\partial y_1} + \nabla_z u_2 \frac{\partial v_0}{\partial y_2} = A \nabla_y v_0, \tag{5.61}
\]

and thus from (5.61) and (5.60) we see that

\[
\nabla_x \cdot \left( \overline{A \overline{AB}^{-1}} \nabla_x p_0 \right) = \nabla_x \cdot \left( \overline{b - A \nabla_z u_0 - A \left( A \nabla_y v_0 \right)} \right). \tag{5.62}
\]

By defining

\[
A_0(x) = \overline{A \overline{AB}^{-1}}, \tag{5.63a}
\]

\[
b_0(x) = \overline{b - A \nabla_z u_0 - A \left( A \nabla_y v_0 \right)}, \tag{5.63b}
\]

and inserting in (5.62), we finally obtain that

\[
\nabla_x \cdot (A_0(x) \nabla_x p_0(x)) = \nabla_x \cdot b_0(x) \quad \text{in } \Omega, \tag{5.64}
\]

\[
p_0(x) = 0 \quad \text{on } \partial \Omega,
\]

where \(A\) is defined as in (5.45) and \(B\) is defined as in (5.58). In other words, (5.64) is the reiterated homogenized boundary value problem corresponding to the deterministic boundary value problem given by (5.1).
Chapter 6

Reiterated homogenization of a nonlinear Reynolds-type equation

6.1 Introduction

In this Chapter some of the previous homogenization results in connection with hydrodynamic lubrication are extended to include two microscopic scales and non-Newtonian fluids. More precisely, we study the limiting behavior as $\varepsilon \to 0$ of the solutions $u_\varepsilon$ of

$$\begin{align*}
\text{div} a_\varepsilon(x, \nabla u_\varepsilon) &= \text{div} b_\varepsilon \quad \text{in } \Omega \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(6.1)

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$ and $a_\varepsilon$ and $b_\varepsilon$ oscillate rapidly on the microscopic scales $\varepsilon$ (meso scale) and $\varepsilon^2$ (micro scale). The idea of reiterated homogenization is that the effects of rapid oscillations upon the solution is averaged out.

As an application we show that for particular choices of $a_\varepsilon$ and $b_\varepsilon$ it is possible to analyze the effects of multiscale surface roughness in some interesting Newtonian and non-Newtonian lubrication models. For example both the stationary incompressible Reynolds equation and the Reynolds-type equation derived in [37] based on the Rabinowitsch constitutive relation, belong to this category. It is well known that the surface micro topography has a significant effect on the hydrodynamic performance in thin film lubrication.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Machined surfaces are never perfectly smooth because of defects in the manufacturing process. Since smoothening of the surfaces in contact with a fluid film bearing may lead to a decrease in performance, bearing designers are also considering artificially textured surfaces. The effects of surface roughness in various lubrication regimes have been studied with homogenization techniques in numerous works, e.g. [13, 14, 18, 20, 31, 39, 76]. The usual assumption is that the roughness is periodic with characteristic wavelength $\varepsilon$. The main result of this Chapter makes it possible to study surface roughness with two distinguishable wavelengths, say $\varepsilon$ and $\varepsilon_2$, in both linear and nonlinear lubrication models.

Reiterated homogenization of $-\text{div} \ a_\varepsilon(x, \nabla u_\varepsilon) = f$ with non-oscillating $f$ has been studied in [62] by the periodic unfolding method. Reiterated homogenization of $-\text{div} \ a_\varepsilon(x, \nabla u_\varepsilon) = f_\varepsilon$, where $f_\varepsilon$ converges strongly in $W^{-1,q}(\Omega)$, has also been studied in [55, 58]. This also differs from the present case in that $\text{div} \ b_\varepsilon$, in general, does not converge strongly in $W^{-1,q}(\Omega)$.

6.2 Preliminaries and notation

Suppose $p, \alpha, \beta, \lambda$ and $\theta$ are constants that obey

$$1 < p < \infty, \quad 0 < \alpha \leq \min\{1, p - 1\}, \quad \max\{2, p\} \leq \beta < \infty, \quad \lambda, \theta > 0.$$  \hfill (6.2)

A function $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is said to belong to the class $M^p_{\alpha, \beta}(\lambda, \theta)$ provided the following conditions are satisfied for any $\xi, \eta \in \mathbb{R}^N$.

$$f(0) = 0,$$  \hfill (6.3a)

$$|f(\xi) - f(\eta)| \leq \lambda(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha,$$  \hfill (6.3b)

$$[f(\xi) - f(\eta)] \cdot (\xi - \eta) \geq \theta \frac{|\xi - \eta|^\beta}{(1 + |\xi| + |\eta|)^{\beta-p}}.$$  \hfill (6.3c)

Moreover, $\mathcal{M}(\lambda, \theta)$ denotes the set of all linear mappings $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $f \in M^p_{\alpha, \beta}(\lambda, \theta)$. The function $a_\varepsilon: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is assumed to be of the form

$$a_\varepsilon(x, \xi) = a \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right) \quad (x \in \Omega, \xi \in \mathbb{R}^N)$$  \hfill (6.4)

where $a: \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a function of Carathéodory type such that

- $a$ is periodic (with respect to the unit cell $Y = Z = [0, 1]^N$) in both the second and the third argument,
Reiterated homogenization of a nonlinear Reynolds-type equation

- there exists constants $p$, $\alpha$, $\beta$, $\lambda$ and $\theta$ satisfying (6.2), such that
  
  $a(x, y, z, \cdot) \in \mathcal{M}_{\alpha, \beta}^p(\lambda, \theta)$ for a.e. $(x, y, z) \in \Omega \times [0, 1]^N \times [0, 1]^N$.

As a consequence of (6.3a–c), for each $\xi \in \mathbb{R}^N$, $a_\varepsilon$ satisfies the growth condition

$$|a_\varepsilon(\cdot, \xi)| \leq \lambda (1 + |\xi|)^{p-1} \quad \text{a.e. in } \Omega$$

and the coercivity condition

$$a_\varepsilon(\cdot, \xi) \cdot \xi \geq 2^{p-\beta} \theta \times \begin{cases} 
|\xi|^\beta & \text{if } |\xi| \leq 1 \\
|\xi|^p & \text{otherwise}
\end{cases} \quad \text{a.e. in } \Omega. \quad (6.6)$$

It is further assumed that $b_\varepsilon$ is of the form

$$b_\varepsilon(x) = b\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)$$

with $b \in L^q(\Omega; C_{\text{per}}(Y \times Z))$.

A weak solution of (6.1) is defined as an element $u_\varepsilon$ of $W^{1,p}_0(\Omega)$ satisfying

$$\int_\Omega a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla \phi \, dx = \int_\Omega b_\varepsilon \cdot \nabla \phi \, dx \quad (6.7)$$

for all $\phi \in W^{1,p}_0(\Omega)$. Let $A_\varepsilon : W^{1,p}_0(\Omega) \to W^{-1,q}(\Omega)$ be defined by

$$\langle A_\varepsilon(u), \phi \rangle = \int_\Omega a_\varepsilon(x, \nabla u) \cdot \nabla \phi \, dx \quad (u, \phi \in W^{1,p}_0(\Omega)).$$

By (6.3b) and Hölder’s inequality, we obtain

$$\|A_\varepsilon(u) - A_\varepsilon(v)\|_{W^{-1,q}(\Omega)} \leq \lambda \|1 + |\nabla u| + |\nabla v|\|_{L^p(\Omega)}^{p-1-\alpha} \|u - v\|_{W^{1,p}_0(\Omega)}^\alpha.$$

Hence $A_\varepsilon$ is continuous. Owing to (6.3c) it can be shown that $A_\varepsilon$ is strictly monotone, i.e.

$$\langle A_\varepsilon(u) - A_\varepsilon(v), u - v \rangle \geq 0$$

with equality if and only if $u = v$. Utilizing (6.6) and Poincaré’s inequality yields

$$\langle A_\varepsilon(u), u \rangle = \int_\Omega a_\varepsilon(x, \nabla u) \cdot \nabla u \, dx \geq \text{const} \|u\|_{W^{1,p}_0(\Omega)}^p - 2^{p-\beta} \theta \text{meas}(\Omega), \quad (6.8)$$

implying that $A_\varepsilon$ is coercive. Thus the hypotheses of the Browder–Minty theorem (see e.g. [78] p. 557) are verified, and we conclude that for each $\varepsilon > 0$, there exists a unique $u_\varepsilon$ that solves (6.7). Moreover, putting $\phi = u_\varepsilon$ in (6.7) and utilizing (6.8) and Young’s inequality we obtain

$$\|u_\varepsilon\|_p \leq \text{const}(\|\text{div } b_\varepsilon\|_{W^{-1,q}(\Omega)}^q + 1). \quad (6.9)$$

Since the sequence $\text{div } b_\varepsilon$ is weak* convergent, and hence bounded, this shows that the sequence of solutions $u_\varepsilon$ is bounded in $W^{1,p}_0(\Omega)$. 
6.3 Three-scale convergence

In 1989 Nguetseng [64] introduced a method for analyzing homogenization problems that was later further developed by Allaire [1] and called two-scale convergence. Two-scale convergence in the setting of Lebesgues spaces $L^p$ with $1 < p < \infty$ is described in [65]. An advantage of the two-scale convergence method is that it is designed to avoid many of the classical difficulties encountered in the homogenization process, such as passing to the limit in the product of two weakly convergent sequences, reducing it to an almost trivial process. A limitation of the method is that one is more or less restricted to the periodic case.

Homogenization of problems with $n$ microscopic scales, for $n > 1$ is referred to as reiterated homogenization, see e.g. [19] and [55]. Two-scale convergence (one microscopic scale) has been generalized to $(n + 1)$-scale convergence or multiscale convergence ($n$ microscopic scales) by Allaire and Briane [4]. As pointed out by the authors of that work, the $n$-scale case is more delicate and for the special case $n = 1$ the proofs are sometimes much simpler compared to the general case. The results of [4] being restricted to $L^2(\Omega)$, it has hitherto not been clear whether a multiscale theory is possible for $L^p(\Omega)$. Nevertheless, many authors have claimed such results, see e.g. Theorem 3.8 in [12] or Theorem 1.7 in [35], without providing any complete proof. An additional result of this Chapter is that we develop such a theory for the case of two microscopic scales ($n = 2$) and $1 < p < \infty$. In particular we give a new proof of Theorem 1.2 (see also Theorem 2.6) in [4]. The assumption that the micro scale is the square of the meso scale corresponds to the notion of well-separated scales defined in [4].

With the appropriate notion of three-scale convergence combined with an additional result concerning three-scale convergence and monotonicity, homogenization of (6.7) becomes a rather short story. We mention that for the case that $a_\varepsilon$ is a matrix, the homogenized equation obtained by three-scale convergence coincides with that of [7], which was obtained by the asymptotic expansion method and can thus be seen as a rigorous justification of this method.

**Definition 6.3.1.** A bounded sequence $u_\varepsilon$ ($\varepsilon > 0$) in $L^p(\Omega)$ is said to three-scale converge to an element $u$ of $L^p(\Omega \times Y \times Z)$ provided

$$
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) \phi \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) dx = \iiint_{\Omega Y Z} u(x, y, z) \phi(x, y, z) dz \ dy \ dx
$$

(6.10)

for every test function $\phi$ of the form $\phi(x, y, z) = \varphi(x) \psi(y) \sigma(z)$, where $\varphi \in \mathbb{R}$.
Reiterated homogenization of a nonlinear Reynolds-type equation

\[ C(\Omega), \psi \in C_{\text{per}}(Y) \text{ and } \sigma \in C_{\text{per}}(Z). \]

We note that we have an equivalent definition of two-scale convergence if we replace the set of test functions by \( D(\Omega; C_\text{per}(Y \times Z)) \), i.e. the space that consists of all functions \( \Omega \rightarrow C_\text{per}(Y \times Z) \) such that for any \( x \in \Omega \), \( u(x, \cdot) \in C_\text{per}(Y \times Z) \) and the mapping \( \Omega \ni x \mapsto u(x, \cdot) \in C_\text{per}(Y \times Z) \) is infinitely differentiable (in the sense of Fréchet) with compact support in \( \Omega \).

For the more general case of periodicity that \( Y \) and \( Z \) are parallelograms in \( \mathbb{R}^N \), Definition 6.3.1 must be modified by dividing the right hand side of (6.10) with \( \text{meas}(Y)\text{meas}(Z) \).

As a direct consequence of the definition of three-scale convergence it is true that any three-scale convergent sequence \( u_\varepsilon \) is also weakly convergent in \( L^p(\Omega) \). More precisely,

\[ u_\varepsilon \rightharpoonup v \text{ weakly}, \quad v(x) = \iint_{YZ} u(x, y, z) \, dz \, dy, \]

whenever \( u_\varepsilon \rightharpoonup u \) three-scale. This follows from taking \( \psi = \sigma = 1 \) in (6.10).

We state below the three most important theorems in the theory of multiscale convergence.

**Theorem 6.3.2 (Three-scale compactness).** For any bounded sequence \( u_\varepsilon \) in \( L^p(\Omega) \), there exists a subsequence that three-scale converges weakly.

**Proof.** The proof is very similar to the two-scale case, see Theorem 7 in [65]. \( \square \)

To prove the next important theorem we need a lemma concerning a special type of convergence for periodic functions. For the sake of brevity, the proof is postponed to the end of the paper.

**Lemma 6.3.3.** Assume \( \psi_1 \in C^\infty_c(\Omega), \psi_2 \in C_\text{per}^\infty(Y) \) and \( f \in L^p_\text{per}(Z) \) satisfies

\[ \int_Z f \, dz = 0. \]

Then the sequence of functions \( f_\varepsilon \), defined for a.e. \( x \in \Omega \) by

\[ f_\varepsilon(x) = \frac{1}{\varepsilon^2} \psi_1(x) \psi_2 \left( \frac{x}{\varepsilon} \right) f \left( \frac{x}{\varepsilon^2} \right), \]

converges weak* to 0 in \( W^{-1,p}(\Omega) \). In particular

\[ \sup_{\varepsilon > 0} \| f_\varepsilon \|_{W^{-1,p}(\Omega)} < \infty. \]
Theorem 6.3.4 (Three-scale convergence of the gradient). Suppose that $u_\varepsilon$ is a sequence in $W^{1,p}_0(\Omega)$ such that

1. $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega)$,
2. $\nabla u_\varepsilon \to \xi \in L^p(\Omega \times Y \times Z; \mathbb{R}^N)$ three-scale.

Then $u_\varepsilon \to u$ three-scale and there exists $u_1 \in L^p(\Omega; W^{1,p}_{per}(Y))$ and $u_2 \in L^p(\Omega \times Y; W^{1,p}_{per}(Z))$ such that

$$\xi(x, y, z) = \nabla u(x) + \nabla_y u_1(x, y) + \nabla_z u_2(x, y, z).$$

Proof.
Step 1. Since $u_\varepsilon$ is a bounded sequence in $L^p(\Omega)$ it is possible to extract a three-scale convergent subsequence (still denoted by $u_\varepsilon$), say $u_\varepsilon \to u'$ three-scale. By integration by parts we obtain the identity

$$\int_\Omega \nabla u_\varepsilon(x) \cdot \Phi(x, x_\varepsilon, x_\varepsilon^2) \, dx = -\int_\Omega u_\varepsilon(x)(\text{div}_x + \varepsilon^{-1}\text{div}_y + \varepsilon^{-2}\text{div}_z)\Phi(x, x_\varepsilon, x_\varepsilon^2) \, dx$$

for all $\Phi \in \mathcal{D}(\Omega; C^\infty_{per}(Y \times Z; \mathbb{R}^N))$. Multiplying (6.11) with $\varepsilon^2$ and passing to the subsequential three-scale limit $u'$ of $u_\varepsilon$ we obtain

$$-\int\int\int_{\Omega Y Z} u'(x, y, z)\text{div}_z \Phi(x, y, z) \, dz \, dy \, dx = 0.$$

It follows that $u'$ does not depend on $z$. Similarly, taking $\Phi$ in (6.11) independent of $z$ and multiplying by $\varepsilon$ yields

$$-\int\int_{\Omega Y} u'(x, y)\text{div}_y \Phi(x, y) \, dy \, dx = 0$$

and we conclude that $u'$ does not depend on $y$ either. The three-scale convergence implies $u_\varepsilon \rightharpoonup u'$ weakly in $L^p(\Omega)$ (always for the same subsequence). Since the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact we also have that $u_\varepsilon \to u$ strongly in $L^p(\Omega)$ (for the whole sequence). It follows that $u' = u$ and consequently the whole sequence $u_\varepsilon$ and not just a subsequence three-scale converges to $u$.

Step 2. From two-scale theory, Theorem 13 in [65], we know that

$$\overline{\xi}(x, y) = \nabla u(x) + \nabla_y u_1(x, y) \quad \text{where} \quad \overline{\xi} = \int_Z \xi \, dz.$$
Reiterated homogenization of a nonlinear Reynolds-type equation

This follows from the fact that \( \nabla u_\varepsilon \to \xi \) two-scale. Next we project \( \xi - \overline{\xi} \) onto the space of \( z \)-gradients. That is, define

\[
L^p_{pot}(Z) = \{ \nabla_z v : v \in L^p(\Omega \times Y; W^{1,p}_{\text{per}}(Z)) \}.
\]

Because of the Poincaré–Wirtinger inequality, \( L^p_{pot}(Z) \) is a closed subspace of \( L^p(\Omega \times Y \times Z; \mathbb{R}^N) \) and the latter being uniformly convex there exists a \( \nabla_z u_2 \in L^p_{pot}(Z) \) which minimizes the distance from \( \xi - \overline{\xi} \) to \( L^p_{pot}(Z) \), i.e.

\[
\|\xi - \overline{\xi} - \nabla_z u_2\| = \min_{\Phi \in L^p_{pot}(Z)} \|\xi - \overline{\xi} - \Phi\|.
\]

(6.12)

Let \( \eta \) be defined by

\[
\xi(x, y, z) = \nabla u(x) + \nabla_y u_1(x, y) + \nabla_z u_2(x, y, z) + \eta(x, y, z).
\]

Computing the first variation of the minimization problem (6.12) we obtain

\[
\iint_{\Omega \times Y \times Z} |\eta|^{p-2} \eta \cdot \Phi \, dz \, dy \, dx = 0
\]

for all \( \Phi \) of the form

\[
\Phi(x, y, z) = \phi(x) \psi(y) \nabla \sigma(z)
\]

with \( \phi \in C^\infty_c(\Omega), \psi \in C^\infty(Y) \) and \( \sigma \in W^{1,p}_{\text{per}}(Z) \). It follows that for a.e. \( x \in \Omega \) and a.e. \( y \in Y \),

\[
\int_{Z} |\eta|^{p-2} \eta(x, y, z) \cdot \nabla \sigma(z) \, dz = 0
\]

for all \( \sigma \in W^{1,p}_{\text{per}}(Z) \).

To summarize we have

\[
\int_{Z} \eta \, dz = 0 \quad \text{and} \quad \text{div}_z (|\eta|^{p-2} \eta) = 0.
\]

(6.13)

Step 3. We show that \( \eta = 0 \). To prove this, we take test functions in (6.11) of the form

\[
\Phi(x, y, z) = \phi(x) \psi(y) \sigma(z)
\]

(6.14)

with \( \phi \in C^\infty_c(\Omega), \psi \in C^\infty_{\text{per}}(Y) \) and \( \sigma \in C^\infty_{\text{per}}(Z; \mathbb{R}^N) \) satisfying

\[
\text{div} \sigma = 0.
\]

(6.15a)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

\[ \int_Z \sigma \, dz = 0. \quad (6.15b) \]

For such \( \Phi \) (6.11) reduces to

\[ \int_{\Omega} \nabla u_\varepsilon \cdot \Phi \, dx = - \int_{\Omega} u_\varepsilon \left( \psi \left( \frac{x}{\varepsilon} \right) \right) \nabla \phi \left( \frac{x}{\varepsilon} \right) + \varepsilon^{-1} \phi(x) \nabla \psi \left( \frac{x}{\varepsilon} \right) \cdot \sigma \left( \frac{x}{\varepsilon^2} \right) \, dx. \]

Letting \( \varepsilon \to 0 \) we obtain

\[ \iint_{\Omega Y Z} \xi \cdot \Phi \, dz \, dy \, dx = \lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} \varphi_\varepsilon u_\varepsilon \, dx \quad (6.16) \]

where

\[ \varphi_\varepsilon(x) = \frac{1}{\varepsilon^2} \phi(x) \nabla \psi \left( \frac{x}{\varepsilon} \right) \cdot \sigma \left( \frac{x}{\varepsilon^2} \right). \]

Applying Lemma 6.3.3, we conclude that \( \varphi_\varepsilon \) is bounded in \( W^{-1,q}(\Omega) \). Since \( u_\varepsilon \) is also bounded in \( W^{1,p}_0(\Omega) \), it holds that

\[ \left| \int_{\Omega} \varphi_\varepsilon u_\varepsilon \, dx \right| = |\langle \varphi_\varepsilon, u_\varepsilon \rangle| \leq \| \varphi_\varepsilon \|_{W^{-1,q}(\Omega)} \| u_\varepsilon \|_{W^{1,p}(\Omega)} \leq \text{const.} \]

Thus (6.16) actually says that

\[ \iint_{\Omega Y Z} \xi \cdot \Phi \, dz \, dy \, dx = 0 \quad \text{implying} \quad \iint_{\Omega Y Z} \eta \cdot \Phi \, dz \, dy = 0 \]

for all \( \Phi \) having the special form (6.14) and satisfying conditions (6.15), but in view of (6.13) it is clear that we can omit condition (6.15b). Thus

\[ \iint_{\Omega Y Z} (\eta \cdot \sigma(z)) \phi(x) \psi(y) \, dz \, dy \, dx = 0, \]

where \( \sigma \) is assumed to satisfy only \( \text{div}_z \sigma = 0 \). It follows by Fubini’s theorem, density etc. that for a.e. \( (x, y) \in \Omega \times Y \)

\[ \int_{Z} \eta(x, y, z) \cdot \sigma(z) \, dz = 0 \]

for all \( \sigma \in L^q_{\text{per}}(Z) \) such that \( \text{div} \sigma = 0 \). Thus, for fixed \( x \) and \( y \), we can take \( \sigma(z) = |\eta|^{p-2} \eta(x, y, z) \). Hence \( \eta = 0 \) a.e. in \( \Omega \times Y \times Z \) and \( \xi = \nabla u + \nabla_y u_1 + \nabla_z u_2 \). \( \square \)
Reiterated homogenization of a nonlinear Reynolds-type equation

We finish this section with the “fundamental theorem of three-scale convergence and monotonicity”. A two-scale version of the statement can be found in [59], Theorem 14. Our proof is very similar, but we include it here for the sake of completeness.

**Theorem 6.3.5 (Three-scale convergence and monotonicity).** Assume $a_\varepsilon$ as in (6.4) and let $v_\varepsilon$ be a bounded sequence in $L^p(\Omega; \mathbb{R}^N)$ such that $v_\varepsilon \rightarrow v$ three-scale and $a_\varepsilon(x, v_\varepsilon) \rightarrow \zeta$ three-scale, for $v, \zeta \in L^p(\Omega \times Y \times Z)$. Then

$$\liminf_{\varepsilon \to 0} \int_\Omega a_\varepsilon(x, v_\varepsilon) \cdot v_\varepsilon \, dx \geq \iiint_{\Omega Y Z} \zeta \cdot v \, dx \, dy \, dz$$

(6.17)

and if equality holds, then $\zeta = a(\cdot, v)$.

**Proof.** Let $\Phi(x, y, z)$ be a linear combination of vector fields of the form $(x, y, z) \mapsto \phi(x)\psi(y)\sigma(z)\nu$ where $\phi \in C^\infty_c(\Omega), \psi \in C^\infty_{per}(Y), \sigma \in C^\infty_{per}(Z)$ and $\nu \in S^{N-1}$ (the unit sphere). By monotonicity

$$\int_\Omega (a_\varepsilon(x, v_\varepsilon) - a_\varepsilon(x, \Phi_\varepsilon)) \cdot (v_\varepsilon - \Phi_\varepsilon) \, dx \geq 0.$$ 

where

$$\Phi_\varepsilon(x) = \Phi \left( x, \frac{x}{\varepsilon}, \frac{x^2}{\varepsilon^2} \right).$$

Rearranging the terms yields

$$\int_\Omega a_\varepsilon(x, v_\varepsilon) \cdot v_\varepsilon \, dx \geq \int_\Omega a_\varepsilon(x, v_\varepsilon) \cdot \Phi_\varepsilon + a_\varepsilon(x, \Phi_\varepsilon) \cdot (v_\varepsilon - \Phi_\varepsilon) \, dx.$$

Since the limit, as $\varepsilon \to 0$, of the right hand side of the above inequality exists and is equal to

$$\iiint_{\Omega Y Z} \zeta \cdot \Phi + a(x, y, z, \Phi) \cdot (v - \Phi) \, dx \, dy \, dz$$

we have

$$\liminf_{\varepsilon \to 0} \int_\Omega a_\varepsilon \cdot v_\varepsilon \, dx \geq \iiint_{\Omega Y Z} \zeta \cdot \Phi + a(x, y, z, \Phi) \cdot (v - \Phi) \, dx \, dy \, dz$$

(6.18)

and by density and continuity this also holds for all $\Phi$ in $L^p(\Omega \times Y \times Z)$. Thus, we establish (6.17) by taking $\Phi = v$. 
Next suppose that equality holds in (6.17), then for some $w \in L^p(\Omega \times Y \times Z; \mathbb{R}^N)$ and $t \in \mathbb{R}$, choose $\Phi = v + tw$ in (6.18) to obtain

$$0 \geq t \iiint_{\Omega Y Z} (\zeta - a(x, y, z, v + tw)) \cdot w \, dx \, dy \, dz.$$ 

Dividing by $t$ and using the continuity of $a$ we let $t \to 0^\pm$, thus obtaining

$$\iiint_{\Omega Y Z} (\zeta - a(x, y, z, v)) \cdot w \, dx \, dy \, dz = 0$$

for all $w \in L^p(\Omega \times Y \times Z; \mathbb{R}^N)$. Hence $\zeta(x, y, z) = a(x, y, z, v(x, y, z))$ almost everywhere.

### 6.4 A three-scale homogenization procedure

Based on Theorems 6.3.2, 6.3.4 and 6.3.5, we outline a homogenization procedure for the problem (6.1).

In view of estimate (6.9) and the following remark, the sequence of solutions $u_\varepsilon$ to (6.7) is bounded in $W^{1,p}_0(\Omega)$. Applying Theorems 6.3.2 and 6.3.4 we can find $u \in W^{1,p}_0(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Y))$, $u_2 \in L^p(\Omega \times Y; W^{1,p}_{\text{per}}(Z))$ and $\zeta \in L^q(\Omega \times Y \times Z; \mathbb{R}^N)$ such that up to a subsequence

1. $u_\varepsilon \to u$ three-scale,
2. $\nabla u_\varepsilon \to \nabla u + \nabla_y u_1 + \nabla_z u_2$ three-scale,
3. $a_\varepsilon(x, \nabla u_\varepsilon) \to \zeta$ three-scale.

Passing to the limit in the weak formulation (6.7) gives

$$\iiint_{\Omega Y Z} \zeta \cdot \nabla \phi \, dz \, dy \, dx = \iiint_{\Omega Y Z} b \cdot \nabla \phi \, dy \, dz \, dx.$$

Let the test function $\phi$ in (6.7) be $\phi(x) = \varepsilon \phi_1(x)w_1(x/\varepsilon)$, where $\phi_1 \in C^\infty_0(\Omega)$, $w_1 \in C^\infty_{\text{per}}(Y)$. Then

$$\int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot (\varepsilon w_1 \nabla \phi_1 + \phi_1 \nabla w_1) \, dx = \int_{\Omega} b_\varepsilon \cdot (\varepsilon w_1 \nabla \phi_1 + \phi_1 \nabla w_1) \, dx.$$

In the limit as $\varepsilon \to 0$ we obtain

$$\iiint_{\Omega Y Z} \zeta \cdot \phi_1(x)\nabla w_1(y) \, dz \, dy \, dx = \iiint_{\Omega Y Z} b \cdot \phi_1(x)\nabla w_1(y) \, dz \, dy \, dx.$$
Reiterated homogenization of a nonlinear Reynolds-type equation

Taking as test function \( \phi(x) = \varepsilon^2 \phi_1(x) \phi_2(x/\varepsilon) w_2(x/\varepsilon^2) \), where \( \phi_1 \in C_c^\infty(\Omega) \), \( \phi_2 \in C^\infty_{\text{per}}(Y) \) and \( w_2 \in C^\infty_{\text{per}}(Z) \), yields

\[
\int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot \left( \varepsilon^2 w_2 \phi_2 \nabla \phi_1 + \varepsilon w_2 \phi_1 \nabla \phi_2 \left( \frac{x}{\varepsilon^2} \right) + \phi_1 \phi_2 \nabla w_2 \left( \frac{x}{\varepsilon^2} \right) \right) \, dx
= \int_{\Omega} b_\varepsilon \cdot \left( \varepsilon^2 w_2 \phi_2 \nabla \phi_1 + \varepsilon w_2 \phi_1 \nabla \phi_2 \left( \frac{x}{\varepsilon^2} \right) + \phi_1 \phi_2 \nabla w_2 \left( \frac{x}{\varepsilon^2} \right) \right) \, dx.
\]

In the limit

\[
\int_{\Omega Y Z} \zeta \cdot \phi_1(x) \phi_2(y) \nabla w_2(z) \, dz \, dy \, dx = \int_{\Omega Y Z} b \cdot \phi_1(x) \phi_2(y) \nabla w_2(z) \, dz \, dy \, dx.
\]

By density it follows that \( \zeta \) satisfies

\[
\int_{\Omega Y Z} \zeta \cdot (\nabla \phi + \nabla_y \phi_1 + \nabla_z \phi_2) \, dz \, dy \, dx
= \int_{\Omega Y Z} b \cdot (\nabla \phi + \nabla_y \phi_1 + \nabla_z \phi_2) \, dz \, dy \, dx \quad (6.19)
\]

for all \( \phi \in W^{1,0}_0(\Omega) \), \( \phi_1 \in L^p(\Omega; W^{1,0}_{\text{per}}(Y)) \), \( \phi_2 \in L^p(\Omega \times Y; W^{1,0}_{\text{per}}(Z)) \). Let us now characterize \( \zeta \). Choosing \( \phi = u \), \( \phi_1 = u_1 \) and \( \phi_2 = u_2 \) in the identity (6.19) gives

\[
\int_{\Omega Y Z} \zeta \cdot (\nabla u + \nabla_y u_1 + \nabla_z u_2) \, dz \, dy \, dx
= \int_{\Omega Y Z} b \cdot (\nabla u + \nabla_y u_1 + \nabla_z u_2) \, dz \, dy \, dx. \quad (6.20)
\]

Taking \( \phi = u_\varepsilon \) in (6.7) yields

\[
\lim_{\varepsilon \to 0} \int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} b_\varepsilon \cdot \nabla u_\varepsilon \, dx
= \int_{\Omega Y Z} b \cdot (\nabla u + \nabla_y u_1 + \nabla_z u_2) \, dz \, dy \, dx. \quad (6.21)
\]

Combining (6.20) and (6.21), we see that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx = \int_{\Omega Y Z} \zeta \cdot (\nabla u + \nabla_y u_1 + \nabla_z u_2) \, dz \, dy \, dx.
\]
According to the fundamental theorem of three-scale convergence and monotonicity (i.e. Theorem 6.3.5) we have that
\[
\lim_{\varepsilon \to 0} \inf \int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \geq \iint_{\Omega Y Z} \zeta \cdot (\nabla u + \nabla_y u_1 + \nabla_z u_2) \, dz \, dy \, dx,
\]
and if (6.22) holds as equality then
\[
\zeta = a(x, y, z, \nabla u + \nabla_y u_1 + \nabla_z u_2).
\]
Inserting (6.23) into (6.19) we obtain the following homogenized variational system
\[
\iint_{\Omega Y Z} a(x, y, z, \nabla u + \nabla_y u_1 + \nabla_z u_2) \cdot (\nabla \phi + \nabla_y \phi_1 + \nabla_z \phi_2) \, dz \, dy \, dx = \iint_{\Omega Y Z} b \cdot (\nabla \phi + \nabla_y \phi_1 + \nabla_z \phi_2) \, dz \, dy \, dx
\]
which is satisfied by \(u, u_1\) and \(u_2\) for all \(\phi \in W^{1,p}_0(\Omega), \phi_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Y)), \phi_2 \in L^p(\Omega \times Y; W^{1,p}_{\text{per}}(Z))\). We have obtained the following partial homogenization result.

**Theorem 6.4.1.** For \(\varepsilon > 0\), let \(u_\varepsilon \in W^{1,p}_0(\Omega)\) denote the solution of
\[
\int_{\Omega} a_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla \phi \, dx = \int_{\Omega} b_\varepsilon \cdot \nabla \phi \, dx \quad \forall \phi \in W^{1,p}_0(\Omega).
\]
Then there exists a subsequence of solutions \(u_{\varepsilon_j}\) such that \(u_{\varepsilon_j} \to u\) weakly and \(\nabla u_{\varepsilon_j} \rightharpoonup \nabla u + \nabla_y u_1 + \nabla_z u_2\) three-scale, as \(\varepsilon_j \to 0\). In addition \(u \in W^{1,p}(\Omega), u_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Y))\) and \(u_2 \in L^p(\Omega \times Y; W^{1,p}_{\text{per}}(Z))\) solve the system (6.24).

A priori, it is not clear that \(u, u_1\) and \(u_2\) are uniquely determined by the fact that they solve the system (6.24). To make the analysis complete it remains to prove this. To this end let \(a^*\) be defined by
\[
a^*(x, y, \xi) = \int_Z a(x, y, z, \xi + \nabla \psi^*) \, dz \quad (x \in \Omega, y, \xi \in \mathbb{R}^N),
\]
where \(\psi^* \in W^{1,p}_{\text{per}}(Z)\) is a solution of the variational problem
\[
\int_Z a(x, y, z, \xi + \nabla \psi^*) \cdot \nabla \psi \, dz = \int_Z b(x, y, z) \cdot \nabla \psi \, dz \quad \forall \psi \in W^{1,p}_{\text{per}}(Z),
\]
and define $a^{**}$ as

$$a^{**}(x, \xi) = \int_Y a^*(x, y, \xi + \nabla \psi^{**}) dy \quad (x \in \Omega, \xi \in \mathbb{R}^N), \quad (6.27)$$

where $\psi^{**} \in W^{1, p}_{\text{per}}(Y)$ solves

$$\int_Y a^*(x, y, \xi + \nabla \psi^{**}) \cdot \nabla \psi dy = \int_Y b^*(x, y) \cdot \nabla \psi dy \quad \forall \psi \in W^{1, p}_{\text{per}}(Y), \quad (6.28)$$

where $b^*(x, y) = \int_Z b(x, y, z) \, dz$. Since $a(x, y, z, \cdot) \in M^{\alpha, \beta}_{\alpha, \beta}(\lambda, \theta)$ it follows by the Browder–Minty theorem, as explained above for (6.7), that for a.e. $(x, y) \in \Omega \times Y$ there exists a solution of (6.26) that is unique up to an additive constant. Thus $a^*$ is well defined, however, we can not say the same for $a^{**}$ unless we know that (6.28) has a solution that is unique (up to a constant). Assume for the moment that existence and uniqueness hold true for (6.28). Next we show how this can be utilized to characterize the homogenized solution $u$ from (6.24).

Taking $\phi = \phi_1 = 0$ and $\phi_2(x, y, z) = \varphi_1(x)\varphi_2(y)\psi(z)$, $\varphi_1 \in C^\infty(\Omega)$, $\varphi_2 \in C^\infty_{\text{per}}(Y)$ and $\psi \in W^{1, p}_{\text{per}}(Z)$, in (6.24) implies that for a.e. $(x, y) \in \Omega \times Y$, $u_2 \in L^p(\Omega \times Y; W^{1, p}_{\text{per}}(Z))$ satisfies

$$\int_Z a(x, y, z, \nabla u(x) + \nabla_y u_1(x, y) + \nabla_z u_2(x, y, z)) \cdot \nabla \psi d z = \int_Z b(x, y, z) \cdot \nabla \psi d z. \quad (6.29)$$

Consequently, by the definition of $a^*$,

$$a^*(x, y, \nabla u(x) + \nabla_y u_1(x, y)) = \int_Z a(x, y, z, \nabla u(x) + \nabla_y u_1(x, y) + \nabla_z u_2(x, y, z)) d z. \quad (6.30)$$

Similarly, by taking $\phi = \phi_2 = 0$ in (6.24) we obtain that for a.e. $x \in \Omega$, $u_1 \in L^p(\Omega; W^{1, p}_{\text{per}}(Y))$ satisfies

$$\int_Y a^*(x, y, \nabla u(x) + \nabla_y u_1(x, y)) \cdot \nabla \psi dy = \int_Y b^*(x, y) \cdot \nabla \psi dy$$

for all $\psi \in W^{1, p}_{\text{per}}(Y)$. Hence

$$a^{**}(x, \nabla u(x)) = \int_Y a^*(x, y, \nabla u(x) + \nabla_y u_1(x, y)) dy.$$
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

\[ \int_{YZ} a(x, y, z, \nabla u(x) + \nabla_y u_1(x, y) + \nabla_z u_2(x, y, z)) \, dz \, dy. \]  (6.31)

Finally, setting \( \phi_1 = \phi_2 = 0 \) in (6.24) and taking (6.25) and (6.27) into account, we see that \( u \in W^{1,p}_0(\Omega) \) satisfies the variational identity

\[ \int_\Omega a^{**}(x, \nabla u) \cdot \nabla \phi \, dx = \int_\Omega b^{**} \cdot \nabla \phi \, dx \quad \forall \phi \in W^{1,p}_0(\Omega), \]  (6.32)

where

\[ b^{**}(x) = \int_{YZ} b(x, y, z) \, dz \, dy. \]  (6.33)

We would like to prove that (6.32) has a unique solution \( u \) and for this we need some properties of \( a^* \) and \( a^{**} \).

Notation. In view of (6.26), any vector field \( \xi : \Omega \times Y \to \mathbb{R}^N \) induces a vector field \( \xi^* : \Omega \times Y \times Z \to \mathbb{R}^N \) defined by

\[ \xi^*(x, y, z) = \xi(x, y) + \nabla \psi^*(z), \]

where \( \psi^* \) is a solution of (6.26). Moreover, if \( \xi \) is a vector field on \( \Omega \), \( \xi^{**} \) is defined as

\[ \xi^{**}(x, y, z) = \xi(x) + \nabla \psi^{**}(y) + \nabla \psi^*(z), \]

where \( \psi^{**} \) solves (6.28) and \( \psi^* \) is a solution of

\[ \int_Z a(x, y, z, \xi(x) + \nabla \psi^{**}(y) + \nabla \psi^*) \cdot \nabla \psi \, dz = \int_Z b(x, y, z) \cdot \nabla \psi \, dz \quad \forall \psi \in W^{1,p}_{\text{per}}(Z). \]

**Theorem 6.4.2.** The function \( a^* \), defined by (6.25), satisfies certain continuity and monotonicity conditions so that existence and uniqueness of (6.28) is guaranteed for a.e. \( x \in \Omega \). Thus the function \( a^{**} \), in (6.27), is well defined. Moreover \( a^{**} \) is sufficiently continuous and monotone in \( \xi \) so that existence and uniqueness of (6.32) follows.

**Sketch of proof.** Set

\[ \alpha^* = \frac{\alpha}{\beta - \alpha} \quad \text{and} \quad \lambda^* = \left( \frac{\lambda}{\theta} \right)^{\alpha^*} \lambda. \]

Utilizing the monotonicity and continuity of \( a \) it follows by some straightforward calculations, for the details of which we refer to [55] (Proposition 3.1) and [60] (Lemma 3.4), that for every \( \xi, \eta \in \mathbb{R}^N \)

\[ |a^*(\cdot, \xi) - a^*(\cdot, \eta)| \leq \lambda^* ||1 + |\xi^*| + |\eta^*||_{L^p(Z)}^{p-1-\alpha^*} |\xi - \eta|^{p^*} \]  (6.34)
Reiterated homogenization of a nonlinear Reynolds-type equation

\[
\left[ a^*(\cdot, \xi) - a^*(\cdot, \eta) \right] \cdot (\xi - \eta) \geq \theta \frac{|\xi - \eta|^\beta}{\|1 + |\xi^*| + |\eta^*|\|^\beta - p}_{L^p(Z)} \tag{6.35}
\]

holds a.e. in \( \Omega \times Y \), and

\[
|a^{**}(\cdot, \xi) - a^{**}(\cdot, \eta)| \leq \lambda^* \|1 + |\xi^{**}| + |\eta^{**}|\|^p - \alpha^*_{L^p(Y \times Z)} |\xi - \eta|^{\alpha^*} \tag{6.36}
\]

\[
\left[ a^{**}(\cdot, \xi) - a^{**}(\cdot, \eta) \right] \cdot (\xi - \eta) \geq \theta \frac{|\xi - \eta|^\beta}{\|1 + |\xi^{**}| + |\eta^{**}|\|^\beta - p}_{L^p(Y \times Z)} \tag{6.37}
\]

holds a.e. in \( \Omega \).

Next we show that there exists a \( c(x) \) such that for any vector field \( \xi \in L^p(Y; \mathbb{R}^N) \) it holds that

\[
\|\xi^*\|\|L^p(Y \times Z) \leq c(x)(1 + \|\xi\|_{L^p(Y)}) \quad \text{with } c(x) < \infty \text{ for a.e. } x \in \Omega. \tag{6.38}
\]

First note the lower bound

\[
\|\xi^*\|_{L^p(Y \times Z)} \geq \|\xi\|_{L^p(Y)}.
\]

Indeed

\[
\|\xi\|_{L^p(Y)}^p = \int_Y \left( \int_Z |\xi|^p \, dz \right) dy \leq \int \|\xi^*\|^p dy.
\]

We now seek to establish an upper bound of \( \|\xi^*\|_{L^p(Y \times Z)} \). Without loss of generality we may assume \( \|\xi^*\|_{L^p(Y \times Z)} \geq 1 \).

On the one hand (6.3c), Hölder’s inequality and the triangle inequality gives

\[
\int \int a(x,y,z,\xi^* \cdot \xi^* \, dz \, dy \geq \theta \int \int \frac{|\xi^*|^\beta}{(1 + |\xi^*|)^\beta - p} \, dz \, dy \geq \theta \frac{2}{\beta - p} \|\xi^*\|_{L^p(Y \times Z)}^p. \tag{6.39}
\]

On the other hand (6.5) and (6.28) implies

\[
\int \int a(x,y,z,\xi^* \cdot \xi^* \, dz \, dy = \int \int a(x,y,z,\xi^* \cdot \xi \, dz \, dy + \int \int b \cdot (\xi^* - \xi) \, dz \, dy \leq \lambda \int \|\xi\|(1 + |\xi^*|)^{p-1} \, dz \, dy + \|b\|_{L^q(Y \times Z)} \|\xi^* - \xi\|_{L^p(Y \times Z)} \leq 2^{p-1} \lambda \|\xi\|_{L^p(Y \times Z)} \|\xi^*\|_{L^p(Y \times Z)}^{-1} + 2 \|b\|_{L^q(Y \times Z)} \|\xi^*\|_{L^p(Y \times Z)} \tag{6.40}
\]

\]
Combining (6.39) and (6.40) we see that
\[ \|\xi^*\|_{L^p(Y \times Z)} \leq \text{const} (\|\xi\|_{L^p(Y)} + \|b\|_{L^q(Y \times Z)}) \tag{6.41} \]

Since \( b \in L^q(\Omega; \mathcal{C}_{\text{per}}(Y \times Z)) \), we have
\[ \|b(x, \cdot)\|_{L^q(Y \times Z)} \leq \|b(x, \cdot)\|_{C(Y \times Z)}^q < \infty \quad \text{for a.e. } x \in \Omega. \]

This establishes (6.38).

By estimating
\[ \iiint a(x, y, z, \xi^{**}) \cdot \xi^{**} \, dz \, dy \, dx \]

similarly, one can show that there exists a constant \( c \) that depends on \( p, \lambda, \theta, \beta \) and \( \|b\|_{L^q(\Omega; \mathcal{C}(Y \times Z))} \) such that for any \( \xi \in L^p(\Omega; \mathbb{R}^N) \)
\[ \|\xi^{**}\|_{L^p(\Omega \times Y \times Z)} \leq c(1 + \|\xi\|_{L^p(\Omega)}) \tag{6.42} \]

Summing up we have the following homogenization result.

**Theorem 6.4.3.** Let \( a^*, a^{**} \) and \( b^{**} \) be defined by (6.25), (6.27) and (6.33) respectively. Both \( a^* \) and \( a^{**} \) are well defined. For \( \varepsilon > 0 \), let \( u_\varepsilon \) denote the solution of (6.1). Then the whole sequence \( u_\varepsilon \) converges weakly to \( u \) as \( \varepsilon \to 0 \), where \( u \) is the unique weak solution of the homogenized problem
\[ \begin{align*}
\text{div} a^{**}(x, \nabla u) &= \text{div} b^{**} \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.\end{align*} \tag{6.43} \]

**6.5 The linear case**

Let us consider the special case that \( a(x, y, z, \cdot) \in \mathcal{M}(\lambda, \theta) \). Since \( a \) is assumed to be linear we write
\[ a(x, y, z, \xi) = a(x, y, z)\xi \quad \text{and} \quad \{a_{ij}(x, y, z)\}_{1 \leq i, j \leq N} \]

denotes the corresponding matrix. Note that conditions (6.3b) and (6.3c) are satisfied if \( a_{ij} \in L^\infty(\Omega; \mathcal{C}_{\text{per}}(Y \times Z)) \) and \( \{a_{ij}\} \) is (uniformly) positive definite, i.e.
\[ \max_{1 \leq i, j \leq N} \|a_{ij}\|_{L^\infty(\Omega; \mathcal{C}_{\text{per}}(Y \times Z))} \leq \lambda \quad \text{and} \]

Reiterated homogenization of a nonlinear Reynolds-type equation

\[
\sum_{1 \leq i, j \leq N} a_{ij}(x, y, z)\xi_j \xi_i \geq \theta |\xi|^2
\]

for all \( \xi \in \mathbb{R}^N \) and a.e. \((x, y, z) \in \Omega \times Y \times Z\). The equation corresponding to (6.1) then becomes

\[
\text{div}(a_\varepsilon \nabla u_\varepsilon) = \text{div} b_\varepsilon \quad \text{in} \ \Omega.
\]

In the linear situation the analysis is essentially simplified in the sense that one only has to solve \( N + 1 \) local problems corresponding to the \( z \)-scale and another \( N + 1 \) problems corresponding to the \( y \)-scale, instead of infinitely many local problems (one for each \( \xi \)), to obtain the homogenized equation (6.43). Let \( \{e_i\}_{i=1, \ldots, N} \) denote the standard basis in \( \mathbb{R}^N \). Due to the linearity of \( a \), as solution \( \psi^*_\xi \) of (6.26), i.e. a solution of

\[
\int_Z a(x, y, z)(\xi + \nabla \psi^*_\xi) \cdot \nabla \psi \, dz = \int_Z b(x, y, z) \cdot \nabla \psi \, dz \quad \forall \psi \in W^{1, p}_{\text{per}}(Z)
\]

can be written in the form

\[
\psi^*_\xi = \psi^*_0 + \sum_{i=1}^N \xi_i \chi^*_i,
\]

where \( \psi^*_0 \) and \( \chi^*_i \) are solutions of

\[
\int_Z a(x, y, z)\nabla \psi^*_0 \cdot \nabla \psi \, dz = \int_Z b(x, y, z) \cdot \nabla \psi \, dz \quad \forall \psi \in W^{1, p}_{\text{per}}(Z) \quad (6.44a)
\]

and

\[
\int_Z a(x, y, z)(e_i + \nabla \chi^*_i) \cdot \nabla \psi \, dz = 0 \quad \forall \psi \in W^{1, p}_{\text{per}}(Z) \quad (i = 1, \ldots, N). \quad (6.44b)
\]

Let \( a_1(x, y) \) be the matrix and \( b_1(x, y) \) the vector defined by

\[
a_1(x, y)e_i = \int_Z a(x, y, z)(e_i + \nabla \chi^*_i) \, dz, \quad (6.45a)
\]

\[
b_1(x, y) = \int_Z b(x, y, z) - a(x, y, z)\nabla \psi^*_0 \, dz. \quad (6.45b)
\]

Then, \( \psi^{**}_\xi \) solving (6.28) is equivalent to \( \psi^{**}_\xi \) solving

\[
\int_Y a_1(x, y)(\xi + \nabla \psi^{**}_\xi) \cdot \nabla \psi \, dy = \int_Y b_1(x, y) \cdot \nabla \psi \, dy \quad \forall \psi \in W^{1, p}_{\text{per}}(Y).
\]
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

By linearity, the solution $\psi^{**}_\xi$ can be written

$$\psi^{**}_\xi = \psi^{**}_0 + \sum_{i=1}^{N} \xi_i \chi^{**}_i,$$

where $\psi^{**}_0$ and $\chi^{**}_i$ are solutions of

$$\int_Y a_1(x, y) \nabla \psi_0^{**} \cdot \nabla \psi \, dy = \int_Y b_1(x, y) \cdot \nabla \psi \, dy \quad \forall \psi \in W^{1, p}_{\text{per}}(Y) \quad (6.46a)$$

and

$$\int_Y a_1(x, y)(e_i + \nabla \chi^{**}_i) \cdot \nabla \psi \, dy = 0 \quad \forall \psi \in W^{1, p}_{\text{per}}(Y) \quad (i = 1, \ldots, N). \quad (6.46b)$$

Let $a_0(x)$ be the matrix and $b_0(x)$ the vector defined by

$$a_0(x) e_i = \int_Y a_1(x, y)(e_i + \nabla \chi^{**}_i) \, dy, \quad (6.47a)$$

$$b_0(x) = \int_Y b_1(x, y) - a_1(x, y) \nabla \psi_0^{**} \, dy. \quad (6.47b)$$

Summing up we have the following homogenization algorithm:

1. Solve the $N+1$ local problems (6.44) on the $z$-scale and use these solutions to compute the matrix $a_1(x, y)$ and the vector $b_1(x, y)$ in (6.45).

2. Solve the $N+1$ local problems (6.46) on the $y$-scale and use these solutions to compute the matrix $a_0(x)$ and the vector $b_0(x)$ in (6.47).

3. Solve the homogenized equation

$$\text{div}(a_0 \nabla u) = \text{div} \, b_0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega. \quad (6.48)$$

### 6.6 Application to hydrodynamic lubrication

The Reynolds equation is a two-dimensional model that describes the flow in a thin film of viscous fluid (lubricant) that is enclosed between two rigid surfaces in relative motion. Reynolds equation is used by engineers to compute the pressure distribution in various situations of hydrodynamic lubrication,
Reiterated homogenization of a nonlinear Reynolds-type equation

e.g. slider bearings or journal bearings. A simple form of Reynolds equation reads

\[
\frac{\partial}{\partial x_1} \left( \frac{h^3}{12 \mu} \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{h^3}{12 \mu} \frac{\partial p}{\partial x_2} \right) = \frac{v}{2} \frac{\partial h}{\partial x_1} \quad \text{in } \Omega, \tag{6.49}
\]

where \( p \) is the unknown pressure distribution, \( \Omega \) is the “bearing domain”, \( h: \Omega \to \mathbb{R} \) denotes the thickness of the film, \( \mu \) is the lubricant viscosity (taken as constant) and \( v \) is the constant speed of the moving surface (the other remains fixed). To study the influence of multiscale surface roughness with two characteristic wavelengths, \( \varepsilon \) and \( \varepsilon^2 \), upon the pressure solution we make the ansatz that the film thickness function \( h_\varepsilon \) is given by

\[
h_\varepsilon(x) = h \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right), \quad \text{where } h: \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}
\]
is assumed to be continuous, \([0,1]^N\)-periodic in the second and third arguments and satisfy \( \theta \leq h^3 \leq \lambda \). Assuming the boundary condition \( p_\varepsilon = 0 \) on \( \partial \Omega \), we can utilize the homogenization result (6.48) with

\[
a_\varepsilon(x) = \begin{pmatrix} h_\varepsilon(x)^3 & 0 \\ 0 & \frac{k h_\varepsilon(x)^3}{320} \end{pmatrix}, \quad b_\varepsilon(x) = \left( 6 \mu v h_\varepsilon, 0 \right) \quad \text{and} \quad u_\varepsilon = p_\varepsilon.
\]

The Reynolds equation (6.49) may be a good approximation for Newtonian fluids, but it does not take into account the particular behaviour of non-Newtonian fluids. Attempts have therefore been made to modify (6.49) so as to capture non-Newtonian effects. For example He [37] has suggested the following Reynolds-type equation for one-dimensional incompressible non-Newtonian lubrication

\[
\frac{d}{dx} \left( \frac{h^3}{12 \mu_0} \frac{dp}{dx} + \frac{k h^5}{320} \left( \frac{dp}{dx} \right)^3 \right) = \frac{v}{2} \frac{dh}{dx} \quad \text{in } \Omega, \tag{6.50}
\]

where \( \mu_0 \) is the zero shear rate viscosity and \( k \geq 0 \) denotes a nonlinear factor accounting for non-Newtonian effects that comes from the Rabinowitsch constitutive relation.

To write (6.50) in the form (6.1), choose

\[
a_\varepsilon(x, \xi) = h_\varepsilon^3 \xi + k_1 h_\varepsilon^5 \xi^3, \quad \text{and} \quad b_\varepsilon = k_2 h_\varepsilon,
\]

where \( k_1 = 3 k \mu_0 / 80 \) and \( k_2 = 6 \mu_0 v \). The homogenization result (6.43) then applies with \( p = 4, \alpha = 1 \) and \( \beta = 4 \).
6.7 A convergence result for periodic functions

The objective of this section is to prove Lemma 6.3.3. First we state a result pertaining to the image of the $\text{div}$-operator.

**Theorem 6.7.1.** For each $f \in L^p_{\text{per}}(Y)$ with mean value zero, there exists a vector field $F \in L^p_{\text{per}}(Y; \mathbb{R}^N)$ such that
\[
\int_{\mathbb{R}^N} f \phi \, dx = - \int_{\mathbb{R}^N} F \cdot \nabla \phi \, dx \tag{6.51}
\]
for all $\phi \in C_c^\infty(\mathbb{R}^N)$. In other words $f = \text{div} F$. Moreover, there exists a constant $C > 0$ such that
\[
\|F\|_{L^p(Y; \mathbb{R}^N)} \leq C \|f\|_{L^p(Y)}.
\]

**Proof.** Consider the periodic $q$-Poisson equation
\[
-\Delta_q u = f \quad \text{in } \mathbb{R}^N
\]
\[
u \text{ is } Y\text{-periodic}, \tag{6.52}
\]
where $\Delta_q$ is the $q$-Laplace operator defined for $q \geq 1$ by
\[
\Delta_q u = \text{div} \left( |\nabla u|^{q-2} \nabla u \right).
\]

By the direct methods of the calculus of variations or the theory for monotone operators it can be shown that, for $f \in L^p_{\text{per}}(Y)$, there exists a weak solution $u \in W^{1,q}_{\text{per}}(Y)$ of (6.52) that satisfies
\[
\int_Y |\nabla u|^{q-2} \nabla u \cdot \nabla v \, dy = \int_Y f v \, dy \quad \forall v \in W^{q,1}_{\text{per}}(\Omega). \tag{6.53}
\]
Thus $F = - |\nabla u|^{q-2} \nabla u \in L^p_{\text{per}}(Y; \mathbb{R}^N)$ satisfies (6.51). Taking $v = u$ in (6.53) we obtain by Hölder's inequality
\[
\int_Y |\nabla u|^q \, dy \leq \left( \int_Y |f|^p \, dy \right)^{\frac{1}{p}} \left( \int_Y |u|^q \, dy \right)^{\frac{1}{q}}.
\]
Noting that $|F|^p = |\nabla u|^q$ combined with The Poincaré–Wirtinger inequality yields
\[
\left( \int_Y |F|^p \, dy \right)^{\frac{1}{p}} \leq C \left( \int_Y |f|^p \, dy \right)^{\frac{1}{p}}.
\]
Reiterated homogenization of a nonlinear Reynolds-type equation

It is well known (and frequently used in homogenization theory) that for \( f: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R} \) continuous and \( Y \)-periodic in the second argument, the function \( f_\varepsilon(x) = f(x, x/\varepsilon) \) converges weakly in \( L^p(\Omega) \) \((1 \leq p < \infty)\) to the function \( x \mapsto \int_Y f(x, y) \, dy \). This result (and its generalizations) is commonly referred to as the “mean value property” or “generalized Riemann–Lebesgue lemma”.

Suppose in addition that for each \( x \in \Omega \), \( f \) satisfies \( \int_Y f(x, y) \, dy = 0 \). Then we can conclude that \( f_\varepsilon \to 0 \) weakly in \( L^p(\Omega) \). The example

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \sin \left( \frac{x}{\varepsilon} \right) \phi \, dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \cos \left( \frac{x}{\varepsilon} \right) \phi' \, dx = 0 \quad (\phi \in C^\infty_c(0, 1)),
\]

i.e. \( \Omega = (0, 1) \) and \( f(x, y) = \sin y \), serves to illustrate that the sequence of functions

\[
\frac{1}{\rho(\varepsilon)} f_\varepsilon
\]

may also converge, in some sense, provided \( \rho(\varepsilon) \to 0 \) sufficiently slowly as \( \varepsilon \to 0 \). The precise statement, along with the obvious generalization to three scales, are given by the following theorems, for the case that \( f(x, y) = \phi(x) \psi(y) \), \( \phi \in C^\infty_c(\overline{\Omega}), \psi \in C^\infty_{per}(Y) \).

For the case \( p = 2 \) similar versions of these theorems can be found in [4].

**Lemma 6.7.2.** Assume \( \psi \in C^\infty_c(\Omega) \) and \( f \in L^p_{per}(Y) \) satisfies

\[
\int_Y f \, dy = 0.
\]

Then the sequence of functions \( f_\varepsilon \), defined for a.e. \( x \in \Omega \) by

\[
f_\varepsilon(x) = \frac{1}{\varepsilon} \psi(x) f \left( \frac{x}{\varepsilon} \right),
\]

converges weak* to 0 in \( W^{-1,p}(\Omega) \).

**Proof.** According to Theorem 6.7.1 there exists \( F \in L^p_{per}(Y; \mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} f \phi \, dx = - \int_{\mathbb{R}^N} F \cdot \nabla \phi \, dx \quad \forall \phi \in C^\infty_c(\mathbb{R}^N).
\]

Choosing as test functions \( x \mapsto \psi(\varepsilon x) \phi(\varepsilon x) \), where \( \phi \in C^\infty_c(\mathbb{R}^N) \) is arbitrary, we obtain by the chain rule and a change of variables

\[
\int_{\mathbb{R}^N} f_\varepsilon \phi \, dx = - \int_{\mathbb{R}^N} F \left( \frac{x}{\varepsilon} \right) \cdot \nabla (\psi \phi) \, dx \quad \forall \phi \in C^\infty_c(\mathbb{R}^N)
\]
By density we can replace $\phi$ with any $v \in W^{1,q}_0(\Omega)$. Taking the limit, we obtain
\[
\lim_{\varepsilon \to 0} (f_\varepsilon, v) = -\int_\Omega \left( \int_Y F \, dy \right) \cdot \nabla (v) \, dx = 0.
\]

The strong generalization of Lemma 6.7.2, by adding the scale $\varepsilon^2$, can not be obtained right away. First we need to establish a weaker result.

**Lemma 6.7.3.** Assume $\psi_1 \in C^\infty_c(\Omega)$, $\psi_2 \in C^\infty_{\text{per}}(Y)$ and $f \in L^p_{\text{per}}(Z)$ satisfies
\[
\int_Z f \, dz = 0.
\]
Then the sequence of functions $f_\varepsilon$, defined for a.e. $x \in \Omega$ by
\[
f_\varepsilon(x) = \frac{1}{\varepsilon} \psi_1(x) \psi_2 \left( \frac{x}{\varepsilon} \right) f \left( \frac{x}{\varepsilon^2} \right),
\]
converges weak$^*$ to 0 in $W^{-1,p}(\Omega)$.

**Proof.** There exists $F \in L^p_{\text{per}}(Z; \mathbb{R}^N)$ such that
\[
\int_{\mathbb{R}^N} f(x) \psi_1(\varepsilon^2 x) \psi_2(\varepsilon x) \phi(\varepsilon^2 x) \, dx = -\int_{\mathbb{R}^N} F(x) \cdot \nabla \left( \psi_1(\varepsilon^2 x) \psi_2(\varepsilon x) \phi(\varepsilon^2 x) \right) \, dx
\]
for all $\phi \in C^\infty_c(\mathbb{R}^N)$. After a change of variables we obtain
\[
\int_{\mathbb{R}^N} f_\varepsilon \phi \, dx = -\varepsilon \int_{\mathbb{R}^N} \psi_2 \left( \frac{x}{\varepsilon^2} \right) F \left( \frac{x}{\varepsilon} \right) \nabla (\psi_1 \phi) \, dx + \int_{\mathbb{R}^N} \left( F \left( \frac{x}{\varepsilon^2} \right) \cdot \nabla \psi_2 \left( \frac{x}{\varepsilon} \right) \right) \psi_1 \phi \, dx.
\]

Thus
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} f_\varepsilon \phi \, dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left( F \left( \frac{x}{\varepsilon^2} \right) \cdot \nabla \psi_2 \left( \frac{x}{\varepsilon} \right) \right) \psi_1 \phi \, dx
\]
\[
= \int_{\mathbb{R}^N} \left( \int_Z F(\varepsilon) \, dz \right) \cdot \left( \int_Y \nabla \psi_2 \, dy \right) \psi_1 \phi \, dx = 0,
\]
where we used the periodicity of $\psi_2$ in the last equality.

We can now give the postponed proof of Lemma 6.3.3.
Proof of Lemma 6.3.3. According to the proof of Lemma 6.7.3 we have

\[ \int_{\mathbb{R}^N} f_\varepsilon \phi \, dx \]

\[ = - \int_{\mathbb{R}^N} \psi_2 \left( \frac{x}{\varepsilon} \right) F \left( \frac{x}{\varepsilon^2} \right) \cdot \nabla (\psi_1 \phi) \, dx + \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left( F \left( \frac{x}{\varepsilon} \right) \cdot \nabla \psi_2 \left( \frac{x}{\varepsilon} \right) \right) \psi_1 \phi \, dx. \]

By Riemann–Lebesgue lemma, the first term tends to

\[ - \int_{\mathbb{R}^N} \left( \int_Y \psi_2 \, dy \right) \left( \int_Z F \, dz \right) \cdot \nabla (\psi_1 \phi) \, dx = 0. \]

Upon noting that the second term can be written as

\[ \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \left( F \left( \frac{x}{\varepsilon} \right) - \mu \right) \cdot \nabla \psi_2 \left( \frac{x}{\varepsilon} \right) \psi_1 \phi \, dx + \int_{\Omega} \frac{1}{\varepsilon} \mu \cdot \nabla \psi_2 \left( \frac{x}{\varepsilon} \right) \psi_1 \phi \, dx, \]

where \( \mu \in \mathbb{R}^N \) denotes the average of \( F \) over \( Z \), we appeal to Lemma 6.7.3 and Lemma 6.7.2 to conclude that

\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} f_\varepsilon \phi \, dx = 0. \]

By density we can take \( \phi \in W^{1,q}_0(\Omega) \) and the proof is complete. \( \square \)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method
Chapter 7

Linear parabolic problems with singular coefficients in non-cylindrical domains

7.1 Introduction

Let us consider in $\mathbb{R}^{N+1}, N \in \mathbb{Z}_+$, the domain

$$Q = \{(x, t) : x \in \Omega_t, 0 < t < T\},$$

(7.1)

where $(0, T)$ is a finite interval, $\Omega_t \in C^{0,1}(\mathbb{R}^N)$ (here, $C^{0,1}(\mathbb{R}^N)$ is a set of all bounded domains in $\mathbb{R}^N$, whose boundary can be locally described by a function from $C^{0,1}(\Delta)$, where $\Delta \subset \mathbb{R}^{N-1}$ is a cube; see [48]) and for every $t, s \in (0, T), t < s$, it is

$$\emptyset \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T$$

(see Figure 7.1).

Let us consider the problem

$$d^\lambda(x, t)\frac{\partial u}{\partial t} + A(x, t)u = f(x, t) \quad \text{in} \quad Q,$$

(7.2)

$$u(x, 0) = 0 \quad \text{in} \quad \Omega_0,$$

(7.3)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

Figure 7.1:

\[ u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = \cdots = \frac{\partial^{k-1} u}{\partial (\nu^{k-1})}(x, t) = 0 \quad 0 < t < T, \quad x \in \partial \Omega_t, \]

(7.4)

where \( \lambda \) is a real parameter, \( d(x, t) = \text{dist}(x, \partial \Omega_t) \) and \( A \) is a linear differential operator of order \( 2k(k \in \mathbb{Z}_+) \) in the divergence form:

\[
(Au)(x, t) = \sum_{|i|, |j| \leq k} (-1)^{|i|} \partial^j (a_{i,j}(x, t) \partial^i u)
\]

for \((x, t) \in Q\) and \( \nu \) the outer normal to \( \partial \Omega_t \). The coefficients \( a_{i,j} \) of this operator may have singularities, which indicates that we have to work with weighted Sobolev spaces \( W^{k,2}(\Omega_t, \lambda) \) (for their properties see [46]), and the right hand side \( f \) is a function defined a.e. in \( Q \).

If \( Q \) is a cylindrical domain, then there exists a number of papers where even more generalized versions of the problem (7.2) – (7.4) has been solved by the method of Rothe. Most of these papers are connected with the names of K. Rektorys, J. Kačur and V. Pluschke (see e.g. [42], [43], [70], [71], [72]). However, in the case when \( Q \) is a non-cylindrical domain this type of problems are much less studied. For example, in [32] and [49] some problems of this type were solved by the transformation method, but with further restriction on the domain \( Q \). Moreover, in [53] the problem (7.2) – (7.4) was solved when \( \lambda = 0 \) and the operator \( A \) is strongly elliptic.
In this paper we will make a more unified approach and prove some generalizations of the results in [53], which also are generalizations of some special cases of the before mentioned papers in the cylindrical case. In particular, we will deal with non-cylindrical domains and permit singularities in the coefficients of the operator \( A \). It is organized as follows: In order not to disturb our discussions later on we present some preliminaries (including a development and discussion of a variant of Rothe’s method suited for our problem) and prove and discuss some auxiliary results in Section 7.2. Our main result (Theorem 7.1) is formulated and proved in Section 7.3. Finally, Section 7.4 is reserved for some concluding examples and results (see Theorems 7.2 and 7.3).

### 7.2 Some preliminaries and auxiliary results

Let us denote by \((\cdot, \cdot)_\tau\) the inner product in \( L_2(\Omega_\tau) \), defined by

\[
((u, v))_{(t, \tau)} = (A(t)u, v)_\tau = \sum_{|i|, |j| \leq k} \int_{\Omega_\tau} a_{i,j}(x, t) \partial^i u(x) \partial^j v(x) dx
\]

the bilinear form associated with the operator \( A \), where we assume that for \( \tau > t \) the coefficients of \( A \) are extended as (cf. Figure 1.1)

\[
a_{i,j}(x, t) = \begin{cases} 
    a_{i,j}(x, t), & x \in \Omega_t, \\
    0, & x \in \Omega_\tau \setminus \Omega_t.
\end{cases}
\]

We need a number of technical assumptions, which we for simplicity collect as follows:

**Assumption 7.1.** Let \( t \in (0, T), M > 0 \) and \( \mu \leq 0 \).

(A1) The parameter \( \lambda \) belongs to \([\mu - k, \mu]\).

The bilinear form \((u, v))_{(t, \tau)}\) satisfies the following conditions:

(A2) Boundedness, i.e.

\[
((u, v))_{(t, \tau)} \leq M \|u\|_{W^{k,2}(\Omega_t, \mu)} \|v\|_{W^{k,2}(\Omega_t, \mu)} \quad \text{for all } u, v \in W^{k,2}(\Omega_t, \mu).
\]

(A3) \( V_t \)-ellipticity, i.e.

\[
((u, u))_{(t, \tau)} \geq \frac{1}{M} \|u\|_{W^{k,2}(\Omega_t, \mu)}^2 \quad \text{for all } u \in V_t := W^{k,2}_0(\Omega_t, \mu).
\]
(A4) $V_t$— symmetry, i.e.

$$((u, v))_{(t,t)} = ((v, u))_{(t,t)} \text{ for all } u, v \in V_t.$$ 

(A5) Lipschitz condition, i.e. for every $h \in (0, T - t)$,

$$\frac{|((u, v))_{(t+h,t)} - ((u, v))_{(t,t)}|}{h} \leq M \|u\|_{W^{k,2}(\Omega_t, \mu)} \|v\|_{W^{k,2}(\Omega_t, \mu)} \text{ for all } u, v \in V_t.$$ 

The function $f$ satisfies the following condition:

(A6) There exists a function $F \in C(I, L_2(\Omega_T)) \cap V_1(I, L_2(\Omega_T))$ such that

$$F(x, t) = f(x, t) d^{-\frac{k}{2}} + k(x, t), \quad (x, t) \in Q$$ 

and we define

$$V(f) := \sup_{t \in (0, T)} \|f(t)\|_{L_2(\Omega_t, -\mu+2k)} + \sup_{\{t_i\}} \sum \|f(t_i) - f(t_{i-1})\|_{L_2(\Omega_{t_{i-1}}, -\mu+2k)}$$

for all finite divisions $\{t_i\}$ of $I$.

The domain $Q$ satisfies the following condition:

(A7) If $\lambda \neq 0$, then there exists a closed subset $S \subset (0, T)$ such that $\nu(S) = 0$ and, for every $\tau \in (0, T) \setminus S$,

$$\lim_{t \searrow \tau} \nu(\Omega_t \setminus \Omega_\tau) = 0,$$

where $\nu$ is the Lebesgue measure in $\mathbb{R}$.

Now we introduce some spaces which we will use in sequel.

Denote by $L_2(I, V_t)$ a subset of the space $L_2(I, V_T)$, i.e.

$$L_2(I, V_t) = \{u(t) \big|_{\Omega_t} \in V_t, \; u(t) \big|_{\Omega_T \setminus \Omega_t} = 0 \; \text{ a.e. in } I \}$$

with the scalar product

$$(u, v)_{L_2(I, V_t)} := \int_0^T (u(t), v(t))_{V_t} dt$$

and similarly we introduce the spaces:

- $L_2(I, L_2(\Omega_t, \mu))$ is a subset of the space $L_2(I, L_2(\Omega_T, \mu))$ with the scalar product
The following useful lemma is a special case of Theorem 8.4 in [46].

**Lemma 7.1.** Let \( \Omega \in C^{0,1}(\mathbb{R}) \) and \( r \) be a nonnegative integer. Then the imbeddings

\[
W^{r,2}_0(\Omega, \mu) \hookrightarrow L_2(\Omega, \mu - 2r)
\]

and

\[
W^{r,2}_0(\Omega, \mu) \hookrightarrow L_2(\Omega, \mu)
\]

hold.

Here and in the sequel the signs \( \hookrightarrow \) and \( \hookrightarrow \hookrightarrow \) mean that we have continuous and compact embedding, respectively.

**Definition 7.1.** Let \( K \) be a subset of \( C(I, L_2(\Omega_t, \mu)) \). Then \( K \) is said to be equicontinuous if to every \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\|u(t) - u(\tau)\|_{L_2(\Omega_t, \mu)} < \varepsilon
\]

holds for all \( u \in K \) and all \( t, \tau \in I \) \((t \leq \tau)\) for which \( \tau - t < \delta \).

**Proposition 7.1.** A set \( K \) in \( C(I, L_2(\Omega_t, \mu)) \) is relatively compact if and only if

- \( K \) is equicontinuous,
- the set \( K(t) = \{u(t); u \in K\} \) is relatively compact in \( L_2(\Omega_t, \mu) \) for any \( t \in I \).

**Proof.** This proposition can be proved analogously as Theorem 1.6.9 in [48] so we omit the details.

For the estimation of the discrete time derivative (Lemma 7.3 below) we need the following nonlinear Gronwall type lemma:
Lemma 7.2. Let $A, B, C, \{d_k\}_{i=1}^j$ and $h$ be given nonnegative constants and let $\alpha_1, \ldots, \alpha_j$ satisfy the conditions: $\alpha_1 \leq A$ and

$$\alpha_i 2 \leq A \alpha_i + Bh \sum_{k=1}^{i-1} \alpha_k^2 + C \sum_{k=1}^{i-1} d_k \alpha_k, \quad i = 2, \ldots, j. \tag{7.5}$$

Then

$$\alpha_i \leq \frac{A}{2} + Ae^{1.5Bjh + \frac{C}{A} \sum_{k=1}^{j} d_k} \quad i = 1, 2, \ldots, j. \tag{7.6}$$

Proof. Let $A > 0$, otherwise there is nothing to prove, since from (7.5) it follows that $\alpha_i = 0$ $i = 1, 2, \ldots, j$. Denote $\beta_k = \frac{\alpha_k}{A} - \frac{1}{2}$ $(k = 1, 2, \ldots, j)$. Then (7.5) takes the form

$$\alpha_i 2 = A2\beta_i 2 + A2\beta_i + A2 \leq A2\beta_i + A2 \frac{4}{2} + Bh \sum_{k=1}^{i-1} A2[\beta_k 2 + \beta_k + \frac{1}{4}] + C \sum_{k=1}^{i-1} d_k A[\beta_k + \frac{1}{2}]$$

and it follows that

$$A2\beta_i 2 \leq \frac{A2}{4} + A2 Bh \sum_{k=1}^{i-1} \beta_k 2 + \sum_{k=1}^{i-1} \beta_k [A2 Bh + ACd_k] + A2B(i-1)h + AC \sum_{k=1}^{i-1} d_k.$$

Hence, by applying the inequality $t \leq t^2 + \frac{1}{4}$ we find that

$$\beta_i 2 \leq \left[ \frac{1}{4} + \frac{B(i-1)h}{4} + \frac{C}{2A} \sum_{k=1}^{i-1} d_k \right]$$

$$+ Bh \sum_{k=1}^{i-1} \beta_k 2 + \sum_{k=1}^{i-1} \beta_k [Bh + \frac{Cd_k}{A}] \leq D + \sum_{k=1}^{i-1} \bar{d}_k \beta_k 2$$

i.e.,

$$\beta_i 2 \leq D + \sum_{k=1}^{i-1} \bar{d}_k \beta_k 2, \tag{7.7}$$

where $D = 1 + Bh + \frac{3C}{4A} \sum_{k=1}^{j} d_k$ and $\bar{d}_k = 2Bh + C \frac{d_k}{A}$. By using (7.7) repeatedly we obtain that

$$\beta_i 2 \leq \bar{d}_{i-1} \beta_{i-1}^2 + D + \sum_{k=1}^{i-2} \bar{d}_k \beta_k 2 \leq \bar{d}_{i-1} (D + \sum_{k=1}^{i-2} \bar{d}_k \beta_k 2) + D + \sum_{k=1}^{i-2} \bar{d}_k \beta_k 2$$
Linear parabolic problems with singular coefficients in non-cylindrical domains

\[
(1 + d_{i-1}) \left( D + \sum_{k=1}^{i-2} d_k \beta_k 2 \right) \leq \ldots \leq (D + d_1) \prod_{k=2}^{i-1} (1 + d_k).
\]

Hence, by using the inequality \( \log(1 + t) \leq t (t > -1) \), we find that

\[
\beta_i 2 \leq e^{\log(D + d_1) + \sum_{k=2}^{i-1} \log(1 + d_k)} \leq e^{D - 1 + \sum_{k=1}^{i-1} d_k}.
\]

This implies that

\[
\alpha_i \leq A \frac{1}{2} e^{\frac{1}{2} (D - 1) + \sum_{k=1}^{i-1} \bar{d}_k},
\]

which, by using (7.7), implies (7.6) and the proof is complete.

We shall prove in the next section that Assumption 7.1 ensures the existence of a weak solution of the problem (7.2) – (7.4) in the sense given in Definition 7.2 and that this weak solution is the limit (in a sense given later) of the sequence \( \{u_n(x,t)\} \) of functions constructed by our version of Rothe’s method described as follows:

**Rothe’s method.** Divide the interval \( I = [0, T] \) into \( n \) subintervals \( I_1, I_2, \ldots, I_n \) (\( I_j = [t_{j-1}, t_j] \), \( t_j = j h \), \( j = 1, 2, \ldots, n \)) of the length \( h = \frac{T}{n} \). According to the initial condition (7.3) we put

\[
z_0(x) = 0, \quad x \in \Omega_T,
\]

for \( t_0 = 0 \) and successively for \( j = 1, 2, \ldots, n \) define functions \( z_j(x) \), which are weak solutions of the following elliptic problems:

\[
\begin{cases}
Az_j + \frac{d}{dt} z_j = f_j + \frac{d}{dt} z_{j-1} & \text{in } \Omega_{t_j}, \\
z_j = \frac{\partial z_j}{\partial x} = \ldots = \frac{\partial^{k-1} z_j}{\partial x^{k-1}} = 0 & \text{on } \partial \Omega_{t_j},
\end{cases}
\]

(7.8)

where \( d_j = d(\cdot, t_j) \) and \( f_j = f(\cdot, t_j) \). We obtain these problems, if we in (7.2) replace the derivative \( \frac{du}{dt} \) by the differential quotient \( \frac{z_{j-1} - z_j}{h} \) in the points \( t = t_j \) and put \( z_{j-1} = 0 \) on \( \Omega_{t_j} \setminus \Omega_{t_{j-1}}, \quad j = 1, 2, \ldots, n \). The weak formulation of the problem (7.8) is as follows:

\[
z_j \in V_{t_j} \quad ((z_j, v))_{(t_j, t_j)} + \frac{1}{h} (d_j^\lambda z_j, v)_{t_j} = (f_j + \frac{d}{dt} z_{j-1}, v)_{t_j} \quad \text{for all } v \in V_{t_j}.
\]

If we define

\[
((u, v)))_{(t_j, t_j)} = ((u, v))_{(t_j, t_j)} + \frac{1}{h} (d_j^\lambda u, v)_{t_j},
\]

Then...
then from (A2), (A3), Lemma 7.1 and the Schwarz inequality it follows that this bilinear form is bounded on \([W^{k,2}(\Omega_{t_j}, \mu)]^2\) and \(V_{t_j}\) - elliptic.

The problem

\[ z_1 \in V_{t_1}; \quad ((z_1, v))_{(t_1, t_1)} = (f_1 + \frac{d}{h} z_0, v)_{t_1} \quad \text{for all} \quad v \in V_{t_1}, \]

has exactly one solution (as a consequence of the theory of elliptic boundary value problems; see, e.g. [71]); here \( f_1 + \frac{d}{h} z_0 = f_1 \in L_2(\Omega_{t_1}, 2k - \mu) \). Further, after redefining again \( z_1 \in V_{t_1} \) as

\[ z_1(x) = \begin{cases} 
  z_1(x), & x \in \Omega_{t_1}, \\
  0, & x \in \Omega_T \setminus \Omega_{t_1},
\end{cases} \]

we get that \( z_1 \in V_T \) for all \( \tau \geq t_1 \). Hence, \( z_1 \in L_2(\Omega_{t_1}, 2k - \mu) \) and

\[ f_2 + \frac{d}{h} z_1 \in L_2(\Omega_{t_2}, 2k - \mu), \quad \text{(here} \quad f_2 + \frac{d}{h} z_1 = (f_2 + \frac{d}{h} z_1)|_{\Omega_{t_2}}). \]

Repeating the above procedure for \( j = 2, 3, ..., n \) we get functions \( z_1, z_2, ..., z_n \).

Now we construct a function \( u_n(x, t) \) called Rothe’s function and define it on \( \Omega_T \times I \), in the following way:

\[
  u_n(x, t) = \begin{cases} 
    z_0(x), & t \in [t_0, t_1], \\
    z_{j-1}(x) + \frac{t-t_j}{t_{j+1}-t_j}(z_j(x) - z_{j-1}(x)), & t \in [t_j, t_{j+1}], \quad j = 1, 2, ..., n - 1.
  \end{cases}
\]

(7.9)

For a fixed \( x \in \Omega_T \), \( u_n(x, t) \) is a piecewise linear function in \( t \) on the interval \( I \) and for \( t = t_j \) it gets values \( z_{j-1}(x) \) as illustrated in Figure 7.2 (for \( n = 10 \)). Hence, we obtain a sequence \( \{u_n(x, t)\}_{n=1}^{\infty} \), which is called Rothe’s sequence of approximate solutions of the problem (7.2) – (7.4).

Intuitively, we can expect that this sequence will converge to some function \( u(x, t) \), which is a solution of the problem (7.2) – (7.4). The next considerations are devoted to derive the necessary preliminaries for the proof of these statements.

**Lemma 7.3.** There exists a number \( C = C(f) \) such that the estimates

\[
  \|u_n\|_{L_2(I, V_1)} \leq C, \quad \text{(7.10)}
\]

\[
  \|\frac{\partial u_n}{\partial t}\|_{L_2(I, L_2(\Omega_{t}, \mu))} \leq C, \quad \text{(7.11)}
\]

hold for all \( n \in \mathbb{N} \).
Proof. Let us consider the integral identity

\[
((z_j, v))(t_j, t_j) + (d^j \frac{z_j - z_{j-1}}{h}, v)t_j = (f_j, v)t_j \quad \text{for all } v \in V_{t_j}, \quad (7.12)
\]

where \(f_j = f(\cdot, t_j), d_j = d(\cdot, t_j)\). We choose \(v = z_j - z_{j-1}\) in the integral identity (7.12), which is reasonable, because \(z_j - z_{j-1} \in V_{t_j}\). Then we get that

\[
((z_j, z_j - z_{j-1}))(t_j, t_j) + (d^j \frac{z_j - z_{j-1}}{h}, z_j - z_{j-1})t_j = (f_j, z_j - z_{j-1})t_j
\]

for \(j = 1, 2, \ldots, n\). After adding both sides of the last equality from \(j = 1\) to \(i\), we find that

\[
\sum_{j=1}^{i}((z_j, z_j - z_{j-1}))(t_j, t_j) + \frac{1}{h} \sum_{j=1}^{i}(d^j \frac{z_j - z_{j-1}}{h}, z_j - z_{j-1})t_j = \sum_{j=1}^{i}(f_j, z_j - z_{j-1})t_j.
\]
Denoting

\[ S_1^i = \sum_{j=1}^{i} ((z_j, z_j - z_{j-1}))(t_j, t_j), \]
\[ S_2^i = \frac{1}{h} \sum_{j=1}^{i} (d_j^i (z_j - z_{j-1}), z_j - z_{j-1})t_j, \]
\[ S_3^i = \sum_{j=1}^{i} (f_j, z_j - z_{j-1})t_j, \]

we can rewrite the last equality as

\[ S_1^i + S_2^i = S_3^i. \] (7.13)

According to (A3) – (A5) we have that

\begin{align*}
S_1^i &= \frac{1}{2} \sum_{j=1}^{i} \{2((z_j, z_j))(t_j, t_j) - 2((z_j, z_{j-1}))(t_j, t_j)\} \\
&= \frac{1}{2} \{((z_i, z_i))(t_i, t_i) + \sum_{j=1}^{i} ((z_j - z_{j-1}, z_j - z_{j-1}))(t_j, t_j) \\
&\quad + \sum_{j=1}^{i} ((z_{j-1}, z_{j-1}))(t_{j-1}, t_{j-1}) - ((z_{j-1}, z_{j-1}))(t_j, t_j)\} \\
&\geq \frac{1}{2} \{((z_i, z_i))(t_i, t_i) + \sum_{j=1}^{i} \{((z_{j-1}, z_{j-1}))(t_{j-1}, t_{j-1}) - ((z_{j-1}, z_{j-1}))(t_j, t_j)\}\}
\end{align*}

\[ \geq \frac{1}{2M} \|z_i\|_{W^{k,2}(\Omega_t, \mu)}^2 - \frac{hM}{2} \sum_{j=1}^{i} \|z_{j-1}\|_{W^{k,2}(\Omega_{t_{j-1}}, \mu)}^2. \]

Hence,

\[ S_1^i \geq \frac{1}{2M} \|z_i\|_{W^{k,2}(\Omega_t, \mu)}^2 - \frac{hM}{2} \sum_{j=1}^{i} \|z_{j-1}\|_{W^{k,2}(\Omega_{t_{j-1}}, \mu)}^2. \] (7.14)

It is easy to see that

\[ S_2^i \geq \frac{C}{h} \sum_{j=1}^{i} \|z_j - z_{j-1}\|_{L^2(\Omega_t, \mu)}^2, \] (7.15)
since $d^k(x, t) \geq Cd^\mu(x, t)$ for all $(x, t) \in Q$, where $C > 0$ is independent on $(x, t)$ (here $\lambda \in [\mu - k, \mu]$, see (A1)).

Moreover, applying the Schwarz inequality and Lemma 7.1 we find that

$$S_i^3 = (f_i, z_i)_{t_i} + \sum_{j=1}^i (f_{j-1} - f_j, z_{j-1})_{t_{j-1}}$$

$$\leq \|f_i\|_{L^2(\Omega_{i, 2k-\mu})}\|z_i\|_{L^2(\Omega_{i, \mu-2k})} + \sum_{j=1}^i \|f_j - f_{j-1}\|_{L^2(\Omega_{j-1, 2k-\mu})}\|z_{j-1}\|_{L^2(\Omega_{j-1, \mu-2k})}$$

$$\leq C \left(\|f_i\|_{L^2(\Omega_{i, 2k-\mu})}\|z_i\|_{W^{k,2}(\Omega_{i, \mu})} + \sum_{j=1}^i \|f_j - f_{j-1}\|_{L^2(\Omega_{j-1, 2k-\mu})}\|z_{j-1}\|_{W^{k,2}(\Omega_{j-1, \mu})}\right),$$

i.e.

$$S_i^3 \leq C \left(\|f_i\|_{L^2(\Omega_{i, 2k-\mu})}\|z_i\|_{W^{k,2}(\Omega_{i, \mu})} + \sum_{j=1}^i \|f_j - f_{j-1}\|_{L^2(\Omega_{j-1, 2k-\mu})}\|z_{j-1}\|_{W^{k,2}(\Omega_{j-1, \mu})}\right).$$

(7.16)

From (7.13) – (7.16) it follows that

$$\frac{1}{2M}\|z_i\|^2_{W^{k,2}(\Omega_{i, \mu})} - \frac{Mh}{2} \sum_{j=1}^i \|z_{j-1}\|^2_{W^{k,2}(\Omega_{j-1, \mu})} \leq S_i^1 \leq S_i^3$$

$$\leq C \left(\|f_i\|_{L^2(\Omega_{i, 2k-\mu})}\|z_i\|_{W^{k,2}(\Omega_{i, \mu})} + \sum_{j=1}^i \|f_j - f_{j-1}\|_{L^2(\Omega_{j-1, 2k-\mu})}\|z_{j-1}\|_{W^{k,2}(\Omega_{j-1, \mu})}\right),$$

and we see that

$$\|z_i\|^2_{W^{k,2}(\Omega_{i, \mu})} \leq 2MV(f)\|z_i\|_{W^{k,2}(\Omega_{i, \mu})}$$

$$+ M^2h \sum_{j=1}^i \|z_{j-1}\|^2_{W^{k,2}(\Omega_{j-1, \mu})} + 2M \sum_{j=1}^i \|f_j - f_{j-1}\|_{L^2(\Omega_{j-1, 2k-\mu})}\|z_{j-1}\|_{W^{k,2}(\Omega_{j-1, \mu})}.$$
and
\[ \alpha_j = \| z_j \|_{W^{k,2}(\Omega_{t_j}, \mu)}, \quad d_j = \| f_j - f_{j-1} \|_{L^2(\Omega_{t_{j-1}}, 2k-\mu)}, \]
we find that
\[ \| z_i \|_{W^{k,2}(\Omega_{t_i}, \mu)} \leq M V(f) + 2 M V(f) e^{M^2 T + 1} \sum_j \| f_j - f_{j-1} \|_{L^2(\Omega_{t_{j-1}}, 2k-\mu)} \]
\[ \leq (1 + 2 e^{1+M^2 T}) M V(f), \]
i.e.
\[ \| z_i \|_{W^{k,2}(\Omega_{t_i}, \mu)} \leq C. \]  \hfill (7.17)
From (7.16) it follows that
\[ S_i,3 \leq C \left[ \sup_I \| f(t) \|_{L^2(\Omega, 2k-\mu)} + \text{Var}(f) \right], \]
where the constant \( C \) is independent of \( i \) and \( n \).

For the estimate of \( S_i,2 \) we note that, according to (7.15), (7.17) and (A5) it follows that
\[ S_i^2 = S_i^3 - S_i^1 \leq C_2 + \sum_{j=1}^i \| (z_{j-1}, z_{j-1})(t_j, t_{j-1}) - ((z_{j-1}, z_{j-1})(t_{j-1}, t_{j-1})) \|_{W^{k,2}(\Omega_{t_{j-1}}, \mu)} \]
\[ \leq C_2 + M h \sum_{j=1}^i \| z_{j-1} \|_{W^{k,2}(\Omega_{t_{j-1}}, \mu)}^2 \]
\[ \leq C_2 + C_1 M h i \leq C_3, \]
i.e.
\[ \frac{1}{h} \sum_{j=1}^i \| z_j - z_{j-1} \|_{L^2(\Omega_{t_j}, \mu)}^2 \leq C_3. \]  \hfill (7.18)
From the definition of the Rothe’s sequence \( \{ u_n(t) \} \) and from (7.17) it follows that
\[ \| u_n(t) \|_{V_i} \leq C \]
for every \( t \in I \) and \( n = 1, 2, \ldots \).

Thus, we have that
\[ \| u_n \|_{L^2(I, V_i)}^2 = \int_0^T \| u_n(t) \|_{V_i}^2 dt \leq C 2 T. \]
Linear parabolic problems with singular coefficients in non-cylindrical domains

It is easy to see that

\[
\frac{\partial u_n(t)}{\partial t} = \begin{cases} 0, & t \in [t_0, t_1], \\ \frac{z_j - z_{j-1}}{h}, & t \in (t_j, t_{j+1}], \quad j = 1, 2, ..., n-1. \end{cases}
\]

From (7.18) it follows that the sequence \( \{\frac{\partial u_n}{\partial t}\}_{n=1}^{\infty} \) is bounded in the space \( L_2(I, L_2(\Omega_t, \mu)) \), i.e.

\[
\| \frac{\partial u_n}{\partial t} \|^2_{L_2(I, L_2(\Omega_t, \mu))} = \int_0^T \| \frac{\partial u_n}{\partial t}(t) \|^2_{L_2(\Omega_t, \mu)} dt = \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} \left\| \frac{z_j - z_{j-1}}{h} \right\|^2_{L_2(\Omega_t, \mu)} dt
\]

\[
= \sum_{j=1}^{n-1} \frac{|z_j - z_{j-1}|^2}{h^2} |t_j - t_{j-1}|
\]

\[
= \frac{1}{h} \sum_{j=1}^{n-1} \frac{|z_{j+1} - z_j|^2}{L_2(\Omega_{t_j}, \mu)} \leq C.
\]

The proof is complete.

Lemma 7.4. The Rothe sequence \( \{u_n(t)\}_{n=1}^{\infty} \) satisfies the inequality

\[
\| u_n(t) - u_n(\tau) \|_{L_2(\Omega_t, \mu)} \leq C (t - \tau)^{\frac{3}{2}} \text{ for all } t, \tau \in I \quad (t \geq \tau), \quad (7.19)
\]

where \( C \) is a constant, which is independent of \( n \).

Proof. Without loss of the generality it can be assumed that \( t \in I_i = [t_{i-1}, t_i] \) and \( \tau \in I_k = [t_{k-1}, t_k] \), where \( 1 \leq k \leq i \leq n \).

- If \( i = k \), then the estimation (7.19) is obtained directly, i.e. according to (7.11), we have that

\[
\| u_n(t) - u_n(\tau) \|_{L_2(\Omega_t, \mu)}^2 = \frac{z_k - z_{k-1}}{h} \| u_n(t) - u_n(\tau) \|_{L_2(\Omega_t, \mu)}^2
\]

\[
= (t - \tau)^{\frac{3}{2}} \left( \frac{z_k - z_{k-1}}{h} \right) (t - \tau) \leq C (t - \tau).
\]

Now we consider the case \( i > k \). We first note that

\[
\| u_n(t) - u_n(\tau) \|_{L_2(\Omega_t, \mu)}^2 = \| u_n(t) - u_n(\tau) \|_{L_2(\Omega_t, \mu)}^2
\]

\[
= \| u_n(t) - u_n(t_{i-1}) \|_{L_2(\Omega_t, \mu)}^2 + \| u_n(t_{i-1}) - u_n(t_k) \|_{L_2(\Omega_t, \mu)}^2 + \| u_n(t_k) - u_n(\tau) \|_{L_2(\Omega_t, \mu)}^2
\]
The proof is complete.

Further, we will use Lemma 7.1 to estimate the last expression in the cases \( i = k + 1 \) and \( i > k + 1 \), separately:

- If \( i = k + 1 \), then we see that the second term is equal to zero and, in view of (7.11), we find that

\[
\|u_n(t) - u_n(\tau)\|_{L^2(\Omega_t, \mu)}^2 \leq 3 \left( \frac{z_i - z_{i-1}}{\sqrt{h}} \right)^2 \left( t - t_{i-1} \right) + \frac{z_k - z_{k-1}}{\sqrt{h}} \|z_{k+1} - z_j\|^2_{L^2(\Omega_t, \mu)} + \frac{t_{k+1} - \tau}{h} (t_{k+1} - \tau) \right)
\]

\[
\leq C (t - \tau).
\]

- If \( i > k + 1 \), then we get that

\[
\|u_n(t) - u_n(\tau)\|_{L^2(\Omega_t, \mu)}^2 \leq 3 \left( \frac{z_i - z_{i-1}}{\sqrt{h}} \right)^2 \left( t - t_{i-1} \right) + \frac{z_k - z_{k-1}}{\sqrt{h}} \|z_{k+1} - z_j\|^2_{L^2(\Omega_t, \mu)} + \frac{t_{k+1} - \tau}{h} (t_{k+1} - \tau) \right)
\]

\[
= 3(t - t_{i-1}) \left( \frac{z_i - z_{i-1}}{\sqrt{h}} \right)^2 \left( t - t_{i-1} \right) + 3 (t_{i-1} - t_k) \sum_{j=k}^{i-2} \|z_{j+1} - z_j\|^2_{L^2(\Omega_t, \mu)} + \frac{t_{k+1} - \tau}{h} (t_{k+1} - \tau) \right)
\]

\[
\leq C (t - \tau).
\]

The proof is complete.
7.3 The main result

In this section we present and prove our main result. But first we give a precise meaning in what sense we understand the solution of our problem.

**Definition 7.2.** A function \( u(t) \) is called a weak solution of the problem \((7.2) - (7.4)\) if the following conditions are fulfilled:

1) \( u \in L_2(I, V_t) \),
2) \( u \in AC(I, L_2(\Omega_t, \mu)) \),
3) \( \frac{\partial u}{\partial t} \in L_2(I, L_2(\Omega_t, \mu)) \),
4) \( u(0) = 0 \),
5) \[ \int_0^T ((u(t), v(t)))_{(I, I)} dt + \int_0^T (d^\lambda(t) \frac{\partial u}{\partial t}(t), v(t))_\Omega dt = \int_0^T (f(t), v(t))_\Omega dt \]

for all \( v \in L_2(I, V_t) \).

We shall now formulate and prove our main result, namely the existence of a unique weak solution of \((7.2) - (7.4)\) and show that this weak solution is the limit of the sequence \( \{u_n(x, t)\} \) of functions constructed by our variant of Rothe’s method.

**Theorem 7.1.** The problem \((7.2) - (7.4)\) has a unique solution in the sense of Definition 7.2, i.e. there exists a unique function which in fact is a weak (strong) limit of the sequence of Rothe’s functions \( u_n(t) \) in the space \( L_2(I, V_t) \) \((C(I, L_2(\Omega_t, \mu)))\).

**Proof.** (Uniqueness). Assume that \( \hat{u}(t) \), \( \tilde{u}(t) \) are weak solutions of the problem \((7.2) - (7.4)\). Denote \( u(t) = \hat{u}(t) - \tilde{u}(t) \). Also this function has the properties 1) - 4) of Definition 7.2 and from 5) it follows that

\[ \int_0^T ((u(t), u(t)))_{(I, I)} dt + \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_\Omega dt = 0. \] (7.20)

Let us define a sequence

\[ d_n(x, t) = d(x, t_i) \quad x \in \Omega_t, \quad t \in (t_{i-1}, t_i]. \]

From this and from (A7) it follows that for every \((x, t) \in Q\)

\[ d_n(x, t) \geq d_{n+1}(x, t) \quad \text{for all } n \in \mathbb{N}. \]
and
\[ \lim_{n \to \infty} d_n(x, t) = d(x, t) \quad \text{a.e.} \quad t \in I \] (7.21)
hold.

It can be seen that
\[ \lim_{n \to \infty} \int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt = \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt. \] (7.22)

In fact,
\[
\left| \int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt - \int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt \right| \\
\leq \int_0^T \left| (d^n \lambda(t) - d^n \lambda(t)) \frac{\partial u(t)}{\partial t}, u(t))_t dt \right| \\
\leq \int_0^T \int_{\Omega_t} \left( 1 - \frac{d^n \lambda(x, t)}{(\Omega_t)} \right) \left| \frac{\partial u(x, t)}{\partial t} \right| |u(x, t)| dx dt,
\]
and by applying the Lebesgue Dominant Convergence Theorem we get (7.22).

Next we write the equality (7.22) in the form
\[
\int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt = \lim_{n \to \infty} \int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt = \frac{1}{2} \lim_{n \to \infty} \sum_{i=1}^n \left\| d^n \lambda(t_{i-1}) u(t_{i-1}) \right\|_{L^2(\Omega_{t_{i-1}})}^2
\]
\[
= \frac{1}{2} \left\| d^n \lambda(T) u(T) \right\|_{L^2(\Omega_T)}^2 + \frac{1}{2} \lim_{n \to \infty} \sum_{i=1}^n \left[ \left\| d^n \lambda(t_{i-1}) u(t_{i-1}) \right\|_{L^2(\Omega_{t_{i-1}})}^2 - \left\| d^n \lambda(t_{i}) u(t_{i-1}) \right\|_{L^2(\Omega_{t_{i-1}})}^2 \right].
\]
The second term is nonnegative, due to monotonicity of \( d(x, \cdot) \). Hence, we have that
\[
\int_0^T (d^n \lambda(t) \frac{\partial u(t)}{\partial t}, u(t))_t dt \geq \frac{1}{2} \left\| d^n \lambda(T) u(T) \right\|_{L^2(\Omega_T)}^2.
\]
From this and from (7.20) it follows that
\[
\int_0^T ((u(t), u(t)))_{(t,t)} dt \leq 0.
\]
Linear parabolic problems with singular coefficients in non-cylindrical domains

Consequently, in view of (A3) we have that
\[
\int_0^T \| u(t) \|^2_{W^{k,2}(\Omega_t,\lambda)} dt = 0
\]
i.e.
\[
u(t) = 0 \text{ in } I, \text{ i.e. } \hat{u}(t) = \tilde{u}(t) \text{ in } I.
\]
Thus, the uniqueness of the solution is proved.

(Existence). According to (7.10) it follows that the Rothe sequence \( \{u_n\}_{n=1}^\infty \) is bounded and has a subsequence, which we again denote by \( \{u_n\}_{n=1}^\infty \), which converges weakly to some function \( u \in L_2(I,V_t) \), and since this space is reflexive we have that
\[
u_n \rightharpoonup u \text{ in } L_2(I,V_t).
\] (7.23)

We will now prove that the function \( u \) is the desired solution. From (7.11) it follows that the sequence \( \{\frac{\partial u_n}{\partial t}\}_{n=1}^\infty \) is bounded in the space \( L_2(I,L_2(\Omega_t,\mu)) \). Therefore we can choose a subsequence \( \{\frac{\partial u_n}{\partial t}\}_{n=1}^\infty \) converging weakly to some function \( U \in L_2(I,L_2(\Omega_t,\mu)) \), i.e.
\[
\frac{\partial u_n}{\partial t} \rightharpoonup U \text{ in } L_2(I,L_2(\Omega_t,\mu)).
\] (7.24)

Thus there exists \( w \) defined by
\[
\omega(t) := \int_0^t U(\tau) d\tau.
\]
The functions \( \frac{\partial u_n}{\partial t}(t) \) and \( u_n(t) \) are connected via the relation
\[
\int_0^t \frac{\partial u_n}{\partial t}(\tau) d\tau = u_n(t).
\]
Hence, according to (7.23) and (7.24) we find that \( w = u \). (To obtain the last equality we need to apply the Lebesgue Dominant Convergence Theorem). Then we find that
\[
u \in AC(I,L_2(\Omega_t,\mu))
\]
and
\[
\frac{\partial u}{\partial t}(t) = U(t) \quad \text{a.e. in } I.
\]
Finally, we obtain that 
\[ u(t) = \int_0^t \frac{\partial u}{\partial t}(\tau)d\tau \]
and it follows that \( u(0) = 0 \).

Denote by \( M \) the set of all functions \( v \in L^2(I, V_t) \), which are equal to some function from \( V_\alpha \) on the interval \([\alpha, \beta] \subset I\) and equal to zero outside this interval. Now denote by \( N \) the set of all linear combinations of functions of \( M \), i.e. the set of all simple functions. The set \( N \) is dense in \( L^2(I, V_t) \).

Next we will show that the function \( u(t) \) satisfies the integral identity (7.2) of Definition 7.2. Consider the integral identity (7.12)
\[
((z_j, v))_{(t_j, t_j)} + (d_\lambda^j z_j - z_{j-1})_{t_j} = (f_j, v)_{t_j} \quad \text{for all } v \in V_t
\]
j = 1, 2, ..., n. Let \( v \in N \). We can write
\[
((\tilde{u}_n(t), v(t)))_{(T_n(t), t)} + (d_n^\lambda(t) \frac{\partial u_n(t)}{\partial t}, v(t))_t = (f_n(t), v(t))_t
\]
for almost all \( t \in I \), where
\[
\tilde{u}_n(t) = \begin{cases} 
    z_0 & t \in [t_0, t_1], \\
    z_j & t \in (t_j, t_{j+1}], \quad j = 1, 2, ..., n - 1,
\end{cases}
\]
\[
T_n(t) = \begin{cases} 
    t_0 & t \in [t_0, t_1], \\
    t_j & t \in (t_j, t_{j+1}], \quad j = 1, ..., n - 1,
\end{cases}
\]
\[
d_n(t) = \begin{cases} 
    d_0 & t \in [t_0, t_1], \\
    d_j & t \in (t_j, t_{j+1}], \quad j = 1, ..., n - 1,
\end{cases}
\]
\[
f_n(t) = \begin{cases} 
    f_0 & t \in [t_0, t_1], \\
    f_j & t \in (t_j, t_{j+1}], \quad j = 1, ..., n - 1.
\end{cases}
\]

After integrating over the interval \( I \), we obtain that
\[
\int_0^T ((\tilde{u}_n(t), v(t)))_{(T_n(t), t)} dt + \int_0^T (d_n^\lambda(t) \frac{\partial u_n(t)}{\partial t}, v(t))_t dt = \int_0^T (f_n(t), v(t))_t dt.
\]
LINEAR PARABOLIC PROBLEMS WITH SINGULAR COEFFICIENTS IN NON-CYLINDRICAL DOMAINS

All these integrals exist and
\[
\int_0^T ((\tilde{u}_n(t), v(t)))_{(T_n(t), t)} dt \to \int_0^T ((u(t), v(t)))_{(t,T)} dt \quad \text{as} \quad n \to \infty
\]
since, by (7.23) we have that
\[
\int_0^T ((\tilde{u}_n(t), v(t)))_{(t,L)} dt \to \int_0^T ((u(t), v(t)))_{(t,T)} dt \quad \text{as} \quad n \to \infty
\]
and, in view of (A2) and (A5),
\[
\left| \int_0^T ((\tilde{u}_n(t), v(t)))_{(T_n(t), t)} dt - \int_0^T ((\tilde{u}_n(t), v(t)))_{(t,T)} dt \right| \\
\leq \int_0^T \left| ((\tilde{u}_n(t), v(t)))_{(T_n(t), t)} - ((\tilde{u}_n(t), v(t)))_{(t,T)} \right| dt \leq C h_n.
\]
Moreover, from (7.24) and (A7) it follows that
\[
\int_0^T (d^\lambda_n(t) \frac{\partial u_n(t)}{\partial t}, v(t))_t dt \to \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t}, v(t))_t dt \quad \text{as} \quad n \to \infty
\]
since
\[
\left| \int_0^T (d^\lambda_n(t) \frac{\partial u_n(t)}{\partial t}, v(t))_t dt - \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t}, v(t))_t dt \right| \\
\leq \left| \int_0^T \left( d^\lambda_n(t) - d^\lambda(t) \right) \frac{\partial u_n(t)}{\partial t}, v(t) \right| dt + \left| \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t} - \frac{\partial u(t)}{\partial t}, v(t))_t dt \right|,
\]
the first integral converges to zero, which follows from (A7) and the boundedness of the sequence \( \{ \frac{\partial u_n(t)}{\partial t} \}_n \) and also the second integral converges to zero, which follows from (7.24).

In view of (A7) and the uniform convergence of the sequence \( \{ f_n \}_n \) it follows that
\[
\int_0^T |(f_n(t) - f(t), v(t))_t| dt \leq \int_0^T \| f_n(t) - f(t) \| L_2(\Omega, 2k-\mu) \| v(t) \| L_2(\Omega, 2k-\mu) dt \\
\leq C \| f_n - f \| L_2(I, L_2(\Omega, 2k-\mu)) \| v \| L_2(I, V).
\]

Altogether, for the fixed function \( v \in M \) and for \( n \to \infty \), it yields that
\[
\int_0^T ((u(t), v(t)))_{(t,T)} dt + \int_0^T (d^\lambda(t) \frac{\partial u(t)}{\partial t}, v(t))_t dt = \int_0^T (f(t), v(t))_T dt.
\]
Since the function \( v(t) \) was an arbitrary element of \( M \), and \( N \) is dense in \( L_2(I,V_t) \), it then implies that the last equality holds for all \( v \in L_2(I,V_t) \).

Thus, we have proved that there exists a subsequence \( \{u_n\}_{n=1}^{\infty} \) of the Rothe sequence \( \{u_n\}_{n=1}^{\infty} \), which converges to some function \( u(t) \) and satisfies all conditions of Definition 7.2. Moreover, we can in the usual manner prove that not only a subsequence, but also the whole sequence, converges to this function. In fact, this follows from the uniqueness theorem, i.e. if we assume that there exists another subsequence \( \{\hat{u}_n\}_{n=1}^{\infty} \) converging to \( \hat{u}(t) \), then this function \( \hat{u}(t) \) solves the problem (7.2) – (7.4). Since we proved that this problem has a unique solution, it follows that

\[
u(t) = \hat{u}(t) \quad \text{for all } t \in I.
\]

Finally, we claim that the Rothe sequence converges uniformly to the solution \( u \). In fact, from Lemma 7.4 it follows that the Rothe sequence \( \{u_n(t)\}_{n=1}^{\infty} \) is equicontinuous, i.e. the first condition of Proposition 7.1 is satisfied. The second condition in the proposition holds according to the fact that

\[
W^{k,2}(\Omega_t, \mu) \hookrightarrow L_2(\Omega_t, \mu),
\]

which is well-known, see e.g. [66]. Hence, our final claim follows from Proposition 7.1 and the proof is complete.

### 7.4 Concluding examples and results

In this section we present some possible generalizations connected with our problem. Let us start with the following simple example

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \sin x & Q, \\
u(x,0) &= 0 & x \in (0, \pi), \\
u(0,t) &= \nu(\pi,t) = 0 & t \in (0,1], \\
u(0,t) &= \nu(2\pi,t) = 0 & t \in (1,2),
\end{aligned}
\]

(7.25)

where \( Q = (0, \pi) \times (0,1] \cup (0,2\pi) \times (1,2) \) is a non-cylindrical domain (see Figure 7.3).

According to Theorem 7.1 this problem has exactly one solution in the sense of Definition 7.2, which is defined in the whole cylinder \( \bar{Q} = (0,2\pi) \times (0,2) \) and it is equal to zero outside \( Q \). When we solve (7.25) we don’t need...
any condition on the solution on $\Omega_1 = (\pi, 2\pi) \times \{1\}$. If we consider the following problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \sin x & \text{in } Q, \\
u(x, 0) &= 0 & x \in (0, \pi), \\
u(x, 1) &= 0 & x \in (\pi, 2\pi), \\
u(0, t) = u(\pi, t) &= 0 & t \in (0, 1], \\
u(0, t) = u(2\pi, t) &= 0 & t \in (1, 2)
\end{aligned}
\]  
(7.26)

then it can be solved by the same approach as in the foregoing case and the definition of the weak solution in Definition 7.2. If we consider the problem (7.26) but replace the condition $u(x, 1) = 0$ for $x \in (\pi, 2\pi)$ by the condition

\[
\nu(x, 1) = (1 - e^{-1}) \sin x & \quad x \in (\pi, 2\pi),
\]  
(7.27)

then we know that the function

\[
u(x, t) = (1 - e^{-t}) \sin x
\]

solves this problem, but this is not a solution in the sense of Definition 7.2. Due to the condition (7.27), we are not able to solve this problem with the same approach as the proof of (7.25).
Definition 7.3. A point \( \tau \in (0, T) \) is called the \textit{expansion point} of \( Q \) if
\[
\mu(\Omega_{\tau+} \setminus \Omega_\tau) > 0,
\]
where \( \Omega_{\tau+} = \bigcap_{t>\tau} \Omega_t \) and \( \mu \) is the Lebesgue measure in \( \mathbb{R}^N \).

For example, the set \( Q \) from problem (7.25) has exactly one expansion point, namely \( \tau = 1 \).

Theorem 7.2. A non-cylindrical domain \( Q \) of the type described in (7.1) has at most countable many expansion points.

Proof. Let \( m \in \mathbb{N} \) be arbitrary and let us consider the set
\[
T_m = \{ t \in (0, T), \ \mu(\Omega_{t+0} \setminus \Omega_t) > \frac{1}{m} \}.
\]
This set contains at most a finite number of points, since
\[
(\Omega_{t+0} \setminus \Omega_t) \cap (\Omega_{\tau+0} \setminus \Omega_\tau) = \emptyset
\]
and
\[
\sum_{t \in T_m} \mu(\Omega_{t+0} \setminus \Omega_t) \leq \mu(\Omega_T).
\]
For the set \( T \) of expansion points we have that
\[
T = \bigcup_{n=1}^{\infty} T_n,
\]
and, hence, \( T \) contains at most countable many elements. The proof is complete. \( \square \)

In the sequel we will deal with domains which have some expansion points. Let us consider the following problem
\[
\begin{cases}
\frac{d^k}{dt^k} u(x,t) \frac{\partial^k u}{\partial x^k} + A(x,t)u = f(x,t) & \text{in } Q, \\
u(x, 0) = u_0(x) & \text{in } \Omega_0,
\end{cases}
\]
where \( \{t_i\}_{i=1}^m \) are the expansion points of \( Q \) and \( \{u_i(x)\}_{i=1}^m \) are given functions from \( W_0^{k,2}(\Omega_{t_i+0} \setminus \Omega_{t_i}) \), respectively.
Definition 7.4. A function \( u \) is called a weak solution of (7.28), if the following conditions are fulfilled:

1) \( u \in AC(I, L^2(\Omega_t)) \),
2) \( u \in L^2(I, V_t) \),
3) \( u(0) = u_0 \),
4) \( u(t_i) = u_i, \quad i = 1, ..., m \),
5) \( \int_0^T ((u(t), v(t)))_{(t, T)} dt + \int_0^T (d^k u(t) \frac{\partial u(t)}{\partial t}, v(t))_T dt = \int_0^T (f(t), v(t))_T dt \)

for all \( v \in L^2(I, V_t) \).

Now we construct the functions \( u_i(x, t) \) as follows:

\[
\begin{aligned}
    u_0(x, t) &= \begin{cases} 
        u_0(x) & (x, t) \in \Omega_0 \times (0, T), \\
        0 & (x, t) \in Q \setminus \Omega_0 \times (0, T),
    \end{cases} \\
    u_i(x, t) &= \begin{cases} 
        u_i(x) & (x, t) \in (\Omega_{t_i+0} \setminus \Omega_{t_i}) \times (t_i, T), \\
        0 & (x, t) \in Q \setminus (\Omega_{t_i+0} \setminus \Omega_{t_i}) \times (t_i, T).
    \end{cases}
\end{aligned}
\]

From the definition it is known that these functions are constant functions on \( t \) and for a fixed \( t \) it is from the Sobolev space \( W^{k,2}_0(\Omega_t) \).

Remark 7.1. The problem (7.28) has a unique solution if and only if the problem

\[
\begin{align*}
    d^k(x, t) \frac{\partial \omega}{\partial t} + A(x, t) \omega &= \bar{f}(x, t) & \text{in } Q, \\
    \omega(x, 0) &= 0 & \text{in } \Omega_0, \\
    \omega(x, t_i) &= 0 & \text{in } \Omega_{t_i+0} \setminus \Omega_t, \quad i = 1, ..., m, \\
    \omega(x, t) &= \frac{\partial \omega}{\partial \nu}(x, t) &= \cdots = \frac{\partial^{k-1} \omega}{\partial \nu^{k-1}}(x, t) &= 0 & 0 < t < T, \quad x \in \partial \Omega_t,
\end{align*}
\]

(7.29)

has a unique weak solution, where \( \bar{f}(x, t) = f(x, t) - \sum_{i=0}^{m} A u_i(x, t) \). We obtain this fact from problem (7.28) via the translation

\[
u(x, t) = \omega(x, t) + \sum_{i=0}^{m} u_i(x, t).
\]

This problem can be solved by the same approach as in previous case and the definition of the weak solution is that in Definition 7.2. Hence, we have the following result.

Theorem 7.3. Consider the problem (7.28) and let Assumption 7.1 be satisfied. Then there exists exactly one weak solution of the problem (7.28).
Chapter 8

Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

8.1 Introduction

Over the years PDEs with periodic rapidly oscillating coefficients have been studied by several authors see e.g. [1], [2], [9], [19], [64], [65], [69] and [76]. These problems were mostly solved e.g. by using the method of multiple scale expansion or some mathematically based homogenization techniques, e.g. $G$-convergence, $\Gamma$-convergence or two scale convergence. However, recently J. Vala (see [74]) used Rothe’s method (for more details on this method see e.g. [42], [43], [72]) and the technique of two scale convergence to solve a non-linear parabolic problem. In that paper the coefficients of the time derivative and that of the differential operator do not depend on time. In this paper we continue this research for the corresponding quasilinear equation and solve the more general case when the coefficients also depend on time. In particular, this requires a partly new technique of proof. Moreover, we derive a corresponding homogenization result (see Theorem 8.1) and homogenization algorithm (see Corollary 8.1), which are useful for concrete numerical solutions of the actual problem.
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

The considered problem has the following form:

\[
\begin{cases}
a(x, x/\varepsilon, t) \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (b(x, x/\varepsilon, t) \nabla u^\varepsilon) = f(x, x/\varepsilon, t, u^\varepsilon) & \text{in} \quad \Omega \times (0, T), \\
u^\varepsilon(x, 0) = u_0 & \text{in} \quad \Omega, \\
u^\varepsilon(x, t) = 0 & \text{on} \quad \partial \Omega \times (0, T),
\end{cases}
\]

(8.1)

where \( \varepsilon > 0 \) is a small parameter, \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary, \( T < \infty \), \( a \) and \( b \) are functions defined in \( \Omega \times \mathbb{R}^3 \times (0, T) \) and the right hand side function \( f \) is defined in \( \Omega \times \mathbb{R}^3 \times (0, T) \times \mathbb{R} \). The function \( u_0 \) is defined in \( \Omega \).

This paper is organized as follows: In Section 8.2 we present the necessary definitions and lemmas, which are connected with two-scale convergence. In addition, we state some necessary assumptions and give a brief description of Rothe’s method. Our main results are stated and discussed in Section 3 and the proofs are given in Section 4.

8.2 Preliminaries

In this section we first give some definitions and lemmas associated with two-scale convergence. Moreover, the space variable is represented by \( x \in \Omega \subset \mathbb{R}^3 \) and \( t \in I = [0, T] \subset \mathbb{R} \) represents the time. The cell of periodicity is denoted by \( Y \) (i.e. the unit cube in \( \mathbb{R}^3 \)).

**Definition 8.1.** Let \( u_0 \) be an element of \( L^2(\Omega \times Y) \) and let \( \varepsilon > 0 \). We say that a sequence \( u^\varepsilon \) from \( L^2(\Omega) \), two-scale converges weakly to \( u_0 \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) \psi(x, x/\varepsilon) dx = \int_{\Omega} \int_Y u_0(x, y) \psi(x, y) dy dx \quad \forall \psi \in C_0^\infty(\Omega, C_0^\infty(Y));
\]

briefly \( u^\varepsilon \rightharpoonup u_0 \).

Let us note that we can replace \( C_0^\infty(\Omega, C_0^\infty(Y)) \) by \( L^2(\Omega, C_0^\infty(Y)) \) in the definition, using the obvious density argument.

**Definition 8.2.** Let \( u_0 \) be an element of \( L^2(\Omega \times Y) \). We say that a sequence \( u^\varepsilon \) from \( L^2(\Omega) \), two-scale converges strongly to \( u_0 \) if \( u^\varepsilon \xrightarrow{2} u_0 \) and, in addition,

\[
\lim_{\varepsilon \to 0} \int_{\Omega} |u^\varepsilon(x)|^2 dx = \int_{\Omega} \int_Y |u_0(x, y)|^2 dy dx;
\]

briefly \( u^\varepsilon \xrightarrow{2} u_0 \).
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

Lemma 8.1. If \( u^\varepsilon \overset{2}{\rightharpoonup} u^0 \) and \( v^\varepsilon \overset{2}{\rightharpoonup} v^0 \), where \( u^0, v^0 \in L_2(\Omega \times Y) \), then also
\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) v^\varepsilon(x) dx = \int_{\Omega} \int_{Y} u^0(x,y) v^0(x,y) dy dx.
\]

Proof. See [1].

In the sequel \( H = L_2(\Omega) \) and \( V = W^{1,2}_0(\Omega) \).

Lemma 8.2. Let \( \{u^\varepsilon\} \) be bounded sequence in the space \( C^{0,1}(I,H) \cap L_\infty(I,V) \). Then there exist \( u \in C^{0,1}(I,H) \cap L_\infty(I,V) \) and \( \tilde{u} \in L_\infty(I,W^{1,2}_{per}(Y)) \) such that up to a subsequence,

a) \( u^\varepsilon(t) \rightharpoonup u(t) \) in \( V \) for every \( t \in I \),

b) \( u^\varepsilon \rightharpoonup u \) in \( C(I,H) \),

c) \( u^\varepsilon(t) \overset{2}{\rightharpoonup} u(t) \) for every \( t \in I \),

d) \( \nabla u^\varepsilon(t) \overset{2}{\rightharpoonup} \nabla u(t) + \nabla_Y \tilde{u}(t) \) for every \( t \in I \),

e) \( \frac{\partial u^\varepsilon}{\partial t}(t) \overset{2}{\rightharpoonup} \frac{\partial u}{\partial t}(t) \) for every \( t \in I \).

Proof. The lemma can be proved analogously as Lemma 5 in [74], and, thus, we leave out the details.

To prove the uniqueness of the solution of the problem we will use the following Gronwall type lemma.

Lemma 8.3. Let \( u \) and \( f \) be continuous and nonnegative functions defined on \( J = [\alpha, \beta] \), and let \( C \) be a nonnegative constant. Then the inequality
\[
\int_{\alpha}^{t} f(s) u(s) ds, \quad t \in J,
\]
implies that
\[
\int_{\alpha}^{t} f(s) ds, \quad t \in J.
\]

Proof. See e.g. [67, Chapter 1, Theorem 1.2.2].

Now we present a brief description of Rothe’s method for the situation at hand. Using this method we can solve the following parabolic problem, which is the weak form of problem (8.1):

\[
\begin{align*}
\frac{\partial u^\varepsilon(t)}{\partial t} + \nabla(t) u^\varepsilon(t), v) = (f^\varepsilon(t), u^\varepsilon(t)), v) \quad \text{for all } v \in V.
\end{align*}
\]
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

\[ u^\varepsilon(0) = u_0, \]

where

\[ u^\varepsilon(t) := u^\varepsilon(x, t), \quad a^\varepsilon(t) := a(x, x/\varepsilon, t), \quad b^\varepsilon(t) := b(x, x/\varepsilon, t), \]

\[ f^\varepsilon(t, u^\varepsilon(t)) := f(x, x/\varepsilon, t, u^\varepsilon(x, t)) \quad (8.3) \]

for a fixed \( \varepsilon \) and \((\cdot, \cdot)\) denotes the scalar product in \( H \) and

\[ \langle b^\varepsilon(t)u^\varepsilon(t), v \rangle = \int_{\Omega} b^\varepsilon(t)\nabla u^\varepsilon(t) \cdot \nabla v dx. \quad (8.4) \]

We also need the following technical assumption for the functions \( a, b, f \) and \( u_0 \) in order to be able to solve problem (8.1).

**Assumption 8.1.** Let \( C_1, C_2 \) be positive numbers and \( y \in \mathbb{R}^3 \). Then

(A1) for all \( t \in (0, T) \), we have that

\[ C_1 \leq a(x, y, t) \leq C_2 \quad \text{for almost all } x \in \Omega, \]

\[ \|a(\cdot, y, t) - a(\cdot, y, \tau)\|_{L^\infty(\Omega)} \leq C_2 |t - \tau| \quad \text{for all } \tau \in (0, T). \]

Also the function \( b \) satisfies the above conditions.

(A2) the function \( f \) satisfies the following condition:

\[ \|f(\cdot, y, t, u) - f(\cdot, y, \tau, v)\|_H \leq C_2 (|t - \tau| + \|u - v\|_H) \]

for all \( t, \tau \in I \) and \( u, v \in H \).

(A3) the function \( u_0 \) from \( V \) is such that:

\[ \nabla \cdot (b(x, y, 0)\nabla u_0) \in H. \]

(Here, we write \( b(x, y, 0) \) which is the limit of \( b(x, y, t) \) as \( t \to 0 \), since the existence of this limit is guaranteed by (A1)). Moreover, it is supposed that all functions \( a, b, f \) are \( Y \)-periodic in the second variable \( y \).

**Rothe’s method.** Let \( h \) be a positive number. We divide the interval \( I = [0, T] \) into subintervals \( I_1, I_2, ..., I_n \) \( I_j = [t_{j-1}, t_j), \quad t_j = jh, \quad j = 1, 2, ..., n - 1, \) and \( I_n = [t_{n-1}, T] \), where \( 0 < T - t_{n-1} \leq h \) such that the interval \( I \) is covered by these intervals. According to the initial condition of problem (8.2) we put

\[ z_0 = u_0, \]
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

for $t_0 = 0$ and successively for $j = 1, 2, ..., n$ define vector functions $z_j$, which are weak solutions of the following elliptic problems:

$$z_j \in V : \frac{1}{h}(a_j z_j, v) + \langle b_j z_j, v \rangle = (f_j(z_{j-1}) + a_j \frac{h}{h} z_{j-1}, v) \quad \text{for all } v \in V,$$

where $a_j = a_\varepsilon(t_j)$, $b_j = b_\varepsilon(t_j)$ and $f_j(z_{j-1}) = f_\varepsilon(t_j, z_{j-1})$. We obtain these problems, if we replace the derivative $\frac{\partial u_\varepsilon(t)}{\partial t}$ by the differential quotient $\frac{z_j - z_{j-1}}{h}$ in the points $t = t_j$, $j = 1, 2, ..., n$, in (8.2).

Let $j = 1$, then problem (8.5) takes the following form

$$z_1 \in V : \frac{1}{h}(a_1 z_1, v) + \langle b_1 z_1, v \rangle = (f_1(z_0) + \frac{a_0}{h} z_0, v) \quad \text{for all } v \in V,$$

and it has exactly one solution (according to Assumption 8.1 and as a consequence of the theory of elliptic boundary value problems; see, e.g. [47]).

Next we solve the problem (8.5) for $j = 2$, i.e.

$$z_2 \in V : \frac{1}{h}(a_2 z_2, v) + \langle b_2 z_2, v \rangle = (f_2(z_1) + \frac{1}{h} a_1 z_1, v) \quad \text{for all } v \in V.$$

Repeating the above procedure for $j = 3, ..., n$ we get functions $z_1, z_2, ..., z_n \in V$, which are uniquely determined. It is thus possible to construct the Rothe function $u_n(t)$ as functions from $I$ to $V$ defined by

$$u_n(t) = z_{j-1} + \frac{t - t_{j-1}}{h}(z_j - z_{j-1}), \quad t \in I_j, \quad j = 1, 2, ..., n. \quad (8.6)$$

Hence, we obtain a sequence $\{u_n(t)\}_{n=1}^\infty$, which is called Rothe’s sequence of approximative solutions of the problem (8.2). Intuitively, we can expect that if $n \to \infty$, then this sequence will converge to some function $u^\varepsilon(t)$, which is a solution of problem (8.2).

In the next Section we will in particular present and prove that this in fact holds in a special sense. Roughly speaking, first we use the Rothe method to prove the existence of $u^\varepsilon(t)$ as $n \to \infty$ (see Theorem 8.2). After that we use the technique of two-scale convergence to prove that $u^\varepsilon(t)$ actually converges to a unique function $u(t)$ as $\varepsilon \to 0$ and this is the approximative (homogenized) solution of (8.1) we are looking for (see Theorem 8.1 (a)). As expected this solution can be calculated by using a homogenization algorithm (see our Theorem 8.1 (b) and Corollary 8.1).

We are now ready to present and prove our main results.
8.3 Main results

In this Section, the notations $\Omega$, $Y$, $V$, $H$ and $I$ are defined as in our previous Sections and, moreover, the function $\tilde{a}(x,t)$ and $\tilde{f}(x,t,u)$ are defined by

$$\tilde{a}(x,t) := \int_Y a(x,y,t)dy, \quad \text{and} \quad \tilde{f}(x,t,u) := \int_Y f(x,y,t,u)dy.$$

Our main result reads:

**Theorem 8.1.** Let Assumption 8.1 be satisfied. Then

(a) problem (8.1) has a unique solution, and this solution can be approximated by another unique function $u \in C^{0,1}_0(I,H) \cap L^\infty(I,V)$ and such that $\tilde{u} \in L^\infty(I,L^2_w(\Omega,W^{1,2}_{\text{per}}(Y)))$ and

$$\int_\Omega \tilde{a}(x,t) \frac{\partial u}{\partial t}(x,t) v(x)dx + \int_\Omega \int_Y b(x,y,t)(\nabla u(x,t) + \nabla_Y \tilde{u}(x,y,t)) \cdot \nabla v(x)dydx = \int_\Omega \tilde{f}(x,t,u(x,t)) v(x)dx$$

for all $v \in V$ and at almost every time $t \in I$ and, $u(x,0) = u_0(x)$ for almost every $x \in \Omega$.

(b) the unique solution $u(x,t)$ in (a) can be obtained by solving the following equation:

$$\tilde{a}(x,t) \frac{\partial u}{\partial t}(x,t) - \nabla \cdot (B(x,t)\nabla u(x,t)) = \tilde{f}(x,t,u(x,t)),$$

where the matrix $B(x,t) = (b_{ij}(x,t))_{i,j=1,2,3}$ is defined by

$$\begin{pmatrix} b_{11}(x,t) \\ b_{21}(x,t) \\ b_{31}(x,t) \end{pmatrix} = \int_Y b(e_1 + \nabla_Y w_1)dy, \quad \begin{pmatrix} b_{12}(x,t) \\ b_{22}(x,t) \\ b_{32}(x,t) \end{pmatrix} = \int_Y b(e_2 + \nabla_Y w_2)dy,$$

$$\begin{pmatrix} b_{13}(x,t) \\ b_{23}(x,t) \\ b_{33}(x,t) \end{pmatrix} = \int_Y b(e_3 + \nabla_Y w_3)dy$$

(8.9)
and \( w_i \in L_\infty \left(I, L_2 \left( \Omega, W^{1,2}_{per}(Y) \right) \right) \), \( i = 1, 2, 3 \), are the solutions of the following local problems

\[
\begin{align*}
\int_Y b(x, y, t) \left( e_1 + \nabla_Y w_1 \right) \cdot \nabla v(y) dy &= 0, \\
\int_Y b(x, y, t) \left( e_2 + \nabla_Y w_2 \right) \cdot \nabla v(y) dy &= 0, \\
\int_Y b(x, y, t) \left( e_3 + \nabla_Y w_3 \right) \cdot \nabla v(y) dy &= 0,
\end{align*}
\]

for all \( v \in C_\infty^{\per}(Y) \), where \( \{e_1, e_2, e_3\} \) is the canonical basis in \( \mathbb{R}^3 \).

**Remark 8.1.** For the case when the coefficients \( a, b \) and the right hand side \( f \) do not depend on \( t \), Theorem 8.1 (a) coincides with that of Vala [74] in the quasilinear case. Moreover, Theorem 8.1 (b), is well suited to directly apply for obtaining a good approximation of the solution of (8.1).

More exactly we obtain the following homogenization algorithm for deriving an approximative solution of equation (8.1):

**Corollary 8.1. Homogenization algorithm**

An approximative solution of equation (8.1) can be obtained in the following way:

- **Step 1:** Solve the local problem (8.10).
- **Step 2:** Insert the solutions of the local problems into (8.9) and compute the homogenized coefficient \( B(x, t) \).
- **Step 3:** Solve the homogenized equation (8.8), which gives the approximative solution \( u(x, t) \) we are looking for.

In order to be able to prove Theorem 8.1 we need the following crucial result of independent interest:

**Theorem 8.2.** Let Assumption 8.1 be satisfied. Then there exists a function \( u^\varepsilon \in C^{0,1}(I, H) \cap L_\infty(I, V) \), which solves (8.1) and has the following properties (for each fixed \( \varepsilon > 0 \)):

1. \( \| u^\varepsilon \|_{L_\infty(I, V)} \leq C, \quad \| \frac{\partial u^\varepsilon}{\partial t} \|_{L_\infty(I, H)} \leq C, \)
2. \( u^\varepsilon(0) = u_0, \)
3. \[
\int_\Omega a(x, x/\varepsilon, t) \frac{\partial u^\varepsilon}{\partial t}(x, t) v(x) dx + \int_\Omega b(x, x/\varepsilon, t) \nabla u^\varepsilon(x, t) \cdot \nabla v(x) dx \\
= \int_\Omega f(x, x/\varepsilon, t, u^\varepsilon(x, t)) v(x) dx
\]

(8.11)
for all \( v \in V = W_0^{1,2}(\Omega) \) and at almost every time \( t \in I \), where \( C \) does not depend on \( \varepsilon \).

8.4 Proofs

Proof of Theorem 8.2. Let us consider the integral identity (8.5), i.e.
\[
\frac{1}{h} (a_j (z_j - z_{j-1}), v) + \langle b_j z_j, v \rangle = (f_j (z_{j-1}), v) \quad \text{for all } v \in V. \tag{8.12}
\]
We choose \( v = z_j - z_{j-1} \) in (8.12). Then we get that
\[
\frac{1}{h} (a_j (z_j - z_{j-1}), z_j - z_{j-1}) + \langle b_j z_j, z_j - z_{j-1} \rangle = (f_j (z_{j-1}), z_j - z_{j-1})
\]
for \( j = 1, 2, \ldots, n \). By applying the Schwarz inequality to the right hand side of the last equality we obtain that
\[
\frac{1}{h} (a_j (z_j - z_{j-1}), z_j - z_{j-1}) + \langle b_j z_j, z_j - z_{j-1} \rangle \leq \|f_j (z_{j-1})\|_H \|z_j - z_{j-1}\|_H.
\]
Hence, according to (A1) and (A2) of Assumption 8.1, we find that
\[
\frac{C_1}{h} \|z_j - z_{j-1}\|_H^2 + \langle b_j z_j, z_j - z_{j-1} \rangle \leq C_2 (j h + \|z_{j-1}\|_H) \|z_j - z_{j-1}\|_H.
\]
By applying first, the trivial inequality \( ab \leq \frac{a^2}{2\theta} + \frac{b^2}{2} \), (for \( \theta = \frac{2C_1}{hC_2} > 0 \)) followed by \( (a + b)^2 \leq 2 (a^2 + b^2) \) to the right hand side of the last estimate we get that
\[
\langle b_j z_j, z_j - z_{j-1} \rangle \leq \frac{hC_1^2}{4C_1} (T + \|z_{j-1}\|_H)^2,
\]
i.e.
\[
\langle b_j z_j, z_j - z_{j-1} \rangle \leq \frac{hC_1^2}{2C_1} (T^2 + \|z_{j-1}\|_H^2). \tag{8.13}
\]
According to the Poincare inequality, (8.4) and (A1) of Assumption 8.1 we have that
\[
\|z_{j-1}\|_H \leq C_s \|z_{j-1}\|_V \leq C_s \left[ \frac{1}{C_1} \langle b_j - b_{j-1}, z_{j-1} - z_{j-1} \rangle \right]^\frac{1}{2}, \tag{8.14}
\]
and
\[
\|((b_{j-1} - b_j) z_{j-1}, z_{j-1})\| \leq \frac{C_2h}{C_1} \langle b_{j-1} z_{j-1}, z_{j-1} \rangle,
\]
Homogenization of linear parabolic problems by the 161
method of Rothe and two-scale convergence

so that

\[-\frac{C_2 h}{C_1} (b_{j-1} z_{j-1}, z_{j-1}) \leq \langle (b_{j-1} - b_j) z_{j-1}, z_{j-1} \rangle. \quad (8.15)\]

Inserting (8.14) into (8.13) we see that

\[\langle b_j z_j, z_j - z_{j-1} \rangle \leq \frac{h C_2^2}{2 C_1} \left( \frac{C_2}{C_1} (b_{j-1} z_{j-1}, z_{j-1}) \right). \quad (8.16)\]

Moreover, we estimate the left hand side of (8.13) as follows:

\[
\begin{align*}
\langle b_j z_j, z_j - z_{j-1} \rangle &= \frac{1}{2} \left[ \langle b_j z_j, z_j \rangle + \langle b_j (z_j - z_{j-1}), z_j - z_{j-1} \rangle - \langle b_j z_{j-1}, z_{j-1} \rangle \right] \\
&= \frac{1}{2} \left[ \langle b_j z_j, z_j \rangle + \langle b_j (z_j - z_{j-1}), z_j - z_{j-1} \rangle + \langle (b_{j-1} - b_j) z_{j-1}, z_{j-1} \rangle \\
&\quad - \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \right] \\
&\geq \frac{1}{2} \left[ \langle b_j z_j, z_j \rangle + \langle (b_{j-1} - b_j) z_{j-1}, z_{j-1} \rangle - \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \right]. \quad (8.17)
\end{align*}
\]

Inserting (8.15) into (8.17) and simplifying we find that

\[\langle b_j z_j, z_j - z_{j-1} \rangle \geq \frac{1}{2} \left[ \langle b_j z_j, z_j \rangle - \left( 1 + \frac{C_2}{C_1} h \right) \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \right]. \quad (8.18)\]

>From (8.16) and (8.18) it follows that

\[
\frac{1}{2} \left[ \langle b_j z_j, z_j \rangle - (1 + \frac{C_2}{C_1} h) \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \right] \leq \frac{h C_2^2}{2 C_1} \left( \frac{C_2}{C_1} \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \right).
\]

By choosing now \( C < \infty \) so that

\[
\frac{C_2^2}{C_1} T^2 \leq C \quad \text{and} \quad \frac{C_2^2 C_3^2 + C_2 C_1}{C_1^2} \leq C,
\]

we find that

\[\langle b_j z_j, z_j \rangle \leq C h + \left( 1 + C h \right) \langle b_{j-1} z_{j-1}, z_{j-1} \rangle.\]

Hence, by using this repeatedly and the fact that \( \ln (1 + t) \leq t, \ t \geq 0 \) it yields that

\[
\begin{align*}
\langle b_j z_j, z_j \rangle &\leq C h + (1 + C h) \langle b_{j-1} z_{j-1}, z_{j-1} \rangle \\
&\leq C h + C h (1 + C h) + (1 + C h)^2 \langle b_{j-2} z_{j-2}, z_{j-2} \rangle \\
&\leq \ldots \leq (1 + C h)^j + (1 + C h)^j \langle b_0 z_0, z_0 \rangle \\
&= e^{j \ln(1+C h)} + e^{j \ln(1+C h)} \langle b_0 z_0, z_0 \rangle \\
&\leq e^{C T} + e^{C T} \langle b_0 z_0, z_0 \rangle.
\end{align*}
\]
According to (A1) and (A3) of Assumption 8.1 we get that
\[ \|z_j\|_V \leq C_3 \quad \text{for all } j = 1, 2, ..., n, \quad (8.19) \]
where \( C_3 \) does not depend on \( j \) and \( n \).

From this estimate we obtain the uniform boundedness of the Rothe sequence, i.e. according to (8.6) and (8.19) it is obvious that
\[ \|u_n(t)\|_V^2 \leq \max_j \|z_j\|_V^2 \leq C_3^2. \quad (8.20) \]
This estimate implies that the first estimate of the theorem holds.

Next we estimate the derivative of the Rothe sequence, i.e. \( \{\frac{\partial u_n(t)}{\partial t}\} \), which is also connected to the proof of the second estimate of the theorem. To this end we consider the identity (8.12) i.e. that
\[ \frac{1}{h} \left( a_j(z_j - z_{j-1}), v \right) + \langle b_j z_j, v \rangle = (f_j(z_{j-1}), v) \quad \text{for all } v \in V. \]

Subtracting from this identity the same identity written for \( j-1 \) and putting \( v = z_j - z_{j-1} \) we obtain that
\[ \frac{1}{h} \left( a_j(z_j - z_{j-1}) - a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1} \right) \]
\[ + \langle b_j z_j - b_{j-1} z_{j-1}, z_j - z_{j-1} \rangle = (f_j(z_{j-1}) - f_{j-1}(z_{j-2}), z_j - z_{j-1}), \quad (8.21) \]
We will separately estimate all terms of (8.21). Let us begin with the first term on the left hand side. To estimate this we use the following inequalities:
\[ (a_{j-1} u, u) - 2(a_{j-1} u, v) + (a_{j-1} v, v) \geq 0 \quad \text{for all } u, v \in V \]
and
\[ |1 - \frac{a_{j-1}(x)}{a_j(x)}| \leq \frac{C_2}{C_1} h \quad \text{for a.e. } x \in \Omega \]
(for all \( j = 1, 2, ..., n \)), which immediately follows (A1) of Assumption 8.1.
By using these estimates we find that
\[ \frac{1}{h} \left( a_j(z_j - z_{j-1}) - a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1} \right) \]
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

\[
\begin{align*}
&= \frac{1}{2h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \frac{1}{2h} \left[ (a_{j-1}(z_j - z_{j-1}), z_j - z_{j-1}) \\
&\quad - 2(a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1}) + (a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \right] \\
&\quad + \frac{1}{2h}((a_j - a_{j-1})(z_j - z_{j-1}), z_j - z_{j-1}) - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \\
&\quad \geq \frac{1}{2h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \frac{1}{2h}(a_j(1 - \frac{a_{j-1}}{a_j})(z_j - z_{j-1}), z_j - z_{j-1}) \\
&\quad - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \\
&\quad \geq \frac{1}{2h}(1 - \frac{C_2}{C_1}h)(a_j(z_j - z_{j-1}), z_j - z_{j-1}) - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}). \\
&\quad \geq 1 \text{ (8.22)}
\end{align*}
\]

Next we use (A1) of Assumption 8.1 to obtain that

\[
\|((b_{j-1} - b_j) z_{j-1}, z_j - z_{j-1})\| \leq C_2 h \int_{\Omega} |\nabla z_{j-1}|| \nabla (z_j - z_{j-1})| \, dx \\
\leq C_2 h \|\nabla z_{j-1}\|_H \| \nabla (z_j - z_{j-1})\|_H \\
\leq C_2 h \|z_{j-1}\|_V \|z_j - z_{j-1}\|_V,
\]

which implies that

\[
- \langle (b_{j-1} - b_j) z_{j-1}, z_j - z_{j-1} \rangle \geq -C_2 h \|z_{j-1}\|_V \|z_j - z_{j-1}\|_V. \quad (8.24)
\]

Thus, by using again (A1) of Assumption 8.1, (8.24) and (8.19) we can estimate the second term on the left hand side of (8.21) as follows:

\[
\begin{align*}
&\langle b_jz_j - b_{j-1}z_{j-1}, z_j - z_{j-1} \rangle \\
&\quad = \langle b_j(z_j - z_{j-1}), z_j - z_{j-1} \rangle - \langle (b_{j-1} - b_j)z_{j-1}, z_j - z_{j-1} \rangle \\
&\quad \geq C_1 \|z_j - z_{j-1}\|_V^2 - C_2 h \|z_{j-1}\|_V \|z_j - z_{j-1}\|_V \\
&\quad = C_1 \left[ \|z_j - z_{j-1}\|_V^2 - \frac{C_2 h}{2C_1} \|z_{j-1}\|_V \right]^2 - C_1 \left[ \frac{C_2 h}{2C_1} \|z_{j-1}\|_V \right]^2 \\
&\quad \geq - \frac{C_2^2 C_3^2}{4C_1^2} h^2. \\
\end{align*}
\]

Moreover, for the right hand side of (8.21) we use the Schwarz inequality,
(A2) in Assumption 8.1, and elementary inequalities to find that
\[
(f_j(z_j-1) - f_{j-1}(z_{j-2}), z_j - z_{j-1}) \\
\leq \|f_j(z_j-1) - f_{j-1}(z_{j-2})\|_H \|z_j - z_{j-1}\|_H \\
\leq C_2(h + \|z_{j-1} - z_{j-2}\|_H) \|z_j - z_{j-1}\|_H \\
\leq C_2 h^2 + C_2 \|z_{j-1} - z_{j-2}\|_H^2 + \frac{C_2}{2} \|z_j - z_{j-1}\|_H^2.
\] (8.26)

By using (A1) of Assumption 8.1, we see that
\[
(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \\
= \int_\Omega (a_{j-1}(z_{j-1} - z_{j-2})(z_{j-1} - z_{j-2})dx \\
\geq C_1 \int_\Omega |z_{j-1} - z_{j-2}|^2 \, dx = C_1 \|z_{j-1} - z_{j-2}\|_H^2,
\]
i.e. that
\[
\|z_{j-1} - z_{j-2}\|_H^2 \leq \frac{1}{C_1}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}).
\] (8.27)

Similarly
\[
\|z_j - z_{j-1}\|_H^2 \leq \frac{1}{C_1}(a_j(z_j - z_{j-1}), z_j - z_{j-1}).
\] (8.28)

Inserting (8.28) and (8.27) into (8.26) we see that
\[
(f_j(z_j-1) - f_{j-1}(z_{j-2}), z_j - z_{j-1}) \\
\leq C_2 h^2 + C_2 \frac{a_j(z_{j-1} - z_{j-2})}{C_1} z_{j-1} - z_{j-2} \\
+ \frac{C_2}{2C_1} \frac{C_2}{2C_1} (a_j(z_j - z_{j-1}), z_j - z_{j-1}).
\] (8.29)

Now, according to (8.22), (8.25), (8.29) and (8.21) we get that
\[
\frac{1}{2h}(1 - \frac{C_2}{C_1} h)(a_j(z_j - z_{j-1}), z_j - z_{j-1}) \\
- \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) - \frac{C_2^2}{4C_1} h^2
\leq C_2 h^2 + C_2 \frac{a_j(z_{j-1} - z_{j-2})}{C_1} z_{j-1} - z_{j-2} \\
+ \frac{C_2}{2C_1} (a_j(z_j - z_{j-1}), z_j - z_{j-1}).
\] (8.30)
Homogenization of linear parabolic problems by the 165 method of Rothe and two-scale convergence

If we denote \( \alpha_j = \frac{1}{h^2}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) \) and insert into (8.30) we find that

\[
(1 - \frac{C_2}{C_1}h)\alpha_j - \alpha_{j-1} - \frac{hC_2^3C_3^2}{2C_1} \leq 2C_2h + \frac{2C_2h}{C_1}\alpha_{j-1} + \frac{C_2h}{C_1}\alpha_j,
\]

i.e.,

\[
\alpha_j - \alpha_{j-1} \leq h \left[ \left( \frac{C_2^3C_3^2}{2C_1} + 2C_2 \right) + \frac{2C_2}{C_1} (\alpha_j + \alpha_{j-1}) \right] \quad (8.31)
\]

where \( C < \infty \) is chosen such that

\[
\frac{C_2^3C_3^2}{2C_1} + 2C_2 \leq C \quad \text{and} \quad \frac{2C_2}{C_1} \leq C.
\]

Simplifying further we see that the last estimate takes the form

\[
\alpha_j \leq 1 + \frac{Ch}{1 - Ch} \alpha_{j-1} + \frac{Ch}{1 - Ch}.
\]

By using this estimate repeatedly and another elementary estimate we get that

\[
\alpha_j \leq \left( \frac{1 + Ch}{1 - Ch} \right)^{j-1} \alpha_1 + \left( \frac{1 + Ch}{1 - Ch} \right)^{j-1} \quad (8.32)
\]

Without loss of generality it can be supposed that \( h \) is less than \( \frac{1}{2C} \), which enables us to make the following estimate:

\[
\frac{1 + Ch}{1 - Ch} = e^{(j-1)ln(1 + \frac{2Ch}{1+h})} \leq e \frac{2C^2(j-1)h}{1+h} \leq e^{\frac{2C^2}{1+h}h} \leq e^{4CT}. \quad (8.33)
\]

Our next goal is to estimate \( \alpha_1 \) in a similar way. First we rewrite the identity (8.12) for \( j = 1 \) and put \( v = z_1 - z_0 \), i.e.

\[
\frac{1}{h}(a_1(z_1 - z_0), z_1 - z_0) + \langle b_1 z_1, z_1 - z_0 \rangle = (f_1(z_0), z_1 - z_0). \quad (8.34)
\]

According to (8.23) for \( j = 1 \) we have that

\[
\langle (b_1 - b_0) z_0, z_1 - z_0 \rangle \geq -C_2 h \| z_0 \|_V \| z_1 - z_0 \|_V. \quad (8.35)
\]

Moreover, by using Green’s formula and the Schwarz inequality we have that

\[
|\langle b_0 z_0, z_1 - z_0 \rangle| = \left| \int_{\Omega} b_0 \nabla z_0 \cdot \nabla (z_1 - z_0) \, dx \right| \quad (8.36)
\]
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method

\[
\begin{align*}
&= \left| - \int_{\Omega} \nabla \cdot (b_0 \nabla z_0) (z_1 - z_0) \, dx \right| \\
&\leq \| \nabla \cdot (b_0 \nabla z_0) \|_H \|z_1 - z_0\|_H \\
&\leq h \| \nabla \cdot (b_0 \nabla z_0) \|_H \frac{\|z_1 - z_0\|_H}{h}.
\end{align*}
\]

Also from (8.28) we see that

\[
\frac{C_1}{h^2} \| z_j - z_{j-1} \|_H^2 \leq \frac{1}{h^2} (a_j (z_j - z_{j-1}), z_j - z_{j-1}) = \alpha_j,
\]

so that, in particular for \( j = 1 \),

\[
\left\| \frac{z_1 - z_0}{h} \right\|_H \leq \sqrt{\frac{\alpha_1}{C_1}}, \tag{8.37}
\]

Also according to (8.19) and Assumption 8.1 we have that

\[
\| z_j \|_V \leq C_3 \quad \text{for all } j = 0, 1, \ldots, n, \tag{8.38}
\]

so that, in particular,

\[
\| z_1 - z_0 \|_V \leq \| z_1 \|_V + \| z_0 \|_V \leq 2C_3. \tag{8.39}
\]

Thus, by using (8.35) - (8.39), we can estimate the second term on the left hand side of (8.34) in the following way:

\[
(b_1 z_1, z_1 - z_0) = \langle b_1 (z_1 - z_0), z_1 - z_0 \rangle + \langle b_1 z_0, z_1 - z_0 \rangle \geq \langle b_1 z_0, z_1 - z_0 \rangle \\
= \langle (b_1 - b_0) z_0, z_1 - z_0 \rangle + \langle b_0 z_0, z_1 - z_0 \rangle \\
\geq -C_2 h \| z_0 \|_V \| z_1 - z_0 \|_V - h \| \nabla \cdot (b_0 \nabla z_0) \|_H \| z_1 - z_0 \|_H \\
\geq -2C_2 C_3^2 h - \frac{C_4}{C_1^2} \alpha_1 \frac{1}{h}.
\]

By using the Schwarz inequality, (A2) of Assumption 8.1 and (8.37) the right hand side of (8.34) can be estimated as follows:

\[
(f_1 (z_0), z_1 - z_0) \leq \| f_1 (z_0) \|_H \| z_1 - z_0 \|_H \leq \frac{C_5}{C_1^2} \frac{1}{\alpha_1^2} h. \tag{8.41}
\]

Inserting (8.40) and (8.41) into (8.34) we see that

\[
\alpha_1 - 2C_2 C_3^2 - \frac{C_4}{C_1^2} \alpha_1 \frac{1}{h} \leq \frac{C_5}{C_1^2} \frac{1}{\alpha_1^2},
\]

Also from (8.28) we see that

\[
\frac{C_1}{h^2} \| z_j - z_{j-1} \|_H^2 \leq \frac{1}{h^2} (a_j (z_j - z_{j-1}), z_j - z_{j-1}) = \alpha_j,
\]

so that, in particular for \( j = 1 \),

\[
\left\| \frac{z_1 - z_0}{h} \right\|_H \leq \sqrt{\frac{\alpha_1}{C_1}}, \tag{8.37}
\]

Also according to (8.19) and Assumption 8.1 we have that

\[
\| z_j \|_V \leq C_3 \quad \text{for all } j = 0, 1, \ldots, n, \tag{8.38}
\]

so that, in particular,

\[
\| z_1 - z_0 \|_V \leq \| z_1 \|_V + \| z_0 \|_V \leq 2C_3. \tag{8.39}
\]

Thus, by using (8.35) - (8.39), we can estimate the second term on the left hand side of (8.34) in the following way:

\[
(b_1 z_1, z_1 - z_0) = \langle b_1 (z_1 - z_0), z_1 - z_0 \rangle + \langle b_1 z_0, z_1 - z_0 \rangle \geq \langle b_1 z_0, z_1 - z_0 \rangle \\
= \langle (b_1 - b_0) z_0, z_1 - z_0 \rangle + \langle b_0 z_0, z_1 - z_0 \rangle \\
\geq -C_2 h \| z_0 \|_V \| z_1 - z_0 \|_V - h \| \nabla \cdot (b_0 \nabla z_0) \|_H \| z_1 - z_0 \|_H \\
\geq -2C_2 C_3^2 h - \frac{C_4}{C_1^2} \alpha_1 \frac{1}{h}.
\]

By using the Schwarz inequality, (A2) of Assumption 8.1 and (8.37) the right hand side of (8.34) can be estimated as follows:

\[
(f_1 (z_0), z_1 - z_0) \leq \| f_1 (z_0) \|_H \| z_1 - z_0 \|_H \leq \frac{C_5}{C_1^2} \frac{1}{\alpha_1^2} h. \tag{8.41}
\]

Inserting (8.40) and (8.41) into (8.34) we see that

\[
\alpha_1 - 2C_2 C_3^2 - \frac{C_4}{C_1^2} \alpha_1 \frac{1}{h} \leq \frac{C_5}{C_1^2} \frac{1}{\alpha_1^2},
\]
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

i.e.

\[ \alpha_1 \leq C_0 + C_0 \alpha_1^{1/2} \]

which implies that

\[ \alpha_1 \leq 2C_0 + C_0^2 \]

(8.42)

where \( C_0 < \infty \) is chosen so that

\[ 2C_2 C_3^2 \leq C_0 \quad \text{and} \quad \frac{C_4 + C_5}{C_1^{1/2}} \leq C_0. \]

Hence, from (8.32), (8.33) and (8.42) we see that

\[ \alpha_j \leq e^{4CT} (1 + C_0)^2 := C_6 < \infty. \]

Therefore, according to (8.28),

\[ C_1 \left\| \frac{z_j - z_{j-1}}{h} \right\|^2_H \leq \frac{1}{h^2} (a_j(z_j - z_{j-1}), z_j - z_{j-1}) = \alpha_j \leq C_6 < \infty. \]

Thus, it yields that

\[ \left\| \frac{z_j - z_{j-1}}{h} \right\|^2_H \leq C^*. \]

The last estimate proves the uniform boundedness of the derivative of Rothe’s functions, i.e. (see (8.6)):

\[ \max_{t \in [0,T]} \left\| \frac{\partial u_n}{\partial t}(t) \right\|_H = \max_{j=1;2;\ldots;n} \left\| \frac{z_j - z_{j-1}}{h} \right\|^2_H \leq C^*. \]  

(8.43)

Next let us introduce the following sequence

\[ v^{\varepsilon_n}(t) = u_n(t), \quad t \in I, \ n = 1, 2, \ldots, \]

where \( \{\varepsilon_n\}_{n=1}^{\infty} \) is a parameter sequence such that \( \varepsilon_n \to 0 \) is equivalent to that \( n \to \infty \). According to (8.20) and (8.43) it follows that the sequence \( v^{\varepsilon_n}(t) \) satisfies the conditions of Lemma 8.2. Therefore, in particular, we obtain that there exists a function \( v \in C^{0,1}(I,H) \cap L_{\infty}(I,V) \) and up to a subsequence,

a) \( v^{\varepsilon_n}(t) \to v(t) \) in \( V \) for every \( t \in I \),

b) \( v^{\varepsilon_n} \to v \) in \( C(I,H) \),

e) \( \frac{\partial v^{\varepsilon_n}}{\partial t}(t) \to \frac{\partial v}{\partial t}(t) \) for every \( t \in I \).

From this and from the definition of \( v^{\varepsilon_n} \) it follows that

a) \( u_n(t) \to u(t) = v(t) \) in \( V \) for every \( t \in I \),
b) $u_n \to u$ in $C(I,H)$,

c) $\frac{\partial u_n}{\partial t}(t) \to \frac{\partial u}{\partial t}(t)$ for every $t \in I$.

The statements a) and b) are obvious. To obtain c) we use the definitions of weak and two scale convergence.

According to (8.20), (8.43) and a) there exist $u \in C^{0,1}(I,H) \cap L_{\infty}(I,V)$ with a time derivative $\frac{\partial u}{\partial t} \in L_{\infty}(I,H)$ which are also bounded by these constants. Moreover, since Rothe’s sequence is uniformly convergent we obtain that $u(0) = u_0$. This implies the correctness of the first two properties of the theorem.

Now we notice that all considerations above have been done for a fixed $\varepsilon$, which implies that the obtained limit function $u(t)$ also depends of $\varepsilon$. Thus, we will in the sequel use the notation $u^\varepsilon(t)$ instead of $u(t)$.

Now we will prove that the function $u^\varepsilon$ also has the third property of the theorem, i.e. that the integral identity (8.11) holds. To this end we introduce the following step functions $\bar{u}_n$, $\bar{a}_n$, and $\bar{b}_n$ defined in $I$ such that

$$z_j = \bar{u}_n(t), \quad a_j = \bar{a}_n(t), \quad b_j = \bar{b}_n(t)$$

and

$$\bar{f}_n(t, \cdot) = f_j(\cdot)$$

for $t \in I_j$, $j = 1, 2, 3, \ldots, n$, and we rewrite the integral identity (8.12) as

$$(\bar{a}_n(t) \frac{\partial u_n(t)}{\partial t}, v(t)) + \langle \bar{b}_n(t) \bar{u}_n(t), v(t) \rangle = (\bar{f}_n(t, \bar{u}_n(t-h)), v(t)), \quad (8.44)$$

where $v \in L_{\infty}(I,V)$. In view of a), b), c) above and Assumption 8.1 we get that

$$(\bar{a}_n(t) \frac{\partial u_n(t)}{\partial t}, v(t)) \to (a \varepsilon(t) \frac{\partial u^\varepsilon(t)}{\partial t}, v(t)),$$

$$(\bar{b}_n(t) \bar{u}_n(t), v(t)) \to (b \varepsilon(t) u^\varepsilon(t), v(t)),$$

$$(\bar{f}_n(t, \bar{u}_n(t-h)), v(t)) \to (f(t, u^\varepsilon(t)), v(t)),$$

as $n \to \infty$, for each fixed $\varepsilon$ and almost all $t \in I$, since $\|\bar{a}_n(t) - a(t)\|_{L_{\infty}(\Omega)} \to 0$, $\|\bar{b}_n(t) - b(t)\|_{L_{\infty}(\Omega)} \to 0$ and $\|\bar{f}_n(t, \bar{u}_n(t-h)) - f(t, u(t))\|_H \to 0$ as $n \to \infty$.

Moreover, to get the foregoing limits (8.45) we use also that $\bar{u}_n(t-h) \to u(t)$ in $H$ which follows from the estimate

$$\|\bar{u}_n(t-h) - u_n(t)\|_H = \|u_n(t_{j-1}) - u_n(t)\|_H$$

$$= \int_{t_{j-1}}^t \frac{\partial u_n(\tau)}{\partial t} \, d\tau \leq \int_{t-h}^t \frac{\partial u_n(\tau)}{\partial t} \, d\tau \leq Ch$$
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

for \( t \in \tilde{I}_j = (t_{j-1}, t_j], \ j = 1, 2, \ldots, n \). Taking the limit on both sides of equality (8.44) we obtain the following identity

\[
(a_\varepsilon(t) \frac{\partial u_\varepsilon}{\partial t}(t), v(t)) + \langle b_\varepsilon(t)u_\varepsilon(t), v(t) \rangle = \langle f_\varepsilon(t, u_\varepsilon(t)), v(t) \rangle,
\]

which is the same as

\[
\int_{\Omega} a(x, x/\varepsilon, t) \frac{\partial u_\varepsilon}{\partial t} (x, t) v(x)dx + \int_{\Omega} b(x, x/\varepsilon, t) \nabla u_\varepsilon(x, t) \cdot \nabla v(x)dx = \int_{\Omega} f(x, x/\varepsilon, t, u_\varepsilon(x, t)) v(x)dx
\]

(8.46)

for all \( v \in V = W^{1,2}_0(\Omega) \) and almost all \( t \in I \). This shows that the function \( u_\varepsilon \) satisfies the integral identity (8.11), and Theorem 8.2 is proved.

**Proof of Theorem 8.7.** (Existence) (a) We will use Lemma 8.1 and Lemma 8.2, which involves the notion of two-scale convergence, to obtain the homogenized equation corresponding to problem (8.1). We note that by Theorem 8.2 and Lemma 8.2 there exists a certain \( u \in C^{0,1}(I, H) \cap L_\infty(I, V) \) with a time derivative \( \frac{\partial u}{\partial t} \in L_\infty(I, H) \) and a certain \( \tilde{u} \in L_\infty(I, L_2(\Omega, W^{1,2}_\text{per}(Y))) \) attained as limits of \( u_\varepsilon \) and \( \frac{\partial u_\varepsilon}{\partial t} \) in the sense of Lemma 8.2. It remains to prove that these limits satisfy the weak formulation (8.7) of the theorem.

Let us choose an arbitrary \( v \in V \) and introduce

\[
\omega_\varepsilon^{\varepsilon}(t) := \omega_\varepsilon(x, t) = a(x, x/\varepsilon, t)v(x),
\]

and

\[
\omega(t) := \omega(x, y, t) = a(x, y, t)v(x).
\]

Evidently \( \omega_\varepsilon^{\varepsilon}(t) \xrightarrow{\varepsilon \to 0} \omega(t) \). By using assertion e) of Lemma 8.2 and Lemma 8.1, we find that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} a(x, x/\varepsilon, t) \frac{\partial u_\varepsilon}{\partial t}(x, t) v(x)dx = \lim_{\varepsilon \to 0} \int_{\Omega} \omega_\varepsilon(x, t) \frac{\partial u_\varepsilon}{\partial t}(x, t)dx
\]

\[
= \int_Y \int_{\Omega} \omega_\varepsilon(x, t) \frac{\partial u_\varepsilon}{\partial t}(x, t)v(x)dydx = \int_Y \int_{\Omega} a(x, y, t) \frac{\partial u}{\partial t}(x, t)v(x)dydx.
\]

This shows that the first integral in (8.46) tends to the corresponding one in (8.7) as \( \varepsilon \to 0 \). Next we evaluate the limit of the right hand side of (8.46) as
$\varepsilon \to 0$ as follows:

$$
\lim_{\varepsilon \to 0} \int_{\Omega} f(x, x/\varepsilon, t, u^\varepsilon(x, t)) v(x) dx \\
= \int_{\Omega} \int_{Y} f(x, y, t, u(x, t)) v(x) dy dx \\
- \lim_{\varepsilon \to 0} \int_{\Omega} (f(x, x/\varepsilon, t, u(x, t)) - f(x, x/\varepsilon, t, u^\varepsilon(x, t))) v(x) dx \\
- \lim_{\varepsilon \to 0} \int_{\Omega} \left( \int_{Y} f(x, y, t, u(x, t)) dy - f(x, x/\varepsilon, t, u(x, t)) \right) v(x) dx;
$$

here the last two integrals converge to zero as $\varepsilon \to 0$. The second integral on the right hand side converges to zero, since $f$ satisfies (A2) of Assumption 8.1 and the sequence $u^\varepsilon(t)$ converges strongly to $u(t)$ in $H$. The convergence to zero of the third integral on the right hand side follows from the definition of two-scale convergence. It also holds that

$$
\lim_{\varepsilon \to 0} \int_{\Omega} b(x, x/\varepsilon, t) \nabla u^\varepsilon(x, t) \cdot \nabla v(x) dx
$$

(8.49)

This statement holds according to Lemma 8.1, since $b_\varepsilon(t) \rightharpoonup b(t)$ which follows from Definitions 8.1 and 8.2, and $\nabla u^\varepsilon(t) \rightharpoonup \nabla u(t) + \nabla_Y \tilde{u}(t)$, which follows from the assertion d) of Lemma 8.2.

By combining (8.46)-(8.49) we find that the function $u$ satisfies the following equality:

$$
\int_{\Omega} \tilde{a}(x, t) \frac{\partial u}{\partial t}(x, t) v(x) dx \\
+ \int_{\Omega} \int_{Y} b(x, y, t)(\nabla u(x, t) + \nabla_Y \tilde{u}(x, y, t)) \cdot \nabla v(x) dy dx
$$

(8.50)

and (8.50) coincides with (8.7) so we are done.

(b) Let us now choose for the test function $v$ in (8.11) (i.e. (8.46)) the function $\varepsilon \psi(x)v(x/\varepsilon)$, where $\psi \in C_0^\infty(\Omega)$ and $v \in C^\infty_{\text{per}}(Y)$. Then we get that

$$
\varepsilon \int_{\Omega} a(x, x/\varepsilon, t) \frac{\partial u^\varepsilon}{\partial t}(x, t) \psi(x)v(x/\varepsilon) dx
$$
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

\[ + \int_{\Omega} (b(x, x/\varepsilon, t) \nabla u^\varepsilon(x, t)) \cdot \nabla [\varepsilon \psi(x) v(x/\varepsilon)] \, dx \]
\[ = \varepsilon \int_{\Omega} f(x, x/\varepsilon, t, u^\varepsilon(x, t)) \psi(x) v(x/\varepsilon) \, dx. \]

Simplifying we see that

\[ \varepsilon \int_{\Omega} a(x, x/\varepsilon, t) \frac{\partial u^\varepsilon}{\partial t}(x, t) \psi(x) v(x/\varepsilon) \, dx \]
\[ + \int_{\Omega} b(x, x/\varepsilon, t) \nabla u^\varepsilon(x, t) \cdot [\psi(x) \nabla v(x/\varepsilon)] \, dx \]
\[ + \varepsilon \int_{\Omega} (b(x, x/\varepsilon, t) \nabla u^\varepsilon(x, t)) \cdot [\nabla \psi(x) v(x/\varepsilon)] \, dx \]
\[ = \varepsilon \int_{\Omega} f(x, x/\varepsilon, t, u^\varepsilon(x, t)) \psi(x) v(x/\varepsilon) \, dx. \]

As \( \varepsilon \to 0 \) we find that

\[ \int_{\Omega} \int_{Y} [b(x, y, t) (\nabla u(x, t) + \nabla_Y \tilde{u}(x, y, t))] \cdot [\psi(x) \nabla v(y)] \, dy \, dx = 0. \]

Since \( \psi \in C^\infty_0(\Omega) \) is arbitrary, we have that (for a.e. \( x \)) \( \tilde{u}(x, y, t) \) is the unique solution of the following periodic problem: Find \( \tilde{u} \in L_{\infty} \left( I, L_2 \left( \Omega, W^{1,2}_{\text{per}}(Y) \right) \right) \)

such that

\[ \int_{Y} [b(x, y, t) (\nabla u(x, t) + \nabla_Y \tilde{u}(x, y, t))] \cdot \nabla v(y) \, dy = 0 \]

for almost all \( x \in \Omega \). Rearranging we find that

\[ \int_{Y} [b(x, y, t) \nabla_Y \tilde{u}(x, y, t)] \cdot \nabla v(y) \, dy = - \int_{Y} [b(x, y, t) \nabla u(x, t)] \cdot \nabla v(y) \, dy \]
\[ = - \int_{Y} b(x, y, t) \frac{\partial u}{\partial x_1} \frac{\partial v(y)}{\partial y_1} \, dy - \int_{Y} b(x, y, t) \frac{\partial u}{\partial x_2} \frac{\partial v(y)}{\partial y_2} \, dy \]
\[ - \int_{Y} b(x, y, t) \frac{\partial u}{\partial x_3} \frac{\partial v(y)}{\partial y_3} \, dy. \]

By linearity

\[ \tilde{u}(x, y, t) = w_1(x, y, t) \frac{\partial u}{\partial x_1} + w_2(x, y, t) \frac{\partial u}{\partial x_2} + w_3(x, y, t) \frac{\partial u}{\partial x_3}, \]
where \( w_i \in L_{\infty} \left( I, L_2 \left( \Omega, W^{1,2}_{per}(Y) \right) \right) \) \((i = 1, 2, 3)\) are the solutions of the following local problems:

\[
\begin{aligned}
\int_Y b(x, y, t) (\nabla_Y w_1 + e_1) \cdot \nabla v(y) dy &= 0, \\
\int_Y b(x, y, t) (\nabla_Y w_2 + e_2) \cdot \nabla v(y) dy &= 0, \\
\int_Y b(x, y, t) (\nabla_Y w_3 + e_3) \cdot \nabla v(y) dy &= 0.
\end{aligned}
\] (8.53)

Finally, to obtain the homogenized equation we insert (8.52) into (8.50) to obtain that

\[
\int_{\Omega} \int_Y a(x, y, t) \frac{\partial u}{\partial t} (x, t) v(x) dy dx + \int_{\Omega} \int_Y b(x, y, t) \left( \nabla u(x, t) + \nabla_Y w_1 \frac{\partial u}{\partial x_1} + \nabla_Y w_2 \frac{\partial u}{\partial x_2} + \nabla_Y w_3 \frac{\partial u}{\partial x_3} \right) \cdot \nabla v(x) dy dx = \int_{\Omega} \int_Y f(x, y, t, u(x, t)) v(x) dy dx.
\] (8.54)

Moreover, we note that the second term on the left hand side can be written as

\[
\int_{\Omega} \int_Y b(x, y, t) \left( \frac{\partial u}{\partial x_1} e_1 + \frac{\partial u}{\partial x_2} e_2 + \frac{\partial u}{\partial x_3} e_3 + \nabla_Y w_1 \frac{\partial u}{\partial x_1} + \nabla_Y w_2 \frac{\partial u}{\partial x_2} + \nabla_Y w_3 \frac{\partial u}{\partial x_3} \right) \cdot \nabla v(x) dy dx
\]

\[
= \int_{\Omega} \left\{ \frac{\partial u}{\partial x_1} \left( \int_Y b(x, y, t) (e_1 + \nabla_Y w_1) dy \right) \\
+ \frac{\partial u}{\partial x_2} \left( \int_Y b(x, y, t) (e_2 + \nabla_Y w_2) dy \right) \\
+ \frac{\partial u}{\partial x_3} \left( \int_Y b(x, y, t) (e_3 + \nabla_Y w_3) dy \right) \right\} \cdot \nabla v(x) dx
\]

\[
= \int_{\Omega} \left\{ \frac{\partial u}{\partial x_1} \begin{pmatrix} b_{11}(x, t) \\ b_{21}(x, t) \\ b_{31}(x, t) \end{pmatrix} + \frac{\partial u}{\partial x_2} \begin{pmatrix} b_{12}(x, t) \\ b_{22}(x, t) \\ b_{32}(x, t) \end{pmatrix} \\
+ \frac{\partial u}{\partial x_3} \begin{pmatrix} b_{13}(x, t) \\ b_{23}(x, t) \\ b_{33}(x, t) \end{pmatrix} \right\} \cdot \nabla v(x) dx
\]
Homogenization of linear parabolic problems by the method of Rothe and two-scale convergence

\[ = \int_{\Omega} (B(x,t)\nabla u(x,t)) \cdot \nabla v(x) dx, \quad (8.55) \]

where the matrix \( B(x,t) = (b_{ij}(x,t))_{i,j=1,2,3} \) is defined by

\[
\begin{pmatrix}
  b_{11}(x,t) \\
  b_{21}(x,t) \\
  b_{31}(x,t)
\end{pmatrix}
= \int_Y b(x,y,t) (e_1 + \nabla_Y w_1) dy,
\]

\[
\begin{pmatrix}
  b_{12}(x,t) \\
  b_{22}(x,t) \\
  b_{32}(x,t)
\end{pmatrix}
= \int_Y b(x,y,t) (e_2 + \nabla_Y w_2) dy
\]

and

\[
\begin{pmatrix}
  b_{13}(x,t) \\
  b_{23}(x,t) \\
  b_{33}(x,t)
\end{pmatrix}
= \int_Y b(x,y,t) (e_3 + \nabla_Y w_3) dy,
\]

and (8.9) is proved. By inserting (8.55) into (8.54) we see that

\[
\int_{\Omega} \int_Y a(x,y,t) \frac{\partial u}{\partial t}(x,t) v(x) dydx + \int_{\Omega} (B(x,t)\nabla u(x,t)) \cdot \nabla v(x) dx
= \int_{\Omega} \int_Y f(x,y,t,u(x,t)) v(x) dydx. \quad (8.56)
\]

If we introduce the notations

\[
\tilde{f}(x,t,u(x,t)) = \int_Y f(x,y,t,u(x,t)) dy
\]

and \( \tilde{a}(x,t) = \int_Y a(x,y,t) dy, \)

then (8.56) takes the form

\[
\int_{\Omega} \tilde{a}(x,t) \frac{\partial u}{\partial t}(x,t) v(x) dx + \int_{\Omega} (B(x,t)\nabla u(x,t)) \cdot \nabla v(x) dx
= \int_{\Omega} \tilde{f}(x,t,u(x,t)) v(x) dx, \quad (8.57)
\]

which is the weak form of

\[
\tilde{a}(x,t) \frac{\partial u}{\partial t}(x,t) - \nabla \cdot (B(x,t)\nabla u(x,t)) = \tilde{f}(x,t,u(x,t)).
\]
The proof of the existence of the solution is complete.

(Uniqueness) Assume that \( u^1 \) and \( u^2 \) are solutions of the problem (8.8), i.e. \( u^i \in C^{0,1}(I, H) \cap L_\infty(I, V) \) such that \( u^i(0) = u_0 \) and

\[
\int_\Omega \tilde{a}(x, t) \frac{\partial u^i}{\partial t}(x, t) v(x) dx + \int_\Omega \left( B(x, t) \nabla u^i(x, t) \right) \cdot \nabla v(x) dx = \int_\Omega \tilde{f}(x, t, u^i(x, t)) v(x) dx,
\]

(8.58)

\((i = 1, 2)\). If we denote \( u(t) = u^1(t) - u^2(t) \) then \( u(0) = 0 \). Subtracting the identity (8.58) written for \( i = 2 \) from the same identity written for \( i = 1 \) and choosing \( v = u \) we get that

\[
\int_\Omega \tilde{a}(x, t) \frac{\partial u}{\partial t}(x, t) u(x, t) dx + \int_\Omega \left( B(x, t) \nabla u(x, t) \right) \cdot \nabla u(x, t) dx = \int_\Omega \left[ \tilde{f}(x, t, u^1(x, t)) - \tilde{f}(x, t, u^2(x, t)) \right] u(x, t) dx.
\]

(8.59)

By using the assumption on \( f \), (see (A2) of Assumption 8.1) we estimate the right hand side as

\[
\int_\Omega \left[ \tilde{f}(x, t, u^1(x, t)) - \tilde{f}(x, t, u^2(x, t)) \right] u(x, t) dx \leq C_2 \int_\Omega u(x, t)^2 dx.
\]

From this and from (8.59) we get that

\[
\int_\Omega \tilde{a}(x, t) \frac{\partial u}{\partial t}(x, t) u(x, t) dx + \int_\Omega \left( B(x, t) \nabla u(x, t) \right) \cdot \nabla u(x, t) dx \leq C_2 \int_\Omega (u(x, t))^2 dx.
\]

From the nonnegativity of the second term (which is guaranteed by (A1) of Assumption 8.1) in the left hand side of the last estimate we obtain that

\[
\int_\Omega \tilde{a}(x, t) \frac{\partial u}{\partial t}(x, t) u(x, t) dx \leq C_2 \int_\Omega (u(x, t))^2 dx.
\]

Now we integrate both sides with respect to \( t \) from 0 to \( \tau \), i.e.

\[
\int_0^\tau \int_\Omega \tilde{a}(x, t) \frac{\partial u}{\partial t}(x, t) u(x, t) dx dt \leq C_2 \int_0^\tau \int_\Omega (u(x, t))^2 dx dt. \quad (8.60)
\]
According to (A1) of Assumption 8.1 we can estimate the left hand side of (8.60) as follows:

\[
\int_0^\tau \int_\Omega \tilde{a}(x,t) \frac{\partial u}{\partial t}(x,t) u(x,t) dx \, dt = \frac{1}{2} \int \int_\Omega \tilde{a}(x,t) \frac{\partial (u(x,t)^2)}{\partial t} dt \, dx \\
= \frac{1}{2} \int \int_\Omega \tilde{a}(x,\tau) u(x,\tau)^2 dx - \frac{1}{2} \int \int_\Omega \frac{\partial \tilde{a}(x,t)}{\partial t} u(x,t)^2 dt \, dx \\
\geq \frac{C_1}{2} \int_\Omega u(x,\tau)^2 dx - \frac{C_2}{2} \int_0^\tau \int_\Omega u(x,t)^2 dx \, dt.
\]

From this and from (8.60) we get that

\[
\int_\Omega u(x,\tau)^2 dx \leq \frac{3C_2}{C_1} \int_0^\tau \int_\Omega u(x,t)^2 dx \, dt,
\]

i.e.

\[
\|u(\tau)\|^2_H \leq \frac{3C_2}{C_1} \int_0^\tau \|u(t)\|^2_H dt.
\]

Hence, by applying Lemma 8.3 we get that

\[
u(t) = 0 \quad \text{i.e. that } u^1(t) = u^2(t), \quad \text{for a.e. in } I.
\]

This proves the uniqueness of the solution of the homogenized equation (8.8). From the uniqueness of the solution it follows that not only some subsequence of \(\{u_\varepsilon\}\) converges to the solution, but also the whole sequence converges. The proof is complete. \(\square\)
Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method
Bibliography


Homogenization of Reynolds equations and of some parabolic problems via Rothe’s method


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